Fat tails and copulas: Limits of diversification revisited*

Rustam Ibragimov
Imperial College Business School
and Innopolis University

Jingyuan Mo
NYU Stern School of Business

Artem Prokhorov
The University of Sydney Business School

Abstract

We consider the problem of portfolio risk diversification in a Value-at-Risk framework with heavy-tailed risks and arbitrary dependence captured by a copula function. We use the power law for modelling the tails of risk distributions and investigate whether the benefits of diversification persist when the risks in consideration are allowed to have extremely heavy tails with tail indices less than one and when their copula describes wide classes of dependence structures. We show that for asymptotically large losses with the Eyraud-Farlie-Gumbel-Morgenstern copula, the threshold value of tail indices at which diversification stops being beneficial is the same as for independent losses. We further extend this result to a wider range of dependence structures which can be approximated using power-type copulas and their approximations. This range of dependence structures includes many well known copula families, among which there are comprehensive, Archimedian, asymmetric and tail dependent copulas. In other words, diversification increases Value-at-Risk for tail indices less than one regardless of the nature of dependence between portfolio components within these classes.

JEL Classification: C58, C13, G11

Keywords: Value at risk, Diversification, Heavy-tailed distributions, Dependence, Copulas

*Helpful comments from participants of CFE2014, CFE2015, the 2015 European Meeting of Statisticians, as well as seminar participants at Imperial College London and the University of Queensland are gratefully acknowledged. We thank Fujie Xia, James Diaz and William Liu for excellent research assistance. Rustam Ibragimov gratefully acknowledges partial support by the Russian Ministry of Education and Science (Innopolis University) and the Russian Government Program of Competitive Growth of Kazan Federal University (Higher Institute of Information Technologies and Information Systems). Correspondence to: Rustam Ibragimov, Imperial College Business School, Tanaka Building, South Kensington Campus, London SW7 2AZ, United Kingdom; *Email:* irustam@imperial.ac.uk

1 Introduction

Level-q Value-at-Risk VaR_q ($q > 0$), also known as the level-q quantile of a distribution of losses, is a commonly used risk measure, whose popularity in a wide range of areas in finance is attributed to the recommendations of the Basel Committee on Banking Supervision. A series of recent papers studied the problem of portfolio optimization in the VaR framework, mostly focusing on the situation when the portfolio components are independent and have a heavy tailed distribution (see, e.g., Embrechts et al., 1997, 2009; Ibragimov and Walden, 2011; Ibragimov et al., 2015). An important conclusion from that work is that if tails of risk or return distributions are extremely heavy then diversification increases their portfolio riskiness in terms of its VaR.

This theoretical property of VaR known as non-subadditivity or non-coherence is often weighted against the practical considerations of the ease of calculation and backtesting and smaller data requirements, compared to subadditive risk measures such as Expected Shortfall - ES (see, e.g., Danielsson et al., 2013; Garcia et al., 2007). Further, importantly, while value at risk is defined for any risk distributions, expected shortfall, obviously, is defined only for risks with finite first moments. Moreover, it is well established in today's finance that in practice risks are dependent in some usually unknown fashion and that the behavior of different commonly used risk measures including ES is closely related to the behavior of the tail of sums of dependent risks used in VaR analysis (see, e.g. Alink et al., 2005). Therefore, a better understanding of when VaR is non-subadditive in non-iid settings is key to continued use of VaR as a robust risk measure.

The literature on VaR for independent risks is very wide and has a long tradition (see, e.g., books by Resnick, 1987, Embrechts et al., 1997, Ibragimov et al., 2015, and references therein). Garcia et al. (2007); Ibragimov and Walden (2007); Ibragimov (2009b) focus on frameworks based on i.i.d. stable random variables (r.v.'s) with infinite variance and show that VaR is subadditive provided the mean of risks in consideration is finite. Similar results are also obtained for asymptotically large losses of portfolios of iid risks with general power law distributions. However, extensions to non-independence are more recent and have been limited to specific cases. For example, Ibragimov and Walden (2007, 2011) consider dependence arising from common multiplicative and additive shocks, Embrechts et al. (2009) and Chen et al. (2012) consider Archimedian copulas, Asmussen and Rojas-Nandayapa (2008) consider the normal copula, Albrecher et al. (2006) consider several copula classes permitting explicit solutions such as Archimedian copulas. An interesting result arising from these studies is that the subadditivity property of VaR is generally affected by both the strength of dependence and the tail behavior of the marginals, however in some cases only heavy tails of the marginals matter.

In this paper, we provide several new results on subadditivity of VaR in non-iid settings. The classes of dependence structures we consider are motivated by several widely used copula families and their approximators. We start with the Eyraud-Farlie-Gumbel-Morgenstern (EFGM) copula family and show that for power law risks whose tail exponent is below one, diversification is suboptimal (ie it increases riskiness) regardless of the value of the dependence parameter provided the loss is large enough. We proceed by providing similar results on copulas that can be viewed as generalizations and first or second order approximations of the EFGM copula. This class of copulas, which we call *power-type*, includes the power copulas of Ibragimov (2009a), polynomial copulas of Drouet Mari and Kotz (2001), copulas with cubic sections of Nelsen et al. (1997), as well as a large number of related copulas with various dependence features such as asymmetry, tail-dependence, comprehensiveness, etc.

The paper is organized as follows. Section 2 sets the stage by introducing heavy-tailed power law distributions and discussing results available for independent risks. Section 3 reviews the basics of copulas, introduces the power-type copula class, and presents the main results on limits of diversification for these dependence structures. Section 4 concludes.

2 Diversification under independence

2.1 Heavy tails and power law family

It has become common in financial econometrics to use the tail index of a distribution to measure its tails (see, e.g., Embrechts et al., 1997; Gabaix, 2009; Ibragimov, 2009b; Ibragimov et al., 2015, and references therein). The tail index characterizes the heaviness, or the rate of decay, of the tails of the relevant univariate distribution of a risk or return X, assuming it obeys a power law. The family of distributions obeying the power law of tail decay is known as the *power law family* and the distributional law is usually written as follows:

$$\mathbb{P}(|X| > x) \sim x^{-\alpha}, \tag{1}$$

where α is the tail index, or tail exponent.[1]

Power law distributions permit modelling rates of tail decay that are slower than the exponential decay of the Gaussian distribution, which is important in financial applications. Such distributions often form the basis of a wider class obtained by introducing a slight disturbance to the tail behavior

[1]Here, "$\sim$" means that the left hand side is asymptotically equivalent to a nonzero constant times the right hand side, where asymptotics is as $x \to \infty$.

in the form of a slowly varying function (see, e.g., Embrechts et al., 1997; Ibragimov and Walden, 2008). The tail index α governs the likelihood of observing outliers or large fluctuations of risks or returns in consideration: a smaller tail index means slower rate of decay of tails of risk distributions, which means that the above likelihood is higher. When the tail index is less than two, the tail decay is so slow that the second moment of the underlying risk or return distribution is infinite; when the tail index is less than one, the first moment is infinite. More generally, the power law distributions have the property that absolute moments of X are finite if and only if their order is less than tail index α. That is,

$$\mathbb{E}|X|^p < \infty \quad \text{if} \quad p < \alpha; \quad \mathbb{E}|X|^p = \infty \quad \text{if} \quad p \geq \alpha.$$

A large number of studies in economics, finance and insurance have documented that financial returns and other important financial and economic variables have heavy-tailed distributions with values of α ranging from significantly lower than one to above four (Jansen and Vries, 1991; Loretan and Phillips, 1994; McCulloch, 1997; Rachev and Mittnik, 2000; Gabaix et al., 2006; Chavez-Demoulin et al., 2006; Silverberg and Verspagen, 2007; Ibragimov et al., 2015, and references therein). Many distributions can be viewed as special cases of power laws, at least for asymptotically large losses. This includes Pareto and Student-t distributions as well as Cauchy, Levy and other stable distributions with the index of stability $\alpha < 2$. We will say that a risk has *extremely* heavy tails if $\alpha < 1$, and *moderately* heavy tails if $\alpha > 1$.

2.2 Limits of diversification under heavy tails and independence

Consider a simple problem of optimal portfolio allocation in the VaR framework with possibly extremely heavy tailed risk distributions. Let $w = (w_1, w_2) \in \mathbb{R}^2$ be the portfolio weights such that $w_1, w_2 \geq 0$, $w_1 + w_2 = 1$. Let X_j represent a loss, $j = 1, 2$ which has a power law distribution with the tail index α_j (throughout the paper, we interpret positive values of r.v.'s X as losses). Let $x > 0$. Consider the tail of the aggregate loss distribution $\mathbb{P}(w_1X_1 + w_2X_2 > x)$, where the weighted average loss $w_1X_1 + w_2X_2$ corresponds to a portfolio of two risks with weights w_1 and w_2. Unless one of the weights is zero, the portfolio is diversified. A 5%-VaR of this portfolio is the value of loss x for which that probability is 0.05. More generally, the q% Value-at-Risk of a portfolio risk Z is $\text{VaR}_q(z) = \inf\{z \in \mathbb{R} : \mathbb{P}(Z > z) \leq q\}$, or the $(1-q)$-th quantile of the loss distribution. The problem of interest is to minimize $\text{VaR}_q(w_1X_1 + w_2X_2)$ over the weights w for a given $q \in (0, 1/2)$.

When X_1 and X_2 are i.i.d. with a stable distribution, it is now well understood that, for all non-zero w's, $\mathbb{P}(w_1X_1 + w_2X_2 > x) \leq \mathbb{P}(X_1 > x)$ if $\alpha_j > 1, j = 1, 2$. Equivalently, the VaR of a diversified portfolio $\text{VaR}_q(w_1X_1 + w_2X_2 > x)$ is not greater than that of a not diversified,

$\text{VaR}_q\,(X_1 > x)$, if $\alpha_j > 1$. In order words, diversification helps to lower the VaR for moderately, but not extremely, heavy tailed risks. If $\alpha_j < 1$, then $\mathbb{P}\,(w_1X_1 + w_2X_2 > x) \geq \mathbb{P}\,(X_1 > x)$; that is, for extremely heavy-tailed risks the benefits of diversification disappear and the least risky portfolio has just one risk. For example, if X_j's are i.i.d. Stable with $\alpha = 1/2$, that is if they are Levy distributed, the aggregate loss of an equally weighted portfolio $\frac{X_1+X_2}{2}$ has the same distribution as $2X_1$ and thus $\text{VaR}_q\left(\frac{X_1+X_2}{2}\right) = 2\text{VaR}_q(X_1) > \text{VaR}_q(X_1)$.

Analogous statements hold for portfolios of any size and asymptotically (for infinitely large losses) even if the distribution of X is not stable. Ibragimov (2009b) showed this in a general context with any number of risks and portfolio weights, using majorization theory. Similar results are available for bounded risks concentrated on a sufficiently large interval: for such cases, VaR-based diversification is suboptimal up to a certain number of risks and then becomes optimal (Ibragimov and Walden, 2007).

There is a growing range of applications of these seemingly counterintuitive results in finance, economics and insurance. Ibragimov et al. (2009) demonstrate how this analysis can be used to explain abnormally low levels of reinsurance among insurance providers in markets for catastrophic insurance. Ibragimov et al. (2011) show how to analyze the recent financial crisis as a case of excessive risk sharing between banks when risks are extremely heavy-tailed. Gabaix (2009) and Ibragimov et al. (2015) provide a review of applications of the above conclusions in different areas of economics and finance.

It follows from the two-risks example above that the limits of diversification results hold for i.i.d. losses regardless of the weights w_j used in construction of a diversified portfolio. In what follows we consider an equally weighted portfolio $w_1 = w_2 = 1/2$. To state the results formally, let $(\xi_1(\beta), \xi_2(\beta))$ denote independent random variables from a power-law distribution with a common tail index β. The following theorem can be easily extended to any diversified portfolio of size n, in which case $\left(\frac{\xi_1(\alpha)+\xi_2(\alpha)}{2}\right)$ in inequalities (2)-(3) is replaced with $\sum_{i=1}^{n} w_i\xi_i(\alpha)$.

Theorem 1 *For sufficiently small loss probability q,*

$$VaR_q\Big(\frac{\xi_1(\alpha)+\xi_2(\alpha)}{2}\Big) < VaR_q(\xi_1(\alpha)), \quad \text{if} \quad \alpha > 1 \tag{2}$$

$$VaR_q\Big(\frac{\xi_1(\alpha)+\xi_2(\alpha)}{2}\Big) > VaR_q(\xi_1(\alpha)), \quad \text{if} \quad \alpha < 1 \tag{3}$$

Proof See Appendix for all proofs.

An interesting boundary case corresponds to $\alpha = 1$. This is when diversification has no effect at all, ie it neither increases nor reduces VaR. For example, if ξ's are i.i.d. Stable with $\alpha = 1$, which means they have a Cauchy distribution, one has that $\sum_{i=1}^{n} w_i\xi_i(\alpha)$ has the same distribution as $\xi_i(\alpha)$, so a diversified and a non-diversified portfolios have identical VaRs.

It is not obvious what happens if we relax the independence assumptions. The two extreme cases, corresponding to a comonotone (extreme positive) and countermonotone (extreme negative) dependence between the components do not present a consistent picture. For example, if we consider extreme positive dependence with $\xi_1 = \xi_2$ (a.s.) then, obviously, $\text{VaR}_q(w_1\xi_1(\alpha) + w_2\xi_2(\alpha)) = \text{VaR}_q(\xi_1(\alpha))$ and so diversification has no effect regardless of the tails; while if we have extreme negative dependence with $\xi_1 = -\xi_2$ (a.s.) then $\text{VaR}_q(w_1\xi_1(\alpha) + w_2\xi_2(\alpha)) = (w_1 - w_2)\text{VaR}_q(\xi(\alpha))$ and it is optimal to fully diversify regardless of the tails.

In the next section we use copulas to allow for arbitrary dependence, which includes the two extreme cases in the limit.

3 Diversification under dependence

3.1 Dependence and copulas

Copulas are joint distributions with uniform marginals. They are useful because given the marginal distributions, they represent the dependence in the joint distribution. Specifically, let $H(x_1, \ldots, x_n)$ and $h(x_1, \ldots, x_n)$ denote the joint cdf and density, respectively, of n random variables $(X_1, \ldots, X_n)$ and suppose that the marginal density and cdf of X_j are $f_j(x_j)$ and $F_j(x_j)$ respectively, $j = 1, \ldots, n$. Then, an n-dimensional copula of $(X_1, \ldots, X_n)$ is a function $C : [0,1]^n \to [0,1]$ such that

(a) $C(u_1, \ldots, u_n)$ is increasing in each $u_i, i = 1, \ldots, n$.

(b) $C(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_n) = 0, i = 1, \ldots, n$.

(c) $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i, i = 1, \ldots, n$.

(d) for any $a_j \le b_j, j = 1, \ldots, n,$,

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C(u_{1i_1}, \ldots, u_{ni_n}) \ge 0,$$

where $u_{j1} = a_j$ and $u_{j2} = b_j$ for all $j = 1, \ldots, n$.

(e) $H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$, or, for absolutely continuous copulas with density $c(u_1, \ldots, u_n)$, $h(x_1, \ldots, x_n) = c(F_1(x_1), \ldots, F_n(x_n)) \prod_{i=1}^{n} f(x_i)$.

It is well known that the copula C (the copula density c) is uniquely determined if the univariate cdf's F_j are continuous. The probability integral transforms $u_j = F_j(X_j), j = 1, 2$, are the uniform random variables that form the marginals of C, c. So, equivalently C can be defined as a joint cdf of n random variables, each of which is uniform on $[0, 1]$. The fact that we can model F_j separately from modelling the dependence between F_j's is what makes copulas natural in the analysis of dependent risks with heavy-tailed power-law marginals.

A well known property of the copula function is that it is bounded by the Frechet-Hoeffding bounds, which correspond to joint cdf's with extreme positive and, in the case $n = 2$, extreme negative dependence. For two risks X_1, X_2, let X_1 be a fixed increasing function of X_2, then the copula of (X_1, X_2) can be written as $\min(u_1, u_2)$ and this is the upper bound for bivariate copulas. Now let X_1 be a fixed decreasing function of X_2; then the copula of (X_1, X_2) can be written as $\max(u_1 + u_2 - 1, 0)$. So the two extreme cases when diversification does not have any effect (comonotonicity) and when it is always beneficial (countermonotonicity) regardless of the heavy-tailedness are nested within the copula framework. Joe (1997) and Nelsen (2006) provide excellent introductions to copulas.

If we return to the two-risk example above, we are interested in how the aggregate loss probability for a diversified portfolio compares to that of a single risk. That is, we are interested in the behavior of

$$\begin{aligned}
\mathbb{P}\left(\frac{X_1 + X_2}{2} > x\right) &= \int\int_{\frac{z_1+z_2}{2}>x} f(z_1;\alpha)f(z_2;\alpha)c\left(F(z_1;\alpha), F(z_2;\alpha);\gamma\right) dz_1 dz_2 \\
&= \mathbb{E}\left\{c\left(F(\xi_1;\alpha), F(\xi_2;\alpha);\gamma\right)\mathbb{I}\left[\frac{\xi_1+\xi_2}{2} > x\right]\right\}
\end{aligned}$$

where $c(u_1, u_2; \gamma)$ is a copula density parameterized by γ, $f(\cdot; \alpha)$'s are power-law marginal densities, $\mathbb{I}[\cdot]$ is the indicator function and ξ_j's are independent copies of X_j's. There is no general way to express this in terms of $\mathbb{P}(X_1 > x)$ and whether diversification decreases or increases VaR depends on the copula family as well as on the interaction between α and γ. However, there exist classes of copulas for which we can make explicit comparisons.

3.2 Power-type copulas

We now discuss a class of copula families which will be used in the paper. The class contains copulas that are multiplicative or additive in powers of the margins, or can be approximated using such copulas. We call this class *power-type*. It is similar but more general than the power copula family and than the polynomial copula family which we discuss below.

The most common family in this class is the Eyraud-Farlie-Gumbel-Morgenstern (EFGM) copula family and its generalizations. The bivariate EFGM copula family can be written as follows:

$$C(u_1, u_2) = u_1 u_2 [1 + \gamma(1 - u_1)(1 - u_2)], \tag{4}$$

where $\gamma \in [-1, 1]$, and its density has the form $c(u_1, u_2) = 1 + g(u_1, u_2)$, where $g(u_1, u_2)$ is an expansion by linear functions $1 - 2\,u_j, j = 1, 2$. This is a non-comprehensive copula in the sense that it has a limited range of dependence it can accommodate. For example, Kendall's τ of an EFGM copula is restricted to $\left[-\frac{2}{9}, \frac{2}{9}\right]$.

The multivariate version of the EFGM copula introduced by Cambanis (1977) has the following form:

$$C(u_1, u_2, ..., u_n) = u_1 u_2 ... u_n \Big[1 + \sum_{c=2}^{n} \sum_{1 \le i_1 < i_2 < ... < i_c \le n} \gamma_{i_1, i_2, ..., i_c} (1 - u_{i_1})(1 - u_{i_2})...(1 - u_{i_c})\Big], \tag{5}$$

where $-\infty < \gamma_{i_1, i_2, ..., i_c} < \infty$ are such that $\sum_{c=2}^{n} \sum_{1 \le i_1 < i_2 < ... < i_c \le n} \gamma_{i_1, i_2, ..., i_c} \delta_{i_1} \cdots \delta_{i_c} \ge -1$ for all $\delta_i \in [-1, 1], i = 1, \dots, n$. This copula family can be viewed as a special case of a wider family of n-dimensional power copulas introduced by Ibragimov (2009a).

The power copula family can be written as follows

$$C(u_1, \dots, u_n) = u_1 u_2 \cdots u_n \Big[1 + \sum_{c=2}^{n} \sum_{1 \le i_1 < i_2 < ... < i_c \le n} \gamma_{i_1, i_2, ..., i_c} (u_{i_1}^l - u_{i_1}^{l+1})(u_{i_2}^l - u_{i_2}^{l+1})...(u_{i_c}^l - u_{i_c}^{l+1})\Big], \tag{6}$$

where $\gamma_{i_1, i_2, ..., i_c} \in (-\infty, \infty)$ are such that

$$\sum_{c=2}^{n} \sum_{1 \le i_1 < i_2 < ... < i_c \le n} |\gamma_{i_1, i_2, ..., i_c}| \le 1.$$

This corresponds to using nonlinear rather than linear functions in the expansion of the copula density function.

Another relevant copula family, of which the EFGM copula in (4) is a special case, is known as a polynomial copula family (see, e.g., Drouet Mari and Kotz, 2001, p. 74). An order m ($m \ge 4$) polynomial copula can be written as follows:

$$C(u, v) = uv \left[1 + \sum_{k \ge 1, q \ge 1}^{k+q \le m-2} \gamma_{kq} (u^k - 1)(v^q - 1)\right], \tag{7}$$

where $\gamma_{kq} = \frac{\theta_{kq}}{(k+1)(q+1)}$ and $0 \le \min\left(\sum_{k \ge 1, q \ge 1}^{k+q \le m-2} q\gamma_{kq}, \sum_{k \ge 1, q \ge 1}^{k+q \le m-2} k\gamma_{kq}\right) \le 1$.

One example of this copula family is Nelsen et al.'s (1997) copula with cubic section, which is written as follows

$$C(u, v) = uv + 2\gamma uv(1 - u)(1 - v)(1 + u + v - 2uv), \tag{8}$$

where $\gamma \in [0, \frac{1}{4}]$.

Several other copula families can be written as approximations of the EFGM copula. For example, it is well known that the EFGM copula is a first-order approximation to the Ali-Mikhail-Haq (AMH) copula family. The AMH copula can be written as follows:

$$C(u_1, \ldots, u_n) = (1-\gamma)\left[\prod_{i=1}^{n}\left(\frac{1-\gamma}{u_i}+\gamma\right)-\gamma\right]^{-1},$$

where $\gamma \in [-1, 1]$.

A less known result is that the Plackett and the Frank copula families are first order Taylor approximations of the EFGM copula at independence (see, e.g., Nelsen, 2006, p. 100, 133). The n-variate Frank copula, which is comprehensive, radially symmetric and Archimedian, can be written as follows

$$C(u_1, \ldots, u_n) = \log_\gamma\left[1+\frac{\prod_{i=1}^{n}(\gamma^{u_i}-1)}{(\gamma-1)^{n-1}}\right],$$

where $\gamma \geq 0$.

The n-variate Plackett copula, which is also comprehensive, is rarely discussed in the literature unless $n = 2$, in which case it has the following form:

$$C(u_1, u_2) = \frac{1}{2(\gamma-1)}\left[1+(\gamma-1)(u_1+u_2)-\sqrt{[1+(\theta-1)(u_1+u_2)]^2-4\gamma(\gamma-1)u_1u_2}\right],$$

where $1 \neq \gamma > 0$. However, a way to generalize to $n > 2$ is presented by Molenberghs and Lesaffre (1994). It is also worth mentioning that for all the three copula families, there exist improved second-order approximations (see, e.g., Nelsen, 2006, p. 83).

An interesting set of approximation results are given by Nelsen et al. (1997), Cuadras (2009) and Cuadras and Diaz (2012). Nelsen et al. (1997) provide a generalization of the bivariate EFGM copula using cubic terms as in (8) and show that it can be used to approximate some well-known families of copulas, both symmetric and not, such as the copulas of Kimeldorf and Sampson (1975) and Lin (1987), as well as the Sarmanov copula. They also show that copulas in (8) are second-degree Maclaurin approximations to members of the Frank and Plackett copula families.

Cuadras (2009) studies the power series class of copulas, obtained as weighted geometric means of the EFGM and AMH copulas, and shows that the Gumbel-Barnett and Cuadras-Auge copulas can be expressed as first-order approximations to that class. Cuadras and Diaz (2012) provide approximations of the tail-dependent Clayton-Oakes copula, which also have the form of a power-type generalization of the EFGM copula.

3.3 Diversification under dependence

We start with the bivariate EFGM copula. Let (X_1, X_2) be random variables with the EFGM copula and power-law marginals with the tail index $\alpha > 0$. Then, for $j = 1, 2$,

$$\begin{aligned} F_j(x) &\sim 1 - x^{-\alpha}, \\ f_j(x) &\sim \alpha x^{-\alpha-1}, \text{as } x \to \infty \\ H(x_1, x_2) &= F_1(x_1)F_2(x_2)\big[1 + \gamma(1 - F_1(x_1))(1 - F_2(x_2))\big], \\ h(x_1, x_2) &= f_1(x_1)f_2(x_2)\big[1 + \gamma(1 - 2F_1(x_1))(1 - 2F_2(x_2))\big]. \end{aligned}$$

As before, let $(\xi_1(\alpha), \xi_2(\alpha))$ be *independent* copies of (X_1, X_2), that is, independent random variables that have the same power-law distributions with tail index α as X_1, X_2. Our key insight is that in the tail, the behavior of products and powers of power-law densities and distributions of X_j's is identical to the behavior of their independent copies. This makes it possible to provide asymptotic (with respect to the loss) comparisons between the VaR of the aggregated loss and that of a single risk. More specifically, the crucial component of $\mathbb{P}\left(\frac{X_1+X_2}{2} > x\right)$ under the EFGM copula can be written as follows

$$\begin{aligned} & \int_{\frac{s+t}{2} > x} \alpha^2 s^{-\alpha-1} t^{-\alpha-1} (2s^{-\alpha} - 1)(2t^{-\alpha} - 1) ds dt \\ = \; & 4\alpha^2 \mathbb{P}\left(\frac{\xi_1(2\alpha) + \xi_2(2\alpha)}{2} > z\right) - 2\alpha^2 \mathbb{P}\left(\frac{\xi_1(2\alpha) + \xi_2(\alpha)}{2} > z\right) \\ & -2\alpha^2 \mathbb{P}\left(\frac{\xi_1(\alpha) + \xi_2(2\alpha)}{2} > z\right) + \alpha^2 \mathbb{P}\left(\frac{\xi_1(\alpha) + \xi_2(\alpha)}{2} > z\right), \end{aligned}$$

where the behavior of the individual summands for large z is driven by the lowest tail index of ξ_j in the portfolio.

We formalize this result in the following theorem, which generalizes to n dependent heavy-tailed random variables $X_1, X_2, ..., X_n$ with multivariate EFGM copula given in (5) and power-law marginals.

Theorem 2 *For an asymptotically large $z > 0$, and any $n, \alpha > 0$,*

$$\mathbb{P}\Big(\sum_{i=1}^{n} X_i > zn\Big) \sim \mathbb{P}\Big(\sum_{i=1}^{n} \xi_i(\alpha) > zn\Big). \tag{9}$$

The result suggests that suboptimality of diversification in the VaR framework for extremely heavy tailed losses carries over from independence to the EFGM copula. That is, diversification increases VaR of dependent extremely heavy tailed risks within this copula family. Specifically, combining the results of Theorems 1 and 2, it is easy to see that the following corollary holds.

Corollary 1 *For dependent losses with the EFGM copula and sufficiently small loss probability q,*

$$VaR_q\Big(\frac{X_1+X_2}{2}\Big) < VaR_q(X_1), \quad \textit{if} \quad \alpha > 1 \tag{10}$$

$$VaR_q\Big(\frac{X_1+X_2}{2}\Big) > VaR_q(X_1), \quad \textit{if} \quad \alpha < 1 \tag{11}$$

Another interesting corollary of Theorem 2 can be obtained by combining this result with Theorem 1 of Sharakhmetov and Ibragimov (2002). The EFGM copula family has restrictive dependence, for example, it is not comprehensive in the sense that it cannot accommodate all possible values of Kendall's τ. Yet, as shown by Sharakhmetov and Ibragimov (2002), it can be used to represent *any* joint distribution of two-valued random variables (see also de la Pena et al., 2006, p. 190). Therefore, for two-valued random variables, our Theorem 2 applies to *all* dependence patterns.

Important generalizations of Theorem 2 arise if we consider the wider class of power-type copulas discussed in Section 3.2. Most popular members of this class such as the polynomial copula of Drouet Mari and Kotz (2001), the copula with cubic section of Nelsen et al. (1997) and the power copula of Ibragimov (2009b) can be written in the following general form

$$C(u_1,\dots,u_n) = \sum_{i_1,\dots,i_n=0,1,\dots} \gamma_{i_1,i_2,\dots,i_n} \cdot u_1^{i_1} \cdot u_2^{i_2} \cdot \dots \cdot u_n^{i_n}, \tag{12}$$

for a multiple index $i = (i_1, i_2, \dots, i_n)$ and a set of corresponding parameters γ_i with appropriate restrictions that make $C(u_1,\dots,u_n)$ a copula. For example, Drouet Mari and Kotz (2001, Section 4.5.2) show how to obtain the polynomial copula in (7) from function $f = u^k v^q$. The key feature of such copulas is that they and their densities can be expressed as powers of u_j's. This allows to apply similar arguments as for EFGM.

Theorem 3 *For dependent losses with a power-type copula in (12) and for an asymptotically large $z > 0$, and any $n, \alpha > 0$, the conclusions of Theorem 2 hold.*

One may argue that the class of copulas in (12) is not sufficiently general. For example, it is not clear whether it can incorporate tail dependence or comprehensive copulas. However, the power-type copulas also include copulas which can approximate or be approximated by the class in (12). And, as discussed in Section 3.2, there are comprehensive and tail-dependent copulas among these copulas. Our next corollary establishes the result for such approximations.

Corollary 2 *For dependent losses with copulas whose Taylor or Maclaurin expansions can be written as (12), for an asymptotically large $z > 0$, and any $n, \alpha > 0$, the conclusions of Theorem 2 hold but only locally at the point of approximation.*

This corollary covers all the copula families discussed in Section 3.2 including the AMH, Plackett, Frank, Clayton-Oakes, Kimeldorf and Sampson, Lin, Gumbel-Barnett and many others, but only to the extent the approximations are valid. That is, the results of Theorem 2 hold for expansions at the point at which we expand, which often coincides with independence. Clearly, they do not have to hold when the approximation error is large. Therefore, applicability of Theorem 2 to a specific copula family needs to be checked on a case-by-case basis but the class of copulas to which it can be potentially applied is quite rich – it includes comprehensive copulas (Plackett, Frank), asymmetric copulas (Nelsen et al.'s copulas with cubic sections) and tail-dependent copulas (Clayton-Oakes).

4 Concluding remarks

We have revisited the limits of diversification for dependent risks. The revisit focused on a wide class of copulas that are additive in powers of margins. This class covers several well known families such as EFGM, power and polynomial families but also contains a number of other copula classes which do not have this form but can be approximated using Taylor-type expansions. So the resulting class we consider is very wide – comprehensive, tail-dependent and asymmetric copula families can be considered within this class.

The main result of the paper is that within the class, diversification increases riskiness in a VaR framework if the tail index of the individual risks falls below one. This makes dependent risks within this class no different from independent in the sense that the same threshold value of the tail index delineates the benefits of diversification.

We have looked at equally weighted portfolios with components having the same tail index. The restriction of equal tail indices can easily be relaxed because the tail behavior of the aggregate loss will be dominated by the component with the lowest tail index. The limits of diversification are determined by whether the lowest index in the portfolio is above or below one. A similar result can be obtained for unequally weighted portfolios but we leave both these extensions for future work.

Similar to the analysis of robustness in economic and finance to heavy-tailedness presented in Ibragimov et al. (2015), the conclusions of this paper can be used in the analysis of robustness of key models in these fields to crises, large fluctuations and dependence. Applications of the obtained results in this direction are currently under by way by authors and will be presented elsewhere.

A Proofs

Proof of Theorem 1

This follows from Theorems 4.1 and 4.2 of Ibragimov (2009b)

Proof of Theorem 2

We start with the case $n = 2$. Due to independence between ξ_1 and ξ_2, we have that

$$\mathbb{P}\Big(\frac{\xi_1(\beta_1) + \xi_2(\beta_2)}{2} > z\Big) = \beta_1\beta_2 \int_{\frac{s+t}{2}>z} s^{-\beta_1-1}t^{-\beta_2-1}dsdt. \tag{13}$$

Now for non-independent (X_1, X_2) under the EFGM copula, we can write using (13):

$$\begin{aligned}
\mathbb{P}\Big(\frac{X_1 + X_2}{2} > z\Big) &= \int_{\frac{s+t}{2}>z} f_1(s)f_2(t)\big[1 + \gamma(1 - 2F_1(s))(1 - 2F_2(t))\big]dsdt \\
&= \mathbb{P}\Big(\frac{\xi_1(\alpha) + \xi_2(\alpha)}{2} > z\Big) \\
&\quad +\gamma \int_{\frac{s+t}{2}>z} f_1(s)f_2(t)(1 - 2F_1(s))(1 - 2F_2(t))dsdt \\
&= \mathbb{P}\Big(\frac{\xi_1(\alpha) + \xi_2(\alpha)}{2} > z\Big) \\
&\quad +\gamma\mathbb{E}(1 - 2F_1(\xi))(1 - 2F_2(\eta))I\Big(\frac{\xi_1 + \xi_2}{2} > z\Big),
\end{aligned}$$

where $I(\cdot)$ denotes the indicator function.

Now consider the last term:

$$\begin{aligned}
\int_{\frac{s+t}{2}>z} f_1(s)f_2(t)(1 - 2F_1(s))(1 - 2F_2(t))dsdt &= \int_{\frac{s+t}{2}>z} \alpha^2 s^{-\alpha-1}t^{-\alpha-1}(2s^{-\alpha} - 1)(2t^{-\alpha} - 1)dsdt \\
&= 4\alpha^2 \int_{\frac{s+t}{2}>z} s^{-2\alpha-1}t^{-2\alpha-1}dsdt \\
&\quad -2\alpha^2 \int_{\frac{s+t}{2}>z} s^{-2\alpha-1}t^{-\alpha-1}dsdt \\
&\quad -2\alpha^2 \int_{\frac{s+t}{2}>z} s^{-\alpha-1}t^{-2\alpha-1}dsdt \\
&\quad +\alpha^2 \int_{\frac{s+t}{2}>z} s^{-\alpha-1}t^{-\alpha-1}dsdt \\
&= 4\alpha^2\mathcal{I}_1 - 2\alpha^2\mathcal{I}_2 - 2\alpha^2\mathcal{I}_3 + \alpha^2\mathcal{I}_4,
\end{aligned}$$

where $\mathcal{I}_1 = \mathbb{P}\Big(\frac{\xi_1(2\alpha)+\xi_2(2\alpha)}{2} > z\Big)$, $\mathcal{I}_2 = \mathbb{P}\Big(\frac{\xi_1(2\alpha)+\xi_2(\alpha)}{2} > z\Big)$, $\mathcal{I}_3 = \mathbb{P}\Big(\frac{\xi_1(\alpha)+\xi_2(2\alpha)}{2} > z\Big)$ and $\mathcal{I}_4 = \mathbb{P}\Big(\frac{\xi_1(\alpha)+\xi_2(\alpha)}{2} > z\Big)$.

Thus we obtain

$$
\begin{aligned}
\mathbb{P}\Big(\frac{X+Y}{2} > z\Big) &= (1+\gamma\alpha^2)\mathbb{P}\Big(\frac{\xi_1(\alpha)+\xi_2(\alpha)}{2} > z\Big) \\
&\quad -2\gamma\alpha^2\mathbb{P}\Big(\frac{\xi_1(\alpha)+\xi_2(2\alpha)}{2} > z\Big) \\
&\quad -2\gamma\alpha^2\mathbb{P}\Big(\frac{\xi_1(2\alpha)+\xi_2(\alpha)}{2} > z\Big) \\
&\quad +4\gamma\alpha^2\mathbb{P}\Big(\frac{\xi_1(2\alpha)+\xi_2(2\alpha)}{2} > z\Big).
\end{aligned}
$$

It is a well-known result in the power law literature (see, among others, Corollary 1.3.2 in Embrechts et al., 1997) that, asymptotically as $z \to \infty$,

$$
\mathbb{P}\Big(\frac{\xi_1(\beta)+\xi_2(\beta)}{2} > z\Big) \sim 2\mathbb{P}(\xi_1(\beta) > 2z) \sim 2^{1-\beta}z^{-\beta} \tag{14}
$$

for all $\beta > 0$. In addition, if $\beta_1 < \beta_2$, then

$$
\mathbb{P}\Big(\frac{\xi_1(\beta_1)+\xi_2(\beta_2)}{2} > z\Big) \sim \mathbb{P}(\xi_1(\beta_1) > 2z) \sim 2^{-\beta_1}z^{-\beta_1} \tag{15}
$$

It follows from (14)-(15) that, as $z \to \infty$,

$$
\begin{aligned}
\mathbb{P}\Big(\frac{X+Y}{2} > z\Big) &\sim (1+\gamma\alpha^2)2^{1-\alpha}z^{-\alpha} - 2\gamma\alpha^2 2^{1-\alpha}z^{-\alpha} + 4\gamma\alpha^2 2^{1-2\alpha}z^{-2\alpha} \qquad (16)\\
&\sim (1-\gamma\alpha^2)2^{1-\alpha}z^{-\alpha} \\
&\sim \mathbb{P}\Big(\frac{\xi_1(\alpha)+\xi_2(\alpha)}{2} > z\Big).
\end{aligned}
$$

We now provide a generalization for any n. Let $X_1, X_2, ..., X_n$ have a multidimensional EFGM copula

$$
C(u_1, u_2, ..., u_n) = u_1 u_2 ... u_n \Big[1 + \sum_{c=2}^{n} \sum_{1\le i_1 < i_2 < ... < i_c \le n} \gamma_{i_1,i_2,...,i_c}(1-u_{i_1})(1-u_{i_2})...(1-u_{i_c})\Big], \tag{17}
$$

where $\gamma_{i_1,i_2,...,i_c}$ are real constants satisfying certain inequalities that guarantee that (17) represents a proper copula.

Let $X_1, X_2, ..., X_n$ have power law distributions with the same parameter $\alpha > 0$. It follows from (17) that the joint cdf of $X_1, X_2, ..., X_n$ has the form

$$
\begin{aligned}
F(x_1, x_2, ..., x_n) &= F_1(x_1)F_2(x_2)...F_n(x_n) \\
&\quad \times \Big[1 + \sum_{c=2}^{n} \sum_{1\le i_1 < i_2 < ... < i_c \le n} \gamma_{i_1,i_2,...,i_c}(1-F_{i_1}(x_{i_1}))(1-F_{i_2}(x_{i_2}))...(1-F_{i_c}(x_{i_c}))\Big],
\end{aligned}
$$

Let, $\xi_1(\beta_1), \xi_2(\beta_2), ..., \xi_n(\beta_n)$ denote the independent random variables with power law distributions with tail indices $\beta_1, \beta_2, ..., \beta_n$, respectively. That is,

$$\mathbb{P}(\xi_i(\beta_i) > x) = x^{-\beta_i}, \tag{18}$$

$x \geq 1$, $i = 1, 2, ..., n,$. In particular, $\xi_1(\alpha), \xi_2(\alpha), ..., \xi_n(\alpha)$ are independent copies of $X_1, X_2, ..., X_n$.

Then, it follows that

$$\begin{aligned} \mathbb{P}\Big(\sum_{i=1}^n X_i > zn\Big) &= \mathbb{P}\Big(\sum_{i=1}^n \xi_i(\alpha) > zn\Big) \\ &+ \sum_{c=2}^n \sum_{1\leq i_1<i_2<...<i_c\leq n} \gamma_{i_1,i_2,...,i_c} \\ &\times \mathbb{E}\Big[\big(1-2F_{i_1}(\xi_{i_1}(\alpha))\big)\big(1-2F_{i_2}(\xi_{i_2}(\alpha))\big)...\big(1-2F_{i_c}(\xi_{i_c}(\alpha))\big) I\Big(\sum_{i=1}^n \xi_i(\alpha) > zn\Big)\Big]. \end{aligned} \tag{19}$$

Thus, since the random variables $\xi_1(\alpha), \xi(\alpha), ..., \xi_n(\alpha)$ are i.i.d.,

$$\begin{aligned} \mathbb{P}\Big(\sum_{i=1}^n X_i > zn\Big) &= \mathbb{P}\Big(\sum_{i=1}^n \xi_i(\alpha) > zn\Big) \\ &+ \Big(\sum_{c=2}^n \sum_{1\leq i_1<i_2<...<i_c\leq n} \gamma_{i_1,i_2,...,i_c}\Big) \\ &\times \mathbb{E}\Big[\big(1-2F_1(\xi_1(\alpha))\big)\big(1-2F_2(\xi_2(\alpha))\big)...\big(1-2F_c(\xi_c(\alpha))\big) I\Big(\sum_{i=1}^n \xi_i(\alpha) > zn\Big)\Big]. \end{aligned} \tag{20}$$

Now consider the last term

$$\begin{aligned} &\mathbb{E}\Big[\big(1-2F_1(\xi_1(\alpha))\big)\big(1-2F_2(\xi_2(\alpha))\big)...\big(1-2F_c(\xi_c(\alpha))\big) I\Big(\sum_{i=1}^n \xi_i(\alpha) > zn\Big)\Big] \\ &= \sum_{s=0}^c \sum_{1\leq j_1<j_2<...<j_s\leq c} (-1)^{c-s} \int_{\sum_{i=1}^n x_i > zn} \prod_{k\in\{j_1,j_2,...,j_s\}} (2\alpha) x_k^{-2\alpha-1} \\ &\qquad \times \prod_{k\in\{1,2,...,n\}\backslash\{j_1,j_2,...,j_s\}} \alpha x_k^{-\alpha-1} dx_1 dx_2 ... dx_n \\ &= \sum_{s=0}^c \sum_{1\leq j_1<j_2<...<j_s\leq c} (-1)^{c-s} \mathbb{P}\Big(\sum_{k\in\{j_1,j_2,...,j_s\}} \xi_k(2\alpha) + \sum_{k\in\{1,2,...,n\}\backslash\{j_1,j_2,...,j_s\}} \xi_k(\alpha) > z\Big), \end{aligned} \tag{21}$$

where $1 \leq j_1 < j_2 < ... < j_s \leq c$, $s = 0, 1, ..., c$, $c = 2, ..., n$, $(s, c) \neq (n, n)$ (and, thus, $(j_1, j_2, ..., j_c)$ is different from $(1, 2, ..., n)$).

Consequently, for large z, we obtain

$$\mathbb{P}\Big(\sum_{k\in\{j_1,j_2,...,j_s\}} \xi_k(2\alpha) + \sum_{k\in\{1,2,...,n\}\backslash\{j_1,j_2,...,j_s\}} \xi_k(\alpha) > z\Big) \sim \mathbb{P}\Big(\sum_{k\in\{1,2,...,n\}\backslash\{j_1,j_2,...,j_s\}} \xi_k(\alpha) > zn\Big). \tag{22}$$

In addition, by Corollary 1.3.2 of Embrechts et al. (1997), we have, for large $z > 0$,

$$\mathbb{P}\Big(\sum_{k\in\{1,2,...,n\}\setminus\{j_1,j_2,...,j_s\}} \xi_k(\alpha) > z\Big) \sim (n-s)\mathbb{P}(\xi_1(\alpha) > zn) \sim \frac{n-s}{z^\alpha n^\alpha}. \tag{23}$$

So, for for $s = c = n$, $(j_1, j_2, ..., j_n) = (1, 2, ..., n)$,

$$\mathbb{P}\Big(\sum_{k=1}^{n} \xi_k(2\alpha) > zn\Big) \sim n\mathbb{P}(\xi_1(2\alpha) > zn) \sim \frac{n}{z^{2\alpha} n^{2\alpha}}. \tag{24}$$

From (21)-(24) it follows that, with $1 \le j_1 < j_2 < ... < j_s \le c$, $s = 0, 1, ..., c$, $c = 2, ..., n$, $(s, c) \ne (n, n)$,

$$\mathbb{E}\Big[\big(1 - 2F_1(\xi_1(\alpha))\big)\big(1 - 2F_2(\xi_2(\alpha))\big)...\big(1 - 2F_c(\xi_c(\alpha))\big)I\Big(\sum_{i=1}^{n} \xi_i(\alpha) > zn\Big)\Big] \sim$$

$$\sum_{s=0}^{c} \sum_{1\le j_1<j_2<...<j_s\le c} (-1)^{c-s}\frac{n-s}{z^\alpha n^\alpha} = \Big(\sum_{s=0}^{c}(-1)^{c-s}C_c^s\Big)z^{-\alpha}n^{1-\alpha} - \Big(\sum_{s=0}^{c}(-1)^{c-s}sC_c^s\Big)z^{-\alpha}n^{-\alpha}, \tag{25}$$

where $C_c^s = c!/(s!(c-s)!)$ denotes binomial coefficients.

Now, by the well-known identity for binomial coefficients,

$$\sum_{s=0}^{c}(-1)^{c-s}C_c^s = \sum_{s=0}^{c}(-1)^{s}C_c^s = 0, \tag{26}$$

$$\sum_{s=0}^{c}(-1)^{c-s}sC_c^s = c\sum_{s=1}^{c}(-1)^{c-s}C_{c-1}^{s-1} = -c\sum_{s=0}^{c-1}(-1)^{c-1-s}C_{c-1}^{s} = 0. \tag{27}$$

It thus follows that $\mathbb{P}\left(\sum_{i=1}^{n} X_i > zn\right) \sim \mathbb{P}\left(\sum_{i=1}^{n} \xi_i(\alpha) > zn\right)$.

Proof of Corollary 1

The result follows trivially by inversion of the cdfs in Theorem 2 for $n = 2$ and applying equations (2)-(3).

Proof of Theorem 3

The density corresponding to (12) is a polynomial of a lower order, which we write in the following generic form:

$$c(u_1, \dots, u_n) = \sum_{k_1,...,k_n=0,1,...} \phi_{k_1,k_2,...,k_n} \cdot u_1^{k_1} \cdot u_2^{k_2} \cdot \dots \cdot u_n^{k_n}, \tag{28}$$

Then, using arguments similar to Theorem 2,

$$\mathbb{P}\Big(\sum_{i=1}^{n} X_i > zn\Big) = \mathbb{E}\Big[\sum_{k_i\in\{0,1,\dots\}} \phi_{k_1,k_2,\dots,k_n} \times F_1^{k_1}(\xi_1(\alpha))F_2^{k_2}(\xi_2(\alpha))...F_n^{k_n}(\xi_n(\alpha))I\Big(\sum_{i=1}^{n}\xi_i(\alpha) > zn\Big)\Big] \quad (29)$$

$$= \mathbb{P}\Big(\sum_{i=1}^{n}\xi_i(\alpha) > zn\Big) + \mathbb{E}\Big[\sum_{k_i\in\{0,1,\dots\}\setminus\{k_i=0\forall i\}} \phi_{k_1,k_2,\dots,k_n} \times F_1^{k_1}(\xi_1(\alpha))F_2^{k_2}(\xi_2(\alpha))...F_n^{k_n}(\xi_n(\alpha))I\Big(\sum_{i=1}^{n}\xi_i(\alpha) > zn\Big)\Big]. \quad (30)$$

Now consider the last term.

$$\mathbb{E}\Big[\sum_{k_i\in\{0,1,\dots\}\setminus\{k_i=0\forall i\}} \phi_{k_1,k_2,\dots,k_n} F_1^{k_1}(\xi_1(\alpha))F_2^{k_2}(\xi_2(\alpha))...F_n^{k_n}(\xi_n(\alpha))I\Big(\sum_{i=1}^{n}\xi_i(\alpha) > zn\Big)\Big] \quad (31)$$

$$= \int_{\sum_{i=1}^{n} s_i > nz} \sum_{k_i\in\{0,1,\dots\}} \psi_{k_1,k_2,\dots,k_n} s_1^{-\alpha(k_1+1)} s_2^{-\alpha(k_2+1)} \dots s_n^{-\alpha(k_n+1)} ds_1 \dots ds_n$$

$$= \sum_{k_i\in\{0,1,\dots\}} \psi_{k_1,k_2,\dots,k_n} \int_{\sum_{i=1}^{n} s_i > nz} s_1^{-\alpha(k_1+1)} s_2^{-\alpha(k_2+1)} \dots s_n^{-\alpha(k_n+1)} ds_1 \dots ds_n$$

$$= \sum_{k_i\in\{0,1,\dots\}} \psi_{k_1,k_2,\dots,k_n} \mathbb{P}\left(\frac{\xi_1(\alpha(k_1+1)) + \dots + \xi_n(\alpha(k_n+1))}{n} > z\right),$$

where the new coefficients ψ's are different from ϕ's because we have expressed $(1-s_i^{-\alpha})^{k_i}$ in terms of powers of s_i^{α}. Now, using the same arguments as for (14)-(15),

$$\mathbb{P}\Big(\frac{\xi_1(\alpha) + \dots + \xi_n(\alpha)}{n} > z\Big) \sim n\mathbb{P}(\xi_1(\alpha) > nz) \sim n^{1-\alpha} z^{-\alpha}$$

$$\mathbb{P}\left(\frac{\xi_1(\alpha(k_1+1)) + \dots + \xi_n(\alpha(k_n+1))}{n} > z\right) \sim \mathbb{P}(\xi_1(\alpha) > nz) \sim n^{-\alpha} z^{-\alpha},$$

for all $k_i \geq 0$. It thus follows that $\mathbb{P}\left(\sum_{i=1}^{n} X_i > zn\right) \sim \mathbb{P}\left(\sum_{i=1}^{n} \xi_i(\alpha) > zn\right)$.

Proof of Corollary 2

Clearly if a copula has a Taylor expansion of the form (12) in a neighborhood of a point, the validity of Theorem 2 will be limited to that neighborhood.

References

Albrecher, H., S. Asmussen, and D. Kortschak (2006): "Tail asymptotics for the sum of two heavy-tailed dependent risks," *Extremes*, 9, 107–130.

Alink, S., M. Loewe, and M. V. Wuethrich (2005): "Analysis of the Expected Shortfall of Aggregate Dependent Risks," *ASTIN Bulletin*, 35, 25–43.

Asmussen, S. and L. Rojas-Nandayapa (2008): "Asymptotics of sums of lognormal random variables with Gaussian copula," *Statistics & Probability Letters*, 78, 2709 – 2714.

Cambanis, S. (1977): "Some properties and generalizations of multivariate Eyraud-Gumbel-Morgenstern distributions," *Journal of Multivariate Analysis*, 7, 551 – 559.

Chavez-Demoulin, V., P. Embrechts, and J. Neslehova (2006): "Quantitative models for operational risk: Extremes, dependence and aggregation," *Journal of Banking and Finance*, 30, 2635–2658.

Chen, D., T. Mao, X. Pan, and T. Hu (2012): "Extreme value behavior of aggregate dependent risks," *Insurance: Mathematics and Economics*, 50, 99 – 108.

Cuadras, C. (2009): "Constructing copula functions with weighted geometric means," *Journal of Statistical Planning and Inference*, 139, 3766 – 3772, special Issue: The 8th Tartu Conference on Multivariate Statistics; The 6th Conference on Multivariate Distributions with Fixed Marginals.

Cuadras, C. and W. Diaz (2012): "Another generalization of the bivariate FGM distribution with two-dimensional extensions," *Acta et Contationes Universitatis Tartuensis de Mathematica*, 16, 3–12.

Danielsson, J., B. N. Jorgensen, G. Samorodnitsky, M. Sarma, and C. G. de Vries (2013): "Fat tails, VaR and subadditivity," *Journal of Econometrics*, 172, 283 – 291, latest Developments on Heavy-Tailed Distributions.

de la Pena, V. H., R. Ibragimov, and S. Sharakhmetov (2006): *Characterizations of joint distributions, copulas, information, dependence and decoupling, with applications to time series*, Beachwood, Ohio, USA: Institute of Mathematical Statistics, vol. Number 49 of *Lecture Notes–Monograph Series*, 183–209.

Drouet Mari, D. and S. Kotz (2001): *Correlation and Dependence*, World Scientific.

Embrechts, P., C. Klüppelberg, and T. Mikosch (1997): *Modelling Extremal Events for Insurance and Finance*, Berlin: Springer-Verlag.

EMBRECHTS, P., J. NESLEHOVA, AND M. V. WUETHRICH (2009): "Additivity properties for Value-at-Risk under Archimedean dependence and heavy-tailedness," *Insurance: Mathematics and Economics*, 44, 164 – 169.

GABAIX, X. (2009): "Power Laws in Economics and Finance," *Annual Review of Economics*, 1, 255–294.

GABAIX, X., P. GOPIKRISHNAN, V. PLEROU, AND H. E. STANLEY (2006): "Institutional Investors and Stock Market Volatility," *Quarterly Journal of Economics*, 121, 461–504.

GARCIA, R., E. RENAULT, AND G. TSAFACK (2007): "Proper Conditioning for Coherent VaR in Portfolio Management," *Management Science*, 53, 483–494.

IBRAGIMOV, M., R. IBRAGIMOV, AND J. WALDEN (2015): *Heavy-Tailed Distributions and Robustness in Economics and Finance*, Springer, New York, Lecture Notes in Statistics 214.

IBRAGIMOV, R. (2009a): "Copula-Based Characterizations of Higher Order Markov Processes," *Econometric Theory*, 25, 819–846.

——— (2009b): "Portfolio diversification and value at risk under thick-tailedness," *Quantitative Finance*, 9, 565 – 580.

IBRAGIMOV, R., D. JAFFEE, AND J. WALDEN (2009): "Non-diversification traps in markets for catastrophic risk," *Review of Financial Studies*, 22, 959–993.

——— (2011): "Diversification Disasters," *Journal of Financial Economics*, 99, 333–348.

IBRAGIMOV, R. AND J. WALDEN (2007): "The limits of diversification when losses may be large," *Journal of Banking and Finance*, 31, 2551–2569.

——— (2008): "Portfolio diversification under local and moderate deviations from power laws," *Insurance: Mathematics and Economics*, 42, 549–599.

——— (2011): "Value at risk and efficiency under dependence and heavy-tailedness: models with common shocks," *Annals of Finance*, 7, 285–318.

JANSEN, D. W. AND C. G. D. VRIES (1991): "On the Frequency of Large Stock Returns: Putting Booms and Busts into Perspective," *The Review of Economics and Statistics*, 73, 18–24.

JOE, H. (1997): *Multivariate models and dependence concepts*, vol. 73 of *Monographs on Statistics and Applied Probability*, Chapman and Hall.

Kimeldorf, G. and A. Sampson (1975): "Uniform representations of bivariate distributions," *Communications in Statistics*, 4, 617–627.

Lin, G. (1987): "Relationships between two extensions of Farlie-Gumbel-Morgenstern distribution," *Annals of the Institute of Statistical Mathematics*, 39, 129–140.

Loretan, M. and P. C. B. Phillips (1994): "Testing the covariance stationarity of heavy-tailed time series: An overview of the theory with applications to several financial datasets," *Journal of Empirical Finance*, 1, 211–248.

McCulloch, J. H. (1997): "Measuring Tail Thickness to Estimate the Stable Index alpha: A Critique," *Journal of Business and Economic Statistics*, 15, 74–81.

Molenberghs, G. and E. Lesaffre (1994): "Marginal Modeling of Correlated Ordinal Data Using a Multivariate Plackett Distribution," *Journal of the American Statistical Association*, 89, pp. 633–644.

Nelsen, R. B. (2006): *An Introduction to Copulas*, vol. 139 of *Springer Series in Statistics*, Springer, 2 ed.

Nelsen, R. B., J. J. Quesada-Molina, and J. A. Rodriguez-Lallena (1997): "Bivariate copulas with cubic sections," *Journal of Nonparametric Statistics*, 7, 205–220.

Rachev, S. and S. Mittnik (2000): *Stable Paretian Models in Finance*, New York: Wiley.

Resnick, S. (1987): *Extreme values, regular variation, and point processes*, New York: Springer.

Sharakhmetov, S. and R. Ibragimov (2002): "A Characterization of Joint Distribution of Two-Valued Random Variables and Its Applications," *Journal of Multivariate Analysis*, 83, 389 – 408.

Silverberg, G. and B. Verspagen (2007): "The size distribution of innovations revisited: An application of extreme value statistics to citation and value measures of patent significance," *Journal of Econometrics*, 139, 318–339.