

Indeterminacy, misspecification and forecastability: Good luck in bad policy?*

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Abstract

This paper explores the consequences of equilibrium indeterminacy in DSGE models for macroeconomic forecasting. Exploiting U.S. data on both the Great Moderation and the preceding era, we first present evidence that (i) higher (absolute) forecastability obtains in the former rather than the latter period for all models considered, and that (ii) the decline in volatility and persistence captured by a finite-order VAR system across the two samples need not be associated with inferior predictive accuracy relative to univariate predictors. Then, using a small-scale New Keynesian monetary DSGE model as laboratory, we generate artificial datasets under either equilibrium regime and investigate numerically whether (absolute and relative) forecastability is improved in the presence of indeterminacy. It is argued that forecasting under indeterminacy with e.g. unrestricted VAR models entails misspecification issues that are generally more severe than those one typically faces under determinacy. Irrespective of the occurrence of non-fundamental (sunspot) noise, for certain values of the arbitrary parameters governing solution multiplicity, the pseudo out-of-sample VAR-based forecasts of inflation and output growth can outperform simple univariate predictors. For other values of these parameters, by contrast, the opposite occurs. In general, it is not possible to establish a one-to-one mapping between indeterminacy and superior forecastability,

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even when sunspot shocks play no role in generating the data. Overall, our analysis points towards a ‘good luck in bad policy’ explanation of the (relative) higher forecastability of macro-econometric models prior to the Great Moderation period.

Keywords: DSGE, Forecasting, Indeterminacy, Misspecification, VAR system.

J.E.L.: C53, C62, E17

1 Introduction

A recent debate in the (inflation) forecasting literature revolves around the inability of elaborate macro-econometric models to improve on simple univariate predictors, since the onset of the so-called Great Moderation. Contributions in the field include, but are not limited to, Atkeson and Ohanian (2001), Orphanides and van Norden (2005), Faust and Wright (2009), Tulip (2009), Rossi and Sekhposyan (2010), Christoffel *et al.* (2010), Edge and Gurkaynak (2010), Trehan (2015). While shown to be robust across a large variety of models – e.g. activity-based Phillips curves (Stock and Watson, 2007) and factor-augmented autoregressions (D’Agostino *et al.*, 2007) –, this finding has been largely associated with the emergence of weakly persistent inflation dynamics, as mostly dominated by transitory rather than permanent components (e.g. Stock and Watson, 2007).

A few authors have investigated this phenomenon in the context of dynamic stochastic general equilibrium (DSGE) models. Benati and Surico (2008) exploit a small-scale monetary New Keynesian model to explore the extent to which the persistence and predictability of inflation vary with the parameters of the monetary rule, and conclude that a more aggressive policy stance towards inflation caused the decline in inflation predictability. In the same vein, Fujiwara and Hirose (2014) suggest that forecast difficulties in the Great Moderation period can be potentially associated with the occurrence of equilibrium determinacy. More specifically, Fujiwara and Hirose (2014) argue that the active monetary policy behavior (e.g., Clarida *et al.*, 2000; Lubik and Schorfheide, 2004; Boivin and Giannoni, 2006; Castelnuovo and Fanelli, 2015) during the Great Moderation episode insulated the economy from nonfundamental shocks and hence prevented excessive business cycle fluctuations. The resulting reduction of the persistence and volatility of inflation and output turned out to penalize the forecastability of macro-econometric models. Conversely, the superior forecastability documented on the Great Inflation period would be, according to this argument, a by-product of equilibrium indeterminacy induced by the ‘passive’ monetary policy conduct of the Fed.¹

¹Kolasa *et al.* (2012) also examine the relative accuracy of DSGE model-based forecasts vis-à-vis professional ones using real-time data. The authors find the DSGE model to be relatively successful in forecasting the U.S.

Indeterminacy in DSGE models is generically associated with a richer correlation structure – e.g., a stronger degree of endogenous persistence – of resulting equilibrium representations.² At its very core, Fujiwara and Hirose’s (2014) exercise involves generating artificially data for output and inflation from a standard (calibrated) DSGE model, without restricting the parameter space to the determinacy region. That is, the DSGE model is assumed to be the actual data generating process, and then forecasts from an univariate autoregressive forecast model for inflation are computed on simulated time-series under determinacy and indeterminacy. The authors’ conclusions then hinge on the enhanced autocovariance of simulated data under indeterminacy, which result in superior predictive power of the indeterminate version of the underlying DSGE model, provided the degree of uncertainty about sunspots shocks is not too large.³

Against this background, this paper explores the relationship between indeterminacy in monetary DSGE models and (relative) forecastability of potentially misspecified macro-econometric predictors. More specifically, we take an econometric approach and ask two distinct though intimately related questions: first, is indeterminacy *per se* bound to favor data predictability *in absolute terms*? Second, can the declining *relative* forecastability across the two historical periods be unambiguously framed in the context of equilibrium indeterminacy?

Our findings point towards a qualified negative answer to both these questions. In fact, we argue that forecasting inflation and output growth, or any other variables of interest, is a non-trivial exercise when the data generating process belongs to the class of indeterminate equilibria associated with a monetary DSGE model. Appropriate forecasting under indeterminacy requires estimating the arbitrary auxiliary parameters that index solution multiplicity, as well as identifying the sunspot shocks that may characterize the dynamics of the system in addition to fundamental shocks. Making inference on both these features of indeterminate equilibria is a complicate task even when the investigator specifies the correct statistical model for the data, e.g. Lubik and Schorfheide (2003, 2004), Fanelli (2012) and Castelnuovo and Fanelli (2015). More generally, in the presence of multiple solutions, practitioners who forecast inflation and output growth by univariate autoregressive (AR) processes or vector autoregressive (VAR) systems, face misspecification issues that are comparatively more serious than those he/she would face under determinacy. As a consequence, the relative predictive ability of macroeconomic predictors across the two regimes will typically depend on the form/degree of misspecification of the economy, for almost all the variables and horizons under investigation.

²This argument is not new to the macro-econometrics literature, see e.g. Broze and Szafarz (1991), Lubik and Schorfheide (2003, 2004).

³The authors also observe that highly volatile sunspot-driven (nonfundamental) dynamics – which can only arise under indeterminacy – may actually thicken the veil of uncertainty surrounding the forecasting exercise and thereby worsen the relative accuracy of forecasts.

set of forecasting models employed. We make this point clear at the very outset of our investigation through a simple example which reproduces, in spirit, the simulation exercise of Fujiwara and Hirose (2014), and next turn to analyzing the qualitative consequence of indeterminacy for macroeconomic forecasting in more general settings.

To this end, an empirical investigation is first performed on U.S. quarterly data compiled on the time span preceding the Great Moderation, conventionally denoted ‘Great Inflation’ period (1954-1984), as well as on the Great Moderation one (1984-2008). We exploit a three-variate VAR for inflation, output growth (or output gap) and the policy interest rate to forecast the first two variables on both regimes, and then evaluate both *absolute* and *relative* forecast accuracy, using univariate ARs and random walk (RW) predictors as benchmarks. We establish a number of basic facts regarding the indeterminacy-forecastability nexus. First, for all models considered, the absolute accuracy of inflation forecasts – as measured by the root-mean-square errors (RMSEs) of the pseudo out-of-sample forecasts – slightly increases across the samples, reflecting the decline of volatility of macroeconomic variables. Roughly the same occurs for output growth or output gap, albeit with qualifications. We confirm that macroeconomic time series have become more easier to forecast in absolute terms, as already pointed out by Stock and Watson (2007). Second, when the output gap is used in the VAR, the forecast performance *relative* to the parsimonious benchmarks is found to deteriorate along the move from the Great Inflation to the Great Moderation period, confirming Stock and Watson’s (2007) and many other authors findings.

We next attempt to interpret this evidence through the lens of a structural DSGE framework capturing the main stylized facts of the U.S. business cycle. Using the small-scale New Keynesian monetary model investigated in Benati and Surico (2008, 2009) as laboratory, we generate artificial datasets under determinacy and indeterminacy and exploit a reduced form VAR system to assess both *absolute* and *relative* inflation and output growth forecasts’ accuracy in either equilibrium regime. Under determinacy, monetary policy is designed to react aggressively to inflation and prevents self-fulfilling sunspots, while passive policy stance leads to equilibrium nonuniqueness. Our analysis shows that in the presence of indeterminate equilibria, irrespective of the presence of sunspot noise, for certain values of the arbitrary parameters that govern solution multiplicity, the pseudo out-of-sample forecasts of inflation and output growth obtained with finite-order VARs exhibit smaller average RMSEs, compared to the forecasts computed with simple univariate AR(1) or RW predictors. For other values of these parameters, by contrast, the opposite occurs. Hence, a stark asymmetry emerges between our empirical and numerical findings in the presence of indeterminacy, in that stronger (absolute and relative) predictive ability of multivariate models proves to depend quite heavily on the selected equilibrium

in the indeterminate set, which is assumed to produce the data. When the employed time-series models are inherently misspecified with respect to this data generating process, persistent dynamics induced by indeterminate equilibria do not necessarily lead to superior forecastability. In particular, our approach shows that ‘more stable’ environments (determinacy) generally enhance forecastability, while the situation is mixed in dynamically more involved situations. Maintaining that our chosen New Keynesian monetary DSGE model represents a reasonable *prima facie* approximation of the post-WWII U.S. business cycle, we read this asymmetry as pointing to a ‘good luck in bad policy’ explanation of the (*relative*) higher forecastability of macro-econometric models prior to the Great Moderation period.

As mentioned, our paper is conceptually related to Benati and Surico (2008) and Fujiwara and Hirose (2014), yet it differs from these works in several respects. First, Benati and Surico (2008) refer to a DSGE-based measure of inflation predictability, whereas we appeal to reduced form forecasting models and hence employ a standard measure of forecast uncertainty (i.e. the RMSEs). Remarkably, Benati and Surico’s (2008)’s conclusions are based on the idea that more (DSGE-based) persistence leads to superior (DSGE-based) predictability. Our results are a natural complement to this argument, as they help clarify the subtle link between indeterminacy (determinacy) of DSGE equilibria and their dynamic (regime-specific) properties on the one hand, and the predictive ability of possibly misspecified forecast models, on the other hand. Second, while following Fujiwara and Hirose (2014) in not restricting attention to unique equilibrium models, our analysis is broad in scope as we are interested in exploring the role of indeterminacy in favoring multivariate forecast models vis-à-vis univariate predictors of macroeconomic time series. Third, from an operational perspective, we run a more comprehensive simulation experiment that fully exploits the time-series representation of equilibria under indeterminacy. In particular, our findings are not restricted to the selection of a particular solution – the orthogonal one (Lubik and Schorfheide, 2003) – from the indeterminate set. One obstacle to the full investigation of indeterminate equilibria in the multivariate context, has been so far the difficulty in generating the data under the multiple solution hypothesis without constraining too heavily the representation of solutions. Our paper fills this gap. Recently, Farmer *et al.* (2015) have provided a method to solve and estimate indeterminate linear rational expectations models using standard software packages, which might be also used as an alternative.

Finally, it is worth remarking that in this paper we are not concerned with the forecast performance of monetary DSGE models vis-à-vis conventional forecasting tools such as univariate time-series models or naive forecasts. We refer to e.g. Giacomini (2015) for a critical review of how theory-based models like DSGE models perform in forecasting macroeconomic variables, see also Gürkaynak *et al.* (2013). In this respect, we are sympathetic with Wickens’s (2014)

observation that DSGE frameworks would only outperform purely backward-looking time-series forecasting models were the theory-implied cross-equation restrictions their structure places on data to prove empirically valid. Unfortunately, testing the correct specification and the determinacy/indeterminacy of DSGE models before forecasting the variables of interest is not what practitioners typically do.

The remainder of this paper is organized as follows. Section 2 provides the main idea of the paper by discussing connections between equilibrium (in)determinacy, misspecification and forecastability in the context of a simple linear rational expectations model. Section 3 presents some empirical facts on the forecast accuracy of time-series models on U.S. quarterly data over both the Great Inflation and Great Moderation periods. Section 4 introduces the reference small-scale New Keynesian structural model and Section 5 reports its state-space representation under either equilibrium regime. Section 6 summarizes the results of our simulation experiments and interpret these results in light of the empirical facts from Section 3. Section 7 offers a few concluding remarks. A technical Appendix focuses on the derivation of the model's equilibria.

2 Background

Does indeterminacy imply superior forecast performance of conventional time-series models, relative to the case of determinacy? To gain insight into this issue, we introduce a simple univariate linear rational expectations model, already used in Lubik and Schorfheide (2004), Lubik and Surico (2010) and Fujiwara and Hirose (2014) for illustrative purposes.

Let x_t be a (scalar) endogenous variable defined on a properly filtered probability space, whose dynamics are governed by the following forward-looking equation

$$x_t = \frac{1}{\theta} E_t x_{t+1} + \omega_t, \quad \omega_t \sim WN(0, \sigma_\omega^2) \quad (1)$$

where $E_t x_{t+1} := E(x_{t+1} | \mathcal{F}_t)$, \mathcal{F}_t represents the conditioning information set at time t , ω_t is a fundamental shock, and θ is a structural parameter. Initial conditions are kept fixed. As is known, any solution to (1) satisfies the recursive equation:

$$x_t = \theta x_{t-1} - \theta \omega_{t-1} + \eta_t \quad (2)$$

where $\eta_t := x_t - E_{t-1} x_t$ is the endogenous expectation error. When $\theta > 1$ (determinacy), the (locally) unique non-explosive solution is given by

$$x_t = \omega_t \quad (3)$$

implying that x_t follows white noise dynamics. When $\theta < 1$ (indeterminacy),⁴ by contrast, the endogenous forecast error is not constrained by stability requirements, hence any covariance-stationary martingale difference sequence (MDS) η_t will deliver an RE stationary equilibrium of the form (2).

The forecast error η_t can be expressed as a linear combination of the model's fundamental disturbance and a conditionally mean-zero sunspot shock, i.e. $\eta_t = \tilde{M}\omega_t + s_t$ (Lubik and Schorfheide, 2003), where \tilde{M} is an arbitrary parameter unrelated to θ and σ_ω^2 . For simplicity, we assume s_t obeys an \mathcal{F}_t -measurable MDS ($E_t s_{t+1} = 0$), with variance σ_s^2 . The full set of solutions under indeterminacy is described by the following ARMA(1,1)-type process

$$x_t = \theta x_{t-1} + \tilde{M}\omega_t - \theta\omega_{t-1} + s_t \quad (4)$$

Simple inspection of Eq.s (3) and (4) reveals that the content of indeterminacy for the forecaster is essentially twofold. First, dynamic properties of the model's equilibrium are richer under indeterminacy as opposed to the determinate (pure noise) case. For $\tilde{M} \neq 1$ and generic σ_s^2 , Eq. (4) gives rise to a large variety of equilibria.⁵ While inducing a richer lag structure and hence persistence in x_t , indeterminacy also implies serial correlation in the composite error term forcing Eq. (4). As a result, the presence of pure beliefs shocks as well as of an arbitrary response to the fundamental shock have the potential to induce higher volatility in data generated under Eq. (4).⁶ Second, indeterminacy involves richly parameterized - - of possibly infinite autoregressive order - time-series representations of equilibria, which are to be estimated for forecasting purposes. A relevant question is how significant the potential for dynamically misspecified forecasting models is when the data are generated according to Eq. (4), even in sunspot-free environments.

Common intuition suggests that x_t in Eq. (3) is less forecastable (predictable) than x_t in Eq. (4) because the variable's dynamics are purely white noise under determinacy, while it exhibits some persistence under indeterminacy which can potentially be exploited for forecasting purposes. As we will illustrate shortly, forecast accuracy under indeterminacy - both in absolute and *relative* terms - crucially depends on the type/degree of dynamic misspecification of the employed forecast model.

⁴Without loss of generality, we abstract from the random walk case, $\theta = 1$, for our focus is on stable (asymptotically stationary) solutions.

⁵For $\tilde{M} = 1$ and $\sigma_s^2 = 0$, and despite $\theta < 1$, Eq. (4) collapses to a Minimum State Variable (MSV) solution which is observationally equivalent to the determinate equilibrium in Eq. (3).

⁶In principle, volatility in x_t may be further amplified by endogenous expectations revisions which are arbitrarily related to fundamentals, whereas the converse might occur for a suitable choice of the reduced form parameter \tilde{M} . Moreover, different dynamic structures of the underlying model, e.g. those featuring lagged expectations, may allow for serially correlate sunspots to arise in equilibrium (e.g. Sorge, 2012).

To see this, we follow Fujiwara and Hirose (2014) and consider the extreme case of a forecaster endowed with data x_1, x_2, \dots, x_T , and wishing to predict the future path of x_t by using the simple AR(1) model:

$$x_t = \beta x_{t-1} + u_t, \quad u_t \sim WN(0, \sigma_u^2), \quad |\beta| < 1, \quad t = 1, \dots, T. \quad (5)$$

which is not theory-based. That is, the forecaster is not aware of the regime-dependent nature of the underlying data generating process (DGP), and also not concerned with the potential misspecification of the employed forecast model. If Eq. (3) is the true DGP, this amounts to overspecify the response dynamics including an irrelevant predictor, whereas an omitted-variable bias arises when Eq. (4) generates the data, irrespective of whether $\sigma_s^2 = 0$ or $\sigma_s^2 \neq 0$. Hence, our naive forecaster is either failing to impose relevant restrictions on the lag structure of the underlying (forecasting) model, or rather forcing the moving average part of the model's solution into the error process. From the point of view of the estimation of the parameters β and σ_u^2 , the misspecification implied by the AR(1) model is almost irrelevant under determinacy, but can be severe under indeterminacy. Which are the consequences on the forecast performance of the AR(1) model?

We answer this question exploiting the following numerical experiment. First, $N = 1000$ artificial samples of length $T = 500$ are generated from the linear rational expectations model in Eq. (1) under determinacy and indeterminacy, respectively. In the first case, for each of the 1000 simulations we generate 600 synthetic observations from Eq. (3), setting the variance of the fundamental shock to $\sigma_\omega^2 := 0.5$; the first 100 observations are then discarded. In the second case, for each of the 1000 simulations we generate 600 synthetic observations from the ARMA(1,1)-type process in Eq. (4) by calibrating the structural parameters to $\theta := 0.95$ and $\sigma_\omega^2 := 0.5$, and selecting the indeterminacy parameter \tilde{M} from the set $\mathcal{M} := \{1, 1.01, 0.98, 0.85, 0.80, 0.015\}$; with no loss of generality with respect to our argument, the sunspot shock is set to zero ($\sigma_s^2 := 0$). Also in this case, the first 100 observations are discarded. The initial values x_0 and ω_0 are fixed to zero. On each simulated dataset, we use the first $T - P$ observations to estimate the parameters β and σ_u^2 of the AR(1) model through OLS, and the last P observations to evaluate its forecast accuracy. Absolute forecastability is measured by the average (across simulations) RMSEs. The forecast performance of the AR(1) model under indeterminacy *relative* to determinacy is assessed by the ratio of the average (across simulations) RMSEs obtained in the two cases. But as is known, absolute evaluation of forecast performance is not likely to be informative, and the issue of evaluating the forecast performance of a given model is best answered by using relative evaluation methods that use a benchmark. To quantify the extent of the misspecification of the AR(1) forecast model estimated with OLS, we consider a 'theory-based' optimal benchmark, represented by a forecaster who perfectly knows the data generating process, i.e. whether she

is forecasting future paths for x_t under determinacy or indeterminacy, as well as the true values of the parameters, namely σ_ω^2 if the data are generated under determinacy, and θ , σ_ω^2 and \tilde{M} if the data are generated under indeterminacy.⁷ In our simple example, the indeterminate (stable) solution has purely backward dynamics, and the variance of (rational) forecast errors grows with the forecasting horizon at any given date t in which predictions are made. Hence, this benchmark provides a theoretical lower-bound on the forecasting performance of less-than-rational forecasting models, e.g. those that are not in the form of the model’s indeterminate solution. The forecast performance of the AR(1) model *relative* to this ‘theory-based’ benchmark is assessed by the ratio of the average (across simulations) RMSEs obtained in the two regimes.

Results for $T=500$, $P=1$, $P=8$ and $P=16$ are reported in Table 1.⁸ The first column of Table 1 collects the absolute (average) RMSEs obtained with the AR(1) model under determinacy and in the six indeterminacy cases. The second column reports the ratio between the average (across simulations) RMSEs obtained under the indeterminacy scenarios on the average (across simulations) RMSEs obtained under determinacy. Finally, the last column reports the forecast performance of the AR(1) model *relative* to the ‘theory-based’ optimal benchmark.

Focusing on the second and third columns of Table 1, we observe that for any considered forecast evaluation window, indeterminacy does not necessarily imply superior forecastability. The average RMSEs obtained under indeterminacy may be lower or higher than the average RMSEs obtained under determinacy, depending on the values of \tilde{M} . As expected, the forecast performance under the MSV equilibrium is the same as under the determinate solution. But for $\tilde{M}=1.01$ and $\tilde{M}=0.98$, which are values for which the indeterminate equilibrium is relatively ‘close’ to the MSV solution, the performance of the AR(1) model may change (albeit slightly) across the two regimes. For values of \tilde{M} that are substantially far from the value that leads to the MSV solution (e.g. $\tilde{M}=0.015$), the forecast performance can be markedly worse.

The fourth column of Table 1 confirms our conjecture about the misspecification of the AR(1) model with respect to the data generating process in either equilibrium regime. Under determinacy, the misspecification of the AR(1) model bears no consequences on the property of the OLS estimators of β and σ_u^2 , and therefore does not affect its forecast performance *relative* to the benchmark. Indeed, the chosen univariate predictor has essentially the same average (across simulation) RMSEs as the ‘theory-based’ benchmark predictor. The same happens under the

⁷This benchmark forecasting model coincides with the actual law of motion of the economy, and empirical forecasts of future endogenous variables will necessarily coincide with model-consistent ones. In fact, the h -step ahead ‘theory-based’ optimal forecasts will be $E_{T-P}x_{T-P+h} = 0$ under determinacy, and $E_{T-P}x_{T-P+h} = \theta(x_{T-P+h-1} - \omega_{T-P+h-1})$ for $h = 1$ and $E_{T-P}x_{T-P+h} = \theta E_{T-P}x_{T-P+h-1}$ for $h \geq 2$ under indeterminacy.

⁸All computations in this paper have been carried out in OxMetrics 5.0. Codes are available upon request to the authors.

MSV indeterminate solution. The picture changes, by contrast, for the other type of indeterminate equilibria, for which we observe a faltering forecast performance for the misspecified AR(1) predictor.

It may be argued that these findings mainly stem from our choice of a first-order autoregressive predictor as benchmark. To understand whether our argument is weakened in the presence of higher-order autoregressive forecasting models, we replace the AR(1) benchmark with an AR(4) predictor, and explore whether this process may deliver better forecasts under indeterminacy relative to determinacy, because of the higher persistence associated with the former regime. From a pure time-series perspective, when the data generating process belongs to the class of ARMA processes for which a stable AR(∞) representation for x_t exists, the AR(4) model indeed provides a better (truncated) approximation to the stable AR(∞) process than the AR(1). In our simulation exercise, the ARMA processes for which the moving average part is invertible (fundamental) and thus x_t has a stable AR(∞) representation, are those for which the indeterminacy parameter \tilde{M} belongs to the subset $\mathcal{M}^f := \{0.85, 0.80, 0.015\} \subset \mathcal{M}$.⁹ The ratio of the (average) RMSEs computed with the AR(1) and AR(4) predictors is reported in the last column of Table 1. Two main results emerge: first, under indeterminacy and for large values of P , the AR(4) tends to outperform (albeit not markedly) the AR(1) benchmark provided \tilde{M} is in \mathcal{M}^f ; second, there seems to be no clear indication of a higher AR(4)-based forecast accuracy (relative to a first-order autoregression) under indeterminacy vis-à-vis determinacy across all evaluation windows and indeterminate equilibria.

The highly stylized experiment discussed in this section shows that (i) it is not necessarily true that forecastability improves if the data generating process belongs to the class of indeterminate equilibria generated by a linear expectational model like Eq. (1); (ii) the endogenous persistence implied by the indeterminate equilibria in Eq. (4) need not help forecasters predict the future path of x_t , once prediction is conducted with misspecified models; (iii) it is certainly true, as argued by Fujiwara and Hirose (2014), that a very high degree of uncertainty stemming from nonfundamental (sunspot) shocks can weaken the predictive ability of macro-econometric models, yet their forecast accuracy under indeterminacy relative to determinacy can prove to

⁹While determinacy involves a one-to-one mapping between the endogenous variable x_t and the structural shock ω_t , indeterminacy may generate non-invertibility, i.e. Eq. (4) might not be inverted to a (possibly infinite-order) autoregressive representation with one-sided lag polynomial (invertibility in the past). More generally, both determinacy and indeterminacy may be associated with nonfundamentalness even when equilibrium reduced forms are only driven by structural (fundamental) shocks. While non-invertibility may hinder the possibility of fully recovering the shock ω_t from an estimated causal AR model of x_t , this issue is immaterial for forecasting purposes, as the MA representation in Eq. (4) is naturally chosen to be fundamental. This argument fully generalizes to VAR-based forecasting (e.g. Alessi *et al.*, 2011).

be inferior even in sunspot-free environments.

There is no reason to think that this simple example is special. From Section 4 onwards, we show that the argument can be generalized to more realistic model-based forecasting frameworks, and fruitfully used to interpret a few empirical facts, to which we turn next.

3 Empirical facts

To shed light on the role of indeterminacy for data forecastability, we implement a simple empirical exercise by which we explore the forecast performance of a set of conventionally employed time-series models on U.S. data. We consider quarterly data, sample 1954Q4-2008Q2, on three observable variables collected in the vector $y_t := (w_t, \pi_t, R_t)'$, where w_t is explained below, π_t is the inflation rate measured by the quarterly growth rate of the GDP deflator, and R_t is the short-term nominal interest rate proxied as the effective Federal funds rate expressed in quarterly terms (averages of monthly values). We use two different measure for w_t : in one case, $w_t := \Delta o_t$ is the growth rate of output, where o_t is the log of the real GDP; in the other case, $w_t := \tilde{o}_t = o_t - o_t^n$ is potential output, where o_t^n is proxied by the Congressional Budget Office measure.¹⁰

Following most of the literature on the Great Moderation, we divide the post-WWII U.S. era in two periods, roughly corresponding to the Great Inflation and the Great Moderation samples. More specifically, our Great Inflation sample covers the period 1954Q4-1984Q2 ($T = 119$), while our Great Moderation sample covers the period 1985Q1-2008Q2 ($T = 94$). This choice is consistent with e.g. D'Agostino *et al.* (2007) and Castelnuovo and Fanelli (2015).

For either of the two regimes and both measures of w_t , we apply the following models:

- (a) Reduced form VAR systems for the observable variables $y_t := (w_t, \pi_t, R_t)'$;
- (b) univariate AR(1) models for π_t and w_t ;
- (c) univariate RW models for π_t and w_t .

The VAR systems in (a) is our reference time-series model through which we investigate forecastability of inflation and output, while the models in (b) and (c) serve as benchmarks (e.g., Atkeson and Ohanian, 2001). Owing to their flexibility, reduced form VAR frameworks have naturally lent themselves for forecasting since their inception. At the same time, theory-based structural VAR methods have been widely used to identify the driving force(s) behind the Great Moderation, see e.g. Stock and Watson (2002), Primiceri (2005), Gambetti *et al.* (2008),

¹⁰The source of the data is the Federal Reserve Bank of St. Louis' web site. Our choice of the sample's chosen length is due to data availability (in particular, of the effective Federal Funds rate), as well as our intention to avoid dealing with the 'zero-lower bound' phase began in December 2008, which triggered a series of non-standard policy moves by the Federal Reserve whose effects are hardly captured by our standard New Keynesian model.

Benati and Surico (2009). The VAR for $y_t := (\tilde{o}_t, \pi_t, R_t)'$ and the VAR for $y_t := (\Delta o_t, \pi_t, R_t)'$ convey different type of information. While the VAR for $y_t := (\tilde{o}_t, \pi_t, R_t)'$ can be interpreted as reduced form of a system that incorporates a Phillips Curve-type equation, the VAR for $y_t := (\Delta o_t, \pi_t, R_t)'$ is essentially used in our empirical forecast exercise because $y_t := (\Delta o_t, \pi_t, R_t)'$ will play the role of vector of observed variables in the monetary New Keynesian DSGE model we simulate in the next sections.

Our VAR specifications include a constant and the lag order is selected combining the Schwarz's criterion with a LM-type vector test for uncorrelated residuals, considering 1 up to 3 lags. In particular, the selected lag order is the first lag chosen by the Schwarz's criterion such that the hypothesis of uncorrelated VAR disturbances is not rejected. The AR(1) predictor in (b) also includes a constant. On both regimes, we use the first $T - P$ observations to estimate the models in (a) and (b) through OLS, and the last P observations to compute pseudo-out-of-sample forecasts and compute the RMSEs for π_t and w_t . Thus, the *absolute* forecast performance of the VAR systems is measured by the RMSEs. Instead, the VAR forecast performance *relative* to the benchmarks in (b)-(c) is computed by the ratio of the corresponding RMSEs. As a measure of persistence, we employ the (absolute value of the) estimated largest root of the VAR companion matrix for the model in (a), and the estimated autoregressive coefficients for the univariate model in (b). Alternative measures of persistence in the multivariate framework have been recently proposed by Cogley and Sargent (2005) and Cogley *et al.* (2010). While Cogley and Sargent's (2005) measure, based on the normalized spectrum at frequency zero, frames naturally in the frequency-domain approach, Cogley's *et al.* (2010) R^2 -like measure of persistence seems particularly suited for the case of VAR systems with drifting-parameters. Given our time-domain approach and the idea of using fixed-parameters VARs on the two subsamples, the largest root of the VAR companion matrix appears a tenable summary measure of the overall persistence of the variables in $y_t := (w_t, \pi_t, R_t)'$.¹¹

Results for the evaluation window $P = 8$ (eight quarters) are summarized in Table 2a ($w_t := \tilde{o}_t$) and Table 2b ($w_t := \Delta o_t$).

The picture that emerges from Table 2a is the following. First, we notice a substantial reduction of the variance (standard deviation) of the model's residuals across the two regimes (fifth column); focusing on the VAR residuals, the magnitude of the estimated standard deviations of inflation and output gap on the Great Moderation sample is roughly half of the magnitude of the estimated variances of the two aggregates on the Great Inflation sample. A similar reduction

¹¹ See also Koop *et al.* (1996) and Fanelli and Paruolo (2010) for a comprehensive treatment of measures of shock persistence in multivariate models like VARs. Instead, a detailed analysis of the persistence of U.S. inflation may be found in e.g. Pivetta and Reis (2007), where alternative measures of persistence are discussed for univariate models. Fuhrer (2010) also explores the notion of inflation persistence in macroeconomic theory.

of volatility is observed with the estimated AR(1) models. Second, the VAR persistence (sixth column) does not vary too much in the move from the Great Inflation (0.9441) sample to the Great Moderation (0.9295) sample. Which are the consequences on the VAR forecastability? Table 2a shows that in *absolute* terms, the VAR-based RMSEs (second column) associated with inflation and potential output are smaller on the Great Moderation period compared to the Great Inflation period. The RW-based and AR(1)-based *absolute* RMSEs confirm a substantial improvement in the *absolute* forecastability of inflation and output gap. This evidence, with partly supports what already observed in Stock and Watson (2007), suggests that from the point of view of VAR practitioners, the risk of inflation forecasts, as measured by RMSEs, has reduced in the move from the Great Inflation to the Great Moderation period. These results are somehow expected: business cycle fluctuations have become considerably more stable during the Great Moderation period relative to the preceding era, hence generally more predictable by VARs.

In *relative* terms, instead, Table 2a shows that as concerns inflation, the forecast performance of the VAR deteriorates, compared to the two benchmarks, in the move from the Great Inflation period to the Great Moderation regime. This supports the findings of e.g. D’Agostino *et al.* (2007), Stock and Watson (2007) and a number of other authors who have used a bunch of models and econometric techniques. As concerns the output gap, instead, the evidence is not clear-cut since the VAR forecast performance deteriorates *relative* to the RW, but not *relative* to the AR model.

Table 2a provides a ‘static’ picture. To accomplish a more ‘dynamic’ evaluation of the VAR forecast performance for inflation *relative* to the AR benchmark, we replicate our forecast exercise in rolling samples of fixed sizes, to comply with the two sub-samples under scrutiny. The first sample includes 119 observations and coincides with the 1954Q4-1984Q2 Great Inflation period analyzed in the upper panel of Table 2a. The second sample includes 94 observations and coincides with the 1985Q1-2008Q2 Great Moderation period analyzed in the lower panel of Table 2a. Figure 1 plots the ratio of the RMSEs obtained with the VAR and the AR(1) on the rolling samples (blue line), and our measure of VAR persistence (red line). While not meant to test for changes in relative forecastability (e.g. Rossi and Sekhposyan, 2010) across the full time span, this exercise makes it possible to grasp the evolution of the relative predictive performance within each historical regime. Figure 1 shows that, in both sub-samples, the VAR persistence remains substantially constant over time, while the VAR *relative* forecast seems to vary with the rolling sample. However, in average, the VAR forecasts tend to be outperformed by the AR benchmark after the eighties and the performance deteriorates markedly since the beginning of the nineties.

Moving to Table 2*b*, we also notice a substantial reduction of the standard deviations of the model’s residuals across the two regimes. A similar reduction of volatility is observed with the estimated AR(1) models. In this case, our measure of persistence reduces substantially in the move from the Great Inflation (0.9632) to the Great Moderation (0.8545) period. Is in this case the observed decline in volatility and persistence associated with inferior VAR forecastability? The answer seems to be in the negative. Table 2*b* shows that in *absolute* terms, the VAR-based RMSEs associated with inflation and output growth are substantially smaller on the Great Moderation period compared to the Great Inflation period. This also holds for the AR(1)-based and (though less markedly) for RW-based absolute RMSEs. The same is observed for output growth. Again, our results support what already observed in Stock and Watson (2007).

Things are slightly different in Table 2*b* when focusing on *relative* forecast performance. In this case, using a VAR for $y_t := (\Delta o_t, \pi_t, R_t)'$ and the chosen lag order specification rule, we do not observe a deteriorating forecast performance on the Great Moderation period compared to the preceding era. Table 2*b* clearly suggests that the decline in volatility and persistence captured by the VAR system in the move from the Great Inflation sample to the Great Moderation sample, is not associated with inferior *relative* forecast accuracy. The differences in results we observe across the VAR for $y_t := (\tilde{o}_t, \pi_t, R_t)'$ (Table 2*a*) and the VAR for $y_t := (\Delta o_t, \pi_t, R_t)'$ (Table 2*b*) are somewhat expected: \tilde{o}_t and Δo_t capture different (albeit related) concepts. Furthermore, this divergence in result seems not at odds with what reported on e.g. Stock and Watson (2007), as our VAR specification relies on a fairly limited information set, which abstracts away from several series and indicators (e.g. interest spreads, monetary aggregates, unemployment) whose predictive content for output growth and inflation has been largely documented in the forecasting literature.

Overall, the joint empirical evidence summarized in Tables 2*a-2b* (and in Figure 1) seems to support the ‘forecast puzzle’ documented in the literature. In the next sections we investigate whether the empirical facts discussed in this section can be reconciled with *absolute* and *relative* VAR-based forecast accuracy associated with data simulated from a New Keynesian DSGE model.

4 Structural model

We consider Benati and Surico's (2009) small-scale DSGE model, given by the three equations:

$$\tilde{o}_t = \gamma E_t \tilde{o}_{t+1} + (1 - \gamma) \tilde{o}_{t-1} - \delta(R_t - E_t \pi_{t+1}) + \omega_{\tilde{y},t} \quad (6)$$

$$\pi_t = \frac{\beta}{1 + \beta\alpha} E_t \pi_{t+1} + \frac{\alpha}{1 + \beta\alpha} \pi_{t-1} + \kappa \tilde{o}_t + \omega_{\pi,t} \quad (7)$$

$$R_t = \rho R_{t-1} + (1 - \rho)(\varphi_{\pi} \pi_t + \varphi_{\tilde{y}} \tilde{o}_t) + \omega_{R,t} \quad (8)$$

where

$$\omega_{x,t} = \rho_x \omega_{x,t-1} + \varepsilon_{x,t} \quad , \quad -1 < \rho_x < 1 \quad , \quad \varepsilon_{x,t} \sim \text{WN}(0, \sigma_x^2) \quad , \quad x = \tilde{y}, \pi, R \quad (9)$$

and expectations are conditional on the information set \mathcal{F}_t , i.e. $E_t := E(\cdot | \mathcal{F}_t)$. The variables \tilde{o}_t , π_t , and R_t stand for the output gap, inflation, and the nominal interest rate, respectively; γ is the weight of the forward-looking component in the intertemporal IS curve; α is price setters' extent of indexation to past inflation; δ is households' intertemporal elasticity of substitution; κ is the slope of the Phillips curve; ρ , φ_{π} , and $\varphi_{\tilde{y}}$ are the interest rate smoothing coefficient, the long-run coefficient on inflation, and that on the output gap in the monetary policy rule, respectively; finally, $\omega_{\tilde{o},t}$, $\omega_{\pi,t}$ and $\omega_{R,t}$ in eq. (9) are the mutually independent, autoregressive of order one disturbances and $\varepsilon_{\tilde{o},t}$, $\varepsilon_{\pi,t}$ and $\varepsilon_{R,t}$ are the structural (fundamental) shocks.

This or similar small-scale models have successfully been employed to conduct empirical analysis concerning the U.S. economy. Clarida *et al.* (2000), Lubik and Schorfheide (2004) and Castelnuovo and Fanelli (2015) have investigated the influence of systematic monetary policy over the U.S. macroeconomic dynamics; Boivin and Giannoni (2006), Benati and Surico (2009), and Lubik and Surico (2010) have replicated the U.S. Great Moderation, Benati (2008) and Benati and Surico (2009) have investigated the drivers of the U.S. inflation persistence; Castelnuovo and Surico (2010) have replicated the VAR dynamics conditional on a monetary policy shock in different sub-samples.

The output gap in Eq. (6) is defined by $\tilde{o}_t := o_t - o_t^n$, where o_t is output and o_t^n is the natural rate of output. We complete the structural model specification by postulating that o_t^n is driven by a technology shock and follows the Random Walk process

$$o_t^n = o_{t-1}^n + \varepsilon_{o^n,t} \quad , \quad \varepsilon_{o^n,t} \sim \text{WN}(0, \sigma_{o^n}^2). \quad (10)$$

Eq. (10) will be used to define the measurement equation associated with the state-space equilibrium representation of our DSGE model, see the Appendix.

It is worth emphasizing that our choice of the small-scale system (6)-(10) as laboratory need not be associated with a comparatively inferior forecast performance vis-à-vis larger scale models like e.g. the Smets and Wouters's (2007) framework. As argued in Herbst and Schorfheide (2012),

the additional features incorporated into larger models do not involve a uniform improvement in the quality of density forecasts and prediction of co-movements of output, inflation, and interest rates, relative to three-equations systems.

5 Equilibria

We compact the structural system composed by Eq.s (6)-(9) in the general representation

$$\Gamma_0 X_t = \Gamma_f E_t X_{t+1} + \Gamma_b X_{t-1} + \omega_t \quad (11)$$

$$\omega_t = \Xi \omega_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \text{WN}(0, \Sigma_\varepsilon) \quad (12)$$

$$\Xi := dg(\rho_{\tilde{o}}, \rho_\pi, \rho_R) \quad , \quad \Sigma_\varepsilon := dg(\sigma_{\tilde{o}}^2, \sigma_\pi^2, \sigma_R^2)$$

where $X_t := (\tilde{o}_t, \pi_t, R_t)'$, $\omega_t := (\omega_{\tilde{o},t}, \omega_{\pi,t}, \omega_{R,t})'$, $\varepsilon_t := (\varepsilon_{\tilde{o},t}, \varepsilon_{\pi,t}, \varepsilon_{R,t})'$ and

$$\Gamma_0 := \begin{pmatrix} 1 & 0 & \delta \\ -\kappa & 1 & 0 \\ -(1-\rho)\varphi_{\tilde{y}} & -(1-\rho)\varphi_\pi & 1 \end{pmatrix}, \quad \Gamma_f := \begin{pmatrix} \gamma & \delta & 0 \\ 0 & \frac{\beta}{1+\beta\alpha} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_b := \begin{pmatrix} 1-\gamma & 0 & 0 \\ 0 & \frac{\alpha}{1+\beta\alpha} & 0 \\ 0 & 0 & \rho \end{pmatrix}.$$

Let $\theta := (\gamma, \delta, \beta, \alpha, \kappa, \rho, \varphi_{\tilde{o}}, \varphi_\pi, \rho_{\tilde{o}}, \rho_\pi, \rho_R, \sigma_{\tilde{o}}^2, \sigma_\pi^2, \sigma_R^2)'$ be the vector of structural parameters. The elements of the matrices Γ_0 , Γ_f , Γ_b and Ξ depend nonlinearly on θ and, without loss of generality, the matrix $\Gamma_0^\Xi := (\Gamma_0 + \Xi\Gamma_f)$ is assumed to be non-singular. The space of all theoretically admissible values of θ is denoted by \mathcal{P} and X_0 and X_{-1} are fixed initial conditions.

A solution to system (11)-(12) is any stochastic process $\{X_t^*\}_{t=0}^\infty$ such that, for $\theta_s \in \mathcal{P}$, $E_t X_{t+1}^* = E(X_{t+1}^* | \mathcal{F}_t)$ exists and it solves the system (11)-(12) at any t , for fixed initial conditions. A reduced form solution is a member of the solution set whose time-series representation is such that X_t can be expressed as a function of ε_t , lags of X_t and ε_t and, possibly, other arbitrary MDSs with respect to \mathcal{F}_t , possibly independent of ε_t , called ‘sunspot shocks’ (e.g. Broze and Szafarz, 1991).

As is known, determinacy/indeterminacy is a system property that depends on all elements in θ , see Lubik and Schorfheide (2004) and Fanelli (2012). The solution properties of the system of Euler equations (11)-(12) depend on whether θ lies in the determinacy or indeterminacy region of the parameter space. Thus, the theoretically admissible parameter space \mathcal{P}_θ is decomposed into two disjoint subspaces, the determinacy region, \mathcal{P}_θ^D , and its complement $\mathcal{P}_\theta^I := \mathcal{P}_\theta \setminus \mathcal{P}_\theta^D$. We assume that for each $\theta \in \mathcal{P}_\theta$, an asymptotically stationary (stable) reduced form solution to system (11)-(12) exists, hence the case of non-stationary possibly ‘explosive’ (unstable) indeterminacy is automatically ruled out. Since we consider only stationary solutions, \mathcal{P}_θ^I contains only values of θ for which multiple stable solutions arise.

A technical discussion of the equilibria associated with the DSGE system (11)-(12) is reported in the Appendix, where the interested reader is referred to. Next we summarize the state-space representation of the DSGE model in the two scenarios, which are the tools through which the data are generated in our experiments.

Under determinacy, the so-called ABCD form (Fernández-Villaverde *et al.* 2007; Ravenna, 2007) of the determinate equilibrium is given by the system

$$\begin{aligned} x_t^d &= A^d(\theta) x_{t-1}^d + B(\theta) u_t^d \\ 2n \times 1 & \quad 2n \times 2n \quad 2n \times 1 \quad 2n \times (n+b) \quad (n+b) \times 1 \\ y_t &= C(\theta) x_{t-1}^d + D(\theta) u_t^d \\ p \times 1 & \quad p \times 2n \quad 2n \times 1 \quad 2n \times (n+b) \quad (n+b) \times 1 \end{aligned} \quad (13)$$

where $x_t^d := (X_t', X_{t-1}')'$ is the state vector, n is the dimension of the state vector X_t in Eq. (11) ($n = 3$ in our specific case), y_t is the vector of observable variables, which in our specific case is given by $y_t := (\Delta o_t, \pi_t, R_t)'$, Δo_t being output growth, $A^d(\theta)$, $B(\theta)$, $C(\theta)$ and $D(\theta)$ are conformable matrices whose elements depend nonlinearly on θ , and $u_t^d := (\varepsilon_t', v_t')'$ is the vector containing all system innovations, i.e. the fundamental shocks ε_t and the innovations associated with the measurement system, v_t , if any. The superscript ‘ d ’ stands for ‘determinacy’.

Under indeterminacy, instead, the ABCD form associated with equilibria is given by

$$\begin{aligned} x_t^{in} &= A^{in}(\theta) x_{t-1}^{in} + B(\theta, \tilde{m}) u_t^{in} \\ 3n \times 1 & \quad 3n \times 3n \quad 3n \times 1 \quad 3n \times (4n+b) \quad (4n+b) \times 1 \\ y_t &= C(\theta) x_{t-1}^{in} + D(\theta, \tilde{m}) u_t^{in} \\ p \times 1 & \quad p \times 3n \quad 3n \times 1 \quad 3n \times (4n+b) \quad (4n+b) \times 1 \end{aligned} \quad (14)$$

where the superscript ‘ in ’ stands for ‘indeterminacy’. Here, the state vector $x_t^{in} := (X_t', X_{t-1}', X_{t-2}')'$ involves an additional lag compared to the case of determinacy, while $y_t := (\Delta o_t, \pi_t, R_t)'$. Notably, the ‘B’ and ‘D’ matrices in Eq. (14) depend not only on the structural parameters θ , but also on a vector of auxiliary parameters, unrelated to θ , that we collect in the vector \tilde{m} . $u_t^{in} := (e_t', v_t')'$ is the vector containing all system innovations, i.e. the shocks e_t and the innovations associated with the measurement system, v_t , if any. Notably, the sub-vector e_t does not contain only the fundamental shocks ε_t , but also additional stochastic terms, collected in the vector ζ_t . More precisely, $e_t := (\varepsilon_t', \zeta_t')'$, where ζ_t is a vector that has the same dimension as ε_t and features a number $n_2 \leq \dim(\varepsilon_t)$ of possibly non-zero stochastic terms independent on ε_t ; the remaining $\dim(\varepsilon_t) - n_2$ elements of ζ_t are equal to zero, see the Appendix.

Thus, while the determinate equilibrium in system (13) depends only on the state variables and the structural parameters θ , the class of indeterminate equilibria summarized by system (14) features (i) higher lag order, (ii) a set of auxiliary parameters (\tilde{m}) in addition to the structural parameters, and (iii) possible additional shocks unrelated to the fundamental shocks (the non-zero elements of ζ_t). As shown in the Appendix, the ‘parametric indeterminacy’ that

springs from \tilde{m} characterizes the moving average part of the VARMA-type reduced form solution for X_t . Such parameters index solution multiplicity and may arbitrarily amplify or dampen the fluctuations of the variables in y_t other those implied by the fundamental shocks. The ‘stochastic indeterminacy’ stems from the non-zero sunspot shocks which enter the vector ζ_t . These shocks may arbitrarily alter the dynamics and volatility of the system, see Lubik and Schorfheide (2003, 2004) and Lubik and Surico (2010) for discussions. A special case of interest is obtained when $\tilde{m} = \text{vec}(I_{n_2})$ and no sunspot shocks drive the reduced form; in this case, despite θ lies in the indeterminacy region of the parameter space, the equilibrium collapses to a MSV solution which admits the same time-series representation as the determinate reduced form solution.

A note of caution about the relationship between persistence and determinacy/indeterminacy in DSGE models is in order. As already noticed in Section 3, persistence in the observables y_t can be measured in different ways. Suppose we measure persistence by the largest root of the companion matrix associated with a finite-order VAR representation for y_t , and assume for simplicity a ‘purely forward-looking’ DSGE model, which corresponds to $\Gamma_b = 0_{n \times n}$ and $\Xi = 0_{n \times n}$ in system (11)-(12). In this case, it is possible to show that system (13) is such that $x_t^d = X_t$, and y_t depends on the innovations u_t^d alone, while system (14) features VARMA(1,1)-type dynamics. Hence, in the ‘purely forward-looking’ case, determinacy implies that the observables y_t are driven by pure noise and have no persistence, while indeterminacy implies more persistence. In more general and complex situations, however, it is not possible to claim that the persistence of y_t under indeterminacy is generally higher than the persistence of y_t under determinacy. This fact will be confirmed by our simulation studies.

The richness and variety of equilibria implied by our reference New-Keynesian model can partly be understood by the graphs in Figure 2. Figure 2 plots simulated paths of inflation of length $T=150$, obtained using the structural model in Eq.s (6)-(10) as data generating process under determinacy and indeterminacy. The red line in Figure 2 is generated under determinacy. The blue line is generated under indeterminacy setting the indeterminacy parameter to $\tilde{m}=1$ and not including sunspot shocks, which implies a MSV solution; the green line is generated under indeterminacy by setting the indeterminacy parameter to $\tilde{m}=1.01$ and not including sunspot shocks; finally, the purple line is generated with $\tilde{m}=1.01$, setting the variance of the sunspot shock to $\sigma_s^2=2$. The appealing feature of Figure 2 is that even in the simple case in which the degree of indeterminacy of the system is only one, there exists a continuum of indeterminate equilibria and corresponding inflation paths indexed by the extra parameters \tilde{m} and σ_s^2 , which are all consistent with the specified structural model. The variety of indeterminate equilibria suggest that forecasting inflation with potentially misspecified time-series models are likely not to lead to unambiguous resulting about data forecastability.

The simulation experiments we discuss in the next section evaluate the forecast performance (and persistence) of VARs for $y_t := (\Delta o_t, \pi_t, R_t)'$ when the data are generated under the null of system (6)-(10) under determinacy and indeterminacy, respectively.

6 Simulation experiment

In this section, we run a set of Monte Carlo experiments to assess the forecast performance of the time-series models in (a), (b) and (c) as detailed in Section 3, when the data are produced by system (6)-(10) under determinacy and indeterminacy, respectively. Specifically, we generate $N=1000$ synthetic datasets in each regime, assuming Gaussian fundamental shocks. To match the features of the empirical forecast exercise, we consider periods of unequal lengths in the two cases: $T=94$ observations under determinacy, and $T=119$ observations under the various indeterminacy scenarios we consider.

Data under determinacy are generated from system (13) for $\theta = \check{\theta} \in \mathcal{P}_\theta^D$, where $\check{\theta}$ is calibrated to the medians of the posterior distributions reported in Table 1 of Benati and Surico (2009), column ‘After the Volcker Stabilization’. Hence, we mimic the scenario documented by Benati and Surico (2009) for the Great Moderation period, namely a structural system characterized by an ‘active’ monetary policy able to prevent self-fulfilling inflation expectations. We label such a scenario the ‘Great Moderation-type’ data generating process. The calibrated $\theta = \check{\theta}$ used in the simulation experiment under determinacy is reported for ease of exposition in the right column of Table 3.

Data under indeterminacy are generated from system (14) for $\theta = \check{\theta} \in \mathcal{P}_\theta^I$, where $\check{\theta}$ is calibrated to the medians of the posterior distributions reported in Table 1 of Benati and Surico (2001), column ‘Before October 1979’. In this case, we mimic the scenario documented by Benati and Surico (2009) for the Great Inflation period, namely a structural system characterized by a ‘passive’ monetary policy that does not react aggressively enough to inflation shocks, inducing multiple (stable) equilibria. In this framework, the degree of indeterminacy of the system is one (i.e. $n_2=1$), hence we also need to calibrate the indeterminacy (scalar) parameter \tilde{m} and the variance of the (univariate) sunspot shock $\zeta_t = s_t$ (which is also assumed Gaussian). We label such a scenario the ‘Great Inflation-type’ data generating process. The calibrated $\theta = \check{\theta}$ used in the simulation experiment under indeterminacy is again reported in the left column of Table 3, along with the indeterminacy parameters, summarized at the bottom.

The differences in the values assumed by the structural parameters θ in the two scenarios are highlighted in bold in Table 3. It can be noticed that the main divergence across the two data generating processes essentially lies in the conduct of monetary policy, namely in the long

run response of the policy rate to output gap and inflation shocks.

On each simulated dataset, we apply the models (a), (b) and (c), where now the VAR refers to the observables $y_t := (\Delta o_t, \pi_t, R_t)'$ ($w_t := \Delta o_t$). Again, the univariate models in (b) and (c) serve as benchmarks. The VAR lag order is selected using Schwarz's criterion, considering 1 up to 3 lags. Observe that all three models are misspecified by construction with respect to the true data generating processes in Eq.s (13) and (14).

The first $T - P$ observations are used to estimate the models in (a) and (b) by OLS, and the last P observations to compute forecasts for π_t and Δo_t and associated RMSEs. In line with the empirical exercise of Section 3, the *absolute* VAR forecast performance is measured by the average (across simulations) RMSEs, while the *relative* VAR forecast performance with respect to the benchmarks in (b)-(c) is computed by the ratio of the corresponding average (across simulations) RMSEs. Our measure of persistence is the modulus of the estimated largest root of the VAR companion matrix; we refer to the estimated autoregressive coefficient for the AR(1) models. Results are summarized in Table 4 for the evaluation windows $P=8$ (eight quarters).

Several interesting findings emerge from the picture.

First, in *absolute* terms, forecast uncertainty of VAR-based forecasts – as measured by their RMSEs – is generally lower under determinacy compared to the case of indeterminacy, for both inflation and output growth.¹² This finding is consistent with the empirical results in Table 2b (and in part in Table 2a), and lines up with our a-priori conjecture: the forecast ability of possibly misspecified macro-econometric models is superior in relatively stable environments in which only fundamental shocks drive fluctuations, and deteriorates in environments in which additional (intrinsic and extrinsic) sources of business cycle fluctuations other than fundamental shocks, not accounted by the forecast model, are at work.

Second, persistence tends to increase as the model's reduced form solution is 'far' from the MSV solution. However, for particular values of the indeterminacy parameters, the largest root of the approximating VAR can be smaller under indeterminacy relative to the case of determinacy. Hence, the richer correlation structure and stronger degree of endogenous persistence featured by indeterminate equilibria need not be associated with superior forecast performance *relative* to the case of determinacy. In fact, a severe model misspecification of the forecast model relative

¹²As expected, the (average) RMSEs associated with the VAR-based forecasts obtained under a MSV indeterminate equilibrium is very close to the (average) RMSE obtained under Great Moderation-type scenario. It is worth remarking that differently from the simple univariate case discussed in Section 2, the MSV solution in this case has the same time-series representation as the determinate solution but is characterized by different values for the structural parameters θ . Hence, the average RMSEs associated with the VAR model under the MSV data generating process, need not to coincide numerically with the average RMSEs associated with the VAR under the determinate data generating process.

to the true DGP – which typically involves quite complex cross-equation restrictions under indeterminacy – may adversely affect forecastability, which is a conditional property, even when the variables of interest feature higher persistence.¹³ The forecast performance of VARs under the Great Inflation-type scenario tends to deteriorate, for fixed indeterminacy parameter, as the uncertainty resulting from sunspot shocks (σ_s^2) increases.

Third, when considering *relative* forecast accuracy, we observe that the VAR-based forecasts are substantially similar to that of the AR(1)-based forecasts under determinacy, but may be inferior or superior to the AR(1)-based forecasts under indeterminacy, depending on the values taken by the indeterminacy parameters \tilde{m} and σ_s^2 . For some values of \tilde{m} and σ_s^2 , the VAR forecasts are inferior to that of AR(1) models. For other values, the converse occurs. A similar property characterizes the VAR-based forecasts relative to the RW's. In general, it is not possible to claim that the relative performance of the VAR model is superior or inferior to that of our univariate benchmarks under indeterminacy.

Summing up, the empirical evidence summarized in Tables 2a-2b can be reconciled with the results of the simulation study in Table 4 under specific assumptions about the indeterminacy parameters that govern solution multiplicity. Assuming, therefore, that the New Keynesian DSGE model as expressed in Eq.s (6)-(10) represents a good *prima facie* approximation of the post-WWII U.S. business cycle, we read the findings of our simulation analyses as pointing towards a ‘good luck in bad policy’ explanation of the (*relative*) higher forecastability of macro-econometric models prior to the Great Moderation period.

7 Concluding remarks

This paper has investigated the consequences of indeterminacy for quantitative forecasting through popular macro-econometric models. Our simulation as well as empirical findings suggest that forecastability through time-series tools like e.g. VARs under indeterminacy may be severely undermined by model misspecification issues, even when extrinsic uncertainty (i.e. sunspot noise) plays no role in generating the data. Although indeterminacy need not imply superior forecastability - however measured - of (potentially misspecified) reduced form predictors, it may well be the case that a particular multivariate model estimated on data generated under indeterminate equilibria produces better forecasts than univariate predictors, regardless of the level of persistence and volatility that characterizes the observed time-series.

¹³Along the same line, Canova and Gambetti (2010) emphasize that, conditional on lags of the endogenous variables, past expectations Granger-cause current values of the latter under ideterminacy but not under determinacy, and hence – irrespective of higher persistence – the omission of (proxies for) such expectations from the forecast model may result in larger prediction errors in the former regime.

According to our analysis, maintaining that the U.S. business cycle can be reasonably approximated by a monetary New Keynesian DSGE models during the post-WWII, and that ‘bad policy’ prevailed during the Great Inflation relative to Great Moderation period, the superior *relative* predictive performance observed in macro-econometric forecasting during the Great Inflation appears to admit a ‘good luck’ explanation.

How should a practitioner address forecasting when she believes that a DSGE model describes the observed economy? Of course, forecasts should be computed from the DSGE model. But in these cases, understanding which regime observed samples belong to, should be regarded as a necessary stage in the process of developing forecasting models suitable for that specific regime. As a minimal requirement, the auxiliary components that characterize business cycle fluctuations under indeterminacy should be properly identified and incorporated in the forecast model to improve on forecast accuracy. Overall, for DSGE model-based forecasting purposes we believe there remains the need to tackle a misspecification-robust approach to testing whether a determinate or an indeterminate equilibrium is supported by the data (Fanelli, 2012).

Appendix: DSGE equilibria and their representation

In this Appendix, we discuss the solutions associated with the DSGE model compacted in Eq.s (11)-(12). To keep exposition as general as possible, throughout this Appendix we denote with n the dimension of the state vector X_t in Eq. (11) (notice that $n = 3$ in our specific case). Moreover, we use the notations ‘ $A(\theta)$ ’ and ‘ $A:=A(\theta)$ ’ to indicate that the elements of the matrix A depend nonlinearly on the structural parameters θ , hence in our setup $\Gamma_0:=\Gamma_0(\theta)$, $\Gamma_f:=\Gamma_f(\theta)$, $\Gamma_b:=\Gamma_b(\theta)$, $\Xi:=\Xi(\theta)$ and $\Gamma_0^\Xi:=\Gamma_0^\Xi(\theta)$. We call ‘stable’ a matrix that has all eigenvalues inside the unit disk and ‘unstable’ a matrix that has at least one eigenvalue outside the unit disk. Thus, denoted with $\lambda_{\max}(\cdot)$ the absolute value of the largest eigenvalue of the matrix in the argument, we have $\lambda_{\max}(A(\theta)) < 1$ for stable matrices and $\lambda_{\max}(A(\theta)) > 1$ for unstable ones. We also consider the partition $\theta:=(\theta'_s, \theta'_\varepsilon)'$, where θ_ε contains the non-zero elements of $vech(\Sigma_\varepsilon)$ and θ_s all remaining elements. The ‘true’ value of θ , $\theta_0:=(\theta'_{0,s}, \theta'_{0,\varepsilon})'$, is assumed to be an interior point of \mathcal{P} . The corresponding partition of the parameter space is given by $\mathcal{P}:=\mathcal{P}_{\theta_s} \times \mathcal{P}_{\theta_\varepsilon}$. This partition is important because the determinacy/indeterminacy of the system depends only on the values taken by θ_s .

A detailed derivation of the time-series representation of the reduced form solutions associated with the New-Keynesian DSGE system (11)-(12) is reported in Castelnuovo and Fanelli (2015). Using the Binder and Pesaran’s (1995) solution method, they show that uniqueness/multiplicity of solutions is governed by the stability/instability of the the matrix $G(\theta_s):=(\Gamma_0^\Xi -$

$\Gamma_f \Phi_1)^{-1} \Gamma_f$, where Φ_1 stems from the solution of a quadratic matrix equation.

Determinacy

For values of θ_s such that the matrix $G(\theta_s) := (\Gamma_0^\Xi - \Gamma_f \Phi_1)$ is stable, i.e. $\lambda_{\max}(G(\theta_s)) < 1$, then $\theta_s \in \mathcal{P}_{\theta_s}^D$ and the reduced form solution to system (11)-(12) can be represented in the form

$$(I_n - \Phi_1(\theta_s)L - \Phi_2(\theta_s)L^2)X_t = u_t \quad , \quad u_t := \Upsilon(\theta_s)^{-1} \varepsilon_t \quad (15)$$

where L is the lag operator ($L^j X_t = X_{t-j}$), $\Phi_1(\theta_s)$, $\Phi_2(\theta_s)$ and $\Upsilon(\theta_s)$ are 3×3 matrices whose elements depend nonlinearly on θ_s and embody the cross-equation restrictions implied by the small New-Keynesian model. The matrices $\Phi_1(\theta_s)$ and $\Phi_2(\theta_s)$ in Eq. (15) are obtained as the unique solution to the second-order quadratic matrix equation

$$\hat{\Phi} = (\hat{\Gamma}_0 - \hat{\Gamma}_f \hat{\Phi})^{-1} \hat{\Gamma}_b \quad (16)$$

where $\hat{\Gamma}_f$, $\hat{\Gamma}_0$, $\hat{\Gamma}_b$ and the stable matrix $\hat{\Phi}$ are respectively given by

$$\hat{\Gamma}_0 := \begin{pmatrix} \Gamma_0^\Xi & 0_{n \times n} \\ 0_{n \times n} & I_n \end{pmatrix}, \quad \hat{\Gamma}_f := \begin{pmatrix} \Gamma_f & 0_{n \times n} \\ 0_{n \times n} & 0_{n \times n} \end{pmatrix}, \quad \hat{\Gamma}_b := \begin{pmatrix} \Gamma_{b,1}^\Xi & \Gamma_{b,2}^\Xi \\ I_n & 0_{n \times n} \end{pmatrix}, \quad \hat{\Phi} := \begin{pmatrix} \Phi_1 & \Phi_2 \\ I_3 & 0_{3 \times 3} \end{pmatrix},$$

and $\Gamma_{b,1}^\Xi := (\Gamma_b + \Xi \Gamma_0)$, $\Gamma_{b,2}^\Xi := -\Xi \Gamma_b$ and $\Upsilon(\theta) := (\Gamma_0 - \Gamma_f \Phi_1(\theta))$. The matrix $\Phi_1 := \Phi_1(\theta_s)$ is the one that enters the definition of $G(\theta_s)$.

A convenient representation of the equilibrium in Eq. (15) is given by

$$\begin{pmatrix} X_t \\ X_{t-1} \\ x_t^d \end{pmatrix} = \begin{pmatrix} \check{\Phi}_1 & \check{\Phi}_2 \\ I_n & 0_{n \times n} \end{pmatrix} \begin{pmatrix} X_{t-1} \\ X_{t-2} \\ x_{t-1}^d \end{pmatrix} + \begin{pmatrix} \check{\Upsilon}^{-1} \\ 0_{n \times n} \\ G^d(\theta_s) \end{pmatrix} \varepsilon_t \quad (17)$$

where $\check{\Phi}_1 = \Phi_1(\theta_s)$, $\check{\Phi}_2 = \Phi_2(\theta_s)$, $\check{\Upsilon} = \Upsilon(\theta_s)$, the matrices $A^d(\theta_s)$ and $G^d(\theta_s)$ are $2n \times 2n$ and $2n \times n$, respectively, and the superscript ‘ d ’ stands for ‘determinacy’. Let $y_t := (y_{1,t}, y_{2,t}, \dots, y_{p,t})'$ be the $p \times 1$ vector of observable variables.

When all variables in X_t are observed, $y_t = X_t$, the state system (17) along with the measurement system: $y_t = H x_t^d$, $H := (I_n : 0_{n \times n})$, give rise to a VAR representation for y_t (X_t) with coefficients that depend on θ through the CER in Eq. (16). In general, however, not all variables in X_t belong to the forecaster’s information set. In this case, the measurement system will take the form

$$y_t = H x_t^d + Q v_t \quad (18)$$

where H is a $p \times 2n$ matrix, v_t a $b \times 1$ vector ($b \leq p$) of measurement errors with covariance matrix Σ_v , and Q is a $p \times b$ selection matrix. For the specific structural model we consider in

the paper, the counterpart of the measurement system (18) is given by

$$\begin{pmatrix} \Delta o_t \\ \pi_t \\ R_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{o}_t \\ \pi_t \\ R_t \\ \tilde{o}_{t-1} \\ \pi_{t-1} \\ R_{t-1} \\ x_t^d \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ Q \end{pmatrix} \varepsilon_{o^n,t} \quad v_t$$

and is obtained by exploiting the RW assumption in Eq. (10). Let $u_t := (\varepsilon_t', v_t')$ be the $(n+b)$ -dimensional vector containing all system innovations. By substituting Eq. (17) into Eq. (18) and using some algebra, one obtains the ABCD form (Fernández-Villaverde *et al.* 2007; Ravenna, 2007) of the determinate equilibrium:

$$\begin{aligned} x_t^d &= A^d(\theta_s) x_{t-1}^d + B(\theta_s) u_t \\ y_t &= C(\theta_s) x_{t-1}^d + D(\theta_s) u_t \end{aligned} \quad (19)$$

where $B(\theta_s) := (G^d(\theta_s) : 0_{2n \times b})$, $C(\theta_s) := HA^d(\theta_s)$ and $D(\theta_s) := (HG^d(\theta_s) : Q)$.

System (19) must be in ‘minimal form’ and must be identified (locally) as discussed in Komunjer and Ng (2011). In general, it is possible to manipulate the state space representation such that an identified system in minimal form is obtained eventually. If the system passes these checks, Eq. (19) can be used as the data generating process implied by our New-Keynesian DSGE model under determinacy. Replacing θ_s with θ it corresponds to system (13) in Section 5 of the paper.

Indeterminacy

For values of θ_s such that the matrix $G(\theta_s) := (\Gamma_0^\Xi - \Gamma_f \Phi_1)^{-1} \Gamma_f$ is unstable, i.e. $\lambda_{\max}(G(\theta_s)) > 1$, then $\theta_s \in \mathcal{P}_{\theta_s}^I$ and the class of reduced form solutions associated with the New-Keynesian system (11)-(12) becomes more involved from a dynamic standpoint. When $\lambda_{\max}(G(\theta_s)) > 1$, the matrix $G(\theta_s)$ can be decomposed in the form

$$G(\theta_s) = P(\theta_s) \begin{pmatrix} \Lambda_1 & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & \Lambda_2 \end{pmatrix} P^{-1}(\theta_s)$$

where $P(\theta_s)$ is a $n \times n$ non-singular matrix, Λ_1 is the $n_1 \times n_1$ ($n_1 < n$) Jordan normal block that collects the eigenvalues of $G(\theta_s)$ that lie inside the unit disk, and Λ_2 is the $n_2 \times n_2$ ($n_2 \leq n$) Jordan normal block that collects the eigenvalues of $G(\theta_s)$ that lie outside the unit disk. Observe that $n_1 + n_2 = n$, where n_2 determines the ‘degree of multiplicity’ of solutions.

The reduced form solutions can be given the VARMA-type representation:

$$(I_n - \Pi(\theta_s)L)(I_n - \Phi_1(\theta_s)L - \Phi_2(\theta_s)L^2)X_t = (\Psi(\theta_s, \tilde{m}) - \Pi(\theta_s)L)V(\theta_s, \tilde{m})^{-1}\varepsilon_t + \tau_t \quad (20)$$

$$\tau_t := (\Psi(\theta_s, \tilde{m}) - \Pi(\theta_s)L)V(\theta_s, \tilde{m})^{-1}P(\theta_s)\zeta_t + P(\theta_s)\zeta_t \quad (21)$$

where the matrices $\Phi_1(\theta_s)$ and $\Phi_2(\theta_s)$ are defined as in the case of determinacy, see Eq. (16), while the matrices $\Pi(\theta_s)$, $\Psi(\theta_s, \tilde{m})$ and $V(\theta_s, \tilde{m})$ are respectively given by

$$\Pi(\theta_s) := P(\theta_s) \begin{pmatrix} 0_{n_1 \times n_1} & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & \Lambda_2^{-1} \end{pmatrix} P^{-1}(\theta_s) \quad , \quad \Psi(\theta_s, \tilde{m}) := P(\theta_s) \begin{pmatrix} I_{n_1} & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & \tilde{M} \end{pmatrix} P^{-1}(\theta_s)$$

$$V(\theta_s, \tilde{m}) := (\Gamma_0(\theta_s) - \Gamma_f(\theta_s)\Phi_1(\theta_s)) - \Xi(\theta_s)\Gamma_f(\theta_s)(I_n - \Psi(\theta_s, \tilde{m})).$$

The $n_2 \times n_2$ sub-matrix \tilde{M} of $\Psi(\theta_s, \tilde{m})$ contains a set of arbitrary auxiliary parameters that do not depend on θ_s ; for ease of reference, we collect these parameters in the vector $\tilde{m} := \text{vec}(\tilde{M})$. The ‘additional’ vector moving average term τ_t depends on an extra source of random fluctuations potentially independent on the fundamental disturbances ε_t , i.e. on the $n \times 1$ vector $\zeta_t := (0'_{n_1 \times 1}, s'_t)'$, where s_t is a $n_2 \times 1$ MDS which collects the ‘sunspot shocks’ featured by the system. We assume, without any loss of generality, that s_t has a time-invariant covariance matrix Σ_s . The sunspot shocks might be also absent from the reduced form solution, i.e. $\Sigma_s = 0_{n_2 \times n_2}$ implying $\zeta_t = 0_{n \times 1}$ a.s. (and $\tau_t = 0_{n \times 1}$ a.s.). This situation will be denoted ‘indeterminacy without sunspots’.

A convenient representation of the class of indeterminate equilibria described by Eq.s (20)-(21) is given by the system

$$\begin{pmatrix} X_t \\ X_{t-1} \\ X_{t-2} \\ x_t^{in} \end{pmatrix} = \begin{pmatrix} -(\check{\Phi}_1 + \check{\Pi}) & (\check{\Phi}_1\check{\Pi} - \check{\Phi}_2) & \check{\Phi}_2\check{\Pi} \\ I_n & 0_{n \times n} & 0_{n \times n} \\ 0_{n \times n} & I_n & 0_{n \times n} \\ A^{in}(\theta_s) & & \end{pmatrix} \begin{pmatrix} X_{t-1} \\ X_{t-2} \\ X_{t-3} \\ x_{t-1}^{in} \end{pmatrix} + \begin{pmatrix} \check{K}_1 & \check{K}_2 \\ 0_{n \times 2n} & 0_{n \times 2n} \\ 0_{n \times 2n} & 0_{n \times 2n} \\ G^{in}(\theta_s, \tilde{m}) \end{pmatrix} \begin{pmatrix} e_t \\ e_{t-1} \\ \varepsilon_t^0 \end{pmatrix} \quad (22)$$

where $\check{\Phi}_1 = \Phi_1(\theta_s)$, $\check{\Phi}_2 = \Phi_2(\theta_s)$, $\check{\Pi} = \Pi(\theta_s)$, $\check{\Psi} = \Psi(\theta_s, \tilde{m})$, $\check{V} = V(\theta_s, \tilde{m})$, $\check{K}_1 := [\check{\Psi}\check{V}^{-1} : (\check{\Psi}\check{V}^{-1} + I_n)\check{P}]$, $\check{K}_2 := [\check{\Pi}\check{V}^{-1} : \check{\Pi}\check{V}^{-1}\check{P}]$ are $n \times 2n$ matrices, $e_t := (\varepsilon'_t, \zeta'_t)'$ is a $2n \times 1$ vector that collects the fundamental and sunspot shocks of the system, the matrices $A^{in}(\theta_s)$ and $G^{in}(\theta_s, \tilde{m})$ are $3n \times 3n$ and $3n \times 4n$, respectively, and the superscript ‘in’ stands for ‘indeterminacy’.

Given the $p \times 1$ vector of observables y_t , the associated measurement system is given by

$$y_t = Hx_t^{in} + Qv_t \quad (23)$$

and, a part from their dimensions, the matrices H , Q and the vector v_t have the same meaning as in Eq. (18). For the specific structural model we consider in the paper, the counterpart of

the measurement system (23) is given by

$$\begin{pmatrix} \Delta o_t \\ \pi_t \\ R_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{o}_t \\ \pi_t \\ R_t \\ \tilde{o}_{t-1} \\ \pi_{t-1} \\ R_{t-1} \\ \tilde{o}_{t-2} \\ \pi_{t-2} \\ R_{t-2} \\ x_t^{in} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ Q \end{pmatrix} \varepsilon_{o^n, t} \quad v_t$$

Upon defining the $(4n + b)$ -dimensional vector $u_t := (e_t, v_t)'$ which contains the complete set of innovations, the ABCD form of the indeterminate equilibria reads

$$\begin{aligned} x_t^{in} &= A^{in}(\theta_s) x_{t-1}^{in} + B(\theta_s, \tilde{m}) u_t \\ y_t &= C(\theta_s) x_{t-1}^{in} + D(\theta_s, \tilde{m}) u_t \end{aligned} \quad (24)$$

where $B(\theta_s, \tilde{m}) := (G^{in}(\theta_s, \tilde{m}) : 0_{2n \times b})$, $C(\theta_s) := H A^{in}(\theta_s)$ and $D(\theta_s, \tilde{m}) := (H G^{in}(\theta_s, \tilde{m}) : Q)$.

Also in this case, provided system (24) is in ‘minimal form’ and identified (locally), it can be used as the data generating process implied by our New-Keynesian DSGE model under indeterminacy. Replacing θ_s with θ it corresponds to system (14) in Section 5 of the paper.

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TABLES

Table 1. RMSEs for x_t computed from Eq. (1) under determinacy and indeterminacy.

Equilibrium regime	AR(1)	Relative to DET.	Relative to BENCH.	Relative to AR(4)
$T=500$ Evaluation window: $P = 1$				
DETERMINACY	0.5561	-	1	0.9958
INDETERMINACY, MSV	0.5561	1	1	0.9958
INDETERMINACY, $\tilde{M} = 1.01$	0.5627	1.0120	1.0114	0.9960
INDETERMINACY, $\tilde{M} = 0.98$	0.5469	0.9830	0.9829	0.9947
INDETERMINACY, $\tilde{M} = 0.85$	0.5393	0.9698	1.1046	0.999
INDETERMINACY, $\tilde{M} = 0.80$	0.5564	1.0005	1.249	1.0157
INDETERMINACY, $\tilde{M} = 0.015$	0.5273	0.948	63.18	0.9932
$T=500$ Evaluation window: $P = 8$				
DETERMINACY	0.6791	-	1	0.9979
INDETERMINACY, MSV	0.6791	1	1	0.9979
INDETERMINACY, $\tilde{M} = 1.01$	0.6866	1.0110	1.0009	0.9972
INDETERMINACY, $\tilde{M} = 0.98$	0.6662	0.9810	1.0008	0.9976
INDETERMINACY, $\tilde{M} = 0.85$	0.6545	0.9638	1.0932	1
INDETERMINACY, $\tilde{M} = 0.80$	0.6799	1.0011	1.1639	1.0144
INDETERMINACY, $\tilde{M} = 0.015$	1.1526	1.6972	1.1360	1.0146
$T=500$ Evaluation window: $P = 16$				
DETERMINACY	0.6919	-	1	0.9996
INDETERMINACY, MSV	0.6919	1	1	0.9996
INDETERMINACY, $\tilde{M} = 1.01$	0.6995	1.011	1.0008	0.9993
INDETERMINACY, $\tilde{M} = 0.98$	0.6785	0.9810	1.0002	0.9993
INDETERMINACY, $\tilde{M} = 0.85$	0.6637	0.9592	1.0604	1.001
INDETERMINACY, $\tilde{M} = 0.80$	0.6890	0.9958	1.1046	1.009
INDETERMINACY, $\tilde{M} = 0.015$	1.4076	2.0344	1.0554	1.0003

NOTES: Results are based on $N=1000$ simulations. Data under determinacy are generated from Eq. (3) with $\sigma_\omega^2=0.5$. Data under indeterminacy are generated from Eq. (4) with $\theta:=0.95$, $\sigma_\omega^2:=0.5$ and $\sigma_s=0$, for different values of \tilde{M} and fixed initial conditions. ‘AR(1)’: average (across simulations) absolute RMSEs obtained with the model in Eq. (5). The first $T - P$ observations are used to estimate the model and the remaining P observations to compute the RMSEs. ‘Relative to DET.’: ratio between the average RMSE obtained with the AR(1) model under indeterminacy and the average RMSEs obtained with the AR(1) model under determinacy. ‘Relative to BENCH.’: ratio between the average AR(1)-based RMSEs and the average RMSEs obtained under a ‘theory-based’ benchmark. ‘Relative to AR(4)’: ratio between the average AR(1)-based RMSEs and the average AR(4)-based RMSEs.

Table 2a. Absolute and relative RMSE for inflation and output gap ($w_t := \tilde{o}_t$) computed on the Great Inflation and Great Moderation samples with time-series models

Great Inflation: 1954Q4-1984Q2		$T = 119$	Evaluation window:	$P = 8$	
RW (c)	Absolute	Relative to RW (b), (a)/(c)	Relative to AR (c), (a)/(b)	stand. dev.	Persistence
inflation	0.3289	-	1.0794		
output gap	0.8524	-	0.4962		
AR(1) (b)					
inflation	0.3039	0.9264	-	0.3743	0.8490
output gap	1.7180	2.0155	-	1.0945	0.9261
VAR (a) 3 lags					
inflation	0.2789	0.8480	0.9177	0.3437	0.9441
output gap	2.0077	2.3553	1.1686	0.9395	
Great Moderation: 1985Q1-2008Q2		$T = 94$	Evaluation window:	$P = 8$	
RW (c)					
inflation	0.3034	-	1.4999		
output gap	0.026	-	1.0412		
AR(1) (b)					
inflation	0.2023	0.6667	-	0.1876	0.6145
output gap	0.8406	32.33	-	0.5066	0.9478
VAR (a) 3 lags					
inflation	0.2596	0.8557	1.2830	0.1753	0.9295
output gap	0.4691	18.04	0.5581	0.4691	

NOTES: Forecasts are computed using the first $T - P$ observations of the sample to estimate the model and the last remaining P

observations to compute one-step ahead forecasts (fixed scheme) of inflation and output growth P times and the corresponding RMSEs. 'RW (c)' stands for univariate random walk; 'AR(1) (b)' stands for univariate autoregressive model of order one which includes a constant; 'VAR (a)' denotes a three-variate VAR system for $y_t := (\tilde{o}_t, \pi_t, R_t)'$ which includes a constant and whose lag order is selected using Schwarz's (SC) information criterion combined with a LM-type vector test for uncorrelated VAR disturbances, considering 1 up to 3 lags. 'stand. dev.' reports the estimated squared root of the variance of residuals for π_t and \tilde{o}_t by the AR(1) and VAR models. 'Persistence' reports the estimated autoregressive coefficient for the AR(1) model and the absolute value of the estimated largest root of the VAR companion matrix.

Table 2*b*. Absolute and relative RMSE for inflation and output growth ($w_t := \Delta o_t$) computed on the Great Inflation and Great Moderation samples with time-series models

Great Inflation: 1954Q4-1984Q2		$T = 119$	Evaluation window:	$P = 8$	
RW (c)	Absolute	Relative to RW (b), (a)/(c)	Relative to AR (c), (a)/(b)	stand. dev.	Persistence
inflation	0.3289	-	1.0794		
output growth	0.6643	-	0.6350		
AR(1) (b)					
inflation	0.3047	0.9264	-	0.3743	0.8490
output growth	1.0462	1.5749	-	1.0935	0.2796
VAR (a) 3 lags					
inflation	0.4078	1.2340	1.3384	0.3552	0.9632
output growth	1.5337	2.3087	1.4660	0.9876	
Great Moderation: 1985Q1-2008Q2		$T = 94$	Evaluation window:	$P = 8$	
RW (c)					
inflation	0.3034	-	1.4999		
output growth	0.4951	-	1.0412		
AR(1) (b)					
inflation	0.2023	0.6667	-	0.1876	0.6145
output growth	0.4755	0.9604	-	0.4989	0.1905
VAR (a) 3 lags					
inflation	0.2538	0.8365	1.2546	0.1754	0.8545
output growth	0.4783	0.9661	1.0059	0.4775	

NOTES: Forecasts are computed using the first $T - P$ observations of the sample to estimate the model and the last remaining

P observations to compute one-step ahead forecasts (fixed scheme) of inflation and output growth P times and the corresponding RMSEs.

'RW (c)' stands for univariate random walk; 'AR(1) (b)' stands for univariate autoregressive model of order one which includes a constant;

'VAR (a)' denotes a three-variate VAR system for $y_t := (\Delta o_t, \pi_t, R_t)'$ which includes a constant and whose lag order is selected using

Schwarz's (SC) information criterion combined with a LM-type vector test for uncorrelated VAR disturbances, considering 1 up to 3 lags.

Δo_t is multiplied by 100. 'stand. dev.' reports the estimated squared root of the variance of residuals for π_t and Δo_t by the AR(1)

and VAR models'. 'Persistence' reports the estimated autoregressive coefficient for the AR(1) model and the absolute value of the estimated largest root of the VAR companion matrix.

Table 3 Values of the structural parameters of the New Keynesian model in Eq.s (6)-(9) used in the simulation experiments of Section 6.

Parameter	Interpretation	Great Inflation	Great Moderation
		data generating process	data generating process
		$\check{\theta}$	$\check{\theta}$
γ	IS: forward looking term	0.744	0.744
δ	IS: inter. elast. of substitution	0.124	0.124
α	NKPC: indexation past inflation	0.059	0.059
κ	NKPC: slope	0.044	0.044
ρ	Rule, smoothing term	0.595	0.834
$\varphi_{\tilde{o}}$	Rule, reaction to output gap	0.527	1.146
φ_{π}	Rule, reaction to inflation	0.821	1.749
$\rho_{\tilde{o}}$	Output gap shock, persistence	0.796	0.796
ρ_{π}	Inflation shock, persistence	0.418	0.418
ρ_R	Policy rate shock, persistence	0.404	0.404
$\sigma_{\tilde{o}}^2$	IS: shock variance	0.055	0.055
σ_{π}^2	NKPC: shock variance	0.391	0.391
σ_R^2	Policy rule: shock variance	0.492	0.492
$\sigma_{o^n}^2$	Natural rate of output: shock variance	0.25	0.25
\tilde{m} ,	Indeterminacy parameters	1; 1.01; 0.98	-
σ_s^2	Variance of sunspot shock	0; 2; 5	-

NOTES: $\check{\theta}$ under determinacy is calibrated to the medians of the posterior distributions reported in Table 1 of Benati and Surico

(2009), column 'After the Volcker Stabilization'. $\check{\theta}$ under indeterminacy is calibrated to the medians of the posterior distributions reported

in Table 1 of Benati and Surico (2009), column 'Before October 1979'. In bold the parameter values that change across the two regimes.

Table 4. Absolute and relative (average) RMSE for inflation and output growth computed from data simulated from the New-Keynesian DSGE model in Eq.s (6)-(10) under determinacy and indeterminacy on different sample lengths.

Evaluation window: $P = 8$				
	Absolute (a)	Relative to RW (a/c)	Relative to AR (a/b)	VAR persistence
$T = 94$				
DETERMINACY				
inflation	1.1714	0.7655	1.0094	0.8496
output growth	1.3580	0.7778	1.0048	
$T = 119$				
INDETERMINACY: MSV solution				
inflation	1.2793	0.7413	1.0029	0.8214
output growth	1.3373	0.7498	1.0001	
INDETERMINACY: $\tilde{m}=1.01, \sigma_s^2=0$				
inflation	2.6674	0.8118	0.9647	0.8454
output growth	5.5618	0.8018	0.9942	
INDETERMINACY: $\tilde{m}=0.98, \sigma_s^2=0$				
inflation	2.4867	0.8467	0.9835	0.8238
output growth	5.5395	0.8014	0.9954	
INDETERMINACY: $\tilde{m}=1.01, \sigma_s^2=2$				
inflation	5.5149	1.0265	1.0064	0.9710
output growth	6.2177	0.8284	0.9995	
INDETERMINACY: $\tilde{m}=1.01, \sigma_s^2=5$				
inflation	7.7233	1.0369	1.0170	0.9729
output growth	7.0334	0.8538	0.9998	

NOTES: Results are based on N=1000 simulations. Data under determinacy are generated by simulating system (13) for $\theta = \check{\theta} \in \mathcal{P}_\theta^D$, where $\check{\theta}$ is calibrated as in the right column of Table 2. Data under indeterminacy are generating by simulating system (14) for $\theta = \check{\theta} \in \mathcal{P}_\theta^I$, where $\check{\theta}$ and the indeterminacy parameters \tilde{m} and σ_s^2 are calibrated as in the left column of Table 2. Forecasts are computed using $T - P$ observations to estimate the model and the last P observations to evaluate forecasts (fixed scheme) and the corresponding RMSEs. (a) denotes the three-variate VAR system for $y_t = (\Delta o_t, \pi_t, R_t)'$ whose lag order is selected using Schwarz's (SC) information criterion, considering 1 up to 4 lags; 'RW' stands for univariate random walk, i.e. model (c); 'AR(1)' stands for univariate autoregressive model of order one, i.e. model (b). 'VAR persistence reports the absolute value of the largest estimated root of the VAR companion matrix'.

FIGURES

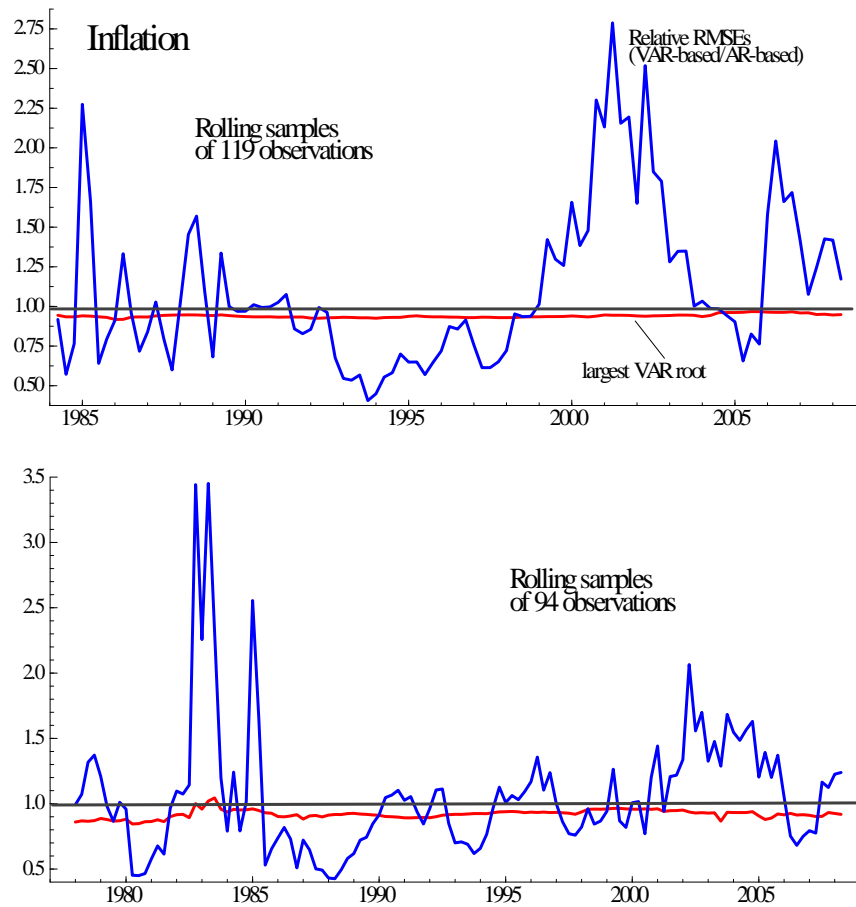


Figure 1. Relative RMSEs (VAR-based/AR_based) relative to inflation (blue line) in rolling samples of 119 (UPPER PANEL) and 94 (LOWER PANEL) observations and estimated VAR largest root (red line). Results are based on the VAR for $y_t := (\tilde{o}_t, \pi_t, R_t)'$, see Section 3. On each rolling sample, the first $T - 8$ observations are used to estimate the model and the last 8 observations to calculate the RMSEs.

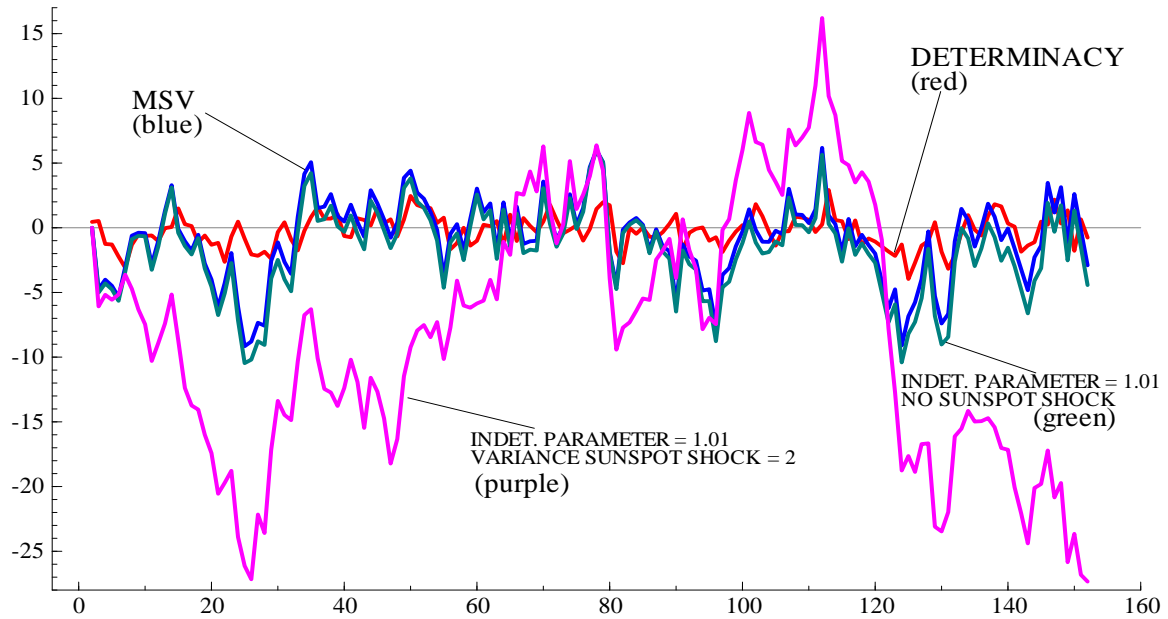


Figure 2. Simulated inflation paths ($T = 150$) obtained using the New-Keynesian DSGE model in Eq.s (6)-(10) as data generating process under determinacy (red line) and indeterminacy (blue, green and purple lines). The red line is obtained by simulating system (13) for $\theta = \check{\theta} \in \mathcal{P}_{\theta}^D$, where $\check{\theta}$ is calibrated to the medians of the posterior distributions reported in Table 1 of Benati and Surico (2001), column ‘After the Volcker Stabilization’. The other lines are obtained from system (14) for $\theta = \check{\theta} \in \mathcal{P}_{\theta}^D$, where $\check{\theta}$ is calibrated to the medians of the posterior distributions reported in Table 1 of Benati and Surico (2009), column ‘Before October 1979’, setting the indeterminacy parameter \tilde{m} and variance of the sunspot shock to different values (see main text).