Autoregressive conditional duration and FIGARCH models: origins of long memory

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Abstract

Although properties of ARCH(\infty) model are well investigated, existence of long memory solutions to FIGARCH and IARCH equations was not established, causing theoretical controversy that other solutions besides the trivial zero one, do not exist. Since ARCH models with non-zero intercept may have only a unique stationary solution and exclude long memory, existence of finite variance FIGARCH and IARCH models and, thus, possibility of long memory in ARCH setting was doubtful. The present paper solves this controversy by showing that FIGARCH and IARCH equations have a non-trivial covariance stationary solution, that always exhibits long memory. Existence and uniqueness of stationary Integrated AR(\infty) processes is also discussed, and their long memory feature established. Summarizing, we show that covariance stationary IARCH, FIGARCH and IAR(\infty) processes exist, their class is wide, and they always have long memory.

Keywords: ARCH, FIGARCH, IARCH, long memory, conditional duration.

JEL classification: C15; C22

1 Introduction

A non-negative random process \{\tau_k\} = \{\tau_k, k \in \mathbb{Z}\} is said to satisfy the ARCH(\infty) equation if there exists a sequence of nonnegative i.i.d. random variables \{\varepsilon_k\} with unit mean \text{E}\varepsilon_0 = 1, nonnegative number \omega \geq 0 and a deterministic sequence \sum_{j=1}^{\infty} b_j \leq 1, such that

(1.1) \quad \tau_k = \varepsilon_k (\omega + \sum_{j=1}^{\infty} b_j \tau_{k-j}), \quad k \in \mathbb{Z}. 
Such ARCH process \( \{\tau_k\} \) is causal, i.e. for any \( k, \tau_k \) can be represented as a measurable function \( f(\varepsilon_k, \varepsilon_{k-1}, \ldots) \) of the present and past values \( \varepsilon_s, s \leq k \). The last property implies that a stationary \( \{\tau_k\} \) process is ergodic, and \( \varepsilon_k \) is independent of \( \tau_s, s < k \). Therefore,

\[
E[\tau_k | \tau_s, s < k] = h_k, \quad h_k := \omega + \sum_{j=1}^{\infty} b_j \tau_{k-j}.
\]

The usual interpretation of \( \tau_k \) in financial econometrics is that of squared returns:

\[
\tau_k = r_k^2, \quad r_k = \zeta_k \frac{1}{\sqrt{h_k}}, \quad h_k = \omega + \sum_{j=1}^{\infty} b_j r_{k-j}^2, \quad k \in \mathbb{Z},
\]

where \( \{\zeta_k\} \) is a standartized i.i.d.\((0,1)\) noise, \( \varepsilon_k = \zeta^2_k \) and \( h_k \) is volatility. Moreover, since typically variables \( \tau_k > 0 \) are almost-surely positive, they may represent random durations between transactions. The class of ARCH\((\infty)\) processes (1.1) includes the parametric stationary ARCH and GARCH models of Engle (1982) and Bollerslev (1986), and the ACD (Autoregressive Conditional Duration) model of Engle and Russel (1998).

An ARCH\((\infty)\) process was introduced by Robinson (1991) and later studied in Kokoszka and Leipus (2000), Giraitis et al. (2000a) (see also the review papers by Giraitis et al. (2007, 2011), Berkes et al. (2004)). In contrast to a standard stationary GARCH\((p,q)\) process whose autocorrelations decay exponentially, an ARCH\((\infty)\) process can have autocovariances \( \text{Cov}(\tau_k, \tau_0) \) decaying to zero at the rate \( k^{-\gamma} \) with \( \gamma > 1 \) arbitrarily close to 1.

In several papers (e.g. Giraitis et al. (2000a), Giraitis and Surgailis (2002), Kazakevičius and Leipus (2002)) it is claimed that a finite variance stationary solution to ARCH equations (1.1), if exists, has short memory /or absolutely summable autocovariances, and its existence requires \( \sum_{j=1}^{\infty} b_j < 1 \). Because of the well known phenomenon of long memory in the squares of financial returns in financial econometrics, the latter finding may be considered as a limitation of ARCH modeling. Subsequently, it initiated and justified the study of other ARCH-type models for which the long memory property can be rigorously established (see Giraitis et al. (2007, 2011)).

The above claims are correct for \( \omega > 0 \). In Theorem 3.2 and Corrolary 2.3 of Giraitis and Surgailis (2002) (below refered to as GS(2002)) it is shown that if a covariance stationary solution \( \tau_k \) of ARCH equations (1.1) exists, it is unique and has summable non-negative auto-covariances. Clearly, that implies \( \sum_{j=1}^{\infty} b_j < 1 \), because \( E\tau_k = (\omega + \sum_{j=1}^{\infty} b_j)E\tau_k > \sum_{j=1}^{\infty} b_j E\tau_k \).

For \( \omega = 0 \), however, the situation is different. Here existence of covariance stationary solution implies \( \sum_{j=1}^{\infty} b_j = 1 \), which leads to integrated ARCH\((\infty)\), or IARCH\((\infty)\) equation

\[
(1.2) \quad \tau_k = \varepsilon_k \left( \sum_{j=1}^{\infty} b_j \tau_{k-j} \right), \quad k \in \mathbb{Z}, \quad \sum_{j=1}^{\infty} b_j = 1.
\]
It obviously has a trivial solution $\tau_k = 0$. There is not much known in the literature about existence of covariance stationary solution of this model. Existence of IARCH process was discussed by Kazakevičius and Leipus (2003) and Douc et al. (2006), assuming that $\omega > 0$ which implies $E\tau_k = \infty$.

A particular case of IARCH process, introduced by Baillie et al. (1996) to capture the long memory effect in volatility $h_k$, is the well-known FIGARCH process defined as a stationary solution to equation

(1.3) $\tau_k = \varepsilon_k \left(1 - (1 - L)^d\right) \tau_k = \varepsilon_k \left(\sum_{j=1}^{\infty} b_j \tau_{k-j}\right), \quad k \in \mathbb{Z}$.

Here $0 < d < 1/2$ is the fractional differencing parameter, $L$ is the backshift operator and the coefficients $b_j$ are determined by the generating function $B(z) = \sum_{j=1}^{\infty} b_j z^j = 1 - (1 - z)^d$. They are positive, $b_j > 0$, satisfy $\sum_{j=1}^{\infty} b_j = 1$ and decay to 0 hyperbolically slowly. Baillie et al. (1996) argue that FIGARCH process, if exist, has long memory but existence of a non-trivial finite variance solution was never shown.

The FIGARCH equation itself was the object of controversial discussion in econometric and statistical literature, see Giraitis et al. (2000a), Kazakevičius and Leipus (2003), Mikosch and Stáríč (2000, 2003), Davidson (2004). Several papers (Giraitis et al. (2000a, 2002), Kazakevičius and Leipus (2003)) claim that the FIGARCH equation has no stationary solution with finite mean $E\tau_k < \infty$ besides the trivial solution $\tau_k = 0$.

The present paper corrects the above claim. We show that for each $\mu > 0$, FIGARCH equation, has a stationary ergodic long memory solution $Y_k$ with the mean $E\tau_k = \mu$, finite variance (and possibly higher moments), and hyperbolically decaying nonsummable autocovariance function $\text{Cov}(\tau_k, \tau_0) \sim c k^{-\gamma}$, $0 < \gamma < 1$, see Theorem 2.3. Note that in the case $\omega > 0$, Douc et al. (2006) and Robinson and Zaffaroni (2006) showed existence a stationary FIGARCH process $\tau_k$ with the infinite mean $E\tau_k = \infty$, but the lack of finite mean does not allow to establish the desired property of long memory.

A possible explanation of these seemingly confusing statements about feasibility of long memory in ARCH($\infty$) model (1.1) is that many papers on ARCH models explicitly or implicitly assume $\omega > 0$. This assumption, together with covariance stationarity of $\{\tau_k\}$ implies $E\tau_k = \omega + E\tau_k (b_1 + b_2 + \ldots)$, or

(1.4) $E\tau_k = \frac{\omega}{1 - \sum_{j=1}^{\infty} b_j}$,

yielding $E\tau_k < \infty$ if $\theta := \sum_{j=1}^{\infty} b_j < 1$, and $E\tau_k = \infty$ if $\theta = 1$. However, for $\omega = 0$ and $\theta = 1$, the r.h.s. of (1.4) is undefined, and therefore it does not contradict $E\tau_k < \infty$. It turns out, that for each $\mu > 0$, FIGARCH equation (1.3) $\mu > 0$, has a covariance stationary solution $\tau_k := Y_k + \mu$ where $\{Y_k\}$ is zero mean stationary ergodic process defined in (2.5), see Theorem 2.3 below. Hence, there exist infinite many FIGARCH solutions parametrized
by the mean value $EY_k = \mu$. The trivial solution $\tau_k = 0$ corresponds to $\mu = 0$ and coincides with the minimal solution of (1.1) defined in the sense of Kazakevičius and Leipus (2003),

$$
\tau_k = \omega \varepsilon_k \left( 1 + \sum_{m=1}^{\infty} \sum_{s_m < \cdots < s_1 < k} b_{k-s_1} \cdots b_{s_m-s_{m-1}} \varepsilon_{s_1} \cdots \varepsilon_{s_m} \right).
$$

In the ARCH literature, the later is the well-know representation of a stationary solution of (1.1) in the case $\omega > 0$. For $\theta = 1$ the series on the r.h.s. does not converges in $L_1$ and therefore the expansion in (1.5) is not useful for studying the existence of $L_1$- or $L_2$-solutions of IARCH($\infty$) equation. For $\omega = 0$, it gives a zero solution.

To explain in brief how IARCH solution with mean $E\tau_k = \mu > 0$ is derived, setting $\zeta_k := (\varepsilon_k - 1)/\sigma$, $\sigma = \text{var}(\varepsilon_1)$, we rewrite (1.2), $\tau_k = (\zeta_k \sigma + 1)(\sum_{j=1}^{\infty} b_j Y_{k-j} + \mu)$, as a bilinear model

$$
Y_k = \zeta_k (\mu \sigma + \sigma \sum_{j=1}^{\infty} b_j Y_{k-j}) + \sum_{j=1}^{\infty} b_j Y_{k-j}
$$

with standartized zero mean i.i.d. innovations $\zeta_k$, $E\zeta_k = 0$. Under assumption $E\varepsilon_1^2 < \infty$, (1.6) belongs to the class of bilinear models studied in Giraitis and Surgailis (2002). That paper provides a necessary and sufficient condition for the existence of a stationary causal solution $Y_k$ of (1.6), $EY_k^2 < \infty$, given by convergent orthogonal Volterra series as in (2.5). In Theorem 2.3 we show that for FIGARCH equation (1.3), the necessary and sufficient condition for the existence of the (long memory) solution $Y_k$ given by (2.5) reduces to condition combining $E\varepsilon_0^2$ and parameter $d$:

$$
E\varepsilon_0^2 < \frac{\Gamma(1-2d)}{\Gamma(1-2d) - \Gamma^2(1-d)}.
$$

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<td>0.45</td>
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</table>

Figure 1: Graph and values of the function $V(d) = \frac{\Gamma(1-2d)}{\Gamma(1-2d) - \Gamma^2(1-d)}$. 


We conclude the paper by discussing stationary integrated AR time series models
\[ x_k - \sum_{j=1}^{\infty} b_j x_{k-j} = \xi_k \quad k \in \mathbb{Z}, \]
where \( b_j \)'s are non-negative, \( \sum_{j=1}^{\infty} b_j = 1 \), and \( \{\xi_k\} \) is stationary ergodic weakly dependent process, in particular, a white noise.

The paper is structured as follows. Section 2 contains the main results, rigorous statements and proofs. In section 2.1 we write ARCH(\(\infty\)) equation (1.1) as a bilinear equation (2.1) with zero mean innovations and investigate existence and uniqueness of its stationary solution. In section 2.2, the results derived for a bilinear model are used to obtain existence and uniqueness of stationary solution of FIGARCH, IARCH and ARCH models. In particular, existence of long memory property of solutions is established. We also show the convergence of normalized partial sums \( \sum_{k=1}^{[nt]} \tau_k \) to a fractional Brownian motion. Theorems 2.2 and 2.3 establish new results on existence and uniqueness of stationary solutions of IARCH and FIGARCH equations. Some results of Theorem 2.4 for ARCH(\(\infty\)) process follow from GS(2002). Our paper extends and complements the results of that work, in particular, correcting the claim of uniqueness of zero ARCH(\(\infty\)) solution of (1.1) for \( \theta = 1 \), \( \omega = 0 \). Section 2.3 contains examples. In Section 2.4 we discuss existence of stationary Integrated AR(\(\infty\)) processes.

In the sequel, we set \( \Pi := [-\pi, \pi] \), and write \( a_n \sim b_n \) if \( a_n/b_n \to 1 \). Moreover, \( \to_p \) and \( \to_D \) denote convergence in probability and distribution, respectively. All (in)equalities involving random variables in this paper are supposed to hold almost surely.

2 Main results

2.1 ARCH(\(\infty\)) model and bilinear equation

This section contains important auxiliary results. We rewrite ARCH(\(\infty\)) process \( \tau_k \), (1.1), generated by a non-negative i.i.d. noise \( \varepsilon_k \) as a bilinear process driven by an i.i.d. zero mean noise and analyse existence of its stationary solution. Bilinear model provides a tool for finding stationary solutions for ARCH(\(\infty\)) model (1.1) in Section 2.2.

More specifically, for ARCH(\(\infty\)) model (1.1) that has constant mean \( E\tau_k = \mu \), we set
\[ \tau_k = (\tau_k - \mu) + \mu = Y_k + \mu, \quad Y_k := \tau_k - \mu, \]
and then write a bilinear equation for the zero mean process \( Y_k \). Let \( \theta := \sum_{j=1}^{n} b_j \). We focus on two cases, a) \( \omega > 0 \) and \( 0 < \theta < 1 \), and b) \( \omega = 0 \) and \( \theta = 1 \). In case a), equality (1.4) implies that \( \mu = E\tau_k = \omega(1 - \theta) \). In case b), (1.4) does not contradict free choice of \( \mu > 0 \).

Summarising, we set
\[ \mu := \begin{cases} \omega/(1 - \theta) & \text{if } \theta < 1 \text{ and } \omega > 0, \\ \text{any real } \mu > 0 & \text{if } \theta = 1 \text{ and } \omega = 0. \end{cases} \]
Introducing zero mean i.i.d. noise \( \zeta_k := (\varepsilon_k - 1)/\sigma \), \( \sigma^2 := \text{var}(\varepsilon_1) \), ARCH(\( \infty \)) equation for \( \tau_k \) of (1.1) can be written as

\[
\tau_k = (\zeta_k \sigma + 1)(\omega + \sum_{j=1}^{\infty} b_j(\tau_{k-j} - \mu) + \theta \mu).
\]

This yields a bilinear equation for \( Y_k = \tau_k - \mu \),

\[
(2.1) \quad Y_k = \zeta_k \left( \mu \sigma + \sigma \sum_{j=1}^{\infty} b_j Y_{k-j} \right) + \sum_{j=1}^{\infty} b_j Y_{k-j}.
\]

To obtain explicit solution of (2.1) and conditions for its existence, following GS(2002), we define on the complex disk \{ \|z\| < 1 \} the functions

\[
(2.2) \quad G(z) := (1 - B(z))^{-1} = \sum_{j=0}^{\infty} g_j z^j, \quad H(z) := \frac{\sigma B(z)}{1 - B(z)} = \sum_{j=1}^{\infty} h_j z^j.
\]

Denote \( \|g\| := (\sum_{j=0}^{\infty} g_j^2)^{1/2} \) and \( \|h\| := (\sum_{j=1}^{\infty} h_j^2)^{1/2} \). Notice that \( h_j = \sigma g_j, \ j \geq 1 \) which implies \( \sigma^2(\|g\|^2 - 1) = \|h\|^2 \). Observe that for \( \theta \leq 1 \), \( B(z) = \sum_{j=0}^{\infty} b_j z^j \) is an analytic function on \{ \|z\| < 1 \}, and \( g_j \)'s in (2.2) are non-negative weights given by

\[
(2.3) \quad g_j = \sum_{m=1}^{\infty} \sum_{0 < s_{m-1} < \ldots < s_1 < j} b_{j-s_1} b_{s_m-s_{m-1}} b_{s_{m-2}-s_{m-1}} \ldots b_{s_1-1}, \quad j \geq 1, \quad g_0 = 0,
\]

which follows from equality \( (1 - B(z))^{-1} = \sum_{m=0}^{\infty} B^m(z) \). For the bilinear model (2.1), Theorem 2.2 proved in GS(2002), implies that under condition

\[
(2.4) \quad \|h\| \equiv \sigma^2 \sum_{j=1}^{\infty} g_j^2 < 1,
\]

there exists a stationary causal ergodic solution \( Y_k \) of equations (2.1) given by the convergent Volterra series

\[
(2.5) \quad Y_k := \mu \sigma \sum_{s \leq k} g_{k-s} \zeta_s v_s, \quad \text{with} \quad v_s := 1 + \sum_{m=1}^{\infty} \left( \sum_{s_m < \ldots < s_1 < s} h_{s_{m-1}-s_m} \ldots h_{s_1-1} \zeta_{s_1} \ldots \zeta_{s_m} \right).
\]

This solution is ergodic, has zero mean \( EY_k = 0 \) and covariance function

\[
(2.6) \quad EY_0 Y_k = \frac{(\mu \sigma)^2}{1 - \|h\|^2} \sum_{j=0}^{\infty} g_j g_{k+j}, \quad \text{corr}(Y_0, Y_k) = \frac{\sum_{j=0}^{\infty} g_j g_{k+j}}{\sum_{j=0}^{\infty} g_j^2}, \quad k \geq 0.
\]

Such solution is a unique zero mean stationary ergodic solution, see Theorem 2.1, below. Without assumption \( EY_k = 0 \), however, uniqueness of bivariate solution \( Y_k \) is not valid for \( \theta = 1 \), since then equation (2.1) admits also the constant solution \( Y_k = -\mu \). In Theorem 2.1 below we re-examine properties of a stationary solution of bilinear equation (2.1), and correct claim of uniqueness of solution in GS(2002, Theorems 2.2) that is invalid for \( \theta = 1 \). Finally, observe, that by (2.6) and Lemma 2.3(a), the process \{\( Y_k \)\} has spectral density

\[
(2.7) \quad f_Y(x) = \frac{\sigma^2 c_f}{2\pi} \left| 1 - \sum_{j=1}^{\infty} b_j e^{ijx} \right|^2, \quad x \in \Pi, \quad c_f := \sigma^2/(1 - \|h\|^2).
\]

6
Notation. We denote by \( \{F_k\}_{k \in \mathbb{Z}} \) the filtration \( F_k := \sigma\{\zeta_s, s \leq k\} = \sigma\{\varepsilon_s, s \leq k\}, k \in \mathbb{Z} \), generated by \( \{\varepsilon_s\} \). A random process \( \{X_k, k \in \mathbb{Z}\} \) is called adapted if \( X_k \) is \( F_k \)-measurable, for each \( k \).

Now, notice that solution of \( Y_k, (2.5) \), of a bilinear equation (2.1) can be written as a weighted sum of stationary ergodic martingale differences \( \eta_s := \zeta_s v_s \),

\[
Y_k := \mu \sigma \sum_{s \leq k} g_{k-s} \eta_s, \tag{2.8}
\]

where \( E[\eta_s | F_{s-1}] = 0 \), \( E[\eta_k^2 | F_{s-1}] = v_s \) and

\[
Ev_s^2 = 1 + \sum_{m=1}^{\infty} \sum_{s_m < \cdots < s_1 < s} h_{s-s_1}^2 h_{s_1-s_2}^2 \cdots h_{s_{m-1}-s_m}^2 = 1 + \sum_{m=1}^{\infty} ||h||^{2m} = 1/(1 - ||h||^2).
\]

By a solution of ARCH equation (1.1) and bilinear equation (2.1) we mean an \( L^2 \)-solution converging in mean square.

Definition 2.1 We say that a stationary adapted process \( \{\tau_k \geq 0\} \) is a solution of (1.1), if it has finite second moment, for each \( k \in \mathbb{Z} \), the series \( \sum_{j=1}^{\infty} b_j \tau_{k-j} \) converges in mean square and (1.1) holds.

Definition 2.2 We say that a stationary adapted process \( \{Y_k\} \) is a solution of (2.1), if it has finite second moment, for each \( k \in \mathbb{Z} \), the series \( \sum_{j=1}^{\infty} b_j Y_{k-j} \) converges in mean square and (2.1) holds.

The following corollary states straightforward 1-1 relationship between a solutions of ARCH(\( \infty \)) equation (1.1) and the bilinear equation (2.1).

Corollary 2.1 Let 0 < \( \mu < \infty \) and \( \theta \in (0, 1] \).

(i) If \( \{\tau_k\} \) is a solution of (1.1), then \( Y_k = \tau_k - \mu \) is a solution of (2.1).

(ii) Conversely, if \( \{Y_k\} \) is a solution of (2.1) such that \( Y_k \geq -\mu \), then \( \tau_k = Y_k + \mu \) is a solution of equation (1.1) with \( \varepsilon_k = \sigma \zeta_k + 1 \) and \( \omega = \mu(1 - \theta) \).

Next we establish sufficient and necessary conditions for existence of zero mean stationary ergodic adaptive solution of bilinear equation (2.1). Recall notation \( \theta = \sum_{k=1}^{\infty} b_k \).

Theorem 2.1 Let 0 < \( \theta \leq 1 \) and \( \mu > 0 \).

(i) The bilinear equation (2.1) has a zero mean finite variance stationary ergodic solution if and only if \( b_j \)'s and \( \sigma^2 \) satisfy condition (2.4). This solution is given by the convergent Volterra series \( Y_k \) in (2.5).

(ii) \( Y_k \) of (2.5) is a unique mean \( \mu \) and finite variance stationary ergodic solution.

(iii) (a) \( Y_k \) of (2.5) satisfies \( Y_k \geq -\mu \); (b) \( \tau_k = \mu + Y_k, k \in \mathbb{Z} \) is a unique zero mean finite variance stationary ergodic solution of ARCH(\( \infty \)) equation (1.1).
Proof. To prove (i), we need to show that condition (2.4) is a) "sufficient", and b) "necessary".

a) In Theorem 2.2 of GS(2002) it is shown under (2.4) equation (2.1) has zero mean finite variance stationary ergodic solution $Y_k$ given by (2.5).

The proof of claim b) is placed for convenience after the proof of (ii).

Proof of (ii). To prove uniqueness of solution $\{Y_k\}$ as in (2.5), assume that there exists another adaptive zero mean finite variance stationary ergodic solution $\{Y'_k\}$. Then the process $\{\hat{Y}_k := Y_k - Y'_k\}, E\hat{Y}_k^2 > 0$, is a zero-mean adaptive stationary ergodic and satisfies equation

$$(2.9) \quad \hat{Y}_k = \sum_{j=1}^{\infty} b_j \hat{Y}_{k-j} + z_k, \quad z_k := \sigma \zeta_k \left( \sum_{j=1}^{\infty} b_j \hat{Y}_{k-j} \right), \quad k \in \mathbb{Z}. $$

The process $\{\hat{Y}_k\}$ is adaptive, and therefore, $z_k$'s are uncorrelated variables with the constant variance $\sigma^2_z := E\hat{z}_k^2 = \sigma^4 E(\sum_{j=1}^{\infty} b_j \hat{Y}_{k-j})^2$. Observe that $E\hat{z}_k^2 > 0$ since otherwise $z_k = \sum_{j=1}^{\infty} b_j \hat{Y}_{k-j} = 0$ and hence, $\hat{Y}_k = 0$ by (2.9), which contradicts $E\hat{Y}_k^2 > 0$.

Let $Y^*_k := G(L)z_k = \sum_{j=0}^{\infty} g_j z_{k-j}, k \in \mathbb{Z}$. We shall show below that

$$(2.10) \quad \text{cov}(Y^*_k, Y^*_j) = \text{cov}({\hat{Y}}_0, \hat{Y}_k), \quad k \in \mathbb{Z}. $$

Since $\sigma^2 \sum_{j=1}^{\infty} b_j Y^*_{k-j} = \sigma B(L)(1 - B(L))^{-1} z_k = H(L)z_k = \sum_{j=1}^{\infty} h_j z_{k-j}$, this yields

$$E\hat{z}_k^2 = \sigma^2 E(\sum_{j=1}^{\infty} b_j \hat{Y}_{k-j})^2 = \sigma^2 E(\sum_{j=1}^{\infty} b_j Y^*_{k-j})^2 = E(\sum_{j=1}^{\infty} h_j z_{k-j})^2 = E[z_k^2] ||h||^2,$$

which contradicts assumption (2.4), $E||h||^2 < 1$, and $E\hat{z}_k^2 > 0$. Therefore, $E\hat{z}_k^2 = 0$, which implies $\sum_{j=1}^{\infty} b_j \hat{Y}_{k-j} = 0$ and together with (2.9) yields $\hat{Y}_k = 0$ proving uniqueness of solution.

It remains to show (2.10). By Bochner’s theorem, $\text{cov}({\hat{Y}}_0, \hat{Y}_k) = \int_{\Pi} e^{ikx} F_{\hat{Y}}(dx)$ and $\text{cov}(Y^*_0, Y^*_k) = \int_{\Pi} e^{ikx} F_{Y^*}(dx), k \in \mathbb{Z}$, where $F_{\hat{Y}}(dx)$ and $F_{Y^*}(dx)$ are the corresponding spectral measures. Hence, it suffices to prove $F_{\hat{Y}} = F_{Y^*}$. Denote $f_{\hat{Y}}(x) = |1 - B(e^{-ix})|^{-2} (\sigma^2_x/2\pi)$, $x \in \Pi$. We divide the proof of the above claim into three steps:

$$(2.11) \quad \begin{aligned}
& (a) \quad F_{Y^*}(dx) = f_{Y^*}(x)dx, \quad x \in \Pi, \\
& (b) \quad F_{\hat{Y}}(dx) = f_{Y^*}(x)dx + \sum_{i=1}^{m} c_i \delta_{x_i}, \quad x \in \Pi; \quad (c) \quad c_i = 0, \quad i = 1, ..., m,
\end{aligned}$$

where $x_i \in \Pi, i = 1, ..., m$ is a finite set of distinct points in spectrum, $\delta_x$ denotes the degenerate measure at $x$ and $c_i$’s are some non-negative constants.

To show (a), observe that by properties of spectral representation, the spectral measure $F_{Y^*}$ of $Y^*_k$ satisfies

$$F_{Y^*}(dx) = |G(e^{ix})|^2 F_z(dx) = (\sigma^2_x/2\pi)|G(e^{ix})|^2 dx = f_{Y^*}(x)dx, \quad x \in \Pi,$$

since $G(e^{ix})$ converges in $L_2$ by assumption (2.4), $F_z(dx) = (\sigma^2_x/2\pi)dx$ and $|G(e^{ix})|^2 = |1 - B(e^{ix})|^{-2}$ by Lemma 2.3(a). Hence, $F_{Y^*}$ has the spectral density $f_{Y^*}$. 8
To show (b), note that summability of $b_j \geq 0$ and equality (2.9), $(1 - B(L))\tilde{Y}_k \equiv \tilde{Y}_k - b_1\tilde{Y}_{k-1} - b_2\tilde{Y}_{k-2} - \ldots = z_k$, imply that

$$
|1 - B(e^{-ix})|F_{\tilde{Y}}(dx) = F_z(dx) = (\sigma^2/2\pi)dx, \quad x \in \Pi,
$$

which together with Lemma 2.1 yields equality $F_{\tilde{Y}}(dx) = f_{\tilde{Y}}(x)dx$ for $x \in \Pi$, $x \neq x_i$, $i = 1, \ldots, m$ except of a finite number of $x_j$’s. Since $F_{\tilde{Y}}$ is non-increasing, the only difference between $F_{\tilde{Y}}$ and $F_{Y^*}$ is a possible jump at the points $x_i$, $i = 1, \ldots, m$, which proves (b).

To show (c), we prove that $c_i = 0$ for each $i = 1, \ldots, m$. Recall that $\tilde{Y}_k$ and $Y^*_k$ have zero mean, and note that by (ii),

$$
\sum_{j=1}^{n-2} \sum_{k=1}^{m} e^{ij(x-x_i)}|c_k| \geq c_i.
$$

Since the process $\tilde{Y}_k$ is stationary ergodic, its covariances satisfy $\text{cov}(\tilde{Y}_0, \tilde{Y}_k) \to 0$ as $k \to \infty$, so that $E|n^{-1}\sum_{j=1}^{n} e^{ijx_k}| \to 0$. On the other hand, existence of the spectral density $f_{\tilde{Y}}$ of $\{Y^*_j\}$ implies $\text{cov}(Y^*_0, Y^*_k) \to 0$ as $k \to \infty$, yielding $E|n^{-1}\sum_{j=1}^{n} e^{ijx_k}| \leq n^{-2}\sum_{j,k=1}^{m} |\text{cov}(Y^*_j, Y^*_k)| \to 0$. Together with (2.13) this implies $r_{n,i} \to 0$, yielding $c_i = 0$. This completes the proof of (2.11)(c).

**Proof of (i)(b).** To verify (b), assume that $\tilde{Y}_k$ is a zero-mean finite variance ergodic adaptive solution of the bilinear equation (2.1),

$$
\tilde{Y}_k = \sum_{j=1}^{\infty} b_j \tilde{Y}_{k-j} + \tilde{z}_k, \quad \tilde{z}_k := \zeta_k(\mu \sigma + \sigma \sum_{j=1}^{\infty} b_j \tilde{Y}_{k-j}), \quad k \in \mathbb{Z}.
$$

Note that by Lemma 2.2(b), $||g|| < \infty$. We need to show that (2.4) holds.

Observe that $\tilde{z}_k$ are uncorrelated variables with the constant variance $\sigma^2 = E\tilde{z}_k^2 = (\mu \sigma)^2 + \sigma^2 E(\sum_{j=1}^{\infty} b_j \tilde{Y}_{k-j})^2 > 0$. The same argument as in the proof of (2.10) implies that the variables $Y^*_k := G(L)z_k = \sum_{j=0}^{\infty} g_j \tilde{z}_{k-j}$, $k \in \mathbb{Z}$ have property (2.10) and satisfy equation $\sigma \sum_{j=1}^{\infty} b_j Y^*_k = \sigma B(L)(1 - B(L))^{-1}z_k = H(L)z_k = \sum_{j=1}^{\infty} h_j \tilde{z}_{k-j}$. Thus,

$$
E\tilde{z}_k^2 = (\mu \sigma)^2 + \sigma^2 E(\sum_{j=1}^{\infty} b_j \tilde{Y}_{k-j})^2 = (\mu \sigma)^2 + \sigma^2 E(\sum_{j=1}^{\infty} b_j Y^*_k)^2 = (\mu \sigma)^2 + \sigma^2 E(\sum_{j=1}^{\infty} h_j \tilde{z}_{k-j})^2 = (\mu \sigma)^2 + E[\tilde{z}_k^2] ||h||^2.
$$

Since $\mu \sigma > 0$, this implies $||h||^2 < 1$, which completes the proof of (i)(b).

**Proof of (iii).** (a) To prove $\tilde{Y}_k \geq -\mu$, let $p \geq 1$. We approximate the solution $Y_k$ in (2.5) by

$$
Y_{k,p} := (\mu \sigma) \sum_{m=1}^{\infty} \left( \sum_{p<s_m<\cdots<s_1<k} g_{k-s_1}s_{s_1-s_2}\cdots h_{s_m-1-s_m} \zeta_{s_1}\cdots \zeta_{s_m} \right).
$$
Observe that $Y_{k,p}$’s satisfy equation (2.5), viz.,

\[
Y_{k,p} = \zeta_k (\mu \sigma + \sigma \sum_{j=1}^{\infty} b_j Y_{k-j,p}) + \sum_{j=1}^{\infty} b_j Y_{k-j,p}, \quad \text{for } k > p,
\]

while $Y_{k,p} = 0$, $k \leq p$. Moreover, by orthogonality of Volterra series,

\[
E(Y_k - Y_{k,p})^2 = (\mu \sigma)^2 \sum_{m=1}^{\infty} y_{kp,m}, \quad y_{kp,m} := \sum_{s_m < \cdots < s_1 \leq k, s_m \leq p} g_{k-s_1} h_{s_1-s_2}^2 \cdots h_{s_m-s_{m-1}}^2.
\]

Notice that $y_{kp,m} \leq \|g\|^2 \|h\|^{2(m-1)}$ is dominated by the converging series, since $\|h\| < 1$, and for each $m \geq 1$, $y_{kp,m} \to 0$ as $p \to -\infty$. Therefore, by the dominated convergence theorem, for each fixed $k$,

\[
\lim_{p \to -\infty} E(Y_k - Y_{k,p})^2 = 0.
\]

Since random variables $Y_k - Y_{k,p}$ are non-negative and monotonically decreasing in $p$, this implies $Y_k - Y_{k,p} \to 0$ a.s. as $p \to -\infty$. Hence, $\lim_{p \to -\infty} (Y_k - Y_{k,p}) \geq 0$ a.s., and to prove that $Y_k \geq -\mu$, it suffices to show that for any $p \in \mathbb{Z}$,

\[
Y_{k,p} \geq -\mu, \quad k \in \mathbb{Z}.
\]

We prove the latter applying induction on $k$. Clearly, (2.15) holds for $k \leq p$ because by definition $Y_{k,p} = 0 > -\mu$ for $k \leq p$. Also, (2.15) holds for $k = p + 1$ since $Y_{p+1,p} = (\mu \sigma) \zeta_{p+1} \geq -\mu$ because $(\mu \sigma) \zeta_j = (\mu \sigma) (\varepsilon_j - 1)/\sigma \geq -\mu$, for $j \in \mathbb{Z}$. Let $k > p + 1$. Assume by induction that (2.15) holds for all $k' = k - j$, $j \geq 1$. Then, by (2.14) and the inductive assumption,

\[
Y_{k,p} = \zeta_k (\mu \sigma) + (\zeta_k \sigma + 1) \left( \sum_{j=1}^{\infty} b_j Y_{k-j,p} \right) \geq \zeta_k (\mu \sigma) + (\zeta_k \sigma + 1) \left( \sum_{j=1}^{\infty} b_j \right) (-\mu)
\]

\[
\geq \zeta_k (\mu \sigma) + (\zeta_k \sigma + 1) (-\mu) \geq -\mu.
\]

This proves the induction step $k - 1 \to k$ and completes the proof of (2.15) and the claim (iii)(a), $Y_k \geq -\mu$.

(b) The claim (iii)(a) together with Theorem 2.1(i)-(ii) and Corollary 2.1 implies that $\tau_k = \mu + Y_k$, $k \in \mathbb{Z}$, is a unique zero mean finite variance stationary ergodic solution of ARCH($\infty$) equation (1.1). □

**Lemma 2.1** Let $\theta \leq 1$. Then, $1 - B(e^{i\xi})$, $x \in \Pi$ has only finite number of zeroes on $\Pi$, including $x = 0$. Moreover, the point $x = 0$ is the only zero if $b_1 > 0$, or $b_kb_{k+1} > 0$ for some $k \geq 1$. In the latter case, for any $\epsilon > 0$,

\[
\sup_{\epsilon \leq x \leq \pi} |1 - B(e^{i\xi})|^{-1} < \infty.
\]
Proof. If $\theta < 1$, then $|B(e^{ix})| \leq \theta < 1$ for all $x \in \Pi$, which clearly implies the claim of the lemma.

Let $\theta = 1$. Suppose that $B(e^{ix_0}) = 1$ for some $x_0 \in \Pi$, and $b_1 = ... = b_{p-1} = 0$, $b_p > 0$. We will prove that

\begin{equation}
(2.17) \quad B(e^{ix_0}) = e^{ipx_0}.
\end{equation}

Then, $1 - B(e^{ix_0}) = 1 - e^{ipx_0} = 0$ yields $px_0 = 0$ (mod 2$\pi$). Hence, $x_0 \in \{0, \pm 2\pi/p, ..., \pm 2\pi k/p, 0 \leq k \leq p/2\}$ which proves (a).

To show (2.17), let $b_{p+1} = ... = b_{p'} = 0$ and $b_p > 0$. (If such $p'$ does not exist, then (2.17) holds.) Then $B(e^{ix_0}) = \sum_{j=p}^{\infty} b_j e^{ijx_0} = e^{ipx_0} \sum_{j=p}^{\infty} b_j e^{ij(j-p)x_0} = e^{ipx_0} (b_p + z + w)$ where $z = b_p e^{ip'(p'-p)x_0}$ and $w := \sum_{j=p'}^{\infty} b_j e^{ij(j-p)x_0}$. Write $z = b_p \cos((p'-p)x_0) + ib_p \sin((p'-p)x_0) =: a + ib$. Then,

\[
1 = |B(e^{ix_0})| = |b_p + z + w| \leq |b_p + z| + |w| \leq b_p + |z| + |w| \leq \sum_{j=1}^{\infty} b_j = 1.
\]

This implies $|b_p + z| = b_p + |z|$, which in turn implies $((b_p + a)^2 + b_p^2)^{1/2} = b_p + (a^2 + b_p^2)^{1/2}$. Taking squares of the both sides gives $a = (a^2 + b_p^2)^{1/2}$, yielding $a \geq 0$ and $b = 0$. Hence $z = b_p$. So, $B(e^{ix_0}) = e^{ipx_0} (b_p + b_p + w)$. Repeated use of the above argument implies $w = b_{p+1} + b_{p+2} + ...$. Hence, $B(e^{ix_0}) = e^{ipx_0} (b_p + b_{p+1} + ...) = e^{ipx_0} (b_1 + b_2 + ...)$, which proves (2.17).

Lemma singles out two cases, when zero $x_0 = 0$ is unique. If $b_1 > 0$, then (2.17) implies that $1 - B(e^{ix_0}) = 1 - e^{ix_0} = 0$, which yields $x_0 = 0$. In addition, if for some $k \geq 1$, $b_kb_{k+1} > 0$, then, with $z' = b_{k+1} e^{ix_0}$ and $w' := \sum_{j=k+1}^{\infty} b_j e^{i(j-k)x_0}$,

\[
1 = |B(e^{ix_0})| = |b_k + z' + w'| \leq |b_k + z'| + |w'| \leq b_k + |z'| + |w'| \leq \sum_{j=1}^{\infty} b_j = 1.
\]

Hence, $|b_k + z'| = b_k + |z'|$, and the same argument as used in the proof of (2.17) implies that $z' = b_{k+1}$. Therefore, $e^{ix_0} = 1$ which implies $x_0 = 0$. In turn, (2.16) follows noting that $1 - B(e^{ix}) = 0$ only for $x = 0$, while for $x \in [\epsilon, 1]$ the function $|1 - B(e^{ix})| > 0$ is continuous and positive. This completes the proof of the lemma. □

Lemma 2.2 Let $\theta = \sum_{j=1}^{\infty} b_j = 1$, where $b_j$'s are non-negative. Assume that $\xi_k$ is a stationary zero mean process which spectral density $f_\xi$ is bounded away from 0 and $\infty$.

Suppose that equation

\begin{equation}
(2.18) \quad Y_k - \sum_{j=1}^{\infty} b_j Y_{k-j} = \xi_k, \quad k \in \mathbb{Z},
\end{equation}

has a stationary zero mean solution $Y_k$ with $0 < EY_k^2 < \infty$. Then,

(a) $A(x) := (1 - B(e^{ix}))^{-1}$ is $L_2$ integrable.

(b) $\|g\| < \infty$.  

11
Proof. (i) First we show (a). Let \( f(x) = |1 - B(e^{-ix})|^2 \xi(x) \), \( x \in \Pi \). Then, by the same argument as in the proof of (2.12), the corresponding spectral measures \( F_Y \) and \( F_\xi \) of stationary processes \( \{Y_k\} \) and \( \{\xi_k\} \) satisfy

\[
|1 - B(e^{-ix})|^2 F_Y(dx) = F_\xi(dx) = f_\xi(x)dx, \quad x \in \Pi,
\]

which together with Lemma 2.1 yields equality (2.11)(b): \( F_Y(dx) = f(x)dx + \sum_{i=1}^{m} c_i \delta_{x_i} \), where \( c_i \geq 0 \) are some non-negative constants. Therefore, \( \infty > EY_k = \int_{\Pi} F_Y(dx) \geq \int_{\Pi} f(x)dx \geq c \int_{\Pi} |1 - B(e^{-ix})|^2 dx \), by since by assumption \( f_\xi(x) \geq c > 0, x \in \Pi \), for some \( c > 0 \), which proves (a).

To prove (b), suppose that \( ||g|| = \infty \). Then by (a), \( ||A|| < \infty \), which by Lemma 2.3(c) implies that \( b_j > 0 \) for infinite number of \( j \)'s. For \( p \geq 1 \), denote \( B'(L) = \sum_{j=1}^{\infty} b_j^p L_j \) where \( b_j' = b_j I(j \leq p) \), and \( G'(L) = (1 - B'(L))^{-1} = \sum_{j=0}^{\infty} g_j^p L_j \). Observe that \( g_j' \geq 0 \) satisfy (2.3) with \( b_j' \) instead of \( b_j \). Rewrite (2.18) as

\[
Y_k - \sum_{j=1}^{p} b_j Y_{k-j} - y_k' = z_k, \quad y_k' := \sum_{j=p+1}^{\infty} b_j Y_{k-j},
\]

or \( (1 - B'(L)) Y_k - y_k' = z_k \). Since \( \theta' = \sum_{j=1}^{p} b_j < 1 \), standard spectral representation argument implies that \( Y_k - G'(L) y_k' = G'(L) z_k \). Setting \( y_k'' := G'(L) y_k' \) and \( z_k' := G'(L) z_k \), we obtain \( z_k' = Y_k - y_k'' \). We will show that under \( ||g|| = \infty \),

\[
(2.19) \quad \text{(c1) } E y_k''^2 \leq EY_k^2; \quad \text{(c2) } E z_k'^2 \to \infty, \quad p \to \infty.
\]

Recall that \( EY_k = 0 \), \( EY_k^2 = EY_1^2 < \infty \). Then \( E y_k''^2 \leq 2 EY_k^2 + 2E y_k'^2 \leq 6EY_1^2 < \infty \) which contradicts (c2). Hence, \( ||g|| < \infty \).

To show (c1), observe that \( y_k'' \) has spectral measure

\[
F_{y''} (dx) = |1 - B'(e^{-ix})|^{-2} \sum_{j=p+1}^{\infty} e^{-ijx} F_Y(dx)
\]

\[
\leq (1 - |B'(e^{-ix})|^{-2}) \sum_{j=p+1}^{\infty} b_j^2 F_Y(dx) \leq F_Y(dx),
\]

because \( 1 - |B'(e^{-ix})| \geq 1 - \theta' = \sum_{j=p+1}^{\infty} b_j \). Hence, \( E y_k''^2 = \int_{\Pi} F_{y''} (dx) \leq \int_{\Pi} F_Y(dx) = EY_k^2 \).

To show (2.19)(c2), notice that \( E z'^2 = E \sum_{j=0}^{\infty} g_j^2 \), and observe that by (2.3), \( \sum_{j=0}^{\infty} g_j^2 \leq \sum_{j=0}^{\infty} \theta'^m = 1/(1 - \theta') < \infty \). Hence, \( ||g''||^2 = \sum_{j=0}^{\infty} g_j^2 \leq (\sum_{j=0}^{\infty} g_j^2)^2 < \infty \). Since \( 0 \leq g_j' \leq g_j \), \( g_j' \) is non-decreasing in \( p \), and \( g_j' \to g_j \) as \( p \to \infty \), then \( ||g''||^2 \to ||g|| = \infty \) as \( p \to \infty \). This completes the proof of (2.19)(c2) and the lemma. \( \square \)

The following lemma shows that the analytic function \( G(z), \{|z| < 1\} \) extends to the unit circle \(|z| = 1\), and such extension coincides with \( A(x) := (1 - B(e^{ix}))^{-1}, x \in \Pi \) in \( L_2 \) norm. Denote \( ||A|| = (\int_{\Pi} |A(x)|^2 dx)^{1/2} \) the \( L_2 \) norm of \( A \).

**Lemma 2.3** Let \( \theta \leq 1 \).

(a) If \( ||g|| < \infty \), then \( ||A|| < \infty \) and \( A(x) = G(e^{ix}) \) a.e. in \( \Pi \).

(b) If \( ||A|| < \infty \), then \( ||g|| < \infty \) and \( A(x) = G(e^{ix}) \) a.e. in \( \Pi \).

(c) If \( b_k = O(k^{-\gamma}), k \to \infty \), for some \( \gamma > 3/2 \), then \( |1 - B(e^{ix})|^{-2} \) is not integrable on \( \Pi \).
Proof. (a) Write, using notation (2.2), \( p(x) := (1 - B(e^{ix}))^{-1} - G(e^{ix}) = (1 - B(e^{ix}))^{-1}(1 - G(e^{ix}) + B(e^{ix})G(e^{ix})) \). By (a), \( (1 - B(e^{ix}))^{-1} \neq 0 \) a.e. On the other hand, \( 1 - G(e^{ix}) + B(e^{ix})G(e^{ix}) = 0 \) in \( L_2 \). Indeed, equality \( B(e^{ix})G(e^{ix}) = \sigma^{-1}H(e^{ix}) \) holds in \( L_2 \) since \( g_j \)'s are square summable and \( b_j \)'s are summable, while \( g_j = \sigma^{-1}h_j, j \geq 1 \) implies \( \sigma^{-1}H(e^{ix}) = G(e^{ix}) - 1 \). This proves \( p(x) = 0 \) a.e. To complete the proof of (a), notice that by Parseval’s identity, \( (2\pi)^{-1}||A||^2 = ||g||^2 < \infty \).

Proof of (b). In view of (a), it suffices to show that \( ||g|| < \infty \). For that we shall show that equation

\[
(2.20) \quad y_k - \sum_{j=1}^{\infty} b_j y_{k-j} = \xi_k, \quad k \in \mathbb{Z},
\]

where \( \xi_k \)'s are i.i.d. random variables with zero mean and unit variance, has a stationary zero mean and finite variance solution, which by Lemma 2.2(i) implies \( ||g|| < \infty \). To construct such solution, denote by \( \xi_k = \int_{\Pi} e^{ikv}Z(\xi)(dv) \) the spectral representation of \( \{\xi_k\} \), and let

\[
(2.21) \quad \tilde{y}_k = \int_{\Pi} e^{ikx}A(x)Z(\xi)(dx), \quad k \in \mathbb{Z}.
\]

Since \( ||A|| < \infty \), \( \tilde{y}_k \) is a stationary process with the spectral measure \( F_y(dx) = |A(x)|^2F(\xi)(dx) = (2\pi)^{-1}|A(x)|^2dx, \) \( E\tilde{y}_k = 0 \) and variance \( E\tilde{y}_k^2 = \int_{\Pi} F_y(dx) < \infty \). To show, that \( \tilde{y}_k \) is solution of (2.20), observe that function \( 1 - \sum_{j=1}^{\infty} b_j e^{-ijx} \) is bounded and, therefore, \( L_2(F_y) \) integrable. Then, by properties of spectral representation of stationary times series, see, e.g., Theorem 2.2.1 in Giraitis, Koul and Surgailis (2012) (below referred to as GKS(2012)), \( \tilde{y}_k = \sum_{j=1}^{\infty} b_j y_{k-j} = \int_{\Pi} e^{ikx}(1 - \sum_{j=1}^{\infty} b_j e^{-ijx})A(x)Z(\xi)(dx) = \int_{\Pi} e^{ikx}Z(\xi)(dx) = \xi_k \), and therefore \( \tilde{y}_k \) is a solution of (2.20). This completes the proof of (b).

Proof of (c). Without restriction of generality assume that \( \gamma < 2 \). Property \( b_k = O(k^{-\gamma}) \), \( k \to \infty \), with \( 3/2 < \gamma < 2 \) implies, see p.128 in GKS(2012), that \( B(e^{ix}) \) is a continuous Lipschitz function from the class \( \Lambda_\beta \) for any \( 0 < \beta < \gamma - 1 \), and therefore, \( |1 - B(e^{ix})| = |B(1) - B(e^{ix})| \leq C|x|^{2\beta} \). Selecting \( 1/2 < \beta_0 < \gamma - 1 \), we obtain \( |1 - B(e^{ix})|^{-2} \geq C|x|^{-2\beta_0} \). where the left hand side is not integrable on \( \Pi \). This proves (c). \( \square \)

2.2 Stationary solutions of FIGARCH, IARCH and ARCH(\( \infty \)) equations

We now are ready to establish existence of stationary IARCH and FIGARCH processes with finite variance. Theorems 2.2 and 2.3 below show that covariance stationary IARCH and FIGARCH processes exist, and that long memory is their inherited feature. The proofs are based on results obtained for a bilinear model (2.1) in Section 2.1.

For convenience we summarize some established facts used in Theorems 2.2-2.4 below. Suppose that \( \tau_k \) is a stationary ergodic adaptive finite variance solution of ARCH(\( \infty \)) equation (1.1) with the mean \( E\tau_k = \mu > 0 \). Then, by Lemma 2.2, \( ||g|| < \infty \) and \( ||A|| < \infty \). By
Theorem 2.1 (iii) such solution \( \tau_k \) can be written as
\[
\tau_k = \mu + Y_{\mu,k}, \quad k \in \mathbb{Z},
\]
where \( Y_{\mu,k} \equiv Y_k = \mu \sigma \sum_{s \leq k} g_{k-s}\eta_s \) is the moving average process (2.8) of stationary ergodic martingale differences \( \eta_s \). Coefficients \( g_j \) can be viewed both as coefficients of the analytic function \( G(z) \), or Fourier coefficients of the transfer function \( A(x) = (1 - B(e^{ix}))^{-1}, x \in \Pi \).

By Lemma 2.3, they coincide. The weights \( g_j \) are non-negative and satisfy (2.3).

By Theorem 2.1, such solution \( \tau_k \) exists if and only if
\[
\sigma^2 \sum_{j=1}^{\infty} g_j^2 < 1.
\]
In turn, \( \sigma^2 = E\epsilon_1^2 - 1 \), and by Parseval’s identity, \( ||g||^2 := \sum_{j=0}^{\infty} g_j^2 = (2\pi)^{-1} \int_{\Pi} |1 - B(e^{ix})|^{-2} dx = (2\pi)^{-1} ||A||^2 \). Thus (2.23) is equivalent to
\[
\sigma^2(||A||^2/(2\pi) - 1) < 1.
\]
If \( b_1 > 0 \), then verification of \( ||A|| < \infty \) reduces to verification of integrability of \( |1 - B(e^{ix})|^{-2} \) in the neighbourhood of \( x = 0 \), see (2.16) of Lemma 2.1.

The process \( \tau_k \) is nonnegative, has mean \( E\tau_k = \mu \), covariance function
\[
\text{cov}(\tau_0, \tau_k) = \mu^2 c_f \sum_{j=0}^{\infty} g_jg_{k+j}, \quad k \geq 0, \quad c_f = \frac{\sigma^2}{1 - \sigma^2(||g||^2 - 1)},
\]
and the spectral density \( f(x) = \mu^2(\sigma_f / 2\pi)|1 - B(e^{ix})|^{-2}, x \in \Pi \), see Theorem 2.1(iii) and (2.7).

Conditional mean \( s_k \) (volatility of ARCH process) of \( \tau_k \) is a stationary ergodic process given by
\[
s_k := E[\tau_k |\mathcal{F}_{k-1}] = \mu + E[Y_{\mu,k} |\mathcal{F}_{k-1}] = \mu + (\mu \sigma) \sum_{s<k} g_{k-s}\eta_s, \quad k \in \mathbb{Z};
\]
it has mean \( Es_k = \mu \) and auto-covariance function
\[
\text{cov}(s_0, s_k) = \mu^2 c_f \sum_{j=1}^{\infty} g_jg_{k+j}, \quad k \geq 0.
\]

The following theorem summarizes findings on existence of a stationary finite variance IARCH(\( \infty \)) process. Its shows that under assumption (2.23), IARCH equations (1.2) has infinite number of IARCH solutions parameterised by the mean parameter \( \mu > 0 \), that are unique for each \( \mu \). Part (iii) of the theorem shows that IARCH process always has long memory, viz. its auto-covariance is not summable, and spectral density is unbounded at zero frequency.
Theorem 2.2 IARCH equation (1.2) has a non-zero stationary ergodic finite variance solution if and only if $\sigma^2 = \text{var}(\varepsilon_1)$ and $b_j$’s satisfy condition (2.23).

(i) Then, for each $\mu > 0$, the process

$$\tau_k = \mu + Y_{\mu,k}$$

where $Y_{\mu,k}$ is given by (2.22) is a unique stationary ergodic solution of (1.2) with the mean $E\tau_k = \mu$.

(ii) Covariance functions $\text{cov}(\tau_0, \tau_k)$, $\text{cov}(s_0, s_k)$, $k \geq 0$ are given by

$$\text{cov}(\tau_0, \tau_k) = \mu^2 c_f \sum_{j=0}^{\infty} g_j g_{k+j}, \quad \text{cov}(s_0, s_k) = \mu^2 c_f \left( \sum_{j=0}^{\infty} g_j g_{k+j} - g_0 g_k \right).$$

Condition (2.23) is equivalent to $\text{var}(\tau_k) < \infty$.

(iii) Covariance function $\text{cov}(\tau_0, \tau_k)$ is non-negative and not summable,

$$\sum_{k \in \mathbb{Z}} \text{cov}(\tau_0, \tau_k) = \infty.$$

Moreover, $\{\tau_k\}$ has spectral density $f(x) = \mu^2(\sigma_f/2\pi)|1-B(e^{ix})|^{-2}$, $x \in \Pi$, that is unbounded at zero frequency.

Proof of Theorem 2.2. The claim about existence of a stationary solution and part (i) follow from Theorem 2.1(iii) and Corollary 2.1, (ii) follows from (2.25) and (2.26), while equivalence of (2.23) to $\text{var}(\tau_k) < \infty$ follows from equality $\text{var}(\tau_0) = \mu^2 c_f ||g||$.

To show (iii), notice that $\text{cov}(\tau_0, \tau_k) \geq 0$ because $g_j$ in (2.3) are non-negative. Observe that for $\theta = 1$, $\sum_{j=0}^{\infty} g_j = \infty$. Indeed, for $r \in (0, 1)$, $(1-B(r))^{-1} = (1-r)^{-1} = \sum_{j=0}^{\infty} g_j r^j \leq \sum_{j=0}^{\infty} g_j \to \infty$ as $r \to 1$. Hence, by (2.27), $\sum_{k=0}^{\infty} \text{cov}(\tau_0, \tau_k) \geq \mu^2 c_f \left( \sum_{j=0}^{\infty} g_j \right)^2 = \infty$. The spectral density $f(x)$ exists by (2.7), and $f(0) = \mu^2(\sigma_f/2\pi)|1-\theta|^{-2} = \infty$. This completes the proof of the theorem. □

The next theorem shows that a stationary ergodic FIGARCH process (1.3) exists and has long memory, viz. its covariances $\text{cov}(\tau_0, \tau_k)$ decays hyperbolically slowly as in (2.30). Hence, following the terminology in GS(2002), the FIGARCH duration process $\{\tau_k\}$ has long memory both in levels and conditional mean.

Theorem 2.3 For FIGARCH model (1.3) with $d \in (0, 1/2)$, condition (2.23) is equivalent to

$$E\varepsilon_0^2 < \frac{\Gamma(1-2d)}{\Gamma(1-2d)^{-1/2(1-d)}}.$$ 

Under (2.29), the claims of Theorem 2.2(i)-(iii) hold. Moreover, as $k \to \infty$, covariances and the spectral density of FIGARCH process $\tau_k$ satisfy

$$\text{cov}(\tau_0, \tau_k) \sim \mu^2 c_f c_g k^{-1+2d}, \quad \text{cov}(s_0, s_k) \sim \mu^2 c_f c_g k^{-1+2d}, \quad g_k \sim c_g k^{-1+d},$$

15
where \( c_\gamma := \Gamma(1-2d)/\{\Gamma(d)\Gamma(1-d)\} \), \( c_g = 1/\Gamma(d) \), and
\[
(2.31) \quad f(x) = \mu^2(\sigma_f/2\pi)|1-e^{ix}|^{-2d} \sim \mu^2(\sigma_f/2\pi)|x|^{-2d}, \quad x \to 0.
\]

**Proof of Theorem 2.4.** Here the functions \( B(z) = 1 - (1 - z)^d = \sum_{j=1}^{\infty} b_j \) and \( G(z) = (1 - z)^{-d} = \sum_{j=1}^{\infty} g_j \) have coefficients
\[
b_j = -\frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)}, \quad g_j = \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(1-d)}, \quad j \geq 1, \quad g_0 = 1,
\]
where \( b_j > 0 \) and \( \sum_{j=1}^{\infty} b_j = 1 \). They satisfy
\[
(2.32) \quad b_j \sim -j^{-d-1}/\Gamma(-d), \quad g_j \sim j^{d-1}/\Gamma(d), \quad j \to \infty.
\]
By (2.25), the covariance \( \text{cov}(\tau_0, \tau_k) = \mu^2 c_f \gamma_k \) is a multiple of that of ARFIMA(0, d, 0) process, viz.,
\[
\gamma_k := \sum_{j=0}^{\infty} g_j g_{k+j} = \frac{\Gamma(k+d)}{\Gamma(k-d+1)\Gamma(1-d)} \frac{\Gamma(1-2d)}{\Gamma(1-d)}.
\]
In particular, \( \gamma_0 := \sum_{j=0}^{\infty} g_j^2 = \Gamma(1-2d)/\Gamma^2(1-d) \), and
\[
\gamma_k = -\frac{\Gamma(k+d)}{\Gamma(k-d+1)\Gamma(1-d)} \frac{\Gamma(1-2d)}{\Gamma(1-d)} \sim c_{\gamma} k^{-1+2d}, \quad c_{\gamma} := \frac{\Gamma(1-2d)}{\Gamma(1-d)},
\]
see, e.g., Chapter 7 of GKS(2012). Hence, condition (2.23) is equivalent to \( \sigma^2(||g||^2 - 1) = \sigma^2(\gamma_0 - 1) < 1 \), i.e. (2.29). Obviously, under (2.29), the claims of Theorem 2.2(i)-(iii) are valid. Since \( g_j \sim j^{-d-1}/\Gamma(d) = o(j^{-1+2d}) \), this together with (2.26) implies \( \text{cov}(s_0, s_k) \sim \text{cov}(\tau_0, \tau_k) \), as \( k \to \infty \), which proves (2.30), while asymptics (2.31) is obvious. \( \square \)

For comparison, the next theorem summarizes results on existence of a stationary finite variance solution of an ARCH(\( \infty \)) equation with \( \theta = \sum_{j=1}^{\infty} b_j < 1 \), obtained in GS(2002). It allows only one permissible value of the mean, \( \mu = \omega/(1 - \theta) \). In sharp contrast to finite variance stationary IARCH process, which can have only long memory, a stationary finite variance process ARCH process when \( \theta < 1 \) always has short memory, viz. its covariance function is non-negative and absolutely summable, see (2.33).

**Theorem 2.4** ARCH(\( \infty \)) equation (1.1) with \( \omega > 0 \) and \( \theta = \sum_{j=1}^{\infty} b_j < 1 \) has a unique stationary ergodic finite variance solution if and only if condition (2.23) is satisfied.

(i) Such solution is given by \( \tau_k = \mu + Y_{\mu,k} \), (2.22), with \( \mu = \omega/(1 - \theta) \). It has finite mean \( E\tau_k = \omega/(1 - \theta) \) and covariance function given by (2.25).

(ii) Covariances \( \text{cov}(\tau_0, \tau_k), \text{cov}(s_0, s_k) \) and weights \( g_k \) are non-negative and absolutely summable:
\[
(2.33) \quad \sum_{k \in \mathbb{Z}} \text{cov}(\tau_0, \tau_k) < \infty, \quad \sum_{k \in \mathbb{Z}} \text{cov}(s_0, s_k) < \infty, \quad \sum_{k \in \mathbb{Z}} g_k < \infty.
\]

**Proof of Theorem 2.4.** (i) follows from Theorem 2.1. To show (ii), notice that \( g_j \geq 0 \) given by (2.3) satisfy \( \sum_{j=0}^{\infty} g_j \leq \sum_{m=0}^{\infty} \theta^m < \infty \) for \( \theta < 1 \). Together with (2.25) and (2.26) this proves (2.33). \( \square \)

In the next proposition, we discuss convergence of partial sum process of \( \tau_k \)'s. Below \( \{B(t), t \in [0,1]\} \) denotes a Brownian motion with variance \( EB^2(t) = t \) and \( \{B_{d+1/2}(t), t \in [0,1]\} \) a fractional Brownian motion with variance \( EB^2_{d+1/2}(t) = t^{2d+1} \).
Proposition 2.1 Suppose that (2.23) holds.

(i) Then solution (2.22) of ARCH(∞) equation (1.1) with \( \omega > 0 \) and \( \theta < 1 \) in Theorem 2.4 satisfies

\[
(2.34) \quad n^{-1/2} \sum_{k=1}^{[nt]} (\tau_k - E\tau_k) \to_{D[0,1]} s^2 B(t), \quad s^2 := \sum_{k \in \mathbb{Z}} \text{cov}(\tau_0, \tau_k).
\]

(ii) Solution (2.22) of FIGARCH equation (1.3) in Theorem 2.3 satisfies

\[
(2.35) \quad n^{-1/2+d} \sum_{k=1}^{[nt]} (\tau_k - E\tau_k) \to_{D[0,1]} s^2_d B_d^{1/2}(t), \quad s^2_d = \mu^2 c_f c_{\gamma} / (d(1 + 2d)).
\]

Proof. (i) The convergence in (2.34) is a consequence of (2.33), the associativity property of \( \{\tau_k\} \) and the functional CLT by Newman and Wright (1981), see Giraitis et al. (2007).

(ii) Since by (2.8) \( \tau_k \) is a moving average of stationary martingale differences, convergence (2.35) follows from Theorem 3.1 in Abadir et al. (2014), see also Theorem 6.2 in GS(2002).

The following proposition provides a sufficient condition for the existence of even moments \( \text{E}\tau_k^{2p} < \infty, p = 2, 3, \ldots \). Denote \( m_p := E\zeta^p_0, \|h\|_p := \sum_{j=1}^{\infty} |h_j|^p = \sum_{j=1}^{\infty} |\sigma g_j|^p, p = 1, 2, \ldots \).

Proposition 2.2 Let \( \{\tau_k := \mu + Y_{\mu,k}\} \) be the solution of (1.1) defined as in (2.22), in cases \( \omega > 0, 0 < \theta < 1 \); or \( \omega = 0, \theta = 1 \). Assume that \( m_{2p} < \infty \) for some integer \( p \geq 1 \) and

\[
\sum_{i=2}^{p} \binom{2p}{i} \|h\|_p^i |m_p| < 1.
\]

Then \( \text{E}\tau_{k,\mu}^{2p} < \infty \).

Proof. This result follows from Proposition 2.3 in Giraitis, Leipus, Surgailis and Robinson (2004). PATIKRINTI □

Remark 2.1 The above results imply that IARCH model (1.2) with \( \omega = 0 \) does not have a stationary solution with finite variance, if weights \( b_j \) decay to zero too "fast", e.g. \( b_j = O(j^{-\gamma}) \) for some \( \gamma > 3/2 \), see Lemma 2.2(a) and Lemma 2.3(c). Hence, covariance stationary IARCH process does not exist if \( b_j \)'s cut off or decay exponentially fast. Its existence requires the integral \( \int_{-\Pi}^{\Pi} |1 - B(e^{-ix})|^{-2} dx \) to be finite, and to satisfy (2.24). FIGARCH model (2.3) with \( d > 1/2 \) also does not have covariance stationary solution since then the function \( |1 - B(e^{-ix})|^{-2} = |1 - e^{-ix}|^{-2d} \sim |x|^{-2d} \) as \( x \to 0 \) is not integrable.

On the contrary, sufficient conditions for existence of a stationary IARCH process with non-zero intercept \( \omega > 0 \) and infinite mean \( \text{E}\tau_k = \infty \), obtained by Kazakevičius and Leipus (2003) and Douc et al. (2006), require the opposite: coefficients \( b_j \) need to cut off or decay to zero exponentially fast.
2.3 Examples

Example 2.1 Autoregressive Conditional Duration model ACD(1, 0) of Engle and Russell (1998) of order 1 is given by

\[ \tau_k = \varepsilon_k (\omega + b\tau_{k-1}), \]

where \( \omega > 0, 0 < b < 1, E\varepsilon_1 = 1 \) and \( \sigma^2 = \text{var}(\varepsilon_1) < \infty. \) Then

\[ B(z) = b z, \quad G(z) = (1 - b z)^{-1} = \sum_{j=0}^{\infty} b^j z^j. \]

Here, \( g_j = b^j, h_j = \sigma b^j, j \geq 1. \) So,

\[ ||h||^2 = \sum_{j=1}^{\infty} h_j^2 = \sigma^2 \sum_{j=1}^{\infty} b^{2j} = \sigma^2 b^2/(1 - b^2). \]

The corresponding bilinear model (2.1) writes as

\[ (2.36) \quad Y_k = \zeta_k (\mu + \sigma b Y_{k-1}) + bY_{k-1}, \quad \mu = \omega/(1 - b), \quad \zeta_k = (\varepsilon_k - 1)/\sigma. \]

Condition \( ||h|| < 1 \) of (2.4) becomes \( \sigma^2 b^2/(1 - b^2) < 1, \) or

\[ E\varepsilon_1^2 < 1/b^2. \]

By (2.6), the covariance of duration process in (2.36) is a multiple of that of AR(1), viz.,

\[ \text{cov}(\tau_0, \tau_k) = \mu^2 c_f \sum_{j=0}^{\infty} g_j g_{k+j} = \mu^2 c_f \sum_{j=0}^{\infty} b^j b^{k+j} = (\frac{\mu^2 c_f}{1 - b^2}) b^k, \quad k \geq 0, \]

where \( c_f = \sigma^2/(1 - ||h||^2). \) Finally, by (2.22), \( \tau_k = \mu + Y_{\mu,k}, \) where \( Y_{\mu,k} \) is defined with corresponding \( g_j \)'s and \( h_j \)'s.

Example 2.2 (IARCH) We consider IARCH model generalizes FIGARCH model (1.3). It allows for additional parametric structure and preserves long memory properties (2.30) and (2.31). Set, for \( d \in (0, 1/2), \)

\[ \tau_k = \varepsilon_k B(L) \tau_k = \varepsilon_k \sum_{j=1}^{\infty} b_j \tau_{k-j}, \quad B(L) = (1 - (1 - L)^d) P(L). \]

Recall that in IARCH model, \( b_j \)'s are non-negative and \( \sum_{j=1}^{\infty} b_j = 1. \) For that, we assume that \( P(z) = \sum_{j=0}^{\infty} p_j z^j \) where \( p_j \)'s are non-negative, \( p_0 > 0 \) and \( \sum_{j=0}^{\infty} p_j = 1. \) Since \( (1 - (1 - L)^d) = \sum_{j=1}^{\infty} \tilde{b}_j L^j \) has non-negative coefficients \( \tilde{b}_j, \) see proof of Theorem 2.3, the latter implies \( \tilde{b}_j \geq 0, j \geq 1, \) and \( \sum_{j=1}^{\infty} b_j = B(1) = P(1) = 1. \) Moreover, \( b_1 = \pi_1 p_0 > 0. \) Assume in addition that \( \sum_{j=0}^{\infty} p_j j < \infty, \) which together with \( P(1) = 1 \) implies \( |P(e^{ix}) - P(1)| \leq |x| \sum_{j=0}^{\infty} p_j j \leq C|x|. \)

First we show that \( ||g|| < \infty. \) Recall that \( ||g||^2 = (2\pi)^{-1} ||A||^2, A(x) = (1 - B(e^{ix}))^{-1}. \)

Since \( b_1 > 0, (2.16) \) implies that \( |A(x)| \) is bounded on \( [\epsilon, \pi] \) for any \( \epsilon > 0. \) Hence, it remains to show that \( |A(x)|^2 \) is integrable at \( x = 0. \) Notice that

\[ 1 - B(e^{ix}) = 1 - (1 - (1 - e^{ix})^d) P(e^{ix}) = (1 - e^{ix})^d h(x), \]

18
where \( h(x) := P(e^{ix}) - (P(e^{ix}) - 1)(1 - e^{ix})^{-d} \). Then, as \( x \to 0 \), \( h(x) \to h(1) = P(1) = 1 \) implies \(|A(x)|^2 \sim |x|^{-2d}, \ x \to 0\). Hence, \(|A|^2\) is integrable for \( d \in (0, 1/2)\).

In addition, if \( \sigma^2(||g||^2 - 1) < 1 \), then the results of Theorem 2.2 hold. In particular, spectral density

\[
(2.37) \quad f(x) = c_0 |1 - B(e^{ix})|^{-2} = c_0 |1 - e^{ix}|^{-2d} h(x)|^{-2}, \quad c_0 := \mu^2(\sigma_f/2\pi)
\]

of \( \{\tau_k\} \) has property \( f(x) \sim c_0 |x|^{-2d} \) as \( x \to 0 \), and is bounded on intervals \([\epsilon, \pi]\) for \( \epsilon > 0 \). Moreover, covariances of \( \tau_k \)'s decay hyperbolically slowly:

\[
\text{cov}(\tau_0, \tau_k) \sim c_0 c_\gamma k^{-1+2d}, \quad c_\gamma := 2\Gamma(1 - 2d) \sin(\pi d).
\]

This claim follows from Lemma 2.3.1 in GKS(2012), noting that the spectral density \( f(x) \) can be written as \( f(x) = c_0 |x|^{-2d} p(x) \), where \( p(x) = (|1 - e^{ix}|/|x|)^{-2d} h(x)|^{-2} \) is a Zygmund slowly varying function at zero, and \( p(0) = 1 \). The latter follows by Proposition 3.1.1 in GKS(2012), noting that derivative \( \dot{p}(x) = (d/dx)g(x) \) satisfies \( \int_{\Pi} |\dot{p}(x)|dx < \infty, p(x) \to p(0) \) and \( x\dot{p}(x) \to 0 \) as \( x \to 0 \).

2.4 Stationary Integrated AR(\(\infty\)) process

In time series literature long memory processes are usually identified with fractional filtering/differencing operation, used by Hosking (1981) and Porter-Hudak (1990) to define ARFIMA\((p,d,q)\) models; for generalised versions of fractional filters see Leipus and Viano (2000).

Below we show that long memory is the inherited feature of stationary Integrated AR\((\infty)\) processes. Its origins lay in itegration rather then in fractional differencing/filtering, which produces specific examples of parametric integrated weights but itself does not explain in full the phenomenon how and why long memory is induced. This often lead to controversies justifying the use of long memory processes and explaining their generating mechanism, see e.g. Lieberman and Phillips (2008) for illustrative analysis of how long memory may arise in realized volatility.

In this section we consider Integrated AR\((\infty)\) time series model

\[
(2.38) \quad x_k - \sum_{j=1}^{\infty} b_j x_{k-j} = \xi_k, \quad k \in \mathbb{Z},
\]

where \( b_j \)'s are non-negative, \( \sum_{j=1}^{\infty} b_j = 1 \), and \( \{\xi_k\} \) is a sequence of stationary ergodic uncorrelated noises with zero mean and finite variance, \( \sigma^2 = E\xi_1^2 < \infty \).

Similarly as in IARCH case, it is of interest to find sufficient and necessary conditions for existence of a stationary finite variance solution to equation (2.38) and investigate its uniqueness and the property of long memory. Contrary to IARCH model, the noise \( \xi_k \) does
not depend on \(\{x_k\}\), so conditions are expected to be less restrictive. As in IARCH case, a stationary solution \(\tilde{x}_k\) of (2.38), if exist, is not unique, since then \(\tilde{x}_k + \mu\) is also a solution of (2.38).

Clearly, \(b_j\)'s cannot cut off or decay to 0 too fast, viz. a unit root model \(x_k - x_{k-1} = \xi_k\) does not have a stationary solution. There are two ways to construct a stationary solution. Firstly, using weights \(g_j\) of analytic function \(G(z) = (1 - B(z)^{-1} = \sum_{j=0}^{\infty} g_j z^j, |z| < 1.\)

They are uniquely defined, and if \(||g|| < \infty\), they define a stationary zero mean and finite variance process,

\[
\tilde{x}_k = \sum_{j=0}^{\infty} g_j \xi_{k-j}, \quad k \in \mathbb{Z}. \tag{2.39}
\]

Alternatively, if transfer function \(A(x) = (1 - B(e^{ix}))^{-1}, x \in \Pi\) is \(L_2\) integrable, \(||A|| < \infty\), then its Fourier coefficients \(g'_j = (2\pi)^{-1} \int_\Pi A(x)e^{-ixj}dx, j \geq 0\) have property \(||g'|| < \infty\) and define a stationary zero mean and finite variance process,

\[
\tilde{x}'_k = \sum_{j=0}^{\infty} g'_j \xi_{k-j}, \quad k \in \mathbb{Z}. \tag{2.40}
\]

The next theorem establishes equivalence of conditions \(||g|| < \infty\) and \(||A|| < \infty\), equality of the weights \(g_j = g'_j\), and \(L_2\) equality of the functions \(G(e^{ix}) = A(x), x \in \Pi\). It discusses existence of a stationary solution of IAR(\(\infty\)) equation, its long memory property and uniqueness.

**Theorem 2.5** Integrated AR equation (2.38) has a stationary ergodic finite variance solution if and only if \(||g|| < \infty\) holds.

(i) \(||g|| < \infty\) implies \(||A|| < \infty\), and \(g_j = g'_j, j \geq 0.\)

(ii) If \(||g|| < \infty\), then \(\tilde{x}_k = \tilde{x}'_k\) in (2.39) and (2.40), and for each real \(\mu,\)

\[
x_{k,\mu} = \mu + \tilde{x}_k, \quad k \in \mathbb{Z},
\]

is a unique stationary ergodic adaptive solution of (2.38) with the mean \(E x_{k,\mu} = \mu.\)

(iii) Solution \(x_{k,\mu}\) has non-negative and not summable covariances,

\[
\text{cov}(x_{0,\mu}, x_{k,\mu}) = \sigma^2 \xi \sum_{j=0}^{\infty} g_j g_{k+j} \geq 0, \quad \sum_{k \in \mathbb{Z}} \text{cov}(x_{0,\mu}, x_{k,\mu}) = \infty. \tag{2.41}
\]

Its spectral density \(f(x) = (\sigma^2 \xi / 2\pi)|1 - B(e^{ix})|^{-2}, x \in \Pi\) is unbounded at zero frequency.
Remark 2.2 As IARCH models, see Remark 2.1, IAR(∞) model (2.38) does not have a stationary finite variance solution, if the weights $b_j$ decay to zero too fast, i.e. $b_j = O(j^{-\gamma})$ for some $\gamma > 3/2$, e.g. in the unit root model $x_k - x_{k-1} = \xi_k$. However, Integrated AR(∞) equation has a stationary process, if a singular “unit root” is distributed over infinite number of $b_j$’s that decay not too fast, so that $|1 - B(e^{-ix})|^{-2}$ is integrable.
There exists a large variety of stationary Integrated AR(\(\infty\)) processes. They always have long memory, viz. their covariances are non-sumable and their spectral density is not bounded at zero frequency. However, their covariances may not decay at hyperbolic rate \(k^{-1+2d}\), and spectral densities may not explode at zero frequency at the rate \(|x|^{-2d}\). The latter are key features of fractionally integrated ARFIMA(0, d, 0) models,

\[(1 - L)^d x_k = x_k - B(L)x_k = x_k - \sum_{j=1}^{\infty} b_j x_{k-j} = \xi_k, \quad 0 < d < 1/2,\]

where \(B(L) = 1 - (1 - L)^d\). Here, \(b_j\)'s are non-negative parameterized by \(d\), \(\sum_{j=1}^{\infty} b_j = 1\) they satisfy (2.32), and the function \(|1 - B(e^{-ix})|^{-2} = |1 - e^{-ix}|^{-2d} \sim |x|^{-2d}\) as \(x \to 0\) is integrable for \(d \in (0, 1/2)\).

A wider class of IAR(\(\infty\)) stationary processes is defined by equation

\[(2.42) \quad x_k - B(L)x_k = \xi_k, \quad B(L) = (1 - (1 - L)^d) P(L), \quad 0 < d < 1/2,\]

where \(P(z)\) is as in Example 2.2. Then in \(B(L) = \sum_{j=1}^{\infty} b_j L^j\) \(b_j\)'s are non-negative, sum up to 1, and the function \(|1 - B(e^{-ix})|^{-2} = |1 - e^{-ix}|^{-2d}|h(x)|^{-2}\) is integrable, see Example 2.2.

Stationary solution \(\{x_k\}\) of (2.42) has spectral density

\[f(x) = (\sigma^2 / 2\pi)|1 - B(e^{-ix})|^{-2} = (\sigma^2 / 2\pi)|1 - e^{-ix}|^{-2d}|h(x)|^{-2}, \quad x \in \Pi,\]

that satisfies \(f(x) \sim (\sigma^2 / 2\pi)|x|^{-2d}\) as \(x \to 0\), and is continuous bounded function of intervals \([\epsilon, \pi]\), for \(\epsilon > 0\), where \(h\) is as in (2.37). Covariances of \(\{x_k\}\) decay hyperbolically slowly:

\[\text{cov}(\tau_{0, \mu}, \tau_{k, \mu}) \sim c_{\gamma} k^{-1+2d}, \quad c_{\gamma} := 2\Gamma(1 - 2d) \sin(\pi d).\]

Notice that IAR(\(\infty\)) model (2.42) defines a parametric class of long memory processes that are different from the standard ARFIMA(\(p, d, q\)) models. For example, the model \(x_k = (1 - (1 - L)^d)(1 + rL)(1 + r)^{-1} x_k + \xi_k\) and ARFIMA(1, d, 0) model \((1 - L)^d(1 + rL)x_k = \xi_k\) have different covariance functions.

Definition of Integrated AR model (2.38) with a white noise \(\{\xi_k\}\) may be restrictive. Next theorem establishes long memory of stationary IAR(\(\infty\)) processes driven by a stationary short memory noise \(\{\xi_k\}\) which spectral density \(f_\xi\) has property

\[(2.43) \quad c_1 \leq f_\xi(x) \leq c_2, \quad x \in \Pi, \quad \exists 0 < c_1 < c_2 < \infty.\]

**Theorem 2.6** Consider IAR(\(\infty\)) equation (2.38) where a white noise process \(\{\xi_k\}\) is replaced by a stationary zero mean ergodic process \(\{\xi_k\}\) which spectral density \(f_\xi(x)\) satisfies (2.43).

Then, Theorem 2.5 remain true with the following modification of (iii):

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22
(iii) Solution \( \{ x_{k,\mu} \} \) has spectral density \( f(x) = |1 - B(e^{ix})|^{-2} f_\xi(x), \quad x \in \Pi \) which is unbounded at zero frequency. In addition, if \( |f_\xi(x) - f_\xi(0)| = O(|x|) \) as \( x \to 0 \), then

\[
\sum_{k \in \mathbb{Z}} \text{cov}(x_{0,\mu}, x_{k,\mu}) = \infty.
\]

**Proof of Theorem 2.6.** We verify (iii)''. (The other claims follow using the same argument as in the proof of Theorem 2.5.)

Existence of the spectral density \( F_x(dx)/dx = |1 - B(e^{ix})|^{-2} f_\xi(x)dx = f(x) \) of the spectral measure \( F_x(dx) \) of \( \{ x_{k,\mu} \} \) follows using similar argument as in the proof of (2.11)(b,c).

To show (2.44), denote \( D_n(x) := \sum_{k=0}^n e^{ikx} \). Then,

\[
\sum_{k=0}^n \text{cov}(x_{0,\mu}, x_{k,\mu}) = \int_{\Pi} D_n(x) f(x) dx = f_\xi(0) \int_{\Pi} D_n(x) |1 - B(e^{ix})|^{-2} dx
\]

\[
+ \int_{\Pi} D_n(x) |1 - B(e^{ix})|^{-2} (f_\xi(x) - f_\xi(0)) dx =: f_\xi(0) i_{n,1} + i_{n,2}.
\]

We prove (2.44) by showing that \( i_{n,1} \to \infty, \quad i_{n,2} = O(1) \). Notice that \( i_{n,1} = \sum_{k=0}^n \gamma_k \) where \( \gamma_k := \sum_{j=1}^\infty g_j g_{j+k} \geq 0 \) are the same as in (2.41). Hence, by (2.41), \( i_{n,1} \to \infty \) as \( n \to \infty \).

To bound \( i_{n,2} \), note that \( |D_n(x)| = |\sin((n+1)x)/2)/\sin(x/2)| \leq C|x|^{-1}, \quad x \in \Pi \). This together with assumption \( |f_\xi(x) - f_\xi(0) = O(|x|) \) implies \( |i_{n,2}| \leq \int_{\Pi} |1 - B(e^{ix})|^{-2} |x(f_\xi(x) - f_\xi(0))| dx \leq C \int_{\Pi} |1 - B(e^{ix})|^{-2} dx < \infty, \) which completes the proof of the theorem. \( \square \)

**References**


