Bond Portfolio Management Using the Dynamic Nelson-Siegel Model

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Abstract

Factor models for the yield curve, such as the dynamic version of the Nelson-Siegel model proposed by Diebold and Li (2006) and its arbitrage-free counterpart proposed by Christensen, Diebold and Rudebusch (2011), have been extensively applied to forecast bond yields. In this paper, we propose a novel utilization of these models in bond portfolio management. More specifically, we derive closed form expressions for the vector of expected bond returns and for its covariance matrix based on a general class of dynamic factor models, and use these estimates to obtain optimal mean-variance and duration-constrained mean-variance bond portfolios. An empirical application involving a large data set of US Treasuries shows that the proposed portfolio policy outperform a broad set of traditional yield curve strategies used in bond desks. Moreover, we find that the investor with a quadratic utility function is willing to pay a performance fee to adopt the proposed mean-variance bond portfolios.

Keywords: Bond indexing, Kalman filter, maximum likelihood estimation, out-of-sample evaluation, portfolio optimization

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1 Introduction

Dynamic factor models for the term structure of interest rates have been extensively and successfully applied in forecasting bond yields. The dynamic version of the Nelson and Siegel (1987) model proposed by Diebold and Li (2006) and its arbitrage-free version proposed by Christensen et al. (2011) have gained popularity among financial market practitioners and central banks (BIS, 2005; Gürkaynak et al., 2007). Existing evidence suggests that these specifications are remarkably well suited both to fit the term structure, as well as to forecast its movements (see, for example, Diebold and Rudebusch, 2011, and the references therein). This evidence makes one tempted to investigate the extent to which these specifications can be applied to other problems. It would be interesting, for instance, to use this approach to support a portfolio policy that exploits the risk-return tradeoff in bond returns. In this sense, the main question of this paper is the following: can we go beyond forecasting and use these specifications to perform bond portfolio selection in a mean-variance context? Even though the mean-variance criterion is well established in building quantitative portfolio strategies, a number of questions need to be addressed in order to motivate its application in fixed income. Is this approach relevant to the fixed income world? A satisfactory answer to these questions depends on an adequate examination of the comparative performance of the resulting mean-variance bond portfolio vis-a-vis traditional portfolio strategies employed in bond desks. Is the comparative performance analysis favorable to the proposed portfolio policy? Moreover, is the investor willing to pay a performance fee in order to switch from a traditional bond portfolio policy to the mean-variance policy? These are fundamental questions that must be answered in order to validate the use of the mean-variance paradigm in the fixed income context.

We address each of these questions by providing a realistic empirical applications involving a data set based on a large panel of monthly time series of U.S. zero-coupon bonds with maturities up to 10 years over the period from January 1970 to December 2009. We initially implement the dynamic version of the Nelson and Siegel (1987) model proposed by Diebold and Li (2006) (hereafter DNS) and its arbitrage-free version proposed by Christensen et al. (2011)

(hereafter AFDNS) specifications along with efficient procedures for estimation of factors and parameters as proposed by Jungbacker and Koopman (2008) in order to generate forecasts of bond returns based on an expanding window. These are then used to solve alternative versions of the mean-variance optimization problem. First, we solve the shortselling-constrained mean-variance problem for alternative values for the risk aversion coefficient. Second, we extend the standard mean-variance formulation and propose the duration-constrained mean-variance problem, thus obtaining an optimal mean-variance portfolio that matches the duration of a given benchmark. The results are benchmarked against alternative bond portfolio strategies widely employed in bond desks, such as spread, barbell, bullet, and ladder strategies.

The empirical evidence shows that the proposed mean-variance bond portfolios seems to be very reasonable alternatives to the portfolios built upon traditional yield curve strategies. Our results show that the shortselling-constrained mean-variance portfolios of US bonds with monthly re-balancing achieved an annualized average gross return ranging from 5.9% to 11.6% and an annualized standard deviation ranging from 1.3% to 12.1%. The risk-adjusted performance measured by the annualized Sharpe ratio ranges from 0.35 to 0.67. Investors with higher (lower) risk-aversion tend to invest in portfolios with lower (higher) duration. We also employ a test for the Sharpe ratio based on the bootstrap procedure of Politis and Romano (1994), which allows us to formally compare portfolio policies in terms of their risk-adjusted returns. The results reveal that in most cases the mean-variance portfolios outperform all benchmark policies in terms of Sharpe ratio. Our results also indicate that an investor with quadratic utility is willing to pay a fee to adopt the mean-variance portfolio policy in all cases, and that this performance fee increases with the investor's risk aversion coefficient. On average, the investor with risk aversion coefficient equal to 1 is willing to pay a performance fee of 2 to 15 basis points per year to switch from traditional bond portfolio strategies to a mean-variance portfolio strategy. We also assess the impacts of transaction costs by implementing the mean-variance portfolios under quarterly re-balancing frequency. We find that lowering the portfolio re-balancing frequency leads to a substantial decrease in portfolio turnover, although accompanied by decreases in the risk-adjusted performance measured by the Sharpe ratio. Nevertheless, for some values of the risk aversion coefficient the mean-variance portfolios under quarterly re-balancing still outperform all benchmark strategies considered in the paper.

We also move forward the literature by considering a novel duration-constrained mean-variance optimization. In this formulation, we solve the shortselling-constrained mean-variance problem by adding a target duration constraint for the bond portfolio. This approach is particularly appealing, since in practice fixed-income portfolios tend to be selected in order to approximate the duration of a benchmark or to replicate the performance of this benchmark in terms of return and volatility. Our results reveal that the risk-adjusted performance of the duration-constrained mean-variance bond portfolios are superior to that of the benchmark strategies with equal duration levels. Moreover, the duration-constrained mean-variance policy yields portfolios with lower risk (standard deviation) in comparison to other benchmark portfolios with similar duration levels.

Our approach to the bond portfolio allocation problem is based on the the mean-variance framework introduced by Markowitz (1952), which is one of the milestones of modern finance theory. In this framework, individuals choose their allocations in risky assets based on the trade-off between expected return and risk. A common criticism of this approach is that it is a myopic single-period portfolio policy, with no concerns involving longer, multi-period investment horizons. However, the the errors coming from solving a complicated dynamic optimization problem can outweigh the expected utility gain from investing optimally as opposed to myopically; see Brandt (2009) for a discussion. Consequently, the application of the mean-variance approach is still widespread among practitioners and academics.

In order to implement the mean-variance optimization in practice, it is common to obtain estimates of the vector of expected returns and its covariance matrix and plug these estimators in an analytical or a numerical solution to the mean-variance problem. We show that factor models for the yield curve can substantially simplify the process of mean-variance bond portfolio selection, since they allow the computation of expected bond returns and their covariance matrix in closed form. Specifically, we show how to obtain these expressions based on a general class of dynamic affine factor models that includes the DNS and AFDNS models as special cases, and use them to obtain optimal bond portfolio allocations based on an active portfolio policy grounded on the mean-variance framework. Two appealing features of our approach are that i) all ingredients necessary to calculating the closed form estimators are easily retrieved from the one-step Kalman filter estimation of the DNS and AFDNS models, and ii) since it is based on factor specifications, our approach is reasonably parsimonious and suitable for high-dimensional applications in which a large number of fixed income securities is involved.

Existing literature shows few references suggesting the use of mean-variance approach to bond portfolio selection. Korn and Koziol (2006) pioneered this literature by employing the Vasicek (1977) model to perform mean-variance bond portfolio selection. Our approach, on the other hand, allows the use of a broad family of affine term structure models to perform bond portfolio selection. In particular, our approach nests the one considered in Korn and Koziol (2006), which is based on the Vasicek (1977) model and extends to other affine specifications with enhanced forecasting power, such as the DNS and AFDNS models.

At least two reasons can be pointed out in order to justify the lack of mean-variance optimization applied to fixed-income portfolios. The first reason is the relative stability and low historical volatility of this asset class, which discouraged the use of sophisticated methods to exploit the risk-return trade-off in fixed-income assets. However, this situation has been changing rapidly in recent years, even in markets where these assets have low default probability (Korn and Koziol, 2006). The recurrence of turbulent episodes in global markets usually brings high volatility to bond prices, which increases the importance of adopting portfolio selection approaches that take into account the risk-return trade-off in bond returns.

The second reason refers to the difficulties in modeling bond returns and the covariance matrix of bond returns (Korn and Koziol, 2006; Puhle, 2008). Fabozzi and Fong (1994) argue that if a covariance matrix of bond returns is available, the process of portfolio optimization using fixed-income securities is similar to that of equity portfolios. However, one should bear in mind that fixed-income securities have finite maturities and promise to pay face value at maturity. In this sense, the end of the year price of a bond with two years to maturity is indeed a random variable. However, the price of that same bond in two years is a deterministic quantity (disregarding the risk of default) given by its face value. This implies that the statistical properties of price and return of a fixed-income security depends on their maturity. Thus, both bond price and bond return are non-ergodic processes, and therefore traditional statistical techniques cannot be used to directly model the expected return and volatility of these assets; see Meucci (2009) for a discussion. One possibility to circumvent this difficulty is to employ factor models for the term structure of interest rates. As argued by Korn and Koziol (2006), the great advantage of using this approach is the possibility to model constant (or fixed) maturity yields. We show in the paper that this allows the estimation of the conditional distribution of bond yields without relying on bond maturities.

The paper is organized as follows. Section 2 describes the factor models used for modeling the term structure and shows how to convert yield forecasts into bond return forecasts. Section 3 discusses the estimation procedure for expected bond returns and for the conditional covariance matrix of bond returns. Section 4 discusses the empirical applications to both portfolio optimization. Finally, Section 5 concludes.

2 Bond returns and dynamic factor models

In this section we show how a general class of dynamic factor models for the yield curve can be used to compute the joint distribution of the log-return of N bonds. Factor models for the term structure of interest rates allow us to obtain closed form expressions for the vector of expected yields and its covariance matrix. From these moments, we show how to compute the distribution of bond prices and bond returns, which are key ingredients to bond portfolio optimization according to the mean-variance approach proposed by Markowitz (1952).

2.1 Dynamic factor models for the yield curve

Dynamic factor models play a major role in econometrics, allowing the explanation of a large set of time series in terms of a small number of unobserved common factors (see, among others, Fama and French, 1993; Stock and Watson, 2002; Jungbacker et al., 2013). Many specifications for the yield curve can be viewed as dynamic factor models with a set of restrictions imposed on the factor loadings.

We consider a set of time series of bond yields with N different maturities $\tau = [\tau_1, \ldots, \tau_N]'$. The yield at time t of a security with maturity τ_i is denoted by $y_{i,t}$ for $t = 1, \ldots, T$ and $i = 1, \ldots, N$. The $N \times 1$ vector of all yields at time t is given by

$$y_t = [y_{1,t},\, y_{2,t},\, \ldots\, ,\, y_{N,t}]', \quad t = 1, \ldots, T.$$

The general specification of the dynamic factor model is given by

$$y_t = \Gamma + \Lambda f_t + \varepsilon_t, \quad \varepsilon_t \sim NID\,(0, \Sigma)\,, \quad t = 1, \ldots, T, \tag{1}$$

where Γ is a $N \times 1$ vector of intercepts, Λ is a $N \times K$ matrix of factor loadings, f_t is a $K-$dimensional stochastic process, ε_t is the $N \times 1$ vector of disturbances with covariance matrix given by Σ. As usual in the yield curve literature, we restrict the covariance matrix Σ to be diagonal (see, for example, Diebold and Li, 2006; Diebold et al., 2006). The dynamic factors f_t are modeled by the following stochastic process:

$$f_t = \mu + \Upsilon f_{t-1} + \eta_t, \quad \eta_t \sim NID\,(0, \Omega)\,, \quad t = 1, \ldots, T, \tag{2}$$

where μ is a $K \times 1$ vector of constants, Υ is the $K \times K$ state-transition matrix, and Ω is the covariance matrix of the disturbance vector η_t, which is independent of the vector of residuals ε_t $\forall t$.

Equations (1) and (2) characterize a linear and Gaussian state space model, thus the Kalman filter can be used to obtain estimates of the unobserved factors, as well as to construct the log-likelihood function. The specification for f_t is fairly general, however, in modeling yield curves the usual specifications for f_t are either a first-order autoregressive process, thus yielding a independent-factor model, or a first order vector autoregressive process, thus yielding a correlated-factor model (see, among others, Diebold et al., 2006).

The yield curve specifications considered in this paper are the main variants of the original formulation of the Nelson and Siegel (1987) factor model, namely the dynamic Nelson-Siegel model proposed by Diebold and Li (2006), and its arbitrage-free version, proposed by Christensen et al. (2011). The alternative specifications considered can be captured in the general dynamic factor model formulation in (1) and (2) with different restrictions imposed on the vector of intercepts Γ, on the loading matrix Λ and on the state-transition equation (2). More specifically, denoting the i^{th} line of Λ by $\Lambda(\tau_i)$, both the dynamic Nelson-Siegel model and its arbitrage-free counterpart imply that

$$\Lambda(\tau_i) \quad = \quad \left[1, \left(\frac{1 - e^{-\lambda\tau_i}}{\lambda\tau_i}\right), \left(\frac{1 - e^{-\lambda\tau_i}}{\lambda\tau_i} - e^{\lambda\tau_i}\right)\right], \tag{3}$$

where the decaying factor λ is assumed to be constant over time. This is in accordance with Diebold and Li (2006) and the common finding that time variations in λ have only a negligible impact on the model's fit and prediction power (see, for example, Haustsch and Ou, 2012). The restrictions in (3) allows us to interpret the three factors as the level of the yield curve, its slope and curvature (Diebold and Li, 2006).

However, the models differ with respect to restrictions imposed on Γ, Υ, and Ω. While the dynamic Nelson-Siegel model of Diebold et al. (2006) sets every element of the vector Γ to zero and does not impose any additional restrictions on Υ or Ω, Christensen et al. (2011) have shown that it is necessary to restrict Γ, Υ, and Ω to ensure that the DNS model is arbitrage-free. We detail in Appendix 1 the derivation of the restrictions in Γ, Υ, and Ω required to ensure an arbitrage-free DNS model.

2.2 The distribution of log-returns

Bond portfolio management requires estimates of the expected return of each bond, as well as estimates of their covariance matrix. However, factor models for the term structure of interest rates are designed to model only bond yields. Nevertheless, it is possible to obtain expressions for the expected bond return and for the conditional covariance matrix of bond returns based on the distribution of the expected yields. We provide these expression below.

Given the system of equations in (1) and (2), the distribution of expected yields $y_{t|t-1}$ is

$N(\mu_{y_{t|t-1}}, \Sigma_{y_{t|t-1}})$, where

$$\mu_{y_{t|t-1}} = \Lambda f_{t|t-1}, \tag{4}$$

and

$$\Sigma_{y_{t|t-1}} = \Lambda \left(\Omega_{t|t-1} + \Upsilon \mathrm{Q}_{t-1} \Upsilon'\right) \Lambda' + \Sigma_{t|t-1}, \tag{5}$$

where $f_{t|t-1} = \mathrm{E}_{t-1}[f_t]$, and $\mathrm{Q}_{t-1} = \mathrm{Var}_{t-1}[f_{t-1}]$ take into account the uncertainty in the estimates of the unobserved factors (see Harvey et al., 1992).

Taking into account that the price of a bond at time t, $P_t^{(\tau)}$, is the present value at time t of \$1 receivable τ periods ahead, we compute the bond price as $P_t^{(\tau)} = \exp\left\{-\tau \cdot y_t^{(\tau)}\right\}$. To compute the unexpected return, $r_t^{(\tau)}$, of holding that bond from $t-1$ to t while its maturity decreases from τ to $\tau - 1$, we use the log-return expression,

$$r_t^{(\tau)} = \log\left(\frac{P_t^{(\tau-1)}}{P_{t-1}^{(\tau)}}\right) = \log P_t^{(\tau-1)} - \log P_{t-1}^{(\tau)} = \tau \cdot y_{t-1}^{(\tau)} - (\tau-1) \cdot y_t^{(\tau-1)}. \tag{6}$$

Letting $y_{t|t-1}^{(\tau-1)}$ denote the one-step-ahead forecast of a continuously compounded zero-coupon nominal yield-to-maturity, together with equations (4)-(6), it is easy to see that the distribution of the vector of expected log-returns, $r_{t|t-1}^{(\tau)}$, is $N\left(\mu_{r_{t|t-1}^{(\tau)}}, \Sigma_{r_{t|t-1}^{(\tau)}}\right)$, where

$$\mu_{r_{t|t-1}^{(\tau)}} = -(\tau-1) \otimes \mu_{y_{t|t-1}^{(\tau-1)}} + \tau \otimes y_{t-1}^{(\tau)}, \tag{7}$$

$$\Sigma_{r_{t|t-1}} = (\tau-1)(\tau-1)' \otimes \left[\underbrace{\Lambda \left(\Omega_{t|t-1} + \Upsilon \mathrm{Q}_{t-1} \Upsilon'\right) \Lambda' + \Sigma_{t|t-1}}_{\Sigma_{y_{t|t-1}^{(\tau-1)}}}\right], \tag{8}$$

and $\otimes$ is the Hadamard (elementwise) multiplication.

The results (7) and (8) show that it is possible to obtain closed form expressions for the expected log-returns of bonds and their covariance matrix based on dynamic factor models for the yield curve. These estimates are key ingredients to the problem of bond portfolio management, as discussed in section 4. It is worth noting that all ingredients necessary to calculating the closed form estimators are easily retrieved from the Kalman filter estimation discussed in Section 3. In particular, the values of $f_{t|t-1}$ and Q_{t-1} in (4) and (5), respectively, are direct products of the Kalman filter recursions and are readily available. Furthermore, the closed form estimators are based on a general formulation for a dynamic factor model for the yield curve, which implies that it is also applicable to other affine term structure models.

3 Estimation procedure

Given the state space formulation of the dynamic factor model presented in (1) and (2), the Kalman filter can be used to obtain the likelihood function via the prediction error decomposition, as well as filtered estimates of the states and of their covariance matrices. However, the computational burden associate with the Kalman filter recursions depends crucially on the dimension of both the state and observation vectors. Moreover, in yield curve models the dimension of the observation vector ($N \times 1$) is often much larger than that of the state vector ($K \times 1$). In these circumstances, Jungbacker and Koopman (2008) have shown that significant computational gains can be achieved by a simple transformation. First, define the $N \times N$ and the $K \times N$ matrices:

$$A = \begin{bmatrix} A^L \\ A^H \end{bmatrix}, \qquad A^L = C\Lambda(\lambda)'\Sigma^{-1},$$

respectively, where C can be any $K \times K$ invertible matrix, and A^H is chosen to guarantee that A is full rank. Selecting $C = \left(\Lambda(\lambda)'\Sigma^{-1}\Lambda(\lambda)\right)^{-1}$ implies:

$$Ay_t = \begin{pmatrix} A^L y_t \\ A^H y_t \end{pmatrix} = \begin{pmatrix} f_t \\ 0 \end{pmatrix} + \begin{pmatrix} A^L \varepsilon_t \\ A^H \varepsilon_t \end{pmatrix}, \begin{pmatrix} A^L \varepsilon_t \\ A^H \varepsilon_t \end{pmatrix} \sim N\left(0, \begin{bmatrix} C & 0 \\ 0 & \Sigma^H \end{bmatrix}\right). \tag{9}$$

The law of motion of the factors in (2) is not affected by the transformation. Note that $A^H y_t$ is neither dependent on f_t, nor correlated with $A^L y_t$, and therefore does not need to be considered for the estimation of the factors. This implies that the Kalman filter only needs to be applied to the low dimensional subvector $A^L y_t$ for signal extraction, generating large computational gains when $N >> K$ (see Table 1 of Jungbacker and Koopman, 2008).

Denote $l(y)$ the log-likelihood function of the untransformed model in (1) and (2), where $y = (y_1', \ldots, y_T')'$. Evaluation of $l(y)$ can also take advantage of the transformations presented above. Jungbacker and Koopman (2008) show that the log-likelihood of the untransformed model can be represented as

$$l(y) = c + l(y^L) - \frac{T}{2} \log \frac{|\Sigma|}{|C|} - \frac{1}{2} \sum_{t=1}^{T} e_t' \Sigma^{-1} e_t, \tag{10}$$

where c is a constant independent of both y and the parameters, $y^L = (A^L y_1', \ldots, A^L y_T')'$, $l\left(y^L\right)$ is the log-likelihood function of the reduced system, and $e_t = y_t - \Lambda(\lambda) f_t$. Note that computation of matrix A^H is not required at any point, as proved in Lemma 2 of Jungbacker and Koopman (2008).

4 Application to bond portfolio management

In this Section we describe the empirical application carried out in the paper. We describe the data set, the mean-variance approach to bond portfolio optimization, the benchmark strategies, and the methodology to assess portfolio characteristics such as performance and composition. Finally, we discuss the results.

4.1 Data

The data set consists of end-of-month continuously compounded yields on U.S. zero-coupon bonds. This panel of monthly time series of yields was constructed from the CRSP unsmoothed Fama and Bliss (1987) forward rates by Jungbacker et al. (2013) and is publicly available on the Journal of Applied Econometrics Data Archive, as part of their supplementary material. The data set consists of 17 maturities over the period from January 1970 to December 2009. The maturities analyzed are $\tau = 3, 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108$

and 120 months. This choice provides us with a panel of 480 monthly observations on 16 different maturities. The 3-month rate is not included in the optimization process and is taken as the risk free rate.

Table 1 provides descriptive statistics for this data set. For each maturity, we report mean, standard deviation, minimum, maximum and one-month, one-year and two-year sample autocorrelation and two-month partial autocorrelation. The summary statistics confirm some stylized facts common to yield curve data: the sample average curve is upward sloping and concave, volatility is decreasing with maturity, and autocorrelations are very high. The estimate of the partial autocorrelation function suggest that autoregressive processes of limited lag order will fit the data well since only the first coefficient is significant for most maturities, while the second lag coefficients are relatively small. Figure 1 depicts the data set and the evolution of the yield in the period analyzed.

[Table 1 about here.]

[Figure 1 about here.]

4.2 Bond portfolio optimization according to the mean-variance framework

To illustrate the applicability of the proposed estimators of expected bond returns and conditional covariance matrix of bond returns defined in Section 2.2, we consider the mean-variance optimization problem of fixed-income portfolios. The formulation of mean-variance portfolio is given by

$$\begin{aligned}
&\min_{w_t} w_t \Sigma_{r_{t|t-1}} w_t - \tfrac{1}{\delta} w_t' \mu_{r_{t|t-1}} \\
&\text{subject to} \\
&w_t' \iota = 1 \\
&w_t \geq 0,
\end{aligned} \tag{11}$$

where $\mu_{r_{t|t-1}}$ is defined in (7), $\Sigma_{r_{t|t-1}}$ is given in (8), w_t is the vector of optimal weights, ι is a vector of ones with dimension $N \times 1$, and δ is the risk aversion coefficient. In our empirical implementation, we solve the mean-variance optimization problem considering four alternative values for the risk aversion coefficient δ, in particular, we evaluate (11) for $\delta = \{1 \times 10^{-4}, 1 \times 10^{-3}, 1 \times 10^{-2}, 1 \times 10^{-1}, 0.5, 1.0\}$. Finally, we consider the case in which shortsales are restricted by adding to (11) a constraint to avoid negative weights, i.e. $w_t \geq 0$. Previous works show that adding such a restriction can substantially improve performance, especially reducing the turnover of the portfolio, see Jagannathan and Ma (2003), among others.

Duration-constrained mean-variance optimization

It is also worth noting that fixed-income portfolio managers commonly employ a strategy known as bond indexing, which consisting in building a portfolio that replicates risk factors of a given benchmark index or portfolio (Fabozzi and Fong, 1994). One of the most common risk factors considered in this strategy is the duration, which is a standard measure of the bond portfolio sensitivity to changes in yields. The bond indexing strategy can be incorporated in the mean-variance context by means of a duration-constrained mean-variance optimization problem. In this setting, we solve the mean-variance optimization problem in (11) with an additional constraint on the duration of the optimal portfolio, thus obtaining an optimal portfolio that matches the duration of a given benchmark. The restriction on the duration of the optimal portfolio is defined as $\tau_p = w'\tau$, where τ_p is the portfolio duration and τ is the vector of individual bond durations. The choice of the target portfolio duration τ_p can be made in order to match the duration of a given benchmark or strategy (e.g. a fixed income index, other fixed income portfolio etc). In our empirical application we obtain duration-constrained mean-variance bond portfolios with a target portfolio duration of $\tau_p = \{1, 3, 5, 7, 9\}$ years for an investor with a risk aversion coefficient $\delta = 1$.

4.3 Methodology for evaluating portfolio performance and implementation details

The performance of optimal mean-variance portfolios is evaluated in terms of average return (R_p), average excess return relative to the risk-free rate ($R_{p,ex}$), standard deviation (volatility) of returns ($\widehat{\sigma}$), Sharpe Ratio (SR), and turnover. These statistics are calculated as follows:

$$R_p = \frac{1}{T-1}\sum_{t=1}^{T-1} w_t' R_{t+1}$$

$$R_{p,ex} = \frac{1}{T-1}\sum_{t=\tau}^{T-1}(w_t' R_{t+1} - Rf_{t+1})$$

$$\hat{\sigma} = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T-1}(w_t' R_{t+1} - \hat{\mu})^2}$$

$$SR = \frac{R_{p,ex}}{\hat{\sigma}}$$

$$\text{Turnover} = \frac{1}{T-1}\sum_{t=1}^{T-1}\sum_{j=1}^{N}(|w_{j,t+1} - w_{j,t}|),$$

where $w_{j,t}$ is the weight of the asset j in the portfolio in period t before the re-balancing, while $w_{j,t+1}$ is the desired weight of the asset j in period $t+1$. As pointed out by DeMiguel et al. (2009), the turnover as defined above, can be interpreted as the average fraction of wealth traded in each period. Rf denotes the risk-free rate. We consider the risk free rate to be the 3-month Treasury.

In order to evaluate the composition of the optimal bond portfolios, we also report the average duration (in years) of the mean-variance portfolios. The average bond portfolio duration is calculated as

$$\text{Average portfolio duration} = \frac{1}{T-1}\sum_{t=1}^{T-1} w_t' \tau,$$

where τ is the vector of individual bond durations.

We also follow Fleming et al. (2001) and Fleming and Kirby (2003) and use a utility-based approach to measure the value of the performance gains associated with employing a given bond portfolio strategy. We assume the investor has a quadratic utility function given by

$$U(R_{p,t}) = W_0\left(1 + R_{p,t} - \frac{\gamma}{2(1+\gamma)}(1+R_{p,t})^2\right),$$

where $R_{p,t} = w_t' R_{t+1}$ is the portfolio return, γ is the investor's relative risk aversion and W_0

is the initial wealth. In order to compare two alternative bond portfolio strategies (R_{p1} and R_{p2}), we determine the maximum performance fee a risk-averse investor would be willing to pay to switch from using one portfolio policy to another. That is, we determine the value of Δ such that

$$\sum_{t=1}^{T-1} U\left(R_{p1,t}\right) = \sum_{t=1}^{T-1} U\left(R_{p1,t} - \Delta\right).$$

This constant represents the maximum return the investor would be willing to sacrifice each month in order to capture the performance gains associated with switching to the second portfolio policy. We report the value of Δ as an annualized basis point fee for each value of the risk aversion coefficient.

In order to implement the models, we employ a recursive estimation strategy based on an expanding estimation window. Departing from the first t observations, all models are estimated and their corresponding one-step-ahead estimate of the vector of expected bond returns and its covariance matrix are computed using the results in (7) and (8) for each of the specifications considered in the paper. Next, we use these expressions to obtain optimal mean-variance portfolios. Finally, we add one observation to the sample, and re-estimate all models to obtain another one-step-ahead estimate of the vector of expected bond returns and its conditional covariance matrix. This process is repeated until the end of the data set is reached. Therefore, we end up with a sample of $T - t$ out-of-sample observations, where T is the length of the data set. We use as initial estimation window a sample of $t = 120$ monthly observations, which yields 360 out-of-sample monthly observations. We adopt a similar procedure to implement the benchmark portfolio strategies discussed in Section 4.4

4.4 Description of the benchmark strategies

To assess the relative performance of the proposed bond portfolio policy based on the mean-variance criterion, we consider a spectrum of popular yield curve strategies employed in the majority of bond desks worldwide.[1] These strategies involve positioning a portfolio to

[1]Jones (1991), Fabozzi and Fong (1994) and Fabozzi (2002) provide a detailed explanation of traditional yield curve strategies.

capitalize on expected changes in the shape of the Treasury yield curve. We consider four alternative yield curve strategies: spread (or slope) strategy, bullet strategy, barbell strategy, and ladder strategy. In the spread strategy, we assume the investor shortsells the 1-year bond and buys the 10-year bond. In the bullet strategy, the portfolio is constructed so that the maturity of the bonds in the portfolio are highly concentrated at one point on the yield curve. We implement the bullet strategy by considering six alternative portfolios invested in the 1-year, 3-year, 5-year, 7-year, 9-year, and 10-year bond. In the barbell strategy, the maturity of the bonds included in the portfolio is concentrated at two extreme maturities. We implement the barbell strategy by considering a portfolio equally-weighted in the 1-year and 10-year bonds. Finally, in a ladder strategy the portfolio is constructed to have approximately equal amounts of each maturity. We implement this strategy by considering a portfolio equally-weighted in the 16 maturities of the data set described in Section 4.1. It is worth noting that each of these strategies will result in different performance when the yield curve shifts. The actual performance will depend on both the type of shift and the magnitude of the shift.

Table 2 reports the performance of the benchmark strategies. The statistics of returns, standard deviation and Sharpe ratio are annualized. We observe that the highest average gross and excess returns are achieved the bullet portfolio invested in the 9-year bond (9.5% and 3.7%, respectively). The strategy with lower portfolio risk in terms of standard deviation is 1-year bond portfolio (2%). In terms of risk-adjusted returns, we find that the 3-year bond portfolio outperforms all other strategies, since it achieves the highest Sharpe ratio (0.447).

[Table 2 about here.]

In order to assess the relative performance of the approach proposed in the paper, we consider as main benchmark strategy the one with highest Sharpe ratio, which is the 3-year bond portfolio. The stationary bootstrap of Politis and Romano (1994) with B=1.000 resamples and block size $b = 5$ was used to test the statistical significance of differences between Sharpe ratios of optimal portfolios relative to the benchmark portfolio. The methodology suggested in Ledoit and Wolf (2008, Note 3.2) was used to obtain p-values.

4.5 Results

Mean-variance bond portfolios

In this section, we present the out-of-sample results of optimal mean-variance portfolios for the data discussed in Section 4.1 and for different levels of risk aversion δ. The specifications used to model expected bond returns are the dynamic version of the 3-factor Nelson-Siegel model (DNS) discussed in Section 2.2 and its arbitrage-free counterpart (AFDNS). To assess the robustness of the results, we consider two alternative specifications to model the factor dynamics (AR(1) and VAR(1)). Optimal portfolio compositions are re-balanced on a monthly basis. Table 3 reports the performance results of mean-variance bond portfolios. The statistics of returns, standard deviation and Sharpe ratio are annualized. We use an asterisk to indicate the instances in which the Sharpe ratio of the mean-variance portfolios is statistically higher than that obtained by the benchmark portfolio strategy with highest Sharpe ratio (3-year bond) at a significance level of 10%.[2] In order to have an idea about portfolio composition, we also report the average duration (in years) of the mean-variance portfolios. All figures are based on out-of-sample observations.

The results in Table 3 show that the optimal mean-variance portfolios of US bonds have risk-return profiles that vary substantially across the levels of risk tolerance considered. The Table shows that the optimal mean-variance portfolio of US bonds achieved an annualized average gross return ranging from 5.9% to 11.6% and an annualized standard deviation ranging from 1.3% to 12.1%. The risk-adjusted performance measured by the annualized Sharpe ratio ranges from 0.35 to 0.67. As expected, we observe that an increase in the risk aversion coefficient δ leads to decreases in portfolio risk (measured by the standard deviation) as well as to decreases in portfolio average return and portfolio turnover. Moreover, we find that an increase in the risk aversion coefficient leads to optimal portfolio composition with lower duration (i.e. invested in short term maturities). This result is also mostly expected, since investors with higher risk aversion can lower portfolio risk by investing in shorter maturities. For instance, the average portfolio duration across specifications for an investor with risk

[2]Table 2 shows the performance of all benchmark bond portfolio strategies.

aversion coefficient $\delta = 1$ is 0.19 years whereas the same figure for an investor with $\delta = 0.01$ is 7.27 years.

A comparison among alternative specifications for mean-variance portfolios suggest that both DNS and AFDNS models deliver good results in comparison to the benchmark strategy. In this sense, it is difficult to identify which one is the best. A similar conclusion is achieved when comparing the performance among alternative specifications for the factor dynamics (AR and VAR).

The most important message in 3 is that the mean-variance bond portfolios seems to be very reasonable alternatives to the portfolios built upon the traditional yield curve strategies discussed in Section 4.4. We observe that several specifications for the mean-variance portfolios achieved higher Sharpe ratios in comparison to all benchmark policies considered. Overall, the best performance in terms of Sharpe ratio (0.675) is achieved by the mean-variance portfolio obtained with the AFDNS/VAR specification with $\delta = 0.1$. This figure is statistically and substantially higher than the one achieved by the benchmark bullet portfolio invested in the 3-year bond (0.447).

It is also worth noting that the differences in the portfolios' risk-return profiles indicate that investors with both low and high risk appetite can benefit from adopting the mean-variance policy. For instance, a mean-variance investor with risk aversion $\delta = 1 \times 10^{-4}$ and employing the DNS/AR specification would select a portfolio invested mostly in long term maturities (average portfolio duration of 6.7 years) thus achieving a Sharpe ratio of 0.574, which is higher that the Sharpe ratios of all benchmark policies. Similarly, a mean-variance investor with risk aversion $\delta = 0.5$ and employing the AFDNS/AR specification would select a portfolio invested mostly in short term maturities (average portfolio duration of 0.7 years) thus achieving a Sharpe ratio of 0.539, which is also higher that the Sharpe ratios of all benchmark policies.

[Table 3 about here.]

We further illustrate the performance of the optimal US mean-variance portfolios by plotting in Figure 2 the cumulative portfolio returns over the out-of-sample period obtained with the AFDNS/AR specification when $\delta = 0.01$. The Figure also plots the cumulative returns of three benchmark bond portfolios strategies (barbell portfolio, 3-year bond, and 10-year bond). We observe a striking difference in favor of the mean-variance portfolio throughout the full the sample period. Moreover, this specification for the mean-variance portfolios yields not only higher cumulative returns, but also higher average returns (both gross and excess) and higher Sharpe ratio in comparison to the benchmark strategies, as indicated in Table 3.

[Figure 2 about here.]

The results reported in Table 3 and illustrated in Figure 2 suggest that the mean-variance portfolios obtained with the proposed estimators for the vector of expected bond returns and its covariance matrix deliver better risk-adjusted performance with respect to traditional bond portfolio strategies. However, it is still unclear how much an investor is willing to pay to switch from a traditional bond portfolio strategy to a mean-variance portfolio strategy. In order to address this question, we report in Table 4 the annualized performance fee an investor with quadratic utility is willing to pay to change from a each of the traditional bond portfolio strategy considered in Section 4.4 to a mean-variance portfolio strategy. The results in Table 4 are unambiguous and corroborate those in Table 3. We find that the investor is willing to pay a fee to adopt the mean-variance portfolio policy in all cases, and that this performance fee increases with the investor's risk aversion coefficient. For instance, an investor with risk aversion coefficient $\delta = 1$ is willing to pay an annualized fee of 34 basis points to change from the spread bond portfolio policy to the mean-variance portfolio policy. On average, the investor is willing to pay a performance fee of 2 to 15 basis points to switch from traditional bond portfolio strategies to a mean-variance portfolio strategy.

[Table 4 about here.]

Duration-constrained mean-variance bond portfolios

The results of the duration-constrained mean-variance bond portfolios are shown in Table 5. We find that an increase in the target portfolio duration leads to optimal portfolios with higher average returns, higher average excess returns and higher standard deviation. This result is expected, since a higher target duration will lead to portfolio compositions invested in longer maturities and, therefore, with higher returns and risk. The most important result of this analysis, however, is the relative performance of the duration-constrained mean-variance bond portfolios with respect to the performance of traditional bond portfolio strategies. For each value of the target portfolio duration considered in Table 5, we compare the risk-adjusted performance of the duration-constrained mean-variance portfolios with that obtained by a portfolio invested in a bond with equal duration. For instance, in the case of a target portfolio duration of 1 year, the performance of the mean-variance portfolios are compared to that obtained by a portfolio invested in the 1-year bond. Finally, we assume an investor with risk aversion coefficient $\delta = 1$. We have also conducted this analysis for alternative values for the risk aversion coefficient δ. The results are very similar to those reported here.

The risk-adjusted performance of the duration-constrained mean-variance bond portfolios are superior to that of the benchmark strategies with equal duration levels in several cases. For instance, when considering a target portfolio duration of 5 years, the duration-constrained mean-variance portfolio obtained with the AFDNS/AR specification obtained a Sharpe ratio of 0.429, whereas the same value for the portfolio invested in the 5-year bond is 0.387. Moreover, we find that an investor with a quadratic utility function is willing to pay a fee to adopt the duration-constrained mean-variance portfolio policy in all cases.

Another important result is that the volatility (standard deviation) of the duration-constrained portfolios is always lower than that of the benchmark portfolio with equal duration. For instance, the annualized standard deviation of the duration-constrained mean-variance portfolio with a 3-year target duration is 4.07, whereas the same value for the benchmark portfolio invested in the 3-year bond is 4.73. Similar results are obtained for other duration levels. This suggests that diversification plays an important role here, since it is possible to achieve a desired duration level while reducing portfolio risk by investing in bonds with different maturities.

[Table 5 about here.]

Summarizing the results reported in Tables 3 to 5, we find that the optimal mean-variance portfolios based on the closed-form estimates for the vector of expected bond returns and the covariance matrix of the bond returns presented in Section 2.2 have superior out-of-sample performance with respect to the benchmark policies from at least two standpoints. First, the risk-adjusted performance of mean-variance portfolios based on dynamic factor models are significantly better than that of the benchmark policies. Second, we find that an investor is willing to pay a performance fee to switch from traditional yield curve strategies to the mean-variance and duration-constrained mean-variance portfolios.

Assessing the impact of portfolio re-balancing frequency

The results discussed in Table 3 assume a monthly re-balancing frequency. The transaction costs involved in this re-balancing frequency might degrade the performance of the portfolios and hinder its implementation in practice. Thus, the performance of optimized portfolios is also evaluated in the case of quarterly re-balancing frequency. A potentially negative effect of adopting a lower re-balancing frequency is that the optimal compositions may become outdated.

Table 6 brings the results of the optimal mean-variance portfolios under quarterly re-balancing frequency. As expected, we find that lowering the portfolio re-balancing frequency leads to a substantial decrease in portfolio turnover. For instance, the average turnover of the mean-variance portfolio under monthly re-balancing across all specifications is 0.719 whereas the same figure for the quarterly re-balancing is 0.233. This result suggests that lowering the portfolio re-balancing frequency can lead to substantial decreases in transaction costs. We also find, however, that, on average, lowering the portfolio re-balancing frequency leads to decreases in the risk-adjusted performance measured by the Sharpe ratio. Nevertheless, for some values of the risk aversion coefficient the mean-variance portfolios obtained with the DNS/VAR specification outperform the benchmark strategy (3-year bond portfolio). For instance, when $\delta = 1 \times 10^{-1}$, the mean-variance portfolios obtained with the DNS/VAR specification achieved a Sharpe ratio of 0.539, whereas the same figure for the benchmark strategy is 0.447. This result suggests that the mean-variance portfolios can outperform traditional yield curve strategies even when optimal portfolio compositions are re-balanced less frequently.

[Table 6 about here.]

5 Concluding remarks

The mean-variance approach introduced by Markowitz (1952) to obtain optimal portfolios has been widely used by market participants and largely documented in the academic literature. However, the use of this methodology for the optimization of portfolios composed of fixed-income securities has received little attention in the literature. In order to address this shortcoming, this paper adopts the mean-variance approach to bond portfolio optimization based on dynamic factor models for the term structure of interest rate, such as the dynamic version of the Nelson-Siegel model proposed by Diebold and Li (2006) and its arbitrage-free version proposed by Christensen et al. (2011). These are standard specifications to model interest rates widely used by market participants and academics. In this paper, we extend their utilization to a broader context of bond portfolio selection.

We show that factor models for the yield curve simplify the process of bond portfolio management, since it allows the computation of the vector expected bond returns and its covariance matrix in closed form. We show how to obtain closed-form expressions for these two moments based on a general class of dynamic factor models, and use them to obtain optimal mean-variance and duration-constrained mean-variance portfolios. Our evidence indicate that the mean-variance portfolios obtained with the proposed estimators for the vector of expected bond returns and its covariance matrix deliver better risk-adjusted performance with respect to traditional bond portfolio strategies. Moreover, an investor with quadratic utility is willing to pay a performance fee to change from a each of the traditional bond portfolio strategies to a mean-variance portfolio strategy.

Appendix: The arbitrage-free Nelson-Siegel model (AFDNS)

The theoretical weakness of the standard DNS model is that is not defined under an arbitrage free setting. Christensen et al. (2011) have extended this model to overcome this by adhering to the standard continuous time affine diffusion processes developed in Duffie and Kan (1996). The resulting class of AFDNS models involves considering the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$ with filtration $(\mathcal{F}_t) = \{\mathcal{F}_t : t \leq 0\}$ satisfying the usual conditions of Williams (1997). Here $\mathbb{Q}$ denotes the risk neutral measure and we will denote the real world probability measure by $\mathbb{P}$. If the resulting risk neutral dynamic factors, generically denoted by state vector $\mathbf{X}_t$, for the DNS term structure model are defined on such a probability space and follow a Markov process defined on set $M \subseteq \mathbb{R}^n$ that is a solution to the stochastic differential equation given by

$$\mathsf{d}\mathbf{X}_t = \mathbf{K}(t)^{\mathbb{P}} \left[\Theta(t)^{\mathbb{P}} - \mathbf{X}_t\right] \mathsf{d}t + \Sigma(t) D(\mathbf{X}_t, t) \mathsf{d}\mathbf{W}_t^{\mathbb{P}}, \tag{12}$$

with $\mathbf{W}^{\mathbb{Q}}$ a standard Brownian Motion in $\mathbb{R}^n$ defined according to the filtration $(\mathcal{F}_t)$. Furthermore, as in Duffie and Kan (1996) one assumes the drifts and dynamics are bounded continuous functions such that $\Theta^{\mathbb{Q}} : [0, T] \to \mathbb{R}^n$, $\mathbf{K}^{\mathbb{Q}} : [0, T] \to \mathbb{R}^{n\times n}$ and volatility $\Sigma : [0, T] \to \mathbb{R}^{n\times n}$. Finally, they assume the diagonal mapping $D : M \times [0, T] \to \mathbb{R}^{n\times n}$ with diagonal elements given by

$$[D]_{ii} = \sqrt{\gamma^i(t) + \delta^i(t) X_t^1 + \ldots + \delta^n(t) X_t^n} \quad \forall i \in \{1, \ldots, n\},$$

where each $\gamma^i : [0, T] \to \mathbb{R}^n$ and $\delta^i : [0, T] \to \mathbb{R}^{n\times n}$ are bounded continuous functions. In addition Duffie and Kan (1996) assume the instantaneous risk neutral rate is an affine function of the state variables given by $r_t = r_0(t) + r_1(t)'\mathbf{X}_t$ with bounded continuous functions $r_0 : [0, T] \to \mathbb{R}$ and $r_1 : [0, T] \to \mathbb{R}^n$. Under this affine formulation of the continuous latent factor dynamics Duffie and Kan (1996) proved that a closed form analytic expression for the zero-coupon bond prices is attained as a linear function of the latent dynamic factors. This gives the dynamic zero-coupon yield at time t for a bond with maturity T as

$$y(t, T) = -\frac{1}{T-t} \log\left(\mathbb{E}^{\mathbb{Q}}\left[-\int_t^T r_u \mathsf{d}u\right]\right) = -\frac{1}{T-t}\left[A(t, T) + B(t, T)'\mathbf{X}_t\right] \tag{13}$$

with $A(t, T)$ and $B(t, T)$ obtained as solutions to the system of ordinary differential equations ode?s with boundary conditions $A(0) = 0$ and $B(0) = 0$ given by

$$\begin{aligned} \frac{\mathsf{d}A(t, T)}{\mathsf{d}t} &= r_1 + \left(\mathbf{K}^{\mathbb{Q}}\right)' B(t, T) - \frac{1}{2}\sum_{i=1}^{n} \left[\Sigma' B(t, T) B(t, T)' \Sigma\right]_{ii} \left(\delta^i\right)' & (14) \\ \frac{\mathsf{d}B(t, T)}{\mathsf{d}t} &= r_0 + B(t, T) \left(\mathbf{K}^{\mathbb{Q}}\right)' \Theta^{\mathbb{Q}} - \frac{1}{2}\sum_{i=1}^{n} \left[\Sigma' B(t, T) B(t, T)' \Sigma\right]_{ii} \left(\gamma^i\right)'. & (15) \end{aligned}$$

Since we would like to work under a model that is the arbitrage-free equivalent model for the DNS specification, we see that by considering the generic affine latent factor dynamic s.d.e. structure proposed in Equation (13), it is clear that to recover the equivalent AFDNS model one requires solutions to this system of odes given by

$$\begin{aligned} B^1(t, T) &= -(T-t) \\ B^2(t, T) &= -\frac{1 - \mathrm{e}^{-\lambda(T-t)}}{\lambda(T-t)} \\ B^3(t, T) &= -\frac{1 - \mathrm{e}^{-\lambda(T-t)}}{\lambda} + (T-t)\,\mathrm{e}^{-\lambda(T-t)} \end{aligned}$$

which results in a solution for $y(t,T)$ given by,

$$y_t(\tau) = X_{1t} + X_{2t}\left[\frac{1-e^{-\lambda(T-t)}}{\lambda(T-t)}\right] + X_{3t}\left[\frac{1-e^{-\lambda(T-t)}}{\lambda(T-t)} - e^{-\lambda(T-t)}\right] - \frac{A(t,T)}{T-t}, \tag{16}$$

where the additional term $-\frac{A(t,T)}{T-t}$ is the correction to the DNS model to ensure it satisfies the theoretical properties of being arbitrage free.

Unlike the DNS-type models, the AFDNS-type models of Christensen et al. (2011) impose particular structure on both the risk neutral and real world latent factor dynamic processes. The ordinary differential equations for the coefficients of $A(t,T)$ and $B(t,T)$ are only functions of r_1 and $\mathbf{K}^{\mathbb{Q}}$. However, under the assumptions of Christensen et al. (2011) in which the mean state variable levels under the risk neutral measure at zero, the additional terms for the drift $\Theta^{\mathbb{Q}}$ and volatility matrix Σ appear in the solution to the yield adjustment term $\frac{A(t,T)}{T-t}$ which is found as a solution of the form,

$$-\frac{A(t,T)}{T-t} = -\frac{1}{2}\frac{1}{T-t}\sum_{i=1}^{3}\int_t^T [\Sigma' B(s,T)B(s,T)'\Sigma]_{ii}\,\mathsf{d}s. \tag{17}$$

Obtaining a solution to the system of equations that satisfy these solutions can be achieved for a large family of possible AFDNS models, such as those discussed in Christensen et al. (2011) in Proposition 1 which presents one such model in which the instantaneous risk neutral rate is given by $r_t = X_{1t} + X_{2t}$ where the latent state variables $\mathbf{X}_t = (X_{1t}, X_{2t}, X_{3t})$ are described by the following system of equations under the risk neutral measure $\mathbb{Q}$ given by

$$\begin{pmatrix} \mathsf{d}X_{1t} \\ \mathsf{d}X_{2t} \\ \mathsf{d}X_{3t} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & \lambda \end{pmatrix}\left[\begin{pmatrix} \theta_1^{\mathbb{Q}} \\ \theta_2^{\mathbb{Q}} \\ \theta_3^{\mathbb{Q}} \end{pmatrix} - \begin{pmatrix} \mathsf{d}X_{1t} \\ \mathsf{d}X_{2t} \\ \mathsf{d}X_{3t} \end{pmatrix}\right]\mathsf{d}t + \Sigma\begin{pmatrix} \mathsf{d}W_{1t}^{\mathbb{Q}} \\ \mathsf{d}W_{2t}^{\mathbb{Q}} \\ \mathsf{d}W_{3t}^{\mathbb{Q}} \end{pmatrix}, \quad \lambda > 0. \tag{18}$$

In addition, it has been shown that the solution to the equation specifying the yield adjustment term $A(t,T)$ results in a restriction to the volatility matrix of the form

$$\Sigma = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}. \tag{19}$$

This completes the specification of the AFDNS model under the risk neutral measure.

Independent and correlated AFDNS models

Opposed to the DNS case, the AFDNS models are formulated in continuous time. To estimate models in this environment a change of probability measure is needed, as the theoretical part above focused on $\mathbb{Q}$-dynamics. By setting up an affine risk specification as in Duffee (2002) the equation in (12) can be formulated to remain affine under $\mathbb{P}$-dynamics (see Christensen et al. (2011) for full details). Because of this, any vector Θ and matrix $\mathbf{K}$ under the $\mathbb{P}$-measure preserve the risk neutral dynamics discussed above. This flexibility guarantees that identical ($\mathbb{P}$-measure) models can be estimated, so our focus is on an independent and correlated model.

AFDNS are an affine model, meaning analytical formula exist for yields. Using the solution already presented in Equation (16) yields most have the following relationship with the state variables

$$\mathbf{y}_t \quad = \quad A + BX_t + \varepsilon_t \tag{20}$$

Note that the matrix B is identical in the DNS and AFDNS models (compare with Equation (**??**) and Equation (16)). The only difference is the addition of the vector A containing the yield-adjustment terms in the AFDNS models. Because the AFNS is a continuous model the "time dimension" is modeled in terms of dynamics instead of a time-series model such as an AR or VAR. However, it does not mean similar models cannot be estimated to ensure comparison, as shown below.

For the independent-factor AFDNS model the dynamics of the state variables (working under the $\mathbb{P}$-measure now)

$$\begin{pmatrix} \mathsf{d}X_{1t} \\ \mathsf{d}X_{2t} \\ \mathsf{d}X_{3t} \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{\mathbb{P}} & 0 & 0 \\ 0 & \kappa_{22}^{\mathbb{P}} & 0 \\ 0 & 0 & \kappa_{33}^{\mathbb{P}} \end{pmatrix} \left[\begin{pmatrix} \theta_1^{\mathbb{P}} \\ \theta_2^{\mathbb{P}} \\ \theta_3^{\mathbb{P}} \end{pmatrix} - \begin{pmatrix} X_{1t} \\ X_{2t} \\ X_{3t} \end{pmatrix} \right] \mathsf{d}t + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} \mathsf{d}W_{1t}^{\mathbb{P}} \\ \mathsf{d}W_{2t}^{\mathbb{P}} \\ \mathsf{d}W_{3t}^{\mathbb{P}} \end{pmatrix}. \quad (21)$$

In the correlated-factor AFDNS model, the three shocks may be correlated, and there may be full interaction among the factors as they adjust to the steady state

$$\begin{pmatrix} \mathsf{d}X_{1t} \\ \mathsf{d}X_{2t} \\ \mathsf{d}X_{3t} \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{\mathbb{P}} & \kappa_{12}^{\mathbb{P}} & \kappa_{13}^{\mathbb{P}} \\ \kappa_{21}^{\mathbb{P}} & \kappa_{22}^{\mathbb{P}} & \kappa_{23}^{\mathbb{P}} \\ \kappa_{31}^{\mathbb{P}} & \kappa_{32}^{\mathbb{P}} & \kappa_{33}^{\mathbb{P}} \end{pmatrix} \left[\begin{pmatrix} \theta_1^{\mathbb{P}} \\ \theta_2^{\mathbb{P}} \\ \theta_3^{\mathbb{P}} \end{pmatrix} - \begin{pmatrix} X_{1t} \\ X_{2t} \\ X_{3t} \end{pmatrix} \right] \mathsf{d}t + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} \mathsf{d}W_{1t}^{\mathbb{P}} \\ \mathsf{d}W_{2t}^{\mathbb{P}} \\ \mathsf{d}W_{3t}^{\mathbb{P}} \end{pmatrix}. \quad (22)$$

From this specification the resemblance to the DNS independent and correlated model is striking. This is the most flexible version of the AFDNS models where all parameters are identified.

An exact expression for the covariance of the continuous-time AFDNS model

In this paper, the models are estimated using maximum likelihood estimation method based on the Kalman filter. The AFDNS model can be formulated in a state space form as follow. For the continuous-time AFDNS models, the conditional mean vector and the conditional covariance matrix are

$$\mathbb{E}\left[\mathbf{X}_T | \mathcal{F}_t\right] = \left[I - \mathrm{e}^{-\mathbf{K}^{\mathbb{P}}\Delta t}\right] \Theta^{\mathbb{P}} + \mathrm{e}^{-\mathbf{K}^{\mathbb{P}}\Delta t} X_t \quad (23)$$

$$\mathbb{V}\left[\mathbf{X}_T | \mathcal{F}_t\right] = \int_0^{\Delta t} \mathrm{e}^{-\mathbf{K}^{\mathbb{P}}s} \Sigma\Sigma' \mathrm{e}^{-(\mathbf{K}^{\mathbb{P}})'s} \mathsf{d}s \quad (24)$$

To estimate AFNS models in the state-space Kalman-filter maximum-likelihood estimation framework proposed by Christensen et al. (2011) we must compute the conditional covariance matrix

$$\mathbb{V}^{\mathbb{P}}\left[\mathbf{X}_t | \mathcal{F}_{t-1}\right] = \int_0^{\Delta t} e^{-\mathbf{K}^{\mathbb{P}}s} \Sigma\Sigma^T e^{-\left(\mathbf{K}^{\mathbb{P}}\right)^T s} \mathsf{d}s \quad (25)$$

of discrete observations. Since the estimation is an intense computational process, we need to provide fast intermediate calculations. One approach is to approximate the integral and the matrix exponential. Another approach uses the diagonalization of $\mathbf{K}^{\mathbb{P}}$ to calculate the integral exactly. Due to the reduced size of the matrix $\mathbf{K}^{\mathbb{P}}$, the latter is significantly faster than the former, at least when compared to naïve numerical integration techniques.

The AFNS state transition equation is

$$\mathbf{X}_t = \left(I - \exp\left(-\mathbf{K}^{\mathbb{P}}\Delta t\right)\right) \Theta^{\mathbb{P}} + \exp\left(-\mathbf{K}^{\mathbb{P}}\Delta t\right) \mathbf{X}_{t-1} + \eta_t,$$

where Δt is the time between the observations at t and $t-1$, with measurement equation

$$\mathbf{y}_t = A + BX_t + \varepsilon_t,$$

and error structure

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix}\right],$$

where H is diagonal, $Q = V^P\left[X_t | \mathcal{F}_{t-1}\right]$ and the transition and measurement errors are assumed orthogonal to the initial state.

The computation of Q is straightforward if we use the diagonalization of $\mathbf{K}^{\mathbb{P}}$

$$\mathbf{K}^{\mathbb{P}} = V\Lambda V^{-1}, \tag{26}$$

where V contains the eigenvectors of $\mathbf{K}^{\mathbb{P}}$, and Λ is a diagonal matrix containing the eigenvalues (λ_i) of $\mathbf{K}^{\mathbb{P}}$. We refer the reader to **?** and **?** for details on linear algebra theory and on the computational aspects of linear algebra, respectively.

Substituting (26) in (25), using

$$\exp(-\mathbf{K}^{\mathbb{P}}s) = V\exp(-\Lambda s)V^{-1},$$

and similarly

$$\exp(-\left(\mathbf{K}^{\mathbb{P}}\right)^T s) = \left(V^{-1}\right)^T \exp(-\Lambda s)V^T,$$

we obtain

$$Q = V\left(\int_0^{\Delta t} \exp(-\Lambda s)\Omega\exp(-\Lambda s)ds\right)V^T,$$

where $\Omega = (\omega_{ij})_{n\times n} = V^{-1}\Sigma\Sigma^T\left(V^{-1}\right)^T$. Since the exponential of a diagonal matrix with entries $-\lambda_i s$ is a diagonal matrix with entries $e^{-\lambda_i s}$, each term of the matrix under the integral is $(\omega_{ij}e^{-(\lambda_i+\lambda_j)s})_{n\times n}$. Integration yields an expression which only involves matrix multiplications

$$Q = V\left(\frac{\omega_{ij}}{\lambda_i+\lambda_j}\left(1-e^{-(\lambda_i+\lambda_j)\Delta t}\right)\right)_{n\times n} V^T. \tag{27}$$

Stationarity of the system under the $\mathbb{P}$-measure is ensured if the real component of all the eigenvalues of $\mathbf{K}^{\mathbb{P}}$ is positive, and this condition is imposed in all estimations. For this reason, we can start the Kalman filter at the unconditional mean, $\mathbf{X}_0 = \Theta^{\mathbb{P}}$, and covariance matrix, Σ_0. In particular, the unconditional variance $\Sigma_0 = \int_0^\infty e^{-K^P s}\Sigma\Sigma^T e^{-\left(K^P\right)^T s}\mathsf{d}s$ used in the initialization of the filter is easily obtained from the above expression. Assuming $\mathbf{K}^{\mathbb{P}}$ has eigenvalues with positive real parts, the integral is convergent to

$$\Sigma_0 = V\left(\frac{\omega_{ij}}{\lambda_i+\lambda_j}\right)_{n\times n} V^T. \tag{28}$$

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Figures

Figure 1: **US term structure dynamics over time**

Note: This figure details the evolution of the US term structure of interest rates over 1970:01-2009:12. We use the following maturities: 3, 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108 and 120 months.

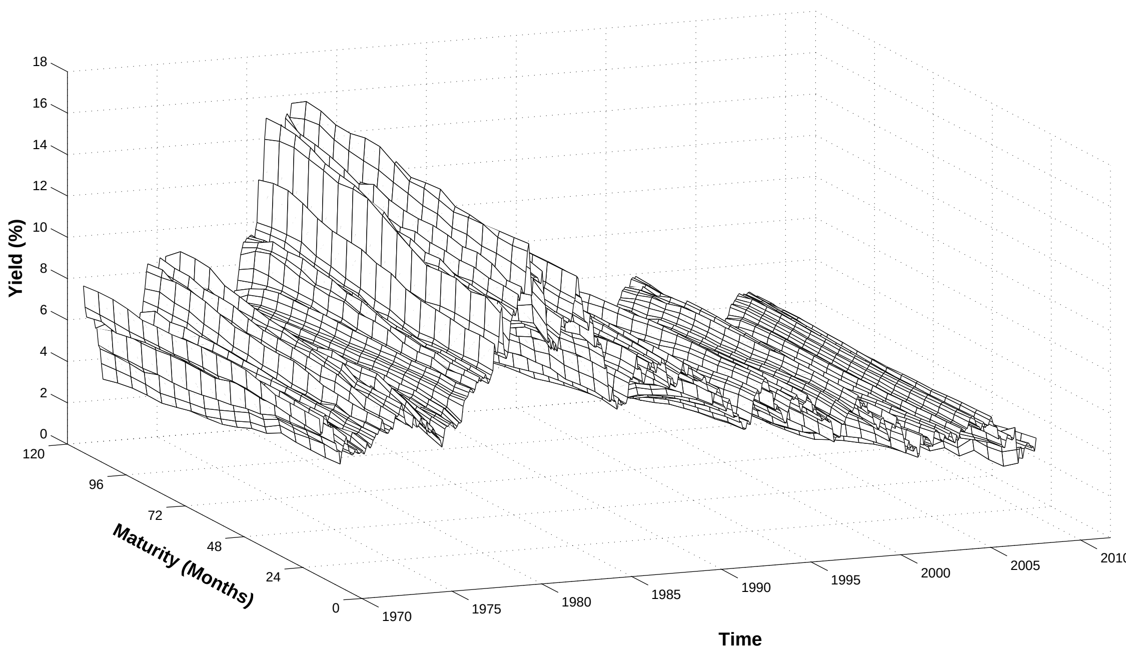
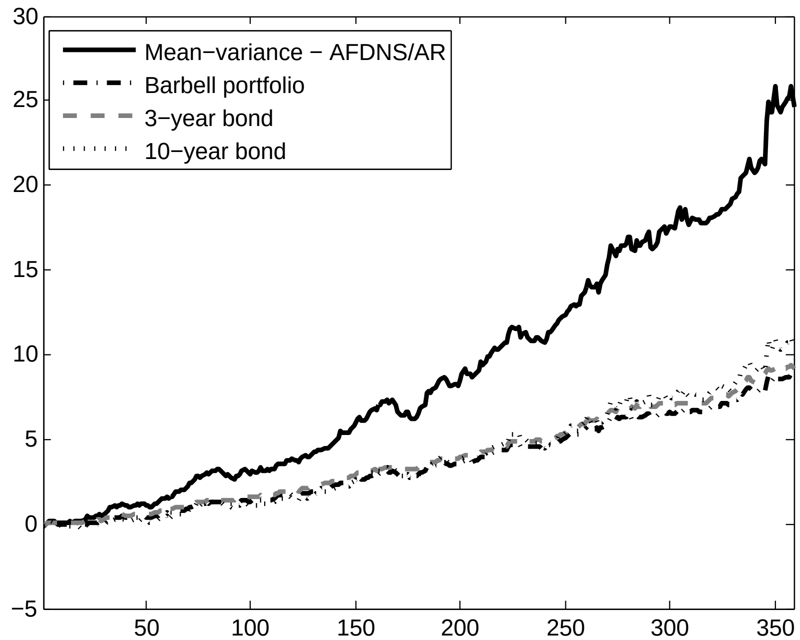


Figure 2: **Cumulative returns for the US mean-variance portfolio**



Tables

Table 1: **Descriptive statistics for the U.S. Treasuries data set**

Summary statistics for US-Treasury data set consisting of monthly yield data from January 1970 to December 2009. We show for each maturity mean, standard deviation, minimum, maximum, skewness, kurtosis, three (1 month, 12 month and 24 month) autocorrelations (*Acf*, $\widehat{\rho}(1)$, $\widehat{\rho}(12)$ and $\widehat{\rho}(24)$)) and two-month partial autocorrelation.

							Acf			Pacf
τ	Mean	Std Dev	Min	Max	Skew	Kurt	$\widehat{\rho}(1)$	$\widehat{\rho}(12)$	$\widehat{\rho}(24)$	$\widehat{\alpha}(2)$
6	5.97	3.10	0.15	16.48	0.67	3.82	0.980	0.763	0.516	-0.126
9	6.08	3.09	0.19	16.39	0.63	3.71	0.981	0.771	0.538	-0.145
12	6.17	3.05	0.25	16.10	0.57	3.59	0.981	0.777	0.552	-0.154
15	6.25	3.03	0.38	16.06	0.52	3.49	0.982	0.785	0.571	-0.149
18	6.32	3.01	0.44	16.22	0.52	3.46	0.983	0.792	0.585	-0.153
21	6.39	2.99	0.53	16.17	0.53	3.46	0.983	0.797	0.598	-0.143
24	6.42	2.94	0.53	15.81	0.52	3.40	0.983	0.799	0.610	-0.163
30	6.51	2.88	0.82	15.43	0.50	3.32	0.983	0.801	0.627	-0.135
36	6.60	2.83	0.98	15.54	0.53	3.35	0.984	0.814	0.642	-0.132
48	6.76	2.75	1.02	15.60	0.57	3.33	0.985	0.823	0.664	-0.112
60	6.85	2.67	1.56	15.13	0.61	3.28	0.986	0.832	0.685	-0.103
72	6.96	2.64	1.53	15.11	0.64	3.26	0.987	0.842	0.702	-0.106
84	7.03	2.57	2.18	15.02	0.71	3.30	0.987	0.841	0.709	-0.124
96	7.07	2.54	2.11	15.05	0.75	3.29	0.988	0.850	0.721	-0.121
108	7.10	2.52	2.15	15.11	0.80	3.33	0.988	0.853	0.724	-0.141
120	7.07	2.46	2.68	15.19	0.86	3.41	0.988	0.843	0.717	-0.117

Table 2: Performance of traditional yield curve strategies

The Table reports performance statistics for the traditional yield curve strategies. The statistics of returns, standard deviation and Sharpe ratio are annualized and the average portfolio duration is measured in years. The excess return is calculated using the 3-month rate as a risk-free asset.

Yield curve strategy	Mean return (%)	Mean excess return (%)	Standard deviation	Sharpe ratio
Spread portfolio	2.253	2.253	11.544	0.195
Barbell portfolio	7.814	2.022	7.184	0.282
Laddered portfolio	7.784	1.992	5.214	0.382
1-year bond	6.688	0.896	2.021	0.443
3-year bond	7.906	2.113	4.728	0.447
5-year bond	8.614	2.822	7.298	0.387
7-year bond	9.232	3.440	9.603	0.358
9-year bond	9.491	3.698	11.738	0.315
10-year bond	8.941	3.149	12.875	0.245

Table 3: Performance of optimal US-Treasuries mean-variance portfolios

The Table reports performance statistics for mean-variance portfolios using US zero-coupon yields with maturities equal to 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108 and 120 months. The yield curve model used is the dynamic version of the Nelson-Siegel 3-factor model (DNS) and its arbitrage-free version (AFDNS). The optimal portfolios are re-balanced on a monthly basis. The statistics of returns, standard deviation and Sharpe ratio are annualized and the average portfolio duration is measured in years. The excess return is calculated using the 3-month rate as a risk-free asset. δ denotes the value of the risk aversion coefficient. Asterisks indicate that the coefficient is statistically higher than that obtained benchmark bond portfolio (3-year bullet portfolio) at a significance level of 10%.

Yield curve model	Factor dynamics	Mean return (%)	Mean excess return (%)	Standard deviation	Sharpe ratio	Turnover	Average duration
		$\delta = 1 \times 10^{-4}$					
DNS	AR	11.656	5.864	10.208	0.574*	0.933	6.735
DNS	VAR	11.547	5.755	10.701	0.538*	0.922	7.459
AFDNS	AR	10.101	4.309	11.181	0.385	0.743	7.638
AFDNS	VAR	11.181	5.389	11.095	0.486*	0.866	7.441
		$\delta = 1 \times 10^{-3}$					
DNS	AR	11.656	5.864	10.191	0.575*	0.950	6.690
DNS	VAR	11.513	5.721	10.685	0.535*	0.931	7.422
AFDNS	AR	10.125	4.333	11.140	0.389	0.752	7.585
AFDNS	VAR	11.149	5.357	11.013	0.486*	0.866	7.386
		$\delta = 1 \times 10^{-2}$					
DNS	AR	11.216	5.424	10.021	0.541*	0.935	6.208
DNS	VAR	11.423	5.631	10.408	0.541*	0.961	7.004
AFDNS	AR	10.144	4.352	10.740	0.405	0.817	7.040
AFDNS	VAR	10.890	5.098	10.591	0.481*	0.924	6.846
		$\delta = 1 \times 10^{-1}$					
DNS	AR	9.057	3.264	5.248	0.622*	0.959	2.179
DNS	VAR	9.516	3.724	5.517	0.675*	1.074	2.829
AFDNS	AR	9.182	3.390	5.771	0.587*	0.931	2.439
AFDNS	VAR	9.477	3.685	6.178	0.596*	1.021	2.654
		$\delta = 0.5$					
DNS	AR	6.734	0.942	1.902	0.495*	0.507	0.685
DNS	VAR	6.695	0.903	1.767	0.511*	0.479	0.724
AFDNS	AR	6.931	1.139	2.114	0.539*	0.544	0.748
AFDNS	VAR	7.181	1.388	2.777	0.500*	0.545	0.786
		$\delta = 1.0$					
DNS	AR	6.334	0.542	1.492	0.363	0.133	0.526
DNS	VAR	6.297	0.505	1.424	0.354	0.104	0.525
AFDNS	AR	6.386	0.594	1.505	0.395	0.160	0.539
AFDNS	VAR	6.539	0.747	1.909	0.391	0.201	0.567

Table 4: Performance fee to switch from traditional yield curve strategies to the mean-variance strategy

The Table reports the annualized performance fee (Δ) in basis points that the investor is willing to pay to change from a traditional bond portfolio strategy to a mean-variance portfolio strategy. δ denotes the value of the risk aversion coefficient.

Yield curve model	Factor dynamics	Δ_{spread}	$\Delta_{barbell}$	Δ_{1Y}	Δ_{3Y}	Δ_{5Y}	Δ_{7Y}	Δ_{9Y}	Δ_{10Y}	Δ_{ladder}
						$\delta = 1 \times 10^{-4}$				
DNS	AR	9.403	3.839	4.963	3.746	3.039	2.424	2.167	2.718	3.868
DNS	VAR	9.293	3.729	4.853	3.636	2.929	2.314	2.057	2.608	3.758
AFDNS	AR	7.848	2.283	3.407	2.190	1.484	0.868	0.611	1.162	2.312
AFDNS	VAR	8.927	3.363	4.487	3.270	2.563	1.948	1.691	2.242	3.392
						$\delta = 1 \times 10^{-3}$				
DNS	AR	9.403	3.809	4.910	3.703	3.012	2.414	2.179	2.741	3.828
DNS	VAR	9.255	3.661	4.762	3.555	2.864	2.266	2.031	2.593	3.680
AFDNS	AR	7.864	2.270	3.372	2.165	1.473	0.876	0.640	1.203	2.289
AFDNS	VAR	8.888	3.294	4.395	3.189	2.497	1.900	1.664	2.226	3.313
						$\delta = 1 \times 10^{-2}$				
DNS	AR	8.987	3.100	3.974	2.866	2.322	1.907	1.878	2.556	3.010
DNS	VAR	9.150	3.264	4.138	3.030	2.486	2.071	2.042	2.720	3.174
AFDNS	AR	7.861	1.975	2.849	1.741	1.197	0.782	0.753	1.431	1.884
AFDNS	VAR	8.609	2.722	3.596	2.488	1.944	1.530	1.500	2.178	2.632
						$\delta = 1 \times 10^{-1}$				
DNS	AR	10.686	2.140	0.949	0.737	1.538	2.784	4.633	6.362	1.060
DNS	VAR	10.940	2.395	1.203	0.992	1.793	3.039	4.887	6.617	1.314
AFDNS	AR	10.530	1.985	0.793	0.582	1.383	2.629	4.478	6.207	0.904
AFDNS	VAR	10.557	2.012	0.820	0.608	1.410	2.656	4.504	6.233	0.931
						$\delta = 0.5$				
DNS	AR	23.977	7.474	0.100	2.572	7.398	13.614	21.083	25.958	3.430
DNS	VAR	24.040	7.537	0.163	2.635	7.462	13.678	21.147	26.021	3.493
AFDNS	AR	23.929	7.426	0.052	2.524	7.351	13.567	21.036	25.910	3.382
AFDNS	VAR	23.508	7.005	-0.369	2.103	6.930	13.146	20.615	25.489	2.961
						$\delta = 1$				
DNS	AR	33.981	12.007	0.383	4.700	12.294	21.927	33.259	40.296	5.926
DNS	VAR	34.021	12.047	0.423	4.740	12.334	21.967	33.299	40.336	5.966
AFDNS	AR	33.983	12.009	0.385	4.702	12.296	21.929	33.262	40.298	5.928
AFDNS	VAR	33.674	11.700	0.076	4.393	11.987	21.620	32.952	39.989	5.619
Average		15.805	5.127	2.279	2.786	4.666	7.244	10.599	13.171	3.252

Table 5: Performance of optimal US-Treasuries duration-constrained mean-variance portfolios

The Table reports performance statistics for duration-constrained mean-variance portfolios using US zero-coupon yields with maturities equal to 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108 and 120 months. The yield curve model used is the dynamic version of the Nelson-Siegel 3-factor model (DNS) and its arbitrage-free version (AFDNS). The optimal portfolios are re-balanced on a monthly basis. The statistics of returns, standard deviation and Sharpe ratio are annualized. The excess return is calculated using the 3-month rate as a risk-free asset. Δ denotes the annualized performance fee (in basis points) that the investor is willing to pay to change from a bond portfolio strategy to a mean-variance portfolio with equal duration. Asterisks indicate that the coefficient is statistically different from that obtained by the bullet bond portfolio strategy with equal duration at a significance level of 10%. We assume an investor with risk aversion coefficient $\delta = 1$.

Yield curve model	Factor dynamics	Mean return (%)	Mean excess return (%)	Standard deviation	Sharpe ratio	Turnover	Δ (b.p.)
		Target portfolio duration=1-year					
DNS	AR	6.595	0.803	1.765	0.455*	0.337	0.221
DNS	VAR	6.571	0.779	1.753	0.444	0.267	0.226
AFDNS	AR	6.626	0.834	1.853	0.450	0.326	0.149
AFDNS	VAR	6.589	0.797	1.874	0.425	0.310	0.121
		Target portfolio duration=3-year					
DNS	AR	7.611	1.818	4.071	0.447	0.214	1.390
DNS	VAR	7.596	1.804	4.071	0.443	0.195	1.388
AFDNS	AR	7.645	1.853	4.102	0.452*	0.241	1.333
AFDNS	VAR	7.720	1.928	4.398	0.438	0.261	0.720
		Target portfolio duration=5-year					
DNS	AR	8.515	2.723	6.603	0.412*	0.163	2.395
DNS	VAR	8.514	2.722	6.605	0.412*	0.166	2.389
AFDNS	AR	8.629	2.837	6.607	0.429*	0.203	2.398
AFDNS	VAR	8.666	2.873	6.886	0.417*	0.252	1.464
		Target portfolio duration=7-year					
DNS	AR	9.409	3.617	9.186	0.394*	0.166	1.972
DNS	VAR	9.407	3.614	9.187	0.393*	0.166	1.969
AFDNS	AR	9.576	3.784	9.162	0.413*	0.207	2.101
AFDNS	VAR	9.628	3.836	9.339	0.411*	0.232	1.288
		Target portfolio duration=9-year					
DNS	AR	9.619	3.827	11.653	0.328	0.067	0.511
DNS	VAR	9.619	3.827	11.653	0.328	0.067	0.510
AFDNS	AR	9.993	4.201	11.669	0.360*	0.126	0.450
AFDNS	VAR	9.927	4.134	11.691	0.354*	0.130	0.319

Table 6: Performance of optimal US-Treasuries mean-variance portfolios under quarterly re-balancing frequency

The Table reports performance statistics for mean-variance portfolios using US zero-coupon yields with maturities equal to 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108 and 120 months. The yield curve model used is the dynamic version of the Nelson-Siegel 3-factor model (DNS) and its arbitrage-free version (AFDNS). The optimal portfolios are re-balanced on a quarterly basis. The statistics of returns, standard deviation and Sharpe ratio are annualized and the average portfolio duration is measured in years. The excess return is calculated using the 3-month rate as a risk-free asset. δ denotes the value of the risk aversion coefficient. Asterisks indicate that the coefficient is statistically higher than that obtained benchmark bond portfolio at a significance level of 10%.

Yield curve model	Factor dynamics	Mean return (%)	Mean excess return (%)	Standard deviation	Sharpe ratio	Turnover	Average duration
				$\delta = 1 \times 10^{-4}$			
DNS	AR	10.568	4.776	9.851	0.485*	0.313	6.774
DNS	VAR	10.608	4.815	10.203	0.472*	0.302	7.448
AFDNS	AR	9.676	3.884	10.615	0.366	0.251	7.306
AFDNS	VAR	9.776	3.984	10.210	0.390	0.296	7.061
				$\delta = 1 \times 10^{-3}$			
DNS	AR	10.566	4.774	9.841	0.485*	0.322	6.729
DNS	VAR	10.609	4.816	10.175	0.473*	0.298	7.393
AFDNS	AR	9.675	3.883	10.613	0.366	0.251	7.306
AFDNS	VAR	9.731	3.939	10.177	0.387	0.294	7.028
				$\delta = 1 \times 10^{-2}$			
DNS	AR	10.109	4.317	9.540	0.453*	0.309	6.209
DNS	VAR	10.655	4.863	9.990	0.487*	0.290	7.015
AFDNS	AR	9.581	3.789	10.222	0.371	0.252	6.874
AFDNS	VAR	9.646	3.854	9.919	0.389	0.294	6.734
				$\delta = 1 \times 10^{-1}$			
DNS	AR	8.122	2.329	4.323	0.539*	0.307	2.218
DNS	VAR	8.276	2.483	4.733	0.525*	0.327	2.845
AFDNS	AR	7.740	1.947	4.894	0.398	0.292	2.486
AFDNS	VAR	7.346	1.553	5.524	0.281	0.287	2.808
				$\delta = 0.5$			
DNS	AR	6.560	0.768	1.701	0.452	0.173	0.697
DNS	VAR	6.563	0.770	1.690	0.456*	0.157	0.736
AFDNS	AR	6.665	0.873	1.949	0.448	0.184	0.775
AFDNS	VAR	6.620	0.828	2.151	0.385	0.177	0.793
				$\delta = 1$			
DNS	AR	6.252	0.460	1.408	0.327	0.048	0.530
DNS	VAR	6.240	0.448	1.409	0.318	0.041	0.529
AFDNS	AR	6.301	0.509	1.458	0.349	0.052	0.549
AFDNS	VAR	6.308	0.516	1.599	0.322	0.072	0.571