Strategic reasoning in $p$-beauty contests

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Abstract

This paper analyzes strategic choice in $p$-beauty contests. First, I show that it is not generally a best reply to guess the expected target value, even in games with $n > 2$ players, and that iterated best response sequences strictly applied do not induce a choice sequence approximating $p^k \cdot 0.5$. Second, I argue that the beliefs and actions of players typically considered to be level 2–4 are captured more accurately using high-level concepts such as quantal response equilibrium and noisy introspection. Third, I analyze this hypothesis econometrically. The results confirm it. In five out of six data sets, the subject pool is represented most accurately as a mixture of quantal response equilibrium types and noisy introspection types.

JEL–Codes: C44, C72

Keywords: beauty contest, logit equilibrium, noisy introspection, level-$k$.

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1 Introduction

In the $p$-beauty contest, $n$ players pick numbers $x_i \in [0,1]$ and the player whose “guess” is closest to $p \in (0,1)$ times the mean of all guesses wins a fixed prize. Nagel (1995) reported the first experimental investigation of such guessing games and advanced a theory of “iterated best responses” to explain her observations, leaning on the level-$k$ model of Stahl and Wilson (1995). Accordingly, the population consists of level-0 types who randomize uniformly, of level-1 types who assume all their opponents are level 0, of level-2 types who assume all their opponents are level 1, and so on. The level-$k$ model has a variety of plausible and attractive features and is currently considered the leading theory of strategic choice in novel situations.\(^1\)

The present study challenges the level-$k$ interpretation of strategic choice in guessing games, however, and presents evidence in favor of high-level concepts such as equilibrium and rationalizability to explain the choices.

The arguably most convincing feature of level-$k$ models is that it is based on the existence of non-strategic players (level 0) and of players assuming their opponents are non-strategic (level 1). The existence of level-1 players is particularly convincing in novel situations, where one acts under uncertainty. Having to formulate subjective beliefs about the opponents’ choice probabilities as a basis for the own choice, the belief that one’s opponents would randomize uniformly follows (if nothing else) from the principle of insufficient reason.

The existence of players on level 2 and higher, i.e. of players who go further and believe all their opponents are at least level 1, is much less obvious, however. I will argue that actions and beliefs of these subjects—who seem to be level 2 and higher—are actually captured more accurately by high-level concepts.

First, level-2 strictly applied does not yield a cluster of choices around 0.222 (see Section 2). Secondly, look at Figure 1, which reviews the data compiled by Bosch-Domenech et al. (2002) in six different sets of $p$-beauty contest experiments (with $p = 2/3$). The most focal observations in the data are probably the modes at .333 and

\(^1\)For example, see Ho et al. (1998), Costa-Gomes et al. (2001), Crawford and Iriberri (2007), and Stahl and Haruvy (2008) for level-$k$ analyses and Camerer et al. (2004), Kübler and Weizsäcker (2004), and Rogers et al. (2009) for related analyses.
Figure 1: Overview of the data from one-shot beauty contests compiled by Bosch-Domenech et al. (2002)

**Laboratory** (86 observations, 15-18 players per game, 5 minutes time)

**Class room** (138 observations, 30-50 players per game, 5 minutes time)

**Take home** (119 observations, 30-50 players per game, 1 week time)

**Theorists** (146 observations, 20-50 players per game, 5 minutes/1 week time)

**Newsgroup** (1 experiment, 150 players, 1 week time)

**Newspaper** (2 experiments, 1476 and 3696 players, 2 weeks/1 week time)

*Note:* The vertical axes are scaled such that the aggregate areas under the histograms are 1. The two additional lines are kernel density estimates, using Gaussian smoothing kernels, either with bandwidth according to Silverman’s rule of thumb (“Kernel-1.0”) or half this value (“Kernel-0.5”). The value of $p$ was $2/3$ in all cases, and I excluded one of the newspaper experiments of Bosch-Domenech et al. (2002), as it restricted the choice set to natural numbers ($\geq 1$).
.222, but they combine for less than 20% of the observations in all cases, they rarely have the highest density in these samples (in most cases, the Nash equilibrium has higher density), and their exact location varies between data sets (see in particular the “Laboratory” data). Most importantly, the strength of the mode around .222 varies drastically between data sets, it is hardly noticeable in any of the density estimates, and higher-level modes do not exist. Thus, it seems negligible to include level-$k$ components for $k \geq 2$ into the strategic model.

Thirdly, as Bosch-Domenech et al. (2002) report, many subjects in $p$-beauty contests explicitly pick numbers they consider “intermediate,” i.e. the subjects themselves believe that some opponents will choose smaller numbers and that others will choose higher numbers. Such beliefs are “balanced” in that they account for the whole distribution of opponents, and they clearly contradict the level-$k$ model for $k > 1$, where players believe that they are exactly one step ahead of their opponents. Balanced beliefs are implied if players behave according to noisy, high-level concepts such as equilibrium or rationalizability. Needless to say, players trying to formulate “balanced” subjective beliefs are likely not to determine a quantal response equilibrium (QRE) exactly, but the overall shape of a QRE may well be closer to their subjective beliefs than the level-$k$ model for $k \geq 2$. For, it captures their objective in formulating beliefs more accurately, and in the end, this may yield a more accurate description of their behavior.

The present paper investigates this hypothesis based on the data set compiled by Bosch-Domenech et al. (2002). Its organization and main results are as follows. Section 2 revisits the computation of iterated best responses and level-$k$ choices in $p$-beauty contests. Section 3 fits level-$k$ models to the data, using the correctly determined best responses in all cases, and shows that level-$k$ fits rather poorly (if restricted to $k \leq 4$). Section 4 considers two models of sophisticated subjective beliefs, quantal response equilibrium (McKelvey and Palfrey, 1995) and noisy introspection (Goeree and Holt, 2004), and examines their fit in relation to level-$k$ models. Both of these models improve upon level-$k$, and the most accurate models of the subject pools are mixtures between them. The most relevant component of level-$k$ models—namely, level 1—is compatible with noisy introspection, and the higher-level actions
are described more accurately using quantal response equilibrium or noisy introspection (depending on context). Thus, the subjective beliefs are actually balanced, as opposed to being iterative as in level-k models. Section 5 concludes. The supplementary material contains, amongst others, all parameter estimates and plots illustrating the goodness-of-fit of all models.

2 The level-k theory of choice in $p$-beauty contests

The set of players is denoted as $N = \{1, \ldots, n\}$, typical players are denoted as $i, j \in N$ and actions as $x_i \in [0, 1]$ for all $i \in N$. In the $p$-beauty contest, all players move simultaneously, and the payoff of $i \in N$, denoted as $\pi_i(x)$ for all $x = (x_i)_{i \in N}$, is

$$\pi_i(x) = \begin{cases} 1/|D(x)|, & \text{if } i \in D(x), \\ 0, & \text{otherwise,} \end{cases}$$

using $D(x) = \arg\min_{j \in N} |x_j - t|$, and $t = \sum_{j \in N} x_j \ast p/n$. \(1\)

The level-k model rests on the assumption that subjective beliefs of (some) inexperienced players about their opponents’ strategies are simple heuristics such as uniform randomization (see Stahl and Wilson, 1994). More sophisticated players may anticipate this and believe that their opponents’ are level 1, level 2, and so on. This basic level-k model is called “rational level-k” in the following, to distinguish it from alternative level-k models defined further below.

**Definition 2.1** (“Rational LevK”). The set of player types is $\mathcal{K} = \{0, \ldots, K\}$; the type shares in the population are $(\rho_k)_{k \in \mathcal{K}}$. Players of type $k = 0$ randomize uniformly on $[0, 1]$, and for all $k \geq 1$, type-$k$ players best respond to level $k - 1$, randomizing uniformly in cases of indifference.

Intuitively, the choices of level-$k$ players approximate the sequence $(p^k/2)_{k \geq 1}$.

\(^2\)Most analyses of beauty contests are based on $(p^k/2)_{k \geq 1}$ or closely related sequences, see e.g. Ho et al. (1998), Stahl (1998), Bosch-Domenech et al. (2002), and Kocher and Sutter (2005), but not all claim that this sequence results from the level-k model (e.g. Nagel, 1995, and Stahl, 1996).
of indifference, which arguably is the most consistent assumption to make in this case. It is the equivalent to assuming uniform randomization at level 0, it is the limiting distribution as random utility perturbations (as in logistic models) disappear, and any refinement of iterated responses implies an incomplete theory as discussed shortly.

**General structure of the level-\(k\) choices**

Assuming uniform randomization at level 0, the approximate level-1 response is 
\[ L_1 = \frac{n-1}{n-p} \cdot p \cdot (0 + 1)/2 \] (for the following illustration, this is sufficiently precise). In response to players choosing \(L_1\), all actions in the interval \((\text{BR}_{\text{inf}}(L_1), L_1)\) are best responses, using

\[
\text{BR}_{\text{inf}}(x_k) = \max \left\{ \frac{n \times (2-1/p) - 2}{n-2p} \cdot px_k, 0 \right\}.
\] (2)

\(\text{BR}_{\text{inf}}(x_k)\) is the smallest number \(x \in [0,1]\) such that, given the target value \(t(x) := p \cdot ((n-1) \cdot x_k + x)/n\), the distance between \(x\) and \(t(x)\) is not greater than the distance between \(x_k\) and \(t(x)\).

Thus, level 2 randomizes uniformly on \((\text{BR}_{\text{inf}}(L_1), L_1)\). In case \(n\) is large, the level-\(k\) choices correspond with \(L_1 = 0.33\) and \(L_2\) randomizing on \((0.11, 0.33)\). The best response of level-3 players is approximately \(L_3 = \frac{n-1}{n-p} \cdot p \cdot (0.11 + 0.33)/2 =\)
0.148, and level 4 randomizes on \((\text{BR}_{3\inf}(L_3), L_3) = (0.0492, 0.148)\). Figure 2 illustrates these choices. Generally, players at odd levels play pure strategies, those at even levels play mixed strategies, and the supports of the strategies of even levels overlap. The pure strategy of level 1 corresponds with the mode in the data sets around 0.333, but modal choices around 0.222 cannot be explained this way.

It seems that refinement of the best responses of even levels yields point predictions approximating \((p^k/2)_{k \geq 1}\). Actually, this does not lead us far. On the one hand, only rather specific refinement assumptions yield this effect. For example, trembling-hand perfection in general does not, and “uniform perfection,” robustness with respect to uniform trembles, does so only if \(n\) is large (for example, the “uniform perfect” response sequence converges above 0.1 if \(n = 16\) as in Laboratory games). On the other hand, point predictions do not explain the actual distribution of choices. Predominantly, subjects do not choose \((p^k/2)_{k \geq 1}\) and in order to explain the actual choices, substantial noise is required. Substantial noise, in turn, dominates any such refinement concept. The limiting distribution as say logistic errors disappear would be the uniform distribution (on the best responses), rather than a refined choice, however. The level-\(k\) model as it is defined above avoids this disconnect and additionally makes for a complete model of beauty-contest choices in its own right (i.e. without the necessity to add errors).

**Best responses to randomizing players**

Next, to illustrate the derivation of best responses to mixed strategies, assume you play a beauty contest against opponents randomizing on \([0, 1]\). If their strategies have mean \(\bar{x}\), the intuitive best response is \(\frac{n - 1}{n - p} \cdot p\bar{x}\), accounting for the weight of the own guess, or \(p \cdot \bar{x}\) for simplicity. That is, \(x_i\) seems to be a best response if it minimizes the distance to the expected target value. Guessing the expected target value is known to be sub-optimal in two-player games, where \(x_i = 0\) is generally optimal (under full support, assuming \(p < 1\)). The following shows that it is not generally optimal for \(n = 3\) either, and similar arguments apply for larger \(n\), although the bias vanishes as \(n\) increases, depending on the distribution of opponents’ choices.
In the following, I focus on the response of level-1 players, but responses to other mixed strategies are derived similarly. That is, fix $n = 3$, $p < 1$, and consider player $i = 1$ in response to two opponents randomizing uniformly on $[0, 1]$. The set of $(x_2, x_3) \in [0, 1]^2$ in response to which a given $x_1$ is closest to the target value is called *win region* of $x_1$. Without loss of generality, assume $x_2 < x_3$. Two cases have to be distinguished. On the one hand, in case $x_1 \leq x_2 < x_3$, player 1 wins (a share of) the prize if the target value is closer to $x_1$ than to $x_2$. Using $\alpha = p/3$, this holds if

$$\alpha(x_1 + x_2 + x_3) \leq \frac{1}{2}(x_1 + x_2) \iff x_3 \leq \frac{1 - 2\alpha}{2\alpha}x_1 + \frac{1 - 2\alpha}{2\alpha}x_2. \quad (3)$$

On the other hand, in case $x_2 < x_1 \leq x_3$, player 1 again wins if the target value is closer to $x_1$ than to $x_2$, but the formal condition is now

$$\alpha(x_1 + x_2 + x_3) \geq \frac{1}{2}(x_1 + x_2) \iff x_3 \geq \frac{1 - 2\alpha}{2\alpha}x_1 + \frac{1 - 2\alpha}{2\alpha}x_2. \quad (4)$$

Note that $p < 1$ implies that the target value is closer to $x_1$ than to $x_3$ in either case. These two cases correspond with two disjoint win regions in $[0, 1]^2$ and are illustrated in Figure 3. A second set of restrictions (and win regions) applies when $x_3 < x_2$. Figure 4 depicts the aggregate win regions in various constellations.

In the case of uniformly randomizing opponents, the expected payoff of guessing $x_1$ simply equates with the aggregate area size of the win regions associated with $x_1$. The expected payoff can be computed for all $x_1$ in closed form, but due to the case distinctions involved, it is relegated to Appendix A. The following results.

**Proposition 2.2.** Assume $n = 3$ and $p \in (0, 1)$. The payoff-maximizing choice in response to two uniformly randomizing opponents is, using $\alpha := p/3$,

$$x_1^* = \begin{cases} \frac{4\alpha}{7 - 16\alpha} & \text{if } p < 0.75, \\ \frac{2\alpha(1 - 2\alpha)}{(4 - 7\alpha)(1 - 3\alpha) + 3\alpha^2} & \text{if } 0.75 \leq p \leq 0.908, \\ \frac{2 - 6\alpha^3}{\alpha(1 - 2\alpha)(4\alpha - 1)} & \text{if } p > 0.908. \end{cases} \quad (5)$$

In the extreme case, as $p$ is equal to 1, player 1 is best off picking the number
Figure 3: Win regions if player 1 chooses $x_1 = 0.5$ (in case $x_2 < x_3$ and $p = 0.9$)

![Win Regions Diagram]

Note: The probability of winning with $x_1 = 0.5$ in this case ($p = 0.9$ and $n = 3$) is 0.5417; the size of the shaded area is half this value (0.2708). The probability of winning with the best response $x_1^* = 0.5217$ in this case is 0.5435.

that is between the opponents’ guesses with highest-possible probability, i.e. 0.5 to maximize the chances to win in case $x_2 < x_1 < x_3$ or $x_3 < x_1 < x_2$. As $p$ decreases, player 1 also has a chance to win when his guess ends up being below both opponents’ guesses. In this case, the boundary of the win regions $\frac{1-2a}{2a}(x_1 + x_2)$, see Figure 3, becomes steeper and the central win region, i.e. winning in case $x_1 \leq x_2 < x_3$, opens up. In these cases, where $0.91 < p < 1$, the players start to gain from emphasizing their chances to win conditional on $x_1 \leq x_2 < x_3$. The $x_1$ results in a tradeoff to win in either of these cases, $x_1 \leq x_2 < x_3$ or $x_2 < x_1 < x_3$ or $x_3 < x_1 < x_2$, and turns out to be decreasing in $p$, i.e. $x_1$ increases as $p$ decreases. In turn, the optimal choice is not monotonically increasing in $p$. Initially, the structure of the win region is akin to Figure 4a (although $x_1 > 0.5$), for $p = 0.908$ it reaches Figure 4b, and as $p$ falls further, the best responses start falling again. The structure of the win regions then
corresponds with either Figure 4c, if $0.75 \leq p \leq 0.908$, or Figure 4d, if $p < 0.75$.

Proposition 2.2 shows that best response functions are considerably more complex than $x_i^* = \frac{n-1}{n-p} \cdot p \bar{x}$. The most counter-intuitive case for $n = 3$ seems to be $p = 0.9$, which happens to have been chosen in three-player treatments by Ho et al. (1998). In case $p = 0.9$, the payoff-maximizing choice in response to uniformly randomizing players is $x_1^* = 0.5217$—the level-1 choice is greater than the average guess of level-0 players in this case. In relation to the expected target value, which is $\alpha^* (0.5 + 0.5 + x_1^*) = 0.457$, the optimal $x_1^*$ is on the “wrong side” of the opponents’ means.

The best response to uniformly randomizing opponents converges to $\frac{n-1}{n-p} \cdot p \bar{x}$, and hence to $p \bar{x}$, as $n$ approaches infinity. This convergence suggests that strategic effects becomes negligible when $n$ is greater than say 10. Ho et al. (1998), however, found that subjects behave as if their opponents’ guesses would be correlated with $\rho = 1$. Hence, the strategic effects inducing deviations from $p \bar{x}$ do not become negligible
as \( n \) grows. Such correlation will be investigated in the next section.

### 3 Do level-\( k \) models fit the data?

In this section, I analyze the fit of level-\( k \) models using the choice structure derived in Section 2. Further below, I will consider models of sophisticated subjective beliefs (variations of rationalizability and equilibrium) and compare the predictions based on these models to those based on the level-\( k \) model.

As Figure 1 shows, the exact location of the empirical “level-1” mode varies across data sets. These shifts of the modes cannot be explained by the rational level-\( k \) model, as \( L_1 \approx 0.33 \) for all \( n \geq 10 \). In order to explain the shifts, we have to weaken the assumption that players are rational, i.e. the assumption that they play best responses to their subjective beliefs. Clearly, reducing the precision of responses, from best responses to say logit responses, does not induce a shift of the mode of choices. The best response(s) would still be chosen most frequently.

An alternative approach to explain the shifts of the modes was proposed by Ho et al. (1998). They argue that subjects determine their (best) responses in thought experiments, to simulate the choices of their opponents, but with a limited number of independent draws. Specifically, they assumed that the opponent’s draws are correlated in the thought experiment. Correlation of truncated random variables (be they uniform, random, or logistic) is difficult to formalize precisely, however, as multivariate distribution functions for arbitrary correlation matrices are not available. Therefore, I adopt an alternative approach to model the thought experiment. In this model, subjects estimate expected payoffs based on a small number \( m \) of independent draws, say \( m = 3 \) or \( m = 4 \), and these draws are then extrapolated to the actual number \( n - 1 \) of opponents. Unbiased behavior results if \( m = n - 1 \), but the hypothesis \( m = n - 1 \) will be rejected in all cases. Notably, in addition to explaining shifts of the mode of choices, this model is compatible with two well-known biases from choice under risk and uncertainty. It is compatible with overconfidence, as \( m < n - 1 \) implies that one overestimates the individual probability of winning, and with the “believe in small numbers,” as such players extrapolate based on small numbers of observations.
Formally, if $i$’s opponents $j \neq i$ randomize according to the density $f_j$ on $[0,1]$, $i$ estimates the expected payoff in a thought experiment using $m \leq n - 1$ draws from $f_j$. Let $f^m(\cdot|f_j)$ denote the joint density of $m$ i.i.d. random variables with density $f_j$, and define $i$’s payoff from $x_i$ as

$$\pi(x_i|m,f_j) = \int_{[0,1]^m} f^m(y|f_j) p_i(x_i,y) dy,$$  \hspace{1cm} (8)$$

where $p_i(x_i,y)$ indicates whether $x_i$ wins in response to $y$

$$p_i(x_i,y) = \begin{cases} 1, & \text{if } |x_i - t(x_i,y)| \leq |y_j - t(x_i,y)| \forall j = 1, \ldots, m, \\ 0, & \text{otherwise,} \end{cases}$$

assuming the target value $t(x_i,y)$ is the weighted average of $x_i$ and $y$

$$t(x_i,y) = \frac{1}{n} \left( x_i + \frac{n-1}{m} \sum_{j \leq m} y_j \right).$$

The expected payoff for real $m \in \mathbb{R}$ is extrapolated as follows: the expected payoff is the weighted average of using $\lceil m \rceil$ and $\lfloor m \rfloor$, with weights $m - \lfloor m \rfloor$ and $\lceil m \rceil - m$, respectively. I assume $m \geq 2$ for computational reasons. Now, the “Biased LevK” model can be defined as follows.

**Definition 3.1** (“Biased LevK”). In contrast to “rational” level-$k$, players of level $k$ respond to level $k - 1$ by maximizing $\pi$ as defined in Eq. (8).

In addition to this bias, I allow for the possibility of logistic errors in the payoff estimates (i.e. for multinomial logit choice functions). The main alternative approach to such random utility models are random behavior models, i.e. the addition of (normal) errors to the response (Stahl, 1996; Ho et al., 1998). I choose the random utility model, because it ensures that the probability of deviations from the best response is decreasing in the loss occurred thereby (as observed by Battalio et al., 2001), and because utility perturbations can be interpreted as mistakes as they occur in computations of expected payoffs, while behavior perturbations are usually interpreted as trembles. The idea of mistakes in the stage of payoff estimation seems more intuitive
Figure 5: Level-k predictions for various precision levels

(a) $\lambda = 3$

(b) $\lambda = 10$

(c) $\lambda = 30$

(d) $\lambda = 100$

(e) $\lambda = 1000$

(f) $\lambda = 1,000,000$

Note: The plots are “proportional” representations of the choice probabilities in the discrete beauty contest with 1001 choices (0, 0.001, 0.002, ..., 1). The actual choice probabilities had been taken to the power of 0.25, as they often are very close to zero and hardly visible on a $[0,1]$ axis.

than trembles during the execution of one’s (correctly optimized) choice.\footnote{Note also that I abstain from truncating the support of higher-level choices to subsets of the strategy set (see e.g. Ho et al., 1998). For, all actions but the upper bound are rationalizable in continuous beauty contests, and all choices but the $k$ highest ones are level-$k$ rationalizable in discrete beauty contests.}

**Definition 3.2 (“Logistic LevK”).** For all $k \geq 1$, players of type $k$ randomize on $[0,1]$ according to the density (using $\lambda_k \geq 0$)

$$f_i(x_i) = \frac{\exp \{\lambda_k \cdot \pi(x_i|m, f_{k-1})\}}{\int_0^1 \exp \{\lambda_k \cdot \pi(\tilde{x}_i|m, f_{k-1})\} \, d\tilde{x}_i}.$$  \hspace{1cm} (9)

Figure 5 illustrates how the logistic level-k model approaches the biased level-k model as $\lambda$ approaches infinity.
Before I show how the various level-

level models fit the data, let me briefly elaborate on the econometric procedure. As usual, the level-

level models are specified as finite-

mixture models (Peel and MacLahlan, 2000). That is, the set of components (“types”) \( K \) is finite. For all \( k \in K \), \( \sigma_k \) denotes the parameter(s) defining type \( k \) and \( \rho_k \) denotes its relative frequency in the population, with \( \sum_{k \in K} \rho_k = 1 \). The log-likelihood of the resulting model \((\sigma_k, \rho_k)_{k \in K}\) given the observations \( o = (o_m)_{m=1,...,M} \) is

\[
LL(\sigma, \rho | o) = \sum_{m=1}^{M} \ln \sum_{k \geq 0} \rho_k \cdot f_k(o_m | \sigma_k), \tag{10}
\]

using \( f_k(x | \sigma_k) \) as the density of the level-

level strategy. Since the density of a pure strategy approximates infinity, likelihood and log-likelihood of any “pure” level-

level model approximate infinity—for almost all parameterizations—if this pure strategy is observed at least once. This renders likelihood maximization in continuous beauty contests impossible, but the continuity assumption does not seem reasonable in the first place. For, subjects picking numbers such as 0.32335 do not seem to deviate from say 0.323 based on expected payoff calculations. Digits beyond the third one seem to be chosen randomly and can therefore be rounded off. In the following, I assume that the smallest choice unit is 0.001 and that the strategy set is \( \{0, 0.001, 0.002, \ldots, 1\} \). Observations off this grid are rounded toward the nearest number on the grid.

The models are estimated by maximizing the full information likelihood jointly over all parameters using the NEWUOA algorithm (Powell, 2008), verifying convergence using a Newton-Raphson algorithm, and ensuring global maximization using a variety of starting values per model (on the order of 100 per model). Standard errors are obtained from the information matrix. Expected payoffs are computed by Quasi Monte Carlo integration (using Niederreiter sequences with 10,000 \( \cdot m^2 \) random numbers, which is rather large in this context). All parameter estimates are provided as supplementary material.

In the analysis, I focus on levels \( k \leq 4 \). This leaves out observations close to the Nash equilibrium, but to explain these observations, the quantal response equilibrium is more appropriate (and much more parsimonious) than including say levels 7-10. Thus, to explain the choices of all subjects, mixtures of concepts, between say level-

level.
Table 1: Level-k models: Log-likelihoods, significance of differences, and pseudo-$R^2$

<table>
<thead>
<tr>
<th></th>
<th>Log-likelihoods</th>
<th>Logistic LevK</th>
<th>$R^2$</th>
<th>$\rho_0$</th>
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Note: The relation signs indicate the level of significance in likelihood ratio tests for non-nested models, where the distribution of the test statistic is bootstrapped. The sign = indicates insignificance, < indicates significance at the .1 level in two-sided tests, << indicates significance at .02, and <<< indicates significance at .001.

The value $R^2$ is the adjusted Pseudo-$R^2$ (following Cox and Snell) of the respective logistic level-k model. That is, $\hat{R}^2 = 1 - \left(\frac{L(\text{null})}{\exp(BIC)}\right)^{2/N}$, with $L(\text{null})$ as the likelihood of the uniform distribution, $BIC = -LL + \#\text{Pars} \cdot \log(N)/2$, and $N$ as the number of observations.

The value $\rho_0$ is the share of level-0 players according to the ML estimates of the logistic level-k model.
and QRE, may be required (see below).

Table 1 summarizes the results on level-\(k\) models. It displays the log-likelihoods of the level-\(k\) models with up to four components, the results of likelihood-ratio tests between “adjacent” models, the adjusted Pseudo-\(R^2\) of the logistic model to assess its relative fit, and the share of level-0 subjects to assess the comprehensiveness of the level-\(k\) models. The likelihood ratio tests employed here make minimal assumptions—by using the test for non-nested models of Vuong (1989) and bootstrapping the distribution of its test statistic—which improves the robustness of the conclusions.\(^4\) The adjusted Pseudo-\(R^2\) adjusts for the number of parameters by using the Bayes information criterion (BIC, Schwarz, 1978) rather than the log-likelihood (for its definition, see Table 1). As a result of this, its values are negative if its explanatory content does not justify its number of parameters, and more generally, \(\hat{R}^2\) puts the BIC into perspective and shows how much of the variance the model actually explains. Note that the model that maximizes \(\hat{R}^2\) simultaneously minimizes BIC.

The results can be summarized as follows. In all data sets but Theorists, the “rational” level-\(k\) model fits worse than at least one of the models weakening rationality. In the Laboratory, Class room, and Newsgroup data, the “biased” model fits significantly better than the “rational” level-\(k\) model, and otherwise the logistic level-\(k\) model is better. The Theorists data are special in that the level-\(k\) model does not fit in any specification considered here (conditional on \(k \leq 4\)). As for the Theorists, the Pseudo-\(R^2\) measures are around zero and the “residual” level-0 component harbors at least 80% of the subjects. However, in the Take home, Newsgroup, and Newspaper data, the residual components still contain about 50% of the subjects. This can also be seen in Figure 6, which plots the predictions of the best logistic models on top the histograms of the data.

Thus, in both the Take home, Newsgroup, and Newspaper experiments, where subjects are given plenty of time to think about the game, and the Theorists experiment, where subjects know the game, the majority of subjects develop beliefs other than level-\(k\) reasoning up to three or four levels. The next section examines to which

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\(^4\)In particular, this procedure ensures that finite mixture models with highly “specialized” components to explain single observations/outliers are not considered to be significantly better than simpler models.
The plots display “adjusted” predictions of the estimated logistic level-$k$ models that maximize the BIC. The number of components of the respective models, the aggregate prediction, and the individual predictions of all components are displayed. The predictions are plotted on top of the histogram of the respective data set, and to ensure comparability of predictions and histogram, the predictions are aggregated into the bins used for the histogram.

degree the choices of these subjects are compatible with high-level concepts such as noisy rationalizability and quantal response equilibrium. In turn, the behavior of subjects from Laboratory and Classroom seems to be modeled fairly comprehensively using level-$k$ models. Here, the models minimizing BIC (i.e. the ones maximizing $\tilde{R}^2$) comprise about 70% of the subjects, as $\rho_0 < 0.3$, but of course, this does not rule out that some of these subjects actually play a noisy equilibrium.

### 4 Equilibrium and noisy introspection

In four of the six data sets, at least 50% of the subjects were classified as level 0, i.e. their choices did not match with the level-$k$ model for $k \leq 4$. In order to explain the choices of these subjects, and possibly to better explain those of the other subjects, models of more sophisticated strategic behavior are required. This section considers the two best-known formalizations of strategic sophistication in this context,
namely quantal response equilibrium (QRE, McKelvey and Palfrey, 1995) and noisy introspection (Nitro, Goeree and Holt, 2004). The QRE concept considered here is the logit equilibrium, which adds (extreme-value distributed) utility perturbations to Nash equilibrium, and Nitro adds utility perturbations to an iterative concept similar to rationalizability. To define these concepts formally, maintain $\pi(x_i|m, \sigma')$ as the expected payoff of choosing $x_i \in X = \{0, 0.001, 0.002, \ldots, 1\}$ if all opponents play according to the strategy $\sigma' \in \Delta(X)$, and define the discrete logit choice probabilities for given $\lambda$ and $m$ as

$$\sigma(x_i|\lambda, \sigma') = \frac{\exp\{\lambda \cdot \pi(x_i|m, \sigma')\}}{\sum_{x'_i \in X} \exp\{\lambda \cdot \pi(x'_i|m, \sigma')\}}. \quad (11)$$

The quantal response equilibrium is defined as follows. (In the analysis, I will consider mixture models with up to $K = 4$ such QRE components.)

**Definition 4.1 (QRE).** Given $\lambda \in \mathbb{R}^+$, the QRE choice probabilities $\sigma \in \Delta(X)$ are the solutions of $\sigma = \sigma(\cdot | \lambda, \sigma)$.

In $p$-beauty contests, it seems plausible to assert uniqueness of QREs, but a general result confirming this assertion is not available. The QREs underlying my analysis are the ones located along the principal branch, which is a result of using the following simple homotopy method (for more elaborate methods, see Turocy, 2005, 2010). Starting at a known solution (i.e. at the one for $\lambda = 0$ initially), I increase $\lambda$ in steps of at most 0.25, and after each increase of $\lambda$, I solve for the respective QRE by function iteration. If the function iteration does not converge, the step size is reduced. Thanks to the simplicity of the beauty contest game, this simple method worked well.

Noisy introspection, in contrast, is defined as follows.

**Definition 4.2 (Nitro).** Fix $\lambda \in \mathbb{R}^+$, $\mu \in [0, 1)$, and for all $k \in \mathbb{N}_0$ define $\lambda_k = \lambda \cdot \mu^k$. The Nitro choice probabilities are $\sigma_0$, where for all $k \geq 0, \sigma_k = \sigma(\cdot | \lambda_k, \sigma_{k-1})$.

Essentially, Nitro represents a continuum of subjective beliefs (and subsequent actions) ranging from uniform randomization ($\mu = 0$) to equilibrium ($\mu \approx 1$). In the analysis, I will assume a maximum of $K = 30$ induction steps and $\mu \leq 0.95$. In order to
Figure 7: QRE and Nitro predictions

(a) QRE for $m = 4$

(b) QRE for $m = 7$

(c) QRE for $m = 10$

(d) Nitro for $\lambda = 10$

(e) Nitro for $\lambda = 20$

(f) Nitro for $\lambda = 30$

Note: Similarly to Figure 5, the choice probabilities are rescaled (by taking them to the power of 0.25) to accentuate the effects on the $[0, 1]$ scale. Further, $m = 4$ in the Nitro models.

model choices based on a larger number of induction steps or larger $\mu$, the equilibrium concept can be adopted virtually without loss.

Figure 7 illustrates the differences between QRE and noisy introspection for various $\lambda$ and $\mu \in \{0.1, 0.3, 0.6, 0.9\}$. Basically, QRE distributions have larger variance than Nitro distributions, and for $m \geq 7$ and intermediate $\lambda$, they exhibit an increasingly pronounced bimodal shape. The Nitro distribution closest to bimodality in Figure 7 is the one for $(\lambda, \mu) = (10, 0.9)$, but as $\lambda$ increases, the Nitro distributions approximate pure strategies rather than the QRE distributions. This difference allows us to distinguish QRE and Nitro types.

Table 2 provides a first overview of the results. It focuses on the goodness-of-fit of logistic level-$k$, Nitro, and QRE models (with up to four components) without
Table 2: Comparison of logistic models with up to four components

<table>
<thead>
<tr>
<th></th>
<th>LevK</th>
<th>Nitro</th>
<th>QRE</th>
<th>Best model</th>
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<th>$\rho_0$</th>
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*Note:* Entries and notation are the same as in Table 1. The additional information given for the “best model” is given for the model maximizing the log-likelihood in the respective row.
mixing these concepts. Mixtures between them are considered below. Table 2 also shows how QRE relates to Nitro in the likelihood ratio tests and indicates whether any of these two improves upon the level-$k$ model.

The most informative likelihood-ratio tests are between the models with the BIC minimizing numbers of components $K$. The BIC minimizing numbers of components are $K = 1$ in Laboratory, $K = 2$ in Class room and Newsgroup, and $K = 4$ in Take home, Theorists, and Newspaper. With respect to these $K$, noisy introspection improves significantly on (logistic) level-$k$ in all cases but Laboratory. Since both noisy introspection and level-$k$ include the possibility that players believe their opponents randomize uniformly, the source of the difference between their goodness-of-fit must be in their fit with respect to the choices of higher-level players. Evidently, the high-level reasoning structure of noisy introspection fits better on aggregate than the intermediate-level structure of level-$k$ models with $k = 2, 3, 4$. The possible explanation that the actual distribution of types is a mixture between intermediate levels (say $k = 2, 3$) and equilibrium types (e.g. Nitro with high $\mu$) is considered below.

Between pure Nitro and pure QRE models, the evidence is mixed. As for Laboratory and Take home data, noisy introspection seems more appropriate, as for Theorists and Newspaper data, QRE seems more appropriate, and with respect to the Class room and Newsgroup data, the differences are insignificant. Combined with the previous observation that noisy introspection seems to fit better than level-$k$, this suggests that typical subject pools may be mixtures of Nitro and QRE types.

In order to understand the compositions of the subject pools, I next considered the possibility that different subjects employ different solution concepts and estimated all three possible combinations of Level-$k$, Nitro, and QRE models. The main results of this analysis can be found in Table 3. The parameter estimates of the best fitting models can be found in Table 4, the remaining estimates are provided as supplementary material. Figure 8 illustrates the attained fit.

The overall picture presented in Table 3 is surprisingly consistent. On the one hand, if the number of level-$k$ components is held constant, modeling the remaining “sophisticated” choices by QRE is never worse and often better than modeling those by noisy introspection (conditional on the imposed restriction $\mu \leq 0.95$). On the other
Table 3: Mixed models: Model comparison

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<td>LevK + QRE</td>
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<td>-41834.63</td>
</tr>
<tr>
<td>(1,2)</td>
<td>-42347.61</td>
<td>&lt;&lt;</td>
<td>&lt;&lt;</td>
<td>-41404.86</td>
</tr>
<tr>
<td>(2,2)</td>
<td>-42269.75</td>
<td>&lt;&lt;</td>
<td>&lt;&lt;</td>
<td>-41097.58</td>
</tr>
</tbody>
</table>

Note: Entries and notation are the same as in Table 1. The notation of the mixtures is such that mixture $(x,y)$ of model family $A+B$ has $x$ $A$-components and $y$ $B$-components. For example, mixture $(2,1)$ of model family LevK+QRE has 2 level-$k$ components and 1 QRE component.
Figure 8: Predictions of the best-fitting models

Note: Similarly to Figure 6, the predictions of the BIC minimizing models are plotted, and the predictions are aggregated to match the bins of the respective histograms.
hand, if the number of QRE components is held constant, the overall distribution is captured more accurately with Nitro components than with level-\(k\) components. For, Nitro can explain level-1 choices just like level-\(k\), and the remaining choices (supposedly level 2–4) evidently fit better with the high-level concept of noisy introspection than with level-\(k\). This improvement is significant for all data sets at the estimated (i.e. \(R^2\) maximizing and BIC minimizing) number of mixture components \(K\).

Thus, I conclude that choices in beauty contests are actually more sophisticated than the level-\(k\) model suggests, or put differently, the underlying subjective beliefs are more “balanced” than suggested by the level-\(k\) model. Aside from the confirmed existence of level-1 players (i.e. Nitro components with \(\mu = 0\), who assert that their opponents randomize uniformly, the observations more closely resemble noisy, high-level thinking than level-\(k\) thinking. These results also seem reliable, as the overall fit of the Nitro and QRE mixture is rather good. With the exception of the Laboratory data, which is the smallest data set and admits an adjusted \(R^2\) of merely 0.46, the \(R^2\) measures vary between 0.63 and 0.87.

Finally, let us look at the posterior classifications of the subjects using the model

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5I have also tested if the Nitro + QRE mixture improves upon the pure Nitro and pure QRE models reviewed in Table 2. The results are positive, i.e. the mixed model is never worse than the other two models (at the BIC minimizing number of components) and often significantly better. Details are provided in the supplementary material.
Figure 9: Posterior classifications according to the best-fitting models

Note: The posterior classifications are computed using Bayes Rule, Eq. (12), and the estimates of the BIC minimizing models.
estimates. Given a data set \( o \) and model estimates \((\rho_k, \sigma_k)\), the probability that subject \( s \) is of type \( k \) conditional on its guess \( o_s \) is

\[
\Pr(\text{type} = k|o_s) = \frac{\rho_k \cdot f_k(o_s)}{\sum_{k' \in \mathcal{K}} \rho_{k'} \cdot f_{k'}(o_s | \sigma_{k'})}.
\] (12)

Figure 9 plots the posterior classifications based on the estimates for all data sets. In general, choices close to zero are equilibrium choices and choices around 0.33 are Nitro choices with \( \mu = 0 \), i.e. level 1. However, the range of choices that classify as level 1 is very thin in most cases. The neighboring choices on both sides of this range are much more likely to have been made by noisy high-level thinkers such as QRE (Laboratory, Classroom, and Take home) or Nitro (Newsgroup and Newspaper).

Also, note that the results underlying Figure 9 falsify an assumption frequently made in the literature: a lower number guessed does not imply (ex post) a higher level of reasoning was used. In all cases, there is a fairly universal QRE or Nitro type dominating the population. Ex ante, the subjects of this type are likely to choose small numbers, but high numbers may also result, as the noise variances of the respective subjects are not small. Thus, the ex post most likely explanation for high numbers is that they result from perturbations of these players’ payoff estimates, as captured by random utility models such as multinomial logit, rather than from non-strategic reasoning such as level 0. The respective subjects do not seem to reason differently than those picking numbers such as 0.15, but their individual perturbations lead them to pick different numbers. The exceptions to this rule are the Theorists and the News- group, in which cases choices greater than 0.5 are extremely rare and most likely have been made by level-0 players.

5 Conclusion

This paper considered \( p \)-beauty contest games and investigated the hypothesis that the actions of subjects that do not classify as level 1 are modeled more accurately using high-level concepts such as equilibrium than using the level-\( k \) concept for \( k > 1 \). The qualitative observations that support this hypothesis are that the mode of the observations around 0.222 is not compatible with level-\( k \) if applied strictly (Figure 2).
that this mode or the supposed modes of higher-level players are not noticeable in the empirical density estimates (Figure 1), and that players choosing numbers such as 0.2 seem to explicitly choose “intermediate” numbers (as reported by Bosch-Domenech et al., 2002). This is incompatible with the level-\(k\) model, where players believe to be exactly one step ahead of their components, while it is compatible with noisy, high-level concepts such as quantal response equilibrium and noisy introspection.

The quantitative analysis confirmed the hypothesis (see Tables 2 and 3). That is, there are players akin to level 1, who seem to hold the subjective believe that the opponents randomize uniformly (when playing the \(p\)-beauty contest for the first time), but the likelihood-ratio tests reject the hypothesis that there are level-\(k\) players for \(k = 2, 3, 4\). The choices of subjects who are classified as levels 2–4 in the existing literature are captured significantly more accurately using quantal response equilibrium or noisy introspection. This has been confirmed independently for five of the six data sets considered here, with the sixth data set, Laboratory, being the smallest one and indecisive with respect to the hypothesis. The estimated models mixing QRE and noisy introspection also fit well to the various data sets, with Pseudo-\(R^2\) between 0.6 and 0.9. Thus, the results can be considered robust.

Further research may investigate whether noisy equilibrium and noisy introspection can be modeled more accurately than using the multinomial logit approach chosen above. Multinomial logit satisfies independence from irrelevant alternatives (IIA), but in beauty contests, the choice sets are ordered and can be segregated into the sub-sets by iteratively eliminating dominated strategies. If subjects do not ignore the orderedness or the nesting structure, their choices violate IIA and can be modeled more accurately than this has been done here. Additionally, one may consider the external validity of the above results, for example by analyzing other games (such as auctions) where level-\(k\) models have been shown to fit well, to examine if say noisy introspection fits better than levels 2 and higher in those games, too.
A Proof of Proposition 2.2

Best responses in case $p \geq 0.75 \iff \alpha \geq 0.25$

Fix $x_1$ and define $f$ and $g$ as follows.

$$f(x) = \frac{1 - 2\alpha}{2\alpha} (x_1 + x) \quad g \in \{x | f(x) = x\} \iff g = \frac{1 - 2\alpha}{4\alpha - 1} x_1$$

(13)

Note that $\alpha \geq 0.25$ implies $\frac{1 - 2\alpha}{2\alpha} \leq 1$. The complementary case $\alpha < 0.25$ is analyzed below.

Both $f$ and $g$ can be interpreted easily in relation to Figure 3. On the one hand, $f$ is the lower-right boundary of the area characterized by Eq. (4) and the upper-left boundary of the one characterized by Eq. (3). On the other hand, $g$ is the fixed point of $f$, i.e. the point where it intersects with the diagonal. These functions greatly simplify the notation relating to the conditions Eqs. (3) and (4). The payoff maximizing $x_1$ is derived in three steps. First, I show that $x_1 \leq \frac{\alpha}{1 - 2\alpha}$, i.e. $x_1$ is not optimal if implies win regions as in Figure 10c.

**Lemma A.1.** The best response satisfies $x_1^* \leq \frac{\alpha}{1 - 2\alpha}$.

**Proof.** If $x_1 \geq \frac{\alpha}{1 - 2\alpha}$, see Figure 10c, the expected payoff is

$$\pi(x_1) = (1 - x_1)^2 + \left[f^{-1}(x_1) + f^{-1}(1)\right] * (1 - x_1).$$

(14)

This equates with

$$\pi(x_1) = \frac{1 - (3 - 8\alpha) * x_1}{1 - 2\alpha} * (1 - x_1)$$

(15)

and implies $\pi'(x_1) < 0$ for all $x_1 \geq \frac{\alpha}{1 - 2\alpha}$.

Second, Lemma A.2 establishes the condition differentiating the cases displayed in Figures 10a and 10b.

**Lemma A.2.** The best response satisfies $x_1^* \leq \frac{4\alpha - 1}{1 - 2\alpha}$ if and only if $\alpha \geq 0.3026$. 

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Figure 10: The cases distinguished in Prop. 2.2 for \( p \geq 0.75 \)

(a) \( x_1 = \frac{(4\alpha - 1)}{(1 - 2\alpha)} \)

(b) \( \cdots \leq x_1 \leq \cdots \)

(c) \( x_1 = \frac{\alpha}{(1 - 2\alpha)} \)

**Proof.** If \( x_1 \leq \frac{4\alpha - 1}{1 - 2\alpha} \), see Figure 10a, the expected payoff is

\[
\pi(x_1) = 2 \cdot x_1 \cdot (1 - x_1) - [x_1 - f^{-1}(x_1)] \cdot [f(x_1) - x_1] + [f(x_1) - x_1] \cdot [g - x_1]
\]

\[
= 2 \cdot x_1 \cdot (1 - x_1) - \frac{(2 - 6\alpha)^2}{(1 - 2\alpha)2\alpha}x_1^2 + \frac{(2 - 6\alpha)^2}{2\alpha(4\alpha - 1)}x_1^2,
\]

and the first derivative of it is positive for all \( x_1 \leq \frac{4\alpha - 1}{1 - 2\alpha} \) if and only if

\[
2\alpha(1 - 2\alpha)^2 + (2 - 6\alpha)^3 - 4\alpha \cdot (1 - 2\alpha)(4\alpha - 1) > 0,
\]

i.e. iff \( \alpha < 0.3026 \). Hence, \( \alpha \leq 0.3026 \) implies \( x_1 \geq \frac{4\alpha - 1}{1 - 2\alpha} \).

If \( \frac{4\alpha - 1}{1 - 2\alpha} \leq x_1 \leq \frac{\alpha}{1 - 2\alpha} \), see Figure 10b, the expected payoff is

\[
\pi(x_1) = 1 - x_1^2 - [x_1 - f^{-1}(x_1)] \cdot [f(x_1) - x_1] - [f^{-1}(1) - x_1] \cdot [1 - f(x_1)]
\]

\[
= 1 - x_1^2 - \frac{(2 - 6\alpha)^2}{(1 - 2\alpha)2\alpha}x_1^2 - \frac{[2\alpha - 2\alpha(1 - 2\alpha)]^2}{(1 - 2\alpha)2\alpha}.
\]

It is easy to verify that its first derivative is non-positive for all \( x_1 \) satisfying \( \frac{4\alpha - 1}{1 - 2\alpha} \leq x_1 \leq \frac{\alpha}{1 - 2\alpha} \) if and only if (17) is not satisfied, i.e. iff \( \alpha \geq 0.3026 \). Hence, \( \alpha \geq 0.3026 \) implies \( x_1 \leq \frac{4\alpha - 1}{1 - 2\alpha} \).

Thus, if \( \alpha \geq 0.3026 \), then the payoff maximizing choice is the zero of the first
Figure 11: The cases distinguished in Prop. 2.2 for \( p < 0.75 \)

(a) \( x_1 < \alpha/(1-2\alpha) \)

(b) \( x_1 > \alpha/(1-2\alpha) \)

The derivative of Eq. (16), which is

\[
x_1^* = 2/ \left( 4 + \frac{(2 - 6\alpha)^2}{(1-2\alpha)\alpha} - \frac{(2 - 6\alpha)^2}{\alpha(4\alpha - 1)} \right) = 2/ \left( 4 - \frac{(2 - 6\alpha)^3}{\alpha(1-2\alpha)(4\alpha - 1)} \right) . \tag{19}
\]

and if \( \alpha < 0.3026 \), it is the zero of the first derivative of Eq. (18), which is

\[
x_1^* = \frac{4\alpha(1-2\alpha)}{(2-6\alpha)^2 + 2(2-3\alpha)(1-2\alpha)} = \frac{2\alpha(1-2\alpha)}{(4-7\alpha)(1-3\alpha) + 3\alpha^2} . \tag{20}
\]

**Best responses in case** \( p < 0.75 \Leftrightarrow \alpha < 0.25 \)

Fix \( x_1 \) and define \( f \) as above, i.e. \( f(x) = \frac{1-2\alpha}{2\alpha} (x_1 + x) \) for all \( x \in [0, 1] \). Now, \( \frac{1-2\alpha}{2\alpha} > 1 \), and the cases to be distinguished are the ones depicted in Figure 11. The proof that choices \( x_1 > \alpha/(1-2\alpha) \) are not optimal is very similar Lemma A.1 and therefore skipped. Thus, \( x_1 \leq \alpha/(1-2\alpha) \), and in this case, the expected payoff is

\[
\pi(x_1) = x_1 \left[ 1 - f(0) + 1 - f(x_1) \right] + (1 - x_1)^2 - \left[ 1 - f(x_1) \right] \left[ f^{-1}(1) - x_1 \right] \\
= x_1 \cdot \left[ 2 - \frac{1-2\alpha}{2\alpha} \cdot 3x_1 \right] + (1 - x_1)^2 - \left[ 1 - \frac{1-2\alpha}{2\alpha} \cdot 2x_1 \right] \left[ \frac{2\alpha}{1-2\alpha} - 2x_1 \right]
\]

By the first-order condition, the optimal choice is the zero of

\[
\left[ 2 \cdot \frac{1-2\alpha}{2\alpha} \cdot 6x_1 \right] - 2(1-x_1) + 2 \left[ 1 - \frac{1-2\alpha}{2\alpha} \cdot 2x_1 \right] + 2 \left[ \frac{2\alpha}{1-2\alpha} - 2x_1 \right]
\]
which yields $x_1 = 4\alpha/(7 - 16\alpha)$.

References


