Dominance Invariant Roommate Problems

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Abstract

Solution concepts in social environments use either a direct or indirect dominance relationship, depending on whether it is assumed that agents are myopic or farsighted. Direct dominance implies indirect dominance, but not the reverse. Hence, the predicted outcomes when assuming myopic (direct) or farsighted (indirect) agents could be very different. In this paper, we characterize dominance invariant roommate problems when preferences are strict. That is, we obtain the conditions on preference profiles such that indirect dominance implies direct dominance in roommate problems and give these an intuitive interpretation. Whenever some of the conditions are not satisfied, it is important to know the kind of agents that are being investigated in order to use the appropriate stability concept. Furthermore, we characterize dominance invariant roommate problems having a non-empty core. Finally, we show that, if the core of a dominance invariant roommate problem is not empty, it contains a unique matching, the dominance invariant stable matching, in which all agents who mutually top rank each other are matched to one another and all other agents remain unmatched.

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1 Introduction

Solution concepts in social environments use either a direct or indirect dominance relationship, depending on whether it is assumed that agents are myopic or farsighted. In solution concepts like the core and the von Neumann-Morgenstern stable set agents are not farsighted in the sense that individual and coalitional deviations cannot be countered by subsequent deviations. These concepts are based on the direct dominance relation and neglect the destabilizing effect of indirect dominance relations introduced by Harsanyi (1974) and formalized by Chwe (1994). Based on the concept of indirect dominance, several solution concepts assume farsighted behavior of the agents in abstract social environments, coalition formation, network formation or matching models.\(^1\) These solution concepts include, among others, the largest consistent set and the von Neumann-Morgenstern farsightedly stable set (Chwe, 1994), the farsighted core (Diamantoudi and Xue, 2003), the farsightedly stable set (Herings et al., 2010), the pairwise farsightedly stable set (Herings et al., 2009) and the path dominance core (Page and Wooders, 2009).

Direct dominance implies indirect dominance. However, indirect dominance does not imply direct dominance. For this reason, any solution concept may give different predictions when considering either myopic or farsighted agents. For instance, in coalition formation games with positive spillovers (e.g. cartel formation with Cournot competition and economies with pure public goods) Herings et al. (2010) and Mauleon and Vannetelbosch (2004) showed that the grand coalition is a farsightedly stable set, a von Neumann-Morgenstern farsightedly stable set and it always belongs to the largest consistent set. However, myopic stability concepts like the α-core, β-core or von Neumann-Morgenstern stable set, do not select the grand coalition as a stable outcome. Regarding the marriage problem, Ehlers (2007) characterized von Neumann-Morgenstern stable sets using the direct dominance relation, if such sets exist. He showed that these can be larger than the core. Mauleon et al. (2011), using Chwe’s (1994) definition of indirect dominance, showed the existence of and completely characterized the von Neumann-Morgenstern farsightedly stable

\(^1\)See for instance Greenberg (1990), Chwe (1994) and Xue (1998) about abstract social environments; Diamantoudi and Xue (2003), Mauleon and Vannetelbosch (2004), and Herings et al. (2010) about coalition formation; Dutta et al. (2005), Page et al. (2005), Herings et al. (2009), and Page and Wooders (2009) about network formation; and Mauleon et al. (2011) and Klaus et al. (2011) about matching models.
sets: a set of matchings is a von Neumann-Morgenstern farsightedly stable set if and only if it is a singleton and belongs to the core. They also showed that the farsighted core, defined as the set of matchings that are not indirectly dominated by other matchings, can be empty.\(^2\) On the contrary, in the formation of free trade networks, the global free trade network is a pairwise (myopically) stable network and also a pairwise farsightedly set in the model of Goyal and Joshi (2006) (see, Zhang et al., 2011).

Recently, some experimental evidence has been provided about the existence of both myopic and farsighted agents.\(^3\) However, it is not obvious to know ex-ante the type of agents that one is facing. Then, an interesting question to investigate is whether there are situations in which one should not care about the kind of agents that are being considered or, in other words, if there are situations where the predicted outcomes do not depend on the kind of agents (myopic or farsighted) that are involved in the studied situation. In the present paper, we characterize dominance invariant roommate problems, i.e., roommate problems for which indirect dominance implies direct dominance when agents have strict preferences. As a consequence, in this kind of roommate problems, any solution concept based on direct or indirect dominance will give the same predictions. Roommate problems (Gale and Shapley, 1962) represent situations in which a finite set of agents has to be partitioned into pairs (roommates) and singletons. Roommate problems are a generalization of the well-known marriage problem and a particular model of hedonic coalition formation (in which coalitions are restricted to have at most two agents) and of network formation (in which each agent is restricted to have at most one link).\(^4\) Hence, roommate problems are a particularly interesting class of matching markets that lie in the intersection of network and coalition formation models. For this reason, by characterizing dominance invariant roommate problems, we provide the basis that could be used for characterizing dominance invariant coalition formation problems and dominance invariant network formation problems. To the best of

\(^2\)The farsighted core only exists when the core contains a unique matching and no other matching indirectly dominates the matching in the core.

\(^3\)Mantovani et al. (2011) have tested whether subjects behave myopically or farsightedly when forming a network. They have shown that behaviors consistent with farsightedness account for 75 percent of the individual observations, while only 6 percent of the individual observations are consistent with myopic behavior.

\(^4\)See Bogomolnaïa and Jackson (2002) and Jackson and Watts (2002).
our knowledge, no characterization of dominance invariance has been provided up to now.

Our main result (Proposition 2) is that a roommate problem is dominance invariant if and only if two conditions are satisfied. When two agents prefer to be matched to one another than being on their own, we say that these two agents are mutually acceptable. The first condition then states that mutually acceptable agents must prefer each other to any other agent (we interpret this condition as ‘reciprocity’). The second condition concerns the position of agents in each individual ranking of the set of mutually acceptable agents. Consider, for instance, agent i’s ranking of her mutually acceptable agents. If agent k (in i’s ranking but not in the last position) prefers another agent l more than i, then agent i places k and l in the last two positions of this rank, with k more preferred than l. Informally, agent i penalizes these two agents: she penalizes agent k for not thinking highly of her and penalizes agent l out of jealousy. (We interpret this condition as ‘extreme jealousy’.) Notice that, with this characterization of dominance invariant roommate problems, we provide two easy to verify conditions that tell us when one should care or not about the kind of agents (myopic or farsighted) that are being investigated. Whenever one of the two conditions for a roommate problem is not satisfied, it is important to understand whether the agents under consideration are myopic or farsighted in order to use the appropriate stability concept.

We subsequently give some properties of agents’ preferences in a roommate problem which is dominance invariant (Proposition 3) and we show (Proposition 4) that a roommate problem with three agents who prefer being matched to being unmatched is always dominance invariant. Such a roommate problem may have an empty core from which we conclude that the notion of dominance invariant has little in common with well-known restrictions on preferences guaranteeing existence and/or uniqueness of the core in the roommate problem such as $\alpha$-reducibility (Alcalde, 1995) or more generally, the weak top coalition property (Banerjee et al. 2001).

Next, we focus on and characterize dominance invariant roommate problems with a nonempty core, or solvable dominance invariant roommate problems. We show (Proposition 5) that a dominance invariant roommate problem is solvable when there does not exist a structure in the preference profile called ring, formed by three agents such that the members of this ring prefer the other agents in the ring to any other agent outside the ring. This allows us to state (Proposition 6) that, if it exists,
the core of a dominance invariant roommate problem contains a unique matching, the dominance invariant stable matching, in which all agents who mutually top rank each other are matched to each other and all other agents are single.

The rest of the paper is organized as follows. Section 2 introduces roommate problems. Section 3 defines dominance invariant roommate problems and contains our main results. Section 4 concludes.

2 Roommate problems

A roommate problem, is a pair \((N, P)\) where \(N\) is a finite set of agents and \(P\) is a preference profile specifying for each agent \(i \in N\) a strict preference ordering over \(N\). That is, \(P = \{P(1),...,P(i),...,P(n)\}\), where \(P(i)\) is agent \(i\)'s strict preference ordering over the agents in \(N\) including herself, which can be interpreted as the prospect of being alone. For instance, \(P(i) = 1,3,i,2,...\) indicates that agent \(i\) prefers agent 1 to agent 3 and she prefers to remain alone rather than get matched to anyone else. We denote by \(R\) the weak orders associated with \(P\). We write \(j \succ_i k\) if agent \(i\) strictly prefers \(j\) to \(k\), \(j \sim_i k\) if \(i\) is indifferent between \(j\) and \(k\), and \(j \succeq_i k\) if \(j \succ_i k\) or \(j \sim_i k\).

A matching \(\mu\) is a function \(\mu : N \rightarrow N\) such that for all \(i \in N\), if \(\mu(i) = j\), then \(\mu(j) = i\). Agent \(\mu(i)\) is agent \(i\)'s mate at \(\mu\); i.e., the agent with whom she is matched to share a room (possibly herself). We denote by \(\mathcal{M}\) the set of all matchings. A matching \(\mu\) is individually rational if each agent is acceptable to his or her partner, i.e. \(\mu(i) \succeq_i i\) for all \(i \in N\). We denote the set of individually rational matchings for a roommate problem \((N, P)\) by \(I(N, P)\). For a given matching \(\mu\), a pair \(\{i,j\}\) (possibly \(i = j\)) is said to form a blocking pair if they are not matched to one another but prefer one another to their partner at \(\mu\), i.e. \(j \succ_i \mu(i)\) and \(i \succ_j \mu(j)\). A matching \(\mu\) is stable if it is not blocked by any individual or any pair of agents. A roommate problem \((N, P)\) is solvable if it has a stable matching. Otherwise, it is called unsolvable.

We extend each agent’s preference over her potential partners to the set of matchings in the following way. We say that agent \(i\) prefers \(\mu'\) to \(\mu\), if and only if agent \(i\) prefers her partner at \(\mu'\) to her partner at \(\mu\), \(\mu'(i) \succ_i \mu(i)\). Abusing notation, we write this as \(\mu' \succ_i \mu\). A coalition \(S\) is a subset of \(N\).\(^5\) For \(S \subseteq N\),

\(^5\)Throughout the paper we use the notation \(\subseteq\) for weak inclusion and \(\subset\) for strict inclusion.
\( \mu(S) = \{ \mu(i) : i \in S \} \) denotes the set of mates of agents in \( S \) at \( \mu \). A matching \( \mu \) is \textit{blocked by a coalition} \( S \subseteq N \) if there exists a matching \( \mu' \) such that \( \mu'(S) = S \) and for all \( i \in S \), \( \mu' \succ_i \mu \). If \( S \) blocks \( \mu \), then \( S \) is called a \textit{blocking coalition} for \( \mu \). Note that if a coalition \( S \subseteq N \) blocks a matching \( \mu \), then there exists a pair \( \{i, j\} \) (possibly \( i = j \)) that blocks \( \mu \). The \textit{core} of a roommate problem, denoted by \( C(N, P) \), consists of all matchings which are not blocked by any coalition. Note that for any roommate problem the set of stable matchings equals the core.

\textbf{Definition 1.} Given a matching \( \mu \), a coalition \( S \subseteq N \) is said to be able to enforce a matching \( \mu' \) over \( \mu \) if the following conditions hold for any agent \( i \in N \): (i) \( \mu'(i) \notin \{ \mu(i), i \} \) implies \( \{i, \mu'(i)\} \subseteq S \) and (ii) \( \mu'(i) = i \neq \mu(i) \) implies \( \{i, \mu(i)\} \cap S \neq \emptyset \).

In other words, this enforceability condition\(^6\) implies both that any new pair in \( \mu' \) that does not exist in \( \mu \) must be formed by agents in \( S \), and that in order to break an existing pair in \( \mu \), one of the two agents involved in that pair must belong to coalition \( S \). Notice that the concept of enforceability is independent of preferences. Furthermore, the fact that coalition \( S \subseteq N \) can enforce a matching \( \mu' \) over \( \mu \) implies that there exists a sequence of matchings \( \mu^0, \mu^1, \ldots, \mu^K \) (where \( \mu^0 = \mu \) and \( \mu^K = \mu' \)) and a sequence of disjoint pairs \( \{i_0, j_0\}, \ldots, \{i_{K-1}, j_{K-1}\} \) (possibly for some \( k \in \{0, 1, \ldots, K-1\} \), \( i_k = j_k \)) such that for any \( k \in \{1, \ldots, K\} \), the pair \( \{i_{k-1}, j_{k-1}\} \in S \) can enforce the matching \( \mu^k \) over \( \mu^{k-1} \).

\textbf{Definition 2.} A matching \( \mu \) is directly dominated by \( \mu' \), denoted by \( \mu < \mu' \), if there exists a coalition \( S \subseteq N \) of agents such that \( \mu' \succ_i \mu \) \( \forall i \in S \) and \( S \) can enforce \( \mu' \) over \( \mu \).

An alternative way of defining the core of a roommate problem is by means of the domination relation. A matching \( \mu \) is in the core if there is no subset of agents who, by rearranging their partnerships only among themselves, possibly dissolving some partnerships of \( \mu \), can all obtain a strictly preferred set of partners. Formally, a matching \( \mu \) is in the core if \( \mu \) is not directly dominated by any other matching \( \mu' \in \mathcal{M} \). Given a profile \( P \), we denote the set of matchings in the core by \( C(N, P) \). Gale and Shapley (1962) showed that the core of a roommate problem may be empty.\(^7\)

\(^6\)This enforceability condition has also been used in Mauleon et al. (2011) and in Klaus et al. (2011).

\(^7\)Several papers are devoted to analyzing the core as solution for this matching problem. See for instance Tan (1991), Chung (2000), Diamantoudi et al. (2004) and Iñarra et al. (2010).
We now introduce the *indirect dominance* relation. A matching \( \mu' \) *indirectly dominates* \( \mu \) if \( \mu' \) can replace \( \mu \) in a sequence of matchings, such that at each matching along the sequence all deviators are strictly better off at the end matching \( \mu' \) compared to the status-quo they face. Formally, indirect dominance is defined as follows.

**Definition 3.** A matching \( \mu \) is indirectly dominated by \( \mu' \), denoted by \( \mu \ll \mu' \), if there exists a sequence of matchings \( \mu^0, \mu^1, \ldots, \mu^K \) (where \( \mu^0 = \mu \) and \( \mu^K = \mu' \)) and a sequence of coalitions \( S^0, S^1, \ldots, S^{K-1} \) such that for any \( k \in \{1, \ldots, K\} \),

(i) \( \mu^K >_i \mu^{k-1} \forall i \in S^{k-1} \), and

(ii) coalition \( S^{k-1} \) can enforce the matching \( \mu^k \) over \( \mu^{k-1} \).

Direct dominance can be obtained directly from Definition 3 by setting \( K = 1 \). Obviously, if \( \mu < \mu' \) then \( \mu \ll \mu' \); i.e., direct dominance implies indirect dominance. Recently, Mauleon et al. (2011) have shown that, in marriage problems (a particular case of the roommate problem where agents are partitioned in two sets), an individually rational matching \( \mu \) indirectly dominates \( \mu' \) if and only if there does not exist a pair \( \{i, \mu'(i)\} \) that blocks \( \mu \). Klaus et al. (2011) have generalized this result for roommate problems, and they have shown that an individually rational matching \( \mu \) indirectly dominates another individually rational matching \( \mu' \) if and only if there does not exist a pair \( \{i, \mu'(i)\} \) that blocks \( \mu \). For proofs we refer to the papers mentioned above.

**Proposition 1** (Klaus et al., 2011). Let \( (N, P) \) be a roommate problem and \( \mu, \mu' \in I(N, P) \). Then, \( \mu \gg \mu' \) if and only if there does not exist a pair \( \{i, \mu'(i)\} \) that blocks \( \mu \).

Diamantoudi and Xue (2003), showed that if a matching belongs to the core, then it indirectly dominates any other matching.

**Lemma 1** (Diamantoudi and Xue, 2003). If \( \mu \in C(N, P) \), then \( \forall \mu' \neq \mu \), it holds that \( \mu' \ll \mu \).

### 3 Dominance Invariant Roommate Problems

We define a roommate problem to be dominance invariant if and only if indirect dominance implies direct dominance.
Definition 4. A roommate problem \((N, P)\) is dominance invariant if the following condition holds:

\[ \mu' \gg \mu \iff \mu' > \mu, \forall \mu, \mu' \in M. \]

Let \((N, P)\) be a roommate problem. Let \(i \in N\). We denote by \(t(i)\) the most preferred partner for agent \(i\). That is, \(t(i) \succ_i j\) for any \(j \in N\). Let \(T\) denote the set of agents who are ranked first by her most preferred agent; i.e.,

\[ T = \{ i \in N : \exists j \in N \text{ such that } j = t(i) \text{ and } i = t(j) \}. \]

Notice that if \(i \in T\), then \(t(t(i)) = i\).

Given the roommate problem \((N, P)\), the set \(A_i\) denotes the set of agents acceptable for agent \(i\), that is \(A_i = \{ j \in N : j \succ_i i \}\) and the set \(M_i\) denotes the set of mutually acceptable agents for \(i\), that is \(M_i = \{ j \in A_i : i \succ_j j \}\). Let \(\omega(i) \in M_i\) denote the least preferred partner for \(i\) in this set; i.e., \(\forall k \in M_i : k \succ_i \omega(i)\). Let \(M^k_i\) denote the set of mutually acceptable agents of \(i\) who are less preferred than \(k\), that is \(M^k_i = \{ j \in M_i : k \succ_i j \}\). Let \(R, Z \subseteq N\) and let \(i \in N\). If agent \(i\) strictly prefers every agent in \(R\) to any agent in \(Z\), then agent \(i\) strictly prefers the set \(R\) to the set \(Z\). This is denoted by \(R \succ_i Z\).

The notion of a ring is a key notion for the existence of stable matchings in roommate problems. A ring \(S = \{ s_1, ..., s_k \} \subseteq N\) is an ordered set of agents such that \(k \geq 3\) and for all \(i \in \{ 1, ..., k \}\), \(s_{i+1} \succ_s s_{i-1} \succ_s s_i\) (subscript modulo \(k\)).

The existence of odd rings in the preference profile is a necessary condition for the emptiness of the core in a roommate problem.

Lemma 2. Let \((N, P)\) be a roommate problem such that \(C(N, P) = \emptyset\). Then, there exists a ring \(S = \{ s_1, ..., s_k \}\), where \(k\) is odd.

This lemma is straightforward from the necessary and sufficient condition provided by Tan (1991) for the emptiness of the core in a roommate problem. We refer the reader to Appendix A for a compilation of definitions and results about the solvability of roommate problems.

Our main result characterizes the dominance invariant roommate problems.

Proposition 2. A roommate problem \((N, P)\) is dominance invariant if and only if the preference relation \(P\) satisfies the following two conditions:

(i) \(\forall i \in N, M_i \succ_i A_i \setminus M_i\),
(ii) \( \forall i \in N, \text{ if } \exists k \in M_i \setminus \{\omega(i)\} \text{ and } \exists l \in M_k \text{ such that } l \succ_k i \text{ then } M^k_i = \{l\}. \)

The proof of this proposition, as well as all other proofs, may be found in Appendix B. The first condition can be seen as ‘reciprocity’, in the sense that agent \( i \) prefers the agents that are mutually acceptable for her to the agents that do not accept her although she accepts them. The second condition says that if two agents \( i, k \) are mutually acceptable, with \( k \neq \omega(i) \), but \( k \) prefers another mutually acceptable agent \( l \) more than \( i \), then, there cannot be any agent mutually acceptable for \( i \) less preferred than \( k \), different from \( l \). In other words, \( l \) is the least preferred potential partner for \( i \) among her mutually acceptable agents. Moreover, there are no agents in agent \( i \)’s preferences less preferred than \( k \) but more preferred than \( l \).

This condition may be interpreted as ‘extreme jealousy’. If agent \( i \) likes agent \( k \) and vice versa, but agent \( k \) likes another mutually acceptable agent \( l \) better (than agent \( i \)), then either \( k \) is the worst mutually acceptable agent for \( i \) (agent \( i \) penalizes \( k \) for not considering her the best), or \( k \) and \( l \) are the worst ranked mutually acceptable agents for \( i \), with \( l \) less preferred than \( k \) (agent \( i \) is jealous of \( k \) and \( l \)).

The next proposition describes some properties of agents’ preferences in a roommate problem which is dominance invariant. They depend on the cardinality of the sets of mutually acceptable agents in the problem.\(^9\)

**Proposition 3.** Let \((N, P)\) be a dominance invariant roommate problem,

\[ \text{P1. Let } i \text{ be an agent such that } |M_i| > 2 \text{ and assume, without loss of generality, that } M_i = \{j_1, \ldots, j_k, \omega(i)\} \text{ such that } j_m \succ_i j_{m+1}, \forall m \in \{1, \ldots, k - 1\} \text{ and } j_k \succ_i \omega(i). \]

\[ \text{a. } \forall j \in M_i \setminus \{j_k, \omega(i)\}, t(j) = i, \]

\[ \text{b. } t(j_k) \in \{i, \omega(i)\} \]

\[ \text{b.1 If } t(j_k) = i \text{ then either } \omega(i) \in T \text{ or } t(\omega(i)) \in \{i, t(i)\}, \text{ and} \]

\[ \text{b.2 If } t(j_k) = \omega(i) \text{ then } t(\omega(i)) = j_k \]

\[ \text{P2. Let } i \text{ be an agent such that } |M_i| \leq 2. \text{ Then either } t(i) \in T \text{ or } i \in S \text{ where } S \text{ is a ring in } P \text{ such that } |S| = 3 \text{ and } \forall s_i \in S, s_{i+1} \succ_s s_i \succ_{s_{i-1}} j \text{ for any } j \in N \setminus \{s_{i+1}, s_{i-1}\}. \]

\(^8\)Notice that \( l \) equals \( \omega(i) \).

\(^9\)From now on, \( |M| \) denotes the cardinality of the set \( M \).
**P3.** For all $i \notin T$, there is no agent $j \notin T$ such that $i \in M_j$, except from those belonging to a ring $S$ in $P$ such that $|S| = 3$ and $\forall s_i \in S$, $s_{i+1} > s_i$, $s_{i-1} > s_i$, $j$ for any $j \in N \setminus \{s_{i+1}, s_{i-1}\}$.

Let us interpret all these properties. Consider any agent $i$ with more than two mutually acceptable agents. Then, by **P1.a**, $i \in T$. The least preferred agent for $i$, agent $\omega(i)$, by **P1.b** either belongs to $T$ (ranking as top choice an agent $k \notin M_i$ (**b.1**) or $j_k$ (**b.2**)) or ranks as top choice either $i$ or $t(i)$ (**b.1**). The remaining agents in $M_i$ rank $i$ as their most preferred agent, except $j_k$ when $t(\omega(i)) = j_k$. Consider now any agent $i$ with at most two mutually acceptable agents. By **P2** either $i \in T$ or her top ranked agent is $j \in T$ with $t(j) \neq i$ or it belongs to a ring formed by 3 agents such that each player in the ring considers acceptable only the other agents in the ring. Properties **P1** and **P2** allow us to determine which agents belong to set $T$. The remaining agents, those who do not belong to set $T$, by **P3** cannot be mutually acceptable among them, except those belonging to a ring formed by 3 agents in which all agent in the ring prefer being matched among themselves to being matched with any other agent.

**Example 1.** The following example of a dominance invariant roommate problem may be useful for clarifying the previous results. Agents that do not appear in the other agent’s preferences are unacceptable.

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In this problem, the set of mutually acceptable agents are $M_1 = \{2, 3, 4, 5\}$, $M_2 = M_3 = \{1\}$, $M_4 = \{5, 1\}$ and $M_5 = \{4, 1\}$, $M_6 = \{7, 8\}$, $M_7 = \{6, 8\}$ and $M_8 = \{6, 7\}$. Notice that the first condition in Proposition 2 is satisfied since these agents are in the first rows of each agent’s preferences. Consider for instance agent 1’s preferences, $P(1)$. Notice that agents 1 and 4 are mutually acceptable and 4 is not the worse agent in $M_1$, however, 5 $\succ_4 1$. Then, by condition (ii) of Proposition 2, agent 5 must be the immediate less preferred agent than 4 for agent 1. Notice that $\{6, 7, 8\}$ form an odd ring in the preferences.
In this example, the only agent satisfying \( |M_i| \geq 2 \) is agent 1 with \( M_1 = \{2, 3, 4, 5\} \) and \( 2 \succ_1 3 \succ_1 4 \succ_1 5 \). We can see that \( \forall j \in \{2, 3\}, t(j) = 1 \) (P1.a in Proposition 3). Moreover, it must happen that \( t(4) \in \{1, 5\} \) (P1.b in Proposition 3). In this case, \( t(4) = 5 \) and therefore \( t(5) = 4 \) (P1.b.2 in Proposition 3).

On the other hand, all the other agents satisfy \( |M_i| \leq 2 \). For \( i \in \{2, 3, 4, 5\} \), we can check that \( t(i) \in T \). Agents in the set \( \{6, 7, 8\} \) form a ring satisfying that \( \forall s_i \in \{6, 7, 8\}, s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} k \) for all \( k \in N \setminus \{s_{i+1}, s_{i-1}\} \).

Notice also that for \( i \in \{1, 2, 3, 4, 5\} \), there is no pair of agents who do not belong to \( T \) such that they are mutually acceptable. In our example, the only agent who is not in \( T \) is agent 3, and there is no agent \( j \) in \( P(3) \) (\( j \notin T \)) such that \( j \succ_3 3 \) and \( j \in M_3 \).

The following result shows that all roommate problems such that \( |N| = 3 \) in which all agents prefer to be matched to being unmatched are dominance invariant.

**Proposition 4.** Let \((N, P)\) be a roommate problem such that \( |N| = 3 \) and \( \forall i \in N: j \succ_i i \) for any \( j \neq i \). Then \((N, P)\) is dominance invariant.

Note that this class of roommate problems can have an empty core when the three agents form an odd ring in \( P \). This then implies that the notion of dominance invariance has little in common with restrictions on preferences which guarantee the existence and/or uniqueness of stable matchings (e.g. \( \alpha \)-reducibility (Alcalde, 1995) or more generally, the weak top coalition property (Banerjee et al., 2001)).

The following proposition characterizes the dominance invariant roommate problems with a non-empty core.

**Proposition 5.** Let \((N, P)\) be a dominance invariant roommate problem. \( C(N, P) \neq \emptyset \) if and only if there is no ring \( S \) in \( P \) such that \( |S| = 3 \) and \( \forall s_i \in S, s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} j \) for any \( j \in N \setminus \{s_{i+1}, s_{i-1}\} \).

The following result, derived from the previous one, states that if a roommate problem is dominance invariant and the core is non-empty, it has a unique stable matching. In this stable matching, all agents who mutually top rank each other are matched to one another and all other agents remain unmatched.

**Proposition 6.** Let \((N, P)\) be a solvable dominance invariant roommate problem. Then, \( C(N, P) = \{\mu_C\} \), where \( \mu_C \) is such that \( \mu_C(i) = t(i) \) for all \( i \in T \), and \( \mu_C(j) = j \) for all \( j \notin T \).
Example 1 (cont.) In this example, we have already seen that there is a ring $S = \{6, 7, 8\}$ in $P$ such that $|S| = 3$ and $\forall s_i \in S$, $s_{i+1} \succ s_i \succ s_{i-1} \succ s_i$ for any $j \in N \setminus \{s_{i+1}, s_{i-1}\}$. Therefore this roommate problem is unsolvable and there is no stable matching.

Consider the roommate problem derived from the previous one such that $N = \{1, 2, 3, 4, 5\}$ and $P = \{P(1), P(2), P(3), P(4), P(5)\}$. In this case, there is no ring in preferences satisfying the conditions above and therefore the problem is solvable. The core, in this case, is formed by the matching $\mu_C = \{\{1, 2\}, \{3\}, \{4, 5\}\}$. □

Finally notice that, when the dominance invariant roommate problem is solvable, the core and the farsighted core (as well as the von Neumann-Morgenstern stable set and the von Neumann-Morgenstern farsightedly stable set) coincide.

4 Conclusion

We have characterized dominance invariant roommate problems when preferences are strict. That is, we have obtained under which conditions on preference profiles indirect dominance implies direct dominance in roommate problems. Hence, we have concluded that, whenever some of the conditions is not satisfied, one should try to know the kind of agents under consideration in order to use the appropriate stability concept. Furthermore, we have characterized solvable dominance invariant roommate problems. This characterization has allowed us to state that the core of such a problem contains a unique matching in which all agents who mutually top rank each other are matched to one another and all other agents remain unmatched.

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Appendix A

Tan (1991) establishes a necessary and sufficient condition for the solvability of roommate problems with strict preferences in terms of stable partitions. This notion, which is crucial in the investigation of the core for these problems, can be formally defined as follows.

Let $A = \{a_1, \ldots, a_k\} \subseteq N$ be an ordered set of agents. The set $A$ is a ring if $k \geq 3$ and for all $i \in \{1, \ldots, k\}$, $a_{i+1} \succ_a a_{i-1} \succ_a a_i$ (subscript modulo $k$). The set $A$ is a pair of mutually acceptable agents if $k = 2$ and for all $i \in \{1, 2\}$, $a_{i-1} \succ_a a_i$ (subscript modulo 2). The set $A$ is a singleton if $k = 1$.

**Definition 5.** A stable partition is a partition $P$ of $N$ such that:

(i) for all $A \in P$, the set $A$ is a ring, a mutually acceptable pair of agents or a singleton, and

(ii) for any sets $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_l\}$ of $P$ (possibly $A = B$), the following condition holds:

$$\text{if } b_j \succ_a a_{i-1} \text{ then } b_{j-1} \succ b_j a_i,$$

for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, l\}$ such that $b_j \neq a_{i+1}$.

Condition (i) specifies the sets contained in a stable partition, and condition (ii) contains the notion of stability to be applied between these sets (and also inside each set).

Note that a stable partition is a generalization of a stable matching. To see this, consider a matching $\mu$ and a partition $P$ formed by pairs of agents and/or singletons. Let $A = \{a_1, a_2 = \mu(a_1)\}$ and $B = \{b_1, b_2 = \mu(b_1)\}$ be sets of $P$. If $P$ is a stable partition then Condition (ii) implies that if $b_1 \succ_{a_2} a_1$ then $b_2 \succ_{b_1} a_2$, which is the usual notion of stability. Hence $\mu$ is a stable matching.

\[\text{10}^\text{Hereafter we omit subscript modulo } k.\]
Remark 1 (Iñarra et al., 2010). (i) A roommate problem \((N, P)\) has no stable matchings if and only if there exists a stable partition with an odd ring.\(^{11}\) (ii) Any two stable partitions have exactly the same odd rings. (iii) Every even ring in a stable partition can be broken into pairs of mutually acceptable agents preserving stability.

Appendix B

Proof. [Proof of Proposition 2]

(⇒) By contradiction, we will show that if one of the conditions (i) or (ii) is not satisfied, then \(\mu \gg \mu' \not\gg \mu > \mu'\).

- Suppose that condition (i) is not satisfied. Then there exists an agent \(i \in N\) such that \(k \succ_i j\) for some \(k \in A_i \setminus M_i\) and some \(j \in M_i\). Let \(\mu_2\) be a matching such that \(\mu_2(i) = k\) and \(\mu_2(s) = s\) for every \(s \neq i, k\), and let \(\mu_1\) be a matching such that \(\mu_1(i) = j\) and \(\mu_1(s) = s\) for every \(s \neq i, j\). Then \(\mu_1 \gg \mu_2\) (since \(k \succ_k i\), agent \(k\) enforces the matching in which every agent is alone, and this matching is blocked by \(\{i, j\}\) enforcing \(\mu_1\)). However, \(\mu_1 \not\gg \mu_2\) since \(\mu_2 \succ_i \mu_1\).

- Suppose that condition (ii) is not satisfied. Then there exists an agent \(k \in M_i \setminus \{\omega(i)\}\) and an agent \(l \in M_k\) such that \(l \succ_k i\) and \(\{l\} \neq M_k^k\). Then it must exist an agent \(j \neq l\) such that \(j \in M_k^k\). Let \(\mu_2\) be a matching such that \(\mu_2(i) = k\) and \(\mu_2(s) = s\) for every \(s \neq i, k\), and let \(\mu_1\) be a matching such that \(\mu_1(k) = l, \mu_1(i) = j\) and where \(\mu_1(s) = s\) for every \(s \neq i, k, l, j\). Then \(\mu_1 \gg \mu_2\) (since \(\{k, l\}\) block \(\mu_2\) enforcing a matching in which \(i\) and \(j\) are alone, and this matching is blocked by \(\{i, j\}\) enforcing \(\mu_1\)). However, \(\mu_1 \not\gg \mu_2\) since \(\mu_2 \succ_i \mu_1\).

(⇐) Now we will prove that if \(\mu_1 \gg \mu_2\) and conditions (i) and (ii) are satisfied, then \(\mu_1 \succ \mu_2\).

Let \(D = \{i \in N : \mu_1 \succ_i \mu_2\}\). First, we prove that for any agent \(i \in N\) such that \(\mu_1(i) \neq \mu_2(i)\) and \(\mu_1(i) \neq i\), we have that \(i \in D\). By contradiction, let \(\mu_1(i) = j\) and let \(\mu_2(i) = k\) and assume that \(k \succ_i j\) (which implies that \(k \neq j\)). Notice that \(j \in M_i\) because otherwise \(\{i, j\}\) would never be formed contradicting \(\mu_1 \gg \mu_2\). Then, by condition (i), \(k \in M_i\). Since \(\mu_1 \gg \mu_2\) and \(i\) prefers \(\mu_2\) to \(\mu_1\), \(k\) must prefer \(\mu_1\) to \(\mu_2\) because, otherwise, \(\{i, k\}\) would be a blocking pair of \(\mu_1\) contradicting that \(\mu_1 \gg \mu_2\).

\(^{11}\)A ring is odd (even) if its cardinality is odd (even).
[see Proposition 1].\textsuperscript{12} Then, the partner of \(k\) at \(\mu_1\), say for instance \(\mu_1(k) = l\) (with \(l \neq k, j\)), by condition (i), also belongs to the set of mutually acceptable agents of agent \(k\) given that \(l \succ_k i\), that is \(l \in M_k\). But then, according to (ii), it must be that \(M^k_i = \{l\}\). But this is a contradiction, since \(j \in M^k_i\). Hence, agent \(i\) should prefer \(\mu_1(i)\) to \(\mu_2(i)\), and, by the same reasoning, \(\mu_1 \succ_j \mu_2\). Therefore \(\{i, \mu_1(i)\} \subseteq D\).

Consider now any agent \(i \in N\) such that \(\mu_1(i) = i \neq \mu_2(i) = k\). Since \(\mu_1 \gg \mu_2\), then either \(\mu_1(k) \succ_k i\) and \(k\) deviates leaving agent \(i\) unmatched (with \(\mu_1(k)\) also preferring \(\mu_1\) to \(\mu_2\)) and then \(\{k, \mu_1(k)\} \subseteq D\), or \(i \succ_i k\) and agent \(i\) individually deviates and therefore \(i \in D\).

Then the coalition \(D\) deviates from \(\mu_2\) enforcing \(\mu_1\) and \(\mu_1 > \mu_2\) as we wanted to prove.

\(\square\)

\textit{Proof.} [Proof of Proposition 3]

We will show the different parts of this proposition by contradiction.

**P1.** Assume that (a) is not satisfied. That is, there exists an agent \(j \in M_i \setminus \{j_k, \omega(i)\}\) such that \(t(j) \neq i\). This implies that \(\exists k \in N\) such that \(t(j) = k\) and \(k \succ_j i\). By condition (i) of Proposition 2, \(k \in M_j\) and then by condition (ii) of Proposition 2, \(M^j_i = \{k\}\). However, this contradicts that \(\{j_k, \omega(i)\} \subseteq M^j_i\).

Now we will show that (b) must be satisfied as well. The fact that \(t(j_k) \in \{i, \omega(i)\}\) is straightforward from condition (ii) of Proposition 2.

In order to prove (b.1), let \(t(j_k) = i\). First, we will show that if \(\omega(i) \notin T\) then either \(t(\omega(i)) = i\) or \(t(\omega(i)) = t(i)\). Let \(\omega(i) \notin T\). Then, there exists an agent \(k\) such that \(t(\omega(i)) = k\). If \(k = i\) we are done, so assume that \(k \neq i\). Since \(k \succ_i \omega(i)\) and \(i \in M(\omega(i))\), by condition (i) of Proposition 2, \(k \in M(\omega(i))\). Moreover, \(t(k) = l \neq \omega(i)\) so \(l \succ_k \omega(i)\) and by condition (i) of Proposition 2, \(l \in M_k\). Thus, \(\exists k \in M(\omega(i)) \setminus \{\omega(\omega(i))\}\) and \(\exists l \in M_k\) such that \(l \succ_k \omega(i)\). Then, by condition (ii) of Proposition 2, it holds that \(\{l\} = M^k_{\omega(i)}\). Since \(i \in M^k_{\omega(i)}\), we have that \(l = i\). Given that \(l \in M_k\) and \(l = i\), it holds that \(k \in M_i\). Let \(k \neq t(i)\), otherwise we are done. Then since \(i \in M_k \setminus \{\omega(k)\}\) and there exists an agent \(k' \in M_i\) such that \(k' \succ_i k\) (remember that \(k \neq t(i)\)), by condition (ii) of Proposition 2, \(\{k'\} = M^k_i\). But this implies that \(k' = \omega(i)\) and this is a

\textsuperscript{12}Although Proposition 1 is stated requiring \(\mu\) and \(\mu'\) to be individually rational, the \(\Rightarrow\)-part of the proposition, which is the one used in this proof, holds for any two different matchings (individually rational or not). See proof in Klaus et al. (2011) (pp. 926-927).
contradiction since $\omega(i) \not\in i, k$. So we have proved that when $\omega(i) \notin T$ either 
$t(\omega(i)) = i$ or $t(\omega(i)) = t(i)$.

Now we will show that if $t(\omega(i)) \notin \{i, t(i)\}$, then $\omega(i) \in T$. Let $t(\omega(i)) = k$
with $k \neq i, t(i)$. By condition (ii) of Proposition 2, either $t(k) = \omega(i)$ and we are done, or there exists an agent $l \in M_k$ such that $l \succ_k \omega(i)$ and $\{l\} = M^k_{\omega(i)}$,
which implies that $l = i$. Following the previous reasoning, we achieve the same contradiction ($\omega(i) \not\in i, k$) and this proves that $\omega(i) \in T$ as desired.

Next, we proceed to prove (b.2). Let $t(j_k) = \omega(i)$. We will prove that in this case $t(\omega(i)) = j_k$. Since $i \in M_{j_k}$, we have that $\omega(i) \in M_{j_k} \setminus \{\omega(j_k)\}$.
By condition (ii) of Proposition 2, either $t(\omega(i)) = j_k$ and we are done, or there exists an agent $k \in M_{\omega(i)}$ such that $k \succ_{\omega(i)} j_k$ and $\{k\} = M^k_{\omega(i)}$. Then,
$k = i$, with $i \in M_{\omega(i)} \setminus \{\omega(\omega(i))\}$. Hence, by condition (ii) of Proposition 2, we have that for any $j \in M_i \setminus \{\omega(i)\}$, $j \succ_i \omega(i)$, then $\{j\} = M^i_{\omega(i)}$. But $|M_i \setminus \{\omega(i)\}| > 1$, and then $j \in M^i_{\omega(i)}$ for all $j \in M_i \setminus \{\omega(i)\}$, contradicting the uniqueness of $M^i_{\omega(i)}$.

P2. Let $i$ be an agent such that $|M_i| \leq 2$. We will prove that either $t(i) \in T$ or agent $i$ belongs to a ring $S$ such that $|S| = 3$ and $\forall s_i \in S, s_{i+1} \succ_s s_i \succ_s r$ for any $r \in N \setminus \{s_{i+1}, s_{i-1}\}$.

Consider first that $M_i = \{j\}$. If $t(j) = i$ we are done, so assume that $t(j) = k$
with $k \neq i$. By the reasoning in [P1.], if $|M_j| > 2$, then $t(k) = j$ and we are done. So let $|M_j| \leq 2$. Since $k \in M_j \setminus \{\omega(j)\}$, by condition (ii) of Proposition 2,
either $t(k) = j$ or there exists an agent $l \in M_k$ such that $l \succ_k j$ and $\{l\} = M^k_j$. However, this implies $l = i$ (since $i \in M^k_j$), which contradicts condition (i)
of Proposition 2 since $k \notin M_i$. Hence, if $M_i = \{j\}$, then either $t(j) = i$ or 
t(j) = k with $t(k) = j$.

Consider now the case that $M_i$ has two elements. Without loss of generality,
let $M_i = \{j, k\}$ with $j \succ k$. Since $j \in M_i \setminus \{\omega(i)\}$ and by condition (ii)
of Proposition 2, we deduce (following the same reasoning as before) that 
$t(j) \in \{i, k\}$. Let assume that $t(j) = k$, otherwise we are done. We will show that either $t(k) = j$ or there exists a ring $S = \{i, j, k\}$ such that $\forall s_i \in S, s_{i+1} \succ_s s_i \succ_s r$ for all $r \in \{s_{i+1}, s_{i-1}\}$.

Assume that there exists an agent $s \in M_j \setminus \{i, k\}$ such that $s \succ_j i$. Since 
j \in M_i \setminus \omega(i)$, by condition (ii) of Proposition 2, $\{s\} = M^i_j$, which implies
Then there exist over being unmatched. There are three possible matchings: directly dominated by all other matchings (because all agents prefer to be matched nor

\[ P3. \]

Proof. Suppose that \( M_j = \{k, i\} \) with \( k \succ_j i \). Then, \( k \in M_j \setminus \omega(j) \) and by condition (ii) of Proposition 2, either \( t(k) = j \) and we are done or there exists an agent \( l \in M_k \) such that \( l \succ_j k \) and \( \{l\} = M_j^k \), which implies \( l = i \). Given that there cannot be any agent between \( i \) and \( j \) in agent \( k \)'s preferences, we have that \( S = \{i, j, k\} \) form a ring in \( P \) such that \( \forall s_i \in S, s_{i+1} \succ s_i, s_{i-1} \succ s_i, r \) for any \( r \in N \setminus \{s_{i+1}, s_{i-1}\} \) as we wanted to prove.

Proof of Proposition 4]

Suppose that \( \forall i \in N, j \succ_i i \) for any \( j \neq i \) but \((N, P)\) is not dominance invariant. Then there exist \( \mu_1 \) and \( \mu_2 \) such that \( \mu_1 \gg \mu_2 \) but \( \mu_1 \not\gg \mu_2 \). First note that neither \( \mu_1 \) nor \( \mu_2 \) can be the matching in which every agent is a singleton, since this matching is directly dominated by all other matchings (because all agents prefer to be matched over being unmatched). There are three possible matchings: \( \mu_i = \{\{i\}, \{j, k\}\} \), \( \mu_j = \{\{j\}, \{i, k\}\} \) and \( \mu_k = \{\{k\}, \{i, j\}\} \). Assume, without loss of generality, that

\[ s = k. \] Therefore, there cannot be any agent different from \( k \) more preferred than \( i \) in agent \( j \)'s preferences.

Consider now the case such that there exists an agent \( s \in M_j \) such that \( i \succ_j s \). Then \( |M_j| > 2 \) and by the reasoning of \([P1.]\), \( t(j) = k \) implies \( t(k) = j \).

Let \( M_j = \{k, i\} \) with \( k \succ_j i \). Then, \( k \in M_j \setminus \omega(j) \) and by condition (ii) of Proposition 2, either \( t(k) = j \) and we are done or there exists an agent \( l \in M_k \) such that \( l \succ_j k \) and \( \{l\} = M_j^k \), which implies \( l = i \). Given that there cannot be any agent between \( i \) and \( j \) in agent \( k \)'s preferences, we have that \( S = \{i, j, k\} \) form a ring in \( P \) such that \( \forall s_i \in S, s_{i+1} \succ s_i, s_{i-1} \succ s_i, r \) for any \( r \in N \setminus \{s_{i+1}, s_{i-1}\} \) as we wanted to prove.

Proof of Proposition 4]

Suppose that \( \forall i \in N, j \succ_i i \) for any \( j \neq i \) but \((N, P)\) is not dominance invariant. Then there exist \( \mu_1 \) and \( \mu_2 \) such that \( \mu_1 \gg \mu_2 \) but \( \mu_1 \not\gg \mu_2 \). First note that neither \( \mu_1 \) nor \( \mu_2 \) can be the matching in which every agent is a singleton, since this matching is directly dominated by all other matchings (because all agents prefer to be matched over being unmatched). There are three possible matchings: \( \mu_i = \{\{i\}, \{j, k\}\} \), \( \mu_j = \{\{j\}, \{i, k\}\} \) and \( \mu_k = \{\{k\}, \{i, j\}\} \). Assume, without loss of generality, that
\( \mu_2 = \mu_k \) and \( \mu_1 = \mu_j \). The same reasoning could be applied for any other pair of matchings satisfying \( \mu_1 \gg \mu_2 \). Since \( \mu_1 \gg \mu_2 \) it must be [by Proposition 1] that \( i \) is better off in \( \mu_1 \) (since \( j \) is worse off being unmatched). Note that \( k \) is also better off in \( \mu_1 \) since she is unmatched in \( \mu_2 \). But then \( i \) and \( k \) can enforce \( \mu_1 \) over \( \mu_2 \) and they are both better off, contradicting that \( \mu_1 \not\gg \mu_2 \). \( \Box \)

**Proof.** [Proof of Proposition 5]

\((\Rightarrow)\) The existence of a ring \( S \) in the preferences with \( |S| = 3 \) and \( \forall s_i \in S, s_{i+1} \succ s_i, s_{i-1} \succ s_i, j \), for any \( j \in N \setminus \{s_{i+1}, s_{i-1}\} \) is a sufficient condition for non-existence of stable matchings in any roommate problem (dominance invariant or not). We prove it as follows:

Let \( \mu \) be a matching such that \( \mu(s_i) = j \) for some \( s_i \in S \) and some \( j \notin S \). This matching is blocked by the pair \( \{s_i, s_{i-1}\} \). Therefore any matching containing a pair formed by an agent in the ring and an agent outside the ring is not stable. Consider then a matching \( \mu' \) satisfying that \( \mu'(s_i) = s_{i+1} \) and \( \mu'(s_{i-1}) = s_{i-1} \) (given that \( |S| = 3 \), maximizing the number of agents in the ring matched among themselves, there is always one agent in the ring who is alone at \( \mu' \)). This matching is blocked by the pair \( \{s_{i-1}, s_{i+1}\} \). Therefore any matching in which agents in \( S \) are matched among themselves is not stable. Hence, there is no matching stable as we wanted to prove.

\((\Leftarrow)\) Now, we will show that if a roommate problem is dominance invariant and unsolvable then there exists a ring \( S \) in \( P \) satisfying that \( |S| = 3 \) and \( \forall s_i \in S, s_{i+1} \succ s_i, s_{i-1} \succ s_i, j \) for any \( j \in N \setminus \{s_{i+1}, s_{i-1}\} \).

Let \( (N, P) \) be unsolvable and dominance invariant. Since \( (N, P) \) is unsolvable, by Lemma 2, there exists a ring \( S = \{s_1, \ldots, s_k\} \subseteq N \) where \( k \) is odd (an odd ring). See Appendix A.

We first show that it must be \( |S| = 3 \). Suppose not, then consider \( \{s_1, \ldots, s_4\} \subseteq S \). Let \( \mu_2 \) be a matching such that \( \mu_2(s_2) = s_3 \) and \( \mu_2(s) = s \) for every \( s \notin \{s_2, s_3\} \), and let \( \mu_1 \) be a matching such that \( \mu_1(s_1) = s_2, \mu_1(s_3) = s_4, \) and \( \mu_1(s) = s \) for every \( s \notin \{s_1, \ldots, s_4\} \). Then \( \mu_1 \gg \mu_2 \) (since \( \{s_3, s_4\} \) block \( \mu_2 \) enforcing a matching in which \( s_1 \) and \( s_2 \) are alone, and this matching is blocked by the pair \( \{s_1, s_2\} \) enforcing \( \mu_1 \)). However, \( \mu_1 \not\gg \mu_2 \) since \( \mu_2 \succ s_2 \mu_1 \), which contradicts that \( (N, P) \) is dominance invariant.
We now show that for any $s_i \in S$ there cannot exist an agent $j \not\in S$ such that $j \in M_{s_i}$.

- Suppose first that $j \in M_{s_i}$ and $s_{i+1} \succ_{s_i} j$. Let $\mu_2$ be a matching such that $\mu_2(s_i) = s_{i+1}$ and $\mu_2(s) = s$ for every $s \not\in \{s_i, s_{i+1}\}$, and let $\mu_1$ be a matching such that $\mu_1(s_{i+1}) = s_{i-1}$, $\mu_1(s_i) = j$, and $\mu_1(s) = s$ for every $s \not\in S \cup \{j\}$. Then $\mu_1 \succ \mu_2$ (since $\{s_{i-1}, s_{i+1}\}$ block $\mu_2$ enforcing a matching in which $s_i$ and $j$ are alone, and this matching is blocked by the pair $\{s_i, j\}$ enforcing $\mu_1$). However, $\mu_1 \not\succ \mu_2$ since $\mu_2 \succ s_i \mu_1$, which contradicts that $(N, P)$ is dominance invariant.

- Suppose instead that $j \in M_{s_i}$ and $j \succ_{s_i} s_{i+1}$. Let $\mu_2$ be a matching such that $\mu_2(s_i) = s_{i-1}$ and $\mu_2(s) = s$ for every $s \not\in \{s_i, s_{i-1}\}$, and let $\mu_1$ be a matching such that $\mu_1(s_i) = j$, $\mu_1(s_{i-1}) = s_{i+1}$, and $\mu_1(s) = s$ for every $s \not\in S \cup \{j\}$. Then $\mu_1 \succ \mu_2$ (since $\{s_i, j\}$ block $\mu_2$ enforcing a matching in which $s_{i+1}$ and $s_{i-1}$ are alone, and this matching is blocked by the pair $\{s_{i+1}, s_{i-1}\}$ enforcing $\mu_1$). However, $\mu_1 \not\succ \mu_2$ since $\mu_2 \succ s_{i-1} \mu_1$, which contradicts that $(N, P)$ is dominance invariant.

By dominance invariance, every agent ranks her mutually acceptable agents higher than any other. Therefore, we can conclude that if the problem is unsolvable there is a ring $S$ in $P$ such that $|S| = 3$ and $\forall s_i \in S, s_{i+1} \succ s_i, s_{i-1} \succ s_i, j$ for any $j \in N \setminus \{s_{i+1}, s_{i-1}\}$. □

**Proof.** [Proof of Proposition 6]

To prove that $C(N, P) = \{\mu_C\}$, consider a dominance invariant roommate problem, $(N, P)$, with more than one matching in the core. By Lemma 1, we know that the matchings in the core dominate each other indirectly. However, by definition of stability, they do not dominate each other directly, contradicting that the problem is dominance invariant. Hence the core must be unique.

Now we prove that $\mu_C(i) = t(i)$ for all $i \in T$ and $\mu_C(j) = j$ for all $j \not\in T$. Suppose that there is an agent $i$ who is not matched in the core to her most preferred partner. Let $j = \mu_C(i)$ such that $t(i) \succ_i j$. Since $j \not\in t(i)$, there exists an agent $k$ such that $k \succ_i j$. Let $\mu_2$ be a matching such that $\mu_2(i) = k$ and the rest of the individuals are unmatched. By Lemma 1, $\mu_C \succ \mu_2$. Assume that $i \succ k$, then $\mu_C \not\succ \mu_2$ since $k \succ_i j$ and the pair $\{i, j\}$ does not block $\mu_2$ and enforce $\mu_C$. Therefore, $k \succ_i k$. Then $\{k\}$
blocks $\mu_2$ enforcing a new matching in which agent $i$ is unmatched. So if $\mu_C > \mu_2$, then $i = j$ proving that either $\mu_C(i) = t(i)$ or $\mu_C(i) = i$. □

References


