Efficiency and Strategic Interdependence*

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Strictly Preliminary: July 4, 2012

Abstract
Makowski and Ostroy [1995] present a revision of the First Theorem of Welfare Economics that identifies two sufficient conditions for social efficiency: the alignment of private awards with social contributions—full appropriation—and the need to prevent coordination failure. Brandenburger and Stuart [2007] obtain similar results using a purely game-theoretic formalism. Both models consider contexts with transferable utility. This paper generalizes these results for non-transferable utility. In both TU and NTU contexts, though, the sufficient conditions for social efficiency have perhaps a surprising consequence: each player will have a dominant strategy. Alternatively, using the contrapositive, if a player’s best choice of action depends upon what other players might do, i.e., if a player faces a game-theoretic decision, it must be the case that either (a) private awards are not aligned with social contributions, (b) there is the possibility of coordination failure, or (c) both.

1 Introduction
Makowski and Ostroy [1995] present a revision of the First Theorem of Welfare Economics that identifies two sufficient conditions for social efficiency: the alignment of private awards with social contributions—full appropriation—and the need to prevent coordination failure. They use a two-stage model. In the first stage, each player makes an ‘occupational’ choice, and the consequences of these choices is a Walrasian equilibrium of an economy modeled as a trade space. Brandenburger and Stuart [2007] obtain similar results. Using a purely game-theoretic formalism—the first stage is a non-cooperative game in strategic form, and the consequences of the first-stage choices are transferable-utility (TU) cooperative games—they make explicit the importance of preventing externalities when aligning individual profits with social contributions. They show that when choosing

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*This paper owes much to joint work with Adam Brandenburger and to many conversations with Joe Ostroy and Louis Makowski. The author thanks John Sutton for posing the question that led to the result linking efficiency to strategic dominance. Financial support from the Stern School of Business is gratefully acknowledged.
TU games, the full appropriation condition relies on both a no-surplus condition and a no externalities condition. In both approaches, utility is assumed to be transferable. This prompts the question whether the sufficient conditions for social efficiency are an artifact of transferable utility. Makowski and Ostroy [1995] predicted an answer of ‘no.’ Using a model in which the consequences of first stage moves are non-transferable utility (NTU) cooperative games, we reach a similar conclusion. This paper provides sufficient conditions for social efficiency in NTU contexts.

To obtain this result, this paper introduces two conditions to accommodate the generalization from TU to NTU games. The first is an ordering condition over efficient frontiers. If players do not agree on what a socially-improving outcome is, it is possible to have a Prisoner’s Dilemma style outcome: despite disagreement over what a socially-improving outcome is, the players end up in an outcome they all agree is inferior. The second condition accounts for the fact that with ordinal utilities, an economy can have multiple no-surplus equilibria. With transferable utility, a no-surplus outcome must be unique, and this fact is exploited in the existing TU results. To obtain NTU results, we limit our analysis to contexts in which no-surplus outcomes are unique (in utility space.) With these additional conditions, we show that the Makowski and Ostroy results can be generalized for NTU contexts.

We also show that in both sets of results—both TU and NTU—the sufficient conditions for efficiency have a significant consequence: each player will have a dominant strategy. Alternatively, using the contrapositive, if a player’s best choice of action depends upon what other players might do, it must be the case that either (a) private awards are not aligned with social contributions, (b) there is the possibility of coordination failure, or (c) both. Loosely, strategic interdependence implies that there is no guarantee of social efficiency.

Section 2 reviews the existing TU results and presents the result about the efficiency conditions implying a lack of strategic interdependence. Section 3 generalizes both the existing results and the interdependence result for non-transferable utility. Although this is logically inefficient (any TU cooperative game is a special case of an NTU cooperative game), it allows for a clearer identification of the issues that arise when utility is non-transferable. The paper concludes with a brief discussion in Section 4. In particular, this paper treats NTU cooperative games as mathematical objects.
It is an open question as to whether restricting the analysis to games derived from well-behaved economies will allow a sharpening of the sufficient conditions for NTU contexts.

2 Preliminaries

Let \((N;v)\) denote a transferable utility (TU) cooperative game, where \(N\) is the player set and \(v\) is the characteristic function, i.e., a mapping \(v : 2^N \rightarrow \mathbb{R}\). For any \(S \subseteq N\), the term \(v(S)\) denotes the maximum economic value that the players in \(S\) can create among themselves, i.e., the maximum gains from trade. An outcome of a TU cooperative game is described by an allocation \(x \in \mathbb{R}^{|N|}\), where component \(x^i\) denotes the value captured by player \(i\). The core of a TU cooperative game \((N;v)\) is the set of allocations satisfying \(\sum_{i \in N} x^i = v(N)\), and for all \(S \subseteq N\), \(\sum_{i \in S} x^i \geq v(S)\).

We consider a model in which players make strategic choices, and the consequences of a profile of choices is a TU game. Brandenburger and Stuart [2007] call such a model a biform game. The following is a slight generalization of their model.

**Definition 2.1** A **TU (transferable utility) biform game** is a collection

\[(N;A^1,\ldots,A^n;v;\succeq^1,\ldots,\succeq^n),\]

where:

1. a finite set \(N\) (the set of players), where \(N = \{1,\ldots,n\}\)
2. for each player \(i \in N\), a finite set \(A^i\) (the player’s strategy set)
3. for each \(a \in A\), where \(A = A^1 \times \ldots \times A^n\), a function \(v(a) : 2^N \rightarrow \mathbb{R}\), with \(v(a)(\emptyset) = 0\) for every \(a \in A\)
4. for each player \(i \in N\), a preference relation \(\succeq^i\) on the class of TU cooperative games with player set \(N\).

Additionally, it is assumed that each \((N;v(a))\) is super-additive, i.e. for any \(S,T \subseteq N\) such that \(S \cap T = \emptyset\), \(v(a)(S) + v(a)(T) \leq v(a)(S \cup T)\).
In this section, we will be analyzing only TU cooperative games with unique core allocations. Because core outcomes are expressed in payoffs, a player’s preferences over the TU cooperative games of interest will be immediate, as shown in Lemma 2.1 below.

Ostroy [1980] and Makowski [1980] show that a perfectly competitive equilibrium can be characterized by a no-surplus allocation. In a TU cooperative game, a no-surplus allocation, say $\mathbf{x}$, satisfies two conditions:

$$ P_{\mathbf{x}} \in \mathcal{P}(\mathbb{N}) $$

and for all $\mathbf{p} \in \mathcal{P}(\mathbb{N})$,

$$ P_{\mathbf{p}} \in \mathcal{P}(\mathbb{N}) \setminus \{\mathbf{x}\} \quad \text{s.t.} \quad P_{\mathbf{x}}(\mathbb{N}) = \mathbf{v}(\mathbb{N}) $$

If the core of the game is non-empty, these conditions imply that

$$ \sum_{i \in \mathbb{N}} [\mathbf{v}(\mathbb{N}) - \mathbf{v}(\mathbb{N} \setminus \{i\})] = \mathbf{v}(\mathbb{N}). $$

Thus, the follow condition may be interpreted as a perfect competition condition.

**Definition 2.2** A TU biform game $(\mathbb{N}; A^1, \ldots, A^n; v; \succeq^1, \ldots, \succeq^n)$ satisfies **adding up** (AU) if for each $a \in A$, the core of $(\mathbb{N}; v(a))$ is non-empty and

$$ \sum_{i \in \mathbb{N}} [v(a)(\mathbb{N}) - v(a)(\mathbb{N} \setminus \{i\})] = v(a)(\mathbb{N}). $$

With adding up, a player’s preferences over TU games is based on its marginal contributions. The following lemma formalizes this.

**Lemma 2.1** Consider a TU biform game $(\mathbb{N}; A^1, \ldots, A^n; v; \succeq^1, \ldots, \succeq^n)$ satisfying AU. For any $a, b \in A$, $(\mathbb{N}; v(a)) \succeq (\mathbb{N}; v(b))$ if, and only if,

$$ v(a)(\mathbb{N}) - v(a)(\mathbb{N} \setminus \{i\}) \geq v(b)(\mathbb{N}) - v(b)(\mathbb{N} \setminus \{i\}). $$

**Proof.** If the core of a game is non-empty, the condition $\sum_{i \in \mathbb{N}} [v(a)(\mathbb{N}) - v(a)(\mathbb{N} \setminus \{i\})] = v(a)(\mathbb{N})$ implies that the core is a singleton with $x_i = v(a)(\mathbb{N}) - v(a)(\mathbb{N} \setminus \{i\})$ for all $i \in \mathbb{N}$. Thus, $v(a)(\mathbb{N}) - v(a)(\mathbb{N} \setminus \{i\}) \geq v(b)(\mathbb{N}) - v(b)(\mathbb{N} \setminus \{i\})$ if, and only if, $x_i(a) \geq x_i(b)$. ■

To guarantee the alignment of private awards with social contributions, we use a condition that rules out externalities. Let $A^{-i}$ denote $A^1 \times \cdots \times A^{i-1} \times A^{i+1} \times \cdots \times A^n$.

**Definition 2.3** A TU biform game $(\mathbb{N}; A^1, \ldots, A^n; v; \succeq^1, \ldots, \succeq^n)$ satisfies **no externalities** (NE) if, for each $i \in \mathbb{N}$, $a^i, b^i \in A^i$, and $a^{-i} \in A^{-i},$

$$ v(b^i, a^{-i})(\mathbb{N} \setminus \{i\}) = v(a^i, a^{-i})(\mathbb{N} \setminus \{i\}). $$

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1 See, for example, Moulin [1995, Remark 2.3].
The no externalities condition ensures that a player cannot affect the total value that can be created when it is not involved. Since the value created without player $i$ is represented by $v(a)(N \setminus \{i\})$, this requires that for any $a^{-i} \in A^{-i}$, the term $v(b^i, a^{-i})(N \setminus \{i\})$ must be constant for all strategy choices of player $i$, i.e. for all $b^i \in A^i$.

When both the adding up and no externalities conditions hold, there is social alignment: any action that improves an individual’s payoff will increase the total value creation. The combination of the AU and NE conditions have the same effect as the full appropriation condition in Makowski and Ostroy [1995].

**Lemma 2.2** Consider a TU biform game $(N; A^1, \ldots, A^n; v; \succeq^1, \ldots, \succeq^n)$ satisfying AU and NE. Then $(N; v(a^i, a^{-i})) \succeq^i (N; v(b^i, a^{-i}))$ if, and only if, $v(a^i, a^{-i})(N) \geq v(b^i, a^{-i})(N)$.

**Proof.** By Lemma 2.1, $(N; v(a^i, a^{-i})) \succeq^i (N; v(b^i, a^{-i}))$ if, and only if,

$$v(a^i, a^{-i})(N) - v(a^i, a^{-i})(N \setminus \{i\}) \geq v(b^i, a^{-i})(N) - v(b^i, a^{-i})(N \setminus \{i\}).$$

By NE, $v(a)(N \setminus \{i\}) = v(b^i, a^{-i})(N \setminus \{i\})$, and the result follows. ■

Despite the alignment of individual and social incentives, there may still be inefficiency due to coordination failure. The final condition addresses the issue of coordination failure. (No coordination is a slight weakening of no complementarities in Makowski and Ostroy [1995].)

**Definition 2.4** A TU biform game $(N; A^1, \ldots, A^n; v; \succeq^1, \ldots, \succeq^n)$ satisfies no coordination (NC) if for each $i \in N$; $a^i, b^i \in S^i$; and $a^{-i}, b^{-i} \in A^{-i}$,

$$v(a^i, a^{-i})(N) > v(b^i, a^{-i})(N) \text{ if and only if } v(a^i, b^{-i})(N) > v(b^i, b^{-i})(N).$$

With the no coordination condition, the sign of the effect on the overall value creation due to a player’s change in strategy choice is independent of the other players’ strategic choices.

The following two definitions are used in the statement of Propositions 2.1 and 2.2.

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2 See, for example, Example 3 in Makowski and Ostroy [1995] or Example 5.2 in Brandenburger and Stuart [2007].
Definition 2.5 In a TU biform game, a strategy profile \(a \in A\) is \textit{efficient} if \(v(a)(N) \geq v(b)(N)\) for all \(b \in A\).

Definition 2.6 A profile of strategies \(a \in A\) is called a \textit{Nash equilibrium} if for all \(i \in N\), for all \(b^i \in A^i\),

\[
(N; v(a)) \succeq^i (N; v(b^i, a^{-i})).
\]

The following two propositions are Propositions 5.1 and 5.2 from Brandenburger and Stuart (2007).

**Proposition 2.1** Consider a biform game \((N; A^1, \ldots, A^n; v; \succeq^1, \ldots, \succeq^n)\) satisfying AU, NE, and NC. Then if a strategy profile \(a \in A\) is a Nash equilibrium, it is efficient.

**Proposition 2.2** Consider a biform game \((N; A^1, \ldots, A^n; v; \succeq^1, \ldots, \succeq^n)\) satisfying AU and NE. Then if a strategy profile \(a \in A\) is efficient, it is a Nash equilibrium.

Proposition 2.1 is the biform version of Theorem 3 in MO [1995]. Makowski and Ostroy call their theorem “A Revision of the First Theorem of Welfare Economics.” This naming is due to the fact that these theorems provide conditions under which equilibrium (in strategic choices) implies efficiency. Using similar reasoning, because Proposition 2.2 provides conditions under which an efficient outcome will be an equilibrium, it can be viewed as game-theoretic ‘Second Theorem’ type of result.

We now show that these efficiency conditions assume away strategic interdependence.\(^4\)

**Proposition 2.3** Consider a TU biform game \((N; A^1, \ldots, A^n; v; \succeq^1, \ldots, \succeq^n)\) satisfying AU, NE, and NC. Then each player has a (strongly) dominant strategy.

**Proof.** Fix a player \(i\) and two strategies \(a^i, b^i \in A^i\). Suppose that \(v(a^i, b^{-i})(N) >^i v(b^i, b^{-i})(N)\) for some \(b^{-i} \in A^{-i}\). Then NC implies \(v(a^i, b^{-i})(N) >^i v(b^i, b^{-i})(N)\) for all \(b^{-i} \in A^{-i}\). Applying NE yields

\[
v(a^i, b^{-i})(N) - v(a^i, b^{-i})(N \setminus \{i\}) > v(b^i, b^{-i})(N) - v(b^i, b^{-i})(N \setminus \{i\})
\]
for all $b^{-i} \in A^{-i}$. By Lemma 2.1, strategy $a^i$ strongly dominates strategy $b^i$.

The remaining case is that $v(a^i, b^{-i})(N) = v(b^i, b^{-i})(N)$ for all $b^{-i} \in A^{-i}$. But then NE yields

$$v(a^i, b^{-i})(N) - v(a^i, b^{-i})(N\{i\}) = v(b^i, b^{-i})(N) - v(b^i, b^{-i})(N\{i\})$$

for all $b^{-i} \in A^{-i}$. By Lemma 2.1, player $i$ is indifferent between strategy $a^i$ and strategy $b^i$. Finiteness of the strategy sets then implies that each player has a strongly dominant strategy (up to repetition of payoff-equivalent strategies).

Proposition 2.3 shows that these efficiency conditions, collectively, are strong. For a game-theorist, the contra-positive shows that these conditions can serve as a classification for the causes of strategic interdependence.

**Corollary 2.1** Suppose at least one player does not have a (strongly) dominant strategy in a TU biform game. Then at least one of the conditions AU, NE, or NC must not be satisfied.

As a further consequence of this corollary, if strategic choices depend upon what other players might do, then there is generally no guarantee that self-interest, even strategic rather than myopic self-interest, will yield a socially optimal outcome.

### 3 Non-Transferable Utility Results

We now model the consequence of players’ strategic choices as non-transferable utility (NTU) cooperative games instead of TU cooperative games. We use a definition of a non-transferable utility (NTU) cooperative game in which the characteristic function is interpreted as a utility possibility set. (See, for example, Mas-Colell et al. [1995, pg. 675].) An NTU cooperative game is a pair $(N; V)$ consisting of a player set $N$ and a correspondence $V$ that assigns to each non-empty subset $S \subseteq N$ a closed, bounded, convex, non-empty subset $V(S) \subseteq \mathbb{R}^S$. For any $S \subseteq N$, the set $V(S)$ contains the outcomes that the players in $S$ can achieve on their own. For an outcome $x \in V(S)$, the $i^{th}$ component of $x$ represents player $i$’s utility for that outcome. The core of the game $(N; V)$ is defined to be

$$\{x \in V(N) : \exists S \subseteq N \text{ such that for some } y \in V(S), y^i > x^i \text{ for all } i \in S\}.$$
In words, an outcome in $V(N)$ is in the core if no coalition of players can achieve a jointly better outcome.

**Definition 3.1** An NTU biform game is a collection  

$$(N; A^1, \ldots, A^n; V; \succeq^1, \ldots, \succeq^n),$$

where:

1. a finite set $N$ (the set of players), where $N = \{1, \ldots, n\}$
2. for each player $i \in N$, a finite set $A^i$ (the player’s strategy set)
3. for each $a \in A$, a correspondence $V(a)$ that assigns to each non-empty subset $S \subseteq N$ a closed, bounded, convex, non-empty subset $V(a)(S) \subseteq \mathbb{R}^S$
4. for each player $i \in N$, a preference relation $\succeq^i$ on the class of NTU cooperative games with player set $N$.

As in the TU case, it is assumed that each $(N; V(a))$ is super-additive: for any $S, T \subseteq N$ such that $S \cap T = \emptyset$, $V(a)(S) \times V(a)(T) \subseteq V(a)(S \cup T)$. Additionally, each set $V(a)(S)$ is assumed to be smooth: for any $x^S, y^S \in V(a)(S)$, if $x^S \geq y^S$ and $x^i > y^i$ for some $i \in S$, then there exists $z^S \in V(a)(S)$ such that $z^S > y^S$. For any $S \subseteq N$ and any $a \in A$, let $\partial V(a)(S)$ denote the pareto-efficient frontier of $V(a)(S)$, i.e.,

$$\partial V(a)(S) = \{x \in V(a)(S) : \exists y \in V(a)(S) \text{ with } y^i \geq x^i \text{ for all } i \in S, \text{ and } y^i > x^i \text{ for some } i \in S\}.$$

Similar to the previous section, we will be analyzing only NTU cooperative games with unique core allocations. Because core outcomes are expressed in utilities, a player’s preferences over NTU cooperative games will again be immediate.

As noted in the Introduction, generalizing the adding up condition for non-transferable utility presents at least two problems. The first is that no-surplus allocations are not necessarily unique, as Example 3.1 will show. In an NTU game, a no-surplus allocation, say $x$, satisfies two conditions: $x \in \partial V(N)$ and for all $i \in N$, $x^{-i} \in \partial V(N \setminus \{i\})$. Consider the following example.
Example 3.1 Consider the following NTU game with $N = \{1, 2, 3, 4\}$, $V(\{i\}) = 0$ for all $i \in N$, and

$$
V(N) = \{x \in \mathbb{R}^4_+ : x^1 + x^2 + x^3 + x^4 \leq 6\} \\
V(N\backslash\{1\}) = \{x \in \mathbb{R}^3_+ : 2x^2 + x^3 + x^4 \leq 6\} \\
V(N\backslash\{2\}) = \{x \in \mathbb{R}^3_+ : 2x^1 + x^3 + x^4 \leq 6\} \\
V(N\backslash\{3\}) = \{x \in \mathbb{R}^3_+ : x^1 + x^2 + 2x^4 \leq 6\} \\
V(N\backslash\{4\}) = \{x \in \mathbb{R}^3_+ : x^1 + x^2 + 2x^3 \leq 6\}.
$$

The core of this game is the set of points $(c, c, 3 - c, 3 - c)$, where $c \in [0, 3]$, and every point is a no-surplus allocation.

As a consequence, for an NTU biform game, we restrict our analysis to situations in which there is a unique, no-surplus allocation (in utility space). But this will still not be enough, in the presence of a no externalities condition, to ensure social alignment. Consider this next example.

Example 3.2 Consider the following NTU biform game with $N = \{1, 2, 3\}$, $A^1 = \{a^1, b^1\}$, $A^2 = \{a^2\}$, and $A^3 = \{a^3\}$. For notational simplicity, let $x, y$, and $z$ denote $x^1$, $x^2$, and $x^3$, respectively.

<table>
<thead>
<tr>
<th>$V(N)$</th>
<th>$(a^1, a^2, a^3)$</th>
<th>$(b^1, a^2, a^3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V(N\backslash{1})$</td>
<td>${(x, y, z) \in \mathbb{R}^3_+ : x + y^2 + y + 2z^2 \leq 38}$</td>
<td>${(b, a^2, a^3) : 0 &lt; b &lt; 10}$</td>
</tr>
<tr>
<td>$V(N\backslash{2})$</td>
<td>${(y, z) \in \mathbb{R}^2_+ : y^2 + z^2 \leq 20}$</td>
<td>${10, \sqrt{10}}$</td>
</tr>
<tr>
<td>$V(N\backslash{3})$</td>
<td>${(x, z) \in \mathbb{R}^2_+ : x^2 + z^2 \leq 16}$</td>
<td>${1, \sqrt{11}}$</td>
</tr>
</tbody>
</table>

(The description of the game $(N; V(b^1, a^2, a^3))$ differs from $(N; V(a^1, a^2, a^3))$ only in the right-hand side constants.)

In this example, each NTU game has a single no-surplus allocation: Core $(N; V(a^1, a^2, a^3)) = \{(0, 2, 4)\}$ and Core $(N; V(b^1, a^2, a^3)) = \{(1, \sqrt{10}, \sqrt{10})\}$. Moreover, though we haven’t defined it yet, the game satisfies no externalities: $V(N\backslash\{1\})$ is the same in both NTU games. Note that for any $x \in \partial V(b^1, a^2, a^3)(N)$, there exists a $y \in \partial V(a^1, a^2, a^3)(N)$ that dominates it. Thus, the choice of $b^1$ over $a^1$ leads to a socially inferior outcome, yet this is what player 1 prefers.

Example 3.2 exploits the fact that in an NTU context, the value of a coalition without a player, say $i$, is not a single quantity. Thus, even though $V(a)(N\backslash\{i\}) = V(b^1, a^{-1})(N\backslash\{i\})$, differences
in player $i$’s marginal contributions can depend upon more than just differences in $V(a)(N)$ and $V(b^i, a^{-i})(N)$. This is in contrast to a TU game, where $v(a)(N\setminus\{i\}) = v(b^i, a^{-i})(N\setminus\{i\})$ yields a single number for calculating the marginal contributions. The following strengthening of (AU) suffices to prevent the problem of Example 3.2.

**Definition 3.2** An NTU biform game $(N; A^1, \ldots, A^n; V; \geq^1, \ldots, \geq^n)$ satisfies **monotonic adding up** (MAU) if for all $a \in A$, the core of $(N; V(a))$ is a singleton, say $x(a)$, with the properties that for all $i \in N$, $x^{-i}(a) \in \partial V(a)(N\setminus\{i\})$, and if, for any $i \in N$ and $a, b \in A$ such that $V(a)(N\setminus\{i\}) = V(b)(N\setminus\{i\})$,

$$V(a)(N) \supset V(b)(N) \Rightarrow x^i(a) > x^i(b)$$

and

$$V(a)(N) = V(b)(N) \Rightarrow x^i(a) = x^i(b).$$

Note that when a TU game is treated as a special case of an NTU game, the MAU condition reduces to the AU condition. Further, the two conditions $V(a)(N) = V(b)(N)$ and $V(a)(N\setminus\{i\}) = V(b)(N\setminus\{i\})$ do not imply that $(N; V(a)) = (N; V(b))$. Loosely, the MAU condition ensures that if a player, say $i$, makes a socially-improving move, the change in gains-from-trade for coalitions of the form $\{i\} \cup S$, where $S \subset N\setminus\{i\}$, do not make player $i$ worse off in the core.

The no externalities condition remains essentially the same.

**Definition 3.3** An NTU biform game $(N; A^1, \ldots, A^n; V; \geq^1, \ldots, \geq^n)$ satisfies **no externalities** (NE) if for any $i \in N$, for any $a^i, b^i \in A^i$ and any $a^{-i} \in A^{-i}$,

$$V(b^i, a^{-i})(N\setminus\{i\}) = V(a^i, a^{-i})(N\setminus\{i\}).$$

The following sufficiency result for social alignment is immediate from the definitions of MAU and NE.

**Lemma 3.1** Consider a NTU biform game $(N; A^1, \ldots, A^n; V; \geq^1, \ldots, \geq^n)$ satisfying MAU and NE. For any $a^i, b^i \in A^i$, and $a^{-i} \in A^{-i}$,

$$V(a)(N) \supset V(b^i, a^{-i})(N) \Rightarrow (N; V(a)) \succ^i (N; V(b^i, a^{-i}))$$

10
and
\[
V(a)(N) = V(b^i, a^{-i})(N) \Rightarrow (N; V(a)) \sim^i (N; V(b^i, a^{-i})).
\]

Lemma 3.1 ensures that a player will not make a socially inefficient choice. But when choosing between two efficient choices, a Prisoner’s Dilemma can arise. Figure 3.1 below depicts the issue.

![Figure 3.1](image)

This picture shows the projection of the four efficient frontiers, \( \partial V(a, a)(N) \), \( \partial V(a, b)(N) \), etc., on the axes for players 1 and 2. The diamonds denote the players’ outcomes in the corresponding NTU games. The above picture comes from a game satisfying both MAU and NE. (The details are in the Appendix.) But because of the ‘crossing’ efficient frontiers—an impossibility with a TU game, NE and MAU do not prevent the outcomes depicted above. In the figure, NE and MAU only imply that each player prefer \((N; V(b, b))\) to \((N; V(a, a))\), which is the case.

To prevent such situations, we use an ordering condition.

**Definition 3.4** An NTU biform game \((N; A^1, \ldots, A^n; V; \succeq^1, \ldots, \succeq^n)\) satisfies **ordering** (ORD) if for any \(a, b \in A\), \(V(a)(N) \subseteq V(b)(N)\) or \(V(b)(N) \supseteq V(a)(N)\).

With ORD, we now have a necessary-and-sufficient social alignment result.
Lemma 3.2 Consider a NTU biform game \((N; A^1, \ldots, A^n; V; \preceq_1, \ldots, \preceq_n)\) satisfying MAU, NE, and ORD. For any \(a^i, b^i \in A^i\), and \(a^{-i} \in A^{-i}\), \((N; V(a)) \succeq^i (N; V(b^i, a^{-i}))\) if, and only if, \(V(a)(N) \supseteq V(b^i, a^{-i})(N)\).

Proof. The ‘if’ part is immediate from Lemma 3.1. For the ‘only if’ part, suppose \((N; V(a)) \succeq^i (N; V(b^i, a^{-i}))\) but \(V(a)(N) \not\supseteq V(b^i, a^{-i})(N)\). By ORD, \(V(a)(N) \subset V(b^i, a^{-i})(N)\). By Lemma 3.1, \((N; V(a)) \prec^i (N; V(b^i, a^{-i}))\), providing a contradiction. ■

In games satisfying ORD, \(V(a)(N) \subset V(b)(N)\) has a natural interpretation of \(V(a)(N)\) being less than \(V(b)(N)\). This motivates our generalization of the NC condition. Furthermore, the following definition implies the TU version of the NC condition.

Definition 3.5 An NTU biform game \((N; A^1, \ldots, A^n; V; \preceq_1, \ldots, \preceq_n)\) satisfies ordinal no coordination (ONC) if for each \(i \in N; a^i, b^i \in A^i\); and \(a^{-i}, b^{-i} \in A^{-i}\),

\[
V(a^i, a^{-i})(N) \supset V(b^i, a^{-i})(N) \text{ if and only if } V(a^i, b^{-i})(N) \supset V(b^i, b^{-i})(N).
\]

The following definitions are used in the propositions.

Definition 3.6 In an NTU biform game, a strategy profile \(a \in A\) is inefficient if there exists a profile \(b \in A\) such that for any \(x \in \partial V(a)(N)\), there exists a \(y \in \partial V(b)(N)\) such that \(y^i \geq x^i\) for all \(i \in N\) and \(y^i > x^i\) for some \(i \in N\). A strategy profile \(a \in A\) is efficient if it is not inefficient.

Note that the above definition implies that if there is a strategy profile \(a \in A\) such that \(V(a)(N) \supseteq V(b)(N)\) for all \(b \in A\), then the profile \(a\) is efficient.

Definition 3.7 A profile of strategies \(a \in A\) is called a Nash equilibrium if for all \(i \in N\), for all \(b^i \in A^i\),

\[
(N; V(a)) \succeq^i (N; V(b^i, a^{-i})).
\]

We now state our main results.
Proposition 3.1 Consider an NTU biform game \((N; A^1,\ldots,A^n; V; \succeq^1,\ldots,\succeq^n)\) satisfying MAU, NE, ORD, and ONC. Then if a strategy profile \(a \in A\) is a Nash equilibrium, it is efficient.

Proof. Consider any \(b \in A\). Since \(a\) is an equilibrium, \((N; V(a)) \succeq^1 (N; V(b^1, a^{-1}))\). By Lemma 3.2, \(V(a)(N) \supseteq V(b^1, a^{-1})(N)\). By ORD, \(V(a^1, b^{-1})(N) \supseteq V(b)(N)\). Next, \((N; V(a)) \succeq^2 (N; V(b^2, a^{-2}))\) implies \(V(a)(N) \supseteq V(b^2, a^{-2})(N)\). By ORD, \(V(a^{12}, b^{-12})(N) \supseteq V(a^1, b^{-1})(N)\). Repeating this reasoning yields the following set of weak inclusions:

\[
\begin{align*}
V(a^1, b^{-1})(N) & \supseteq V(b)(N) \\
V(a^{12}, b^{-12})(N) & \supseteq V(a^1, b^{-1})(N) \\
V(a^{123}, b^{-123})(N) & \supseteq V(a^{12}, b^{-12})(N) \\
& \vdots \\
V(a)(N) & \supseteq V(a^{-n}, b^n)(N)
\end{align*}
\]

Thus, \(V(a)(N) \supseteq V(b)(N)\) for all \(b \in A\), implying that \(a\) is efficient.

Proposition 3.2 Consider an NTU biform game \((N; A^1,\ldots,A^n; V; \succeq^1,\ldots,\succeq^n)\) satisfying MAU, NE, and ORD. Then if a strategy profile \(a \in A\) is efficient, it is a Nash equilibrium.

Proof. Consider an efficient \(a \in A\). By ORD, for all \(i \in N\), \(V(a)(N) \succeq V(b^i, a^{-i})(N)\) for all \(b^i \in A^i\). By Lemma 3.2, \((N; V(a)) \succ^i (N; V(b^i, a^{-i}))\). Thus, for all \(i \in N\), \((N; V(a)) \succeq^i (N; V(b^i, a^{-i}))\) for all \(b^i \in A^i\).

Note that Proposition 3.2 does not require a complete ordering. As noted above, if there is a ‘largest’ NTU game, then the profile generating that game will be an efficient profile. The following corollary makes this precise.

Corollary 3.1 Consider an NTU biform game \((N; A^1,\ldots,A^n; V; \succeq^1,\ldots,\succeq^n)\) satisfying MAU and NE. If there exists a strategy profile \(a \in A\) such that \(V(a)(N) \supseteq V(b)(N)\) for all \(b \in A\), then it is a Nash equilibrium.

Finally, we have the dominance result as well.
Proposition 3.3 Consider an NTU biform game \((N; A^1, \ldots, A^n; V; \succeq^1, \ldots, \succeq^n)\) satisfying MAU, NE, ORD, and ONC. Then each player has a (strongly) dominant strategy.

Proof. Fix a player \(i\) and two strategies \(a^i, b^i \in A^i\). Suppose that \(V(a^i, b^{-i})(N) \supset V(b^i, b^{-i})(N)\) for some \(b^{-i} \in A^{-i}\). Then NC implies \(V(a^i, b^{-i})(N) \supset V(b^i, b^{-i})(N)\) for all \(b^{-i} \in A^{-i}\). By Lemma 3.1, \((N; V(a^i, b^{-i})) \succ_i^1 (N; V(b^i, b^{-i}))\) for all \(b^{-i} \in A^{-i}\).

By ORD, the other case is that \(V(a^i, b^{-i})(N) = V(b^i, b^{-i})(N)\) for some \(b^{-i} \in A^{-i}\). Then NC implies \(V(a^i, b^{-i})(N) = V(b^i, b^{-i})(N)\) for all \(b^{-i} \in A^{-i}\). By Lemma 3.1, \((N; V(a^i, b^{-i})) \sim_i^1 (N; V(b^i, b^{-i}))\) for all \(b^{-i} \in A^{-i}\). Finiteness of the strategy sets then implies that each player has a dominant strategy (up to repetition of payoff-equivalent strategies). \(\blacksquare\)

Corollary 3.2 Consider an NTU biform game satisfying ORD. Suppose at least one player does not have a dominant strategy. Then at least one of the conditions MAU, NE, or ONC must not be satisfied.

4 Discussion

Non-cooperative game theory studies situations in which players have well-defined moves. In such contexts, it is natural to ask an ‘invisible hand’ type of question: will self-interested behavior lead to socially efficient outcomes? With transferable utility, Makowski and Ostroy identify conditions under which this question has a positive answer. In this paper, we show that their results can be generalized for non-transferable utility provided that two additional conditions are met: an ordering of the sets corresponding to the value of the grand coalitions, and a restriction to games with ‘well-behaved’ no-surplus outcomes.

It is an open question whether these additional conditions allow for NTU games that are meaningful different from TU games, or whether these conditions effectively are selecting what might be called ‘TU-like’ NTU games. For instance, the ordering condition might seem particularly strong, yet as Figure 3.1 demonstrates, differing preferences over efficient frontiers can be problematic. Ideally, these additional conditions should be revisited from the perspective of specific types of economies, rather than from perspective of the (abstract) characteristic function.
In both the TU and NTU cases, though, there is the dominance result. The sufficient conditions for efficiency are assuming away strategic interdependence. It is entirely possible that there are weaker sufficient conditions. For example, with TU games, the adding up condition can be replaced with a condition that a player’s profits are increasing in its marginal contribution. But taking the existing conditions as given, the results of this paper imply that if there is strategic interdependence in a game, then there are three possible reasons, not necessarily mutually exclusive: profits not aligning with marginal contributions, externalities, or coordination failure. And if utility is not transferable, there is a fourth possible reason: differences in preferences.
References


Appendix

The game for Figure 3.1 has a player set $N = \{1, 2, 3\}$, with $A^1 = A^2 = \{a, b\}$. Player 3 has a singleton strategy set, and we suppress the notation.

<table>
<thead>
<tr>
<th>$V(N)$</th>
<th>$(a, a)$</th>
<th>$(b, a)$</th>
<th>$(a, b)$</th>
<th>$(b, b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V(N) = {x \in \mathbb{R}_+^3 : x^1 + x^2 + x^3 \leq 16}$</td>
<td>$(a, a)$</td>
<td>$(b, a)$</td>
<td>$(a, b)$</td>
<td>$(b, b)$</td>
</tr>
<tr>
<td>$V(N \setminus {1}) = {(x^2, x^3) \in \mathbb{R}_+^2 : x^2 + x^3 \leq 12}$</td>
<td>$(b, a)$</td>
<td>$(a, b)$</td>
<td>$(b, b)$</td>
<td>$(7, 7, 4)$</td>
</tr>
<tr>
<td>$V(N \setminus {2}) = {(x^1, x^3) \in \mathbb{R}_+^2 : x^1 + x^3 \leq 12}$</td>
<td>$(a, b)$</td>
<td>$(a, b)$</td>
<td>$(a, b)$</td>
<td>$(7, 7, 4)$</td>
</tr>
<tr>
<td>$V(N \setminus {3}) = {(x^1, x^2) \in \mathbb{R}_+^2 : x^1 + x^2 \leq 8}$</td>
<td>$(a, b)$</td>
<td>$(a, b)$</td>
<td>$(a, b)$</td>
<td>$(7, 7, 4)$</td>
</tr>
</tbody>
</table>

Outcome: $(4, 4, 8)$

Outcome: $(3, 8, 2)$

Outcome: $(8, 3, 4)$

Outcome: $(7, 7, 4)$