Efficiency and lack of commitment in an overlapping generations model with endowment shocks

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Abstract

This paper considers a pure exchange stochastic overlapping generations model in which, on each date, an economy faces an endowment shock. In one shock a young agent is relatively rich compared to an old agent, and in the other shock an old agent is relatively rich compared to a young agent. On each date, a young agent and an old agent simultaneously decide how much of their respective endowments to transfer to the other agent; however, a young agent cannot make promises about how much she will give when she gets old. In this sense, an economy faces a limited commitment constraint. This paper studies an efficient risk sharing allocation that satisfies a limited commitment constraint, and compares the implications from the model with an infinite-lived-agents model such as in [Thomas and Worrall (1988), Kocherlakota (1996) and so on.

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1 Introduction

The overlapping generations model, which was formulated by Samuelson (1958) and Diamond (1965), is one of the basic models in macroeconomics and public finance. There is a more recent literature, concerned with insurance, that is formulated in models of contemporaneous, infinite-lived agents, and is constrained by the inability to make some contractual commitments (see, for example, Thomas and Worrall (1988), Kocherlakota (1996), Alvarez and Jermann (2000) and so on). This paper considers insurance under a limited commitment constraint in a stochastic overlapping generations economy. The situation in this environment is substantially different from that in contemporaneous, infinite-lived agent economies. This paper’s main purpose is to figure out the difference between the model comprised of contemporaneous, infinite-lived agents and the overlapping generations model.

The model in this paper considers a pure exchange overlapping generations economy that faces an endowment shock on every date. Each generation consists of one single agent. After the shock is realized, a new young agent is born. As for the endowment shock, in one shock a young agent is relatively rich compared to an old agent, while in the other shock an old agent is relatively rich compared to a young agent. Both a young agent and an old agent decide how much of their respective endowments to transfer to the other agent, doing so both simultaneously and independently. After the transfer is made, each agent consumes and derives utility, and an old agent dies.

Interim Pareto efficiency considers an agent’s expected lifetime utility calculated on each agent’s birth date given the histories at that moment. Since a young agent is born after a current shock is realized, it is natural for the agent to be distinguished after different histories, even though she has the same name. Because of the limited commitment constraint, an equilibrium concept requires subgame perfection. Hence, an equilibrium concept of this paper is a subgame perfect equilibrium.

In the analysis, to make the comparison of a contemporaneous, infinite-lived agents model and an overlapping generations model clear, the paper imposes one assumption and focuses on some featured allocations. One important result of the contemporaneous, infinite-lived agents model is that history-dependent behavior improves agents’ welfare. That is, the autarkic allocation is not efficient. Thus, the paper imposes the assumption that the autarkic allocation is not interim Pareto efficient. Another important result of the contemporaneous, infinite-lived agents model is that the first-best allocation is subgame perfect when the discount factor is large enough. In an overlapping generations environment, it is difficult to define the first-best allocation well. Among all interim Pareto efficient allocations, this paper focuses on golden-rule type allocations. For this purpose and the following previous literature, this

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For instance, another possible welfare concept is “ex ante” Pareto efficiency. Ex ante Pareto efficiency considers an agent’s expected lifetime utility calculated on the initial date of the economy, that is, before a shock is realized. Under ex ante Pareto efficiency, “the” agent who lives on a particular date is regarded as one individual.

There are several papers that investigate subgame perfect, efficient allocations in a deterministic, two-period-lived-agents overlapping generations model. See, for example, Hammond (1975) and Bhaskar (1998). In a stochastic environment, although the setting is different from that in this paper, Messner and Polborn (2003) extend Cremer (1986) by adding a stochastic cooperation cost.
An allocation is a golden-rule type allocation if the allocation maximizes the weighted sum of the young agent’s conditional expected lifetime utility. Notice that a young agent who has different histories is distinguished. Although this golden-rule type allocation does not exactly correspond to the first-best allocation in the contemporaneous, infinite-lived agents model, the study of a golden-rule type allocation can provide some insight into the comparison of two models.

The main result of this paper is that whenever the autarkic allocation is not interim Pareto efficient, there are golden-rule type allocations that satisfy the limited commitment constraint if agents condition their behavior on past history. The first implication of this result is that history-dependent behavior improves welfare, which is also derived in the contemporaneous, infinite-lived agents model. In the contemporaneous, infinite-lived agents model, however, the first-best allocation might not be subgame perfect even if the autarkic allocation is not efficient, because of the low discount factor. In the overlapping generations model studied in this paper, as long as the autarkic allocation is not interim Pareto efficient, we can find some golden-rule type allocation that is subgame perfect. As already mentioned, it may not be the right comparison of the two models in this respect. However, this result implies that it is probable that the allocation is “first best” in the framework of the overlapping generations model, whereas it is not in the framework of the contemporaneous, infinite-lived agents model. That is, the allocation observed in the real world could be “first best” in the sense of this paper.

The papers such as Thomas and Worrall (1988) and Kocherlakota (1996) also provide the characterization result of the subgame-perfect constrained efficient allocations. These papers describe the dynamics of each agent’s subgame-perfect constrained allocation in a simple manner. More precisely, these papers find the switching rule of the labor wage/consumption goods, which has a Markov property. Since the paper focuses on a stationary allocation and each agent lives only for two periods, it is not possible to derive the similar characterization results as in the contemporaneous, infinite-lived agents model. However, there is a difference from the contemporaneous, infinite-lived agents model. The difference is seen when the shock in which the young agent is richer does not occur frequently. In this case, it is possible that the allocation achieved by transfer from only the rich young agent to the poor old agent is not subgame perfect even though it is a golden-rule type allocation. In that case, the golden-rule type allocation satisfying the limited commitment constraint is an allocation that is achieved by transfers from both the rich young agent and the poor young agent. This observation cannot be found in an infinite-lived agents model. In an infinite-lived agents model such as Kocherlakota (1996) a poor agent transfers nothing to a rich agent and a rich agent transfers some of her own endowment to a poor agent in an efficient, subgame perfect allocation. The reason is as follows. In an infinite-lived agents model each

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3 One difficulty in an overlapping generations model is that it is not easy to study non-stationary allocations. Because of this difficulty, quite a few papers focus on a stationary allocation. For instance, see Demange and Laroque (1999) and Gottardi and Kubler (2011).

4 The probability can be low, but the assumption that the autarkic allocation is not interim Pareto efficient must hold. If the probability is too low, then in an extreme case, the young agent is poorer than the old agent. In this case, the autarkic allocation is interim Pareto efficient.

5 Kocherlakota (1996) considers an endowment shock in an infinite-lived agents model. In his model, when one agent is rich, the other agent is poor.
agent plays a transfer game with the same agent forever. However, in an overlapping generations model every agent plays a transfer game with a different player. Furthermore, every agent stops playing after playing a transfer game twice. In an infinite-lived agents model, if a poor agent today transfers some of her endowment to a rich agent today, then a rich agent today needs to compensate a poor agent today at some point in the future, and since agents are risk averse, this is more costly than the case in which there is no transfer from a poor agent to a rich agent. In an overlapping generations model, even if a rich old agent receives a transfer from a poor young agent, this old agent does not need to compensate the poor young agent in the future because she dies after today. In addition, a poor young agent is willing to transfer some of her endowment to a rich old agent because a currently young agent does not transfer any of her endowment when she becomes old in a subgame perfect equilibrium.

One related research area of this paper is, as I said before, risk sharing with a limited commitment constraint. Another strand of this paper is to study the pay-as-you-go social security system. Since seminal papers by Samuelson (1958) and Diamond (1965), an overlapping generations model has been used to understand the pay-as-you-go social security system which is adopted in many countries. A recent trend in this research area is to consider aggregate uncertainty explicitly to understand the intergenerational risk sharing role of the pay-as-you-go social security. The difference between this paper and the other literature is the cause of the market incompleteness. Papers such as Krueger and Kubler (2002) and Krueger and Kubler (2006) consider an economy in which the financial market is incomplete, while the incompleteness in this paper is caused by a young agent’s inability to make some contractual commitments. Without commitment ability, it is possible that a young agent will break the promise about her future behavior made when she is young once she becomes old. In the pay-as-you-go social security framework, it is not rare to see that the young agree with low payment but they protest against the low payment when they become old. If the pay-as-you-go social security system satisfies the limited commitment constraint, it is probable to avoid such a protest. The lack of a commitment constraint should be considered as one of essential constraints when we discuss the optimal pay-as-you-go social security design.

The remainder of this paper is organized as follows. Section 2 sets up the model and gives several definitions. Section 3 characterizes the golden-rule type allocation that satisfies the limited commitment constraint. Section 4 provides some discussions about the model, and also compares the model with a model with fiat money as well as a political economy model that are often used in the analysis of an overlapping generations model. Section 5 concludes.

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7 For instance, we can consider the “American Association of Retired Persons (AARP)”, which is one of the largest interest groups in the U.S. According to the web site, “OpenSecrets.org”, in 2008, they spent $27,900,000 and it was the 3rd largest amount of money spent.
2 The Model

In this section, I set up the model and give definitions of efficiency and the equilibrium.

2.1 Environment

Time is discrete and infinite, \( t = 1, 2, \ldots \). On each date \( t \), a single agent is born. Call an agent born on date \( t \) a \textit{generation-} \( t \) agent. An agent born on date \( t = 0 \) is called \textit{initial old}. Agents live for two periods, \textit{young} and \textit{old}.

On each date \( t \), the economy faces an endowment shock. Let \( S := \{1, 2\} \) denote a set of shocks with generic element \( s \). The probability that a shock is \( s \in S \) is denoted by \( \pi(s) \), where \( \pi(s) > 0 \) for all \( s \in S \). Endowments for a young agent and an old agent are determined by a shock. \( e_y(s) \) denotes a young agent’s endowment when a shock is \( s \in S \) and \( e_o(s) \) is an old agent’s endowment given the current shock \( s \in S \). The endowments are assumed to satisfy

\[
e_y(1) = e_y(2)
\]

and

\[
e_y(1) > e_o(1) > 0 \text{ and } 0 < e_y(2) < e_o(2)
\]

In words, in state 1 the young agent has a larger endowment (or is “richer”) than the old agent, and in state 2 the reverse applies. The shock can be considered to be a random capital share in a production economy. If the production function is a Cobb-Douglas function, then it is described as \( f(k) = k^\alpha \) and \( \alpha \) is a random variable. Let \( e(s) := e_y(s) + e_o(s) \) be a total endowment in shock \( s \in S \). Notice that \( e(1) < e(2) \). Let \( s^t := (s_1, s_2, \ldots, s_t) \in S^t \) be a history of shocks up to date \( t \). I assume that the stochastic process, \( \{s_t\} \), is i.i.d.

An \textit{allocation for generation-} \( t \) \((\geq 1)\) is denoted by a pair of mappings, \( c_t := (c^y_t, c^{o+1}_t) \), where \( c^y_t : S^t \to \mathbb{R}_+ \) and \( c^{o+1}_t : S^{t+1} \to \mathbb{R}_+ \). An \textit{allocation for the initial old} is denoted by a mapping, \( c^o_1 : S \to \mathbb{R}_+ \). Let \( c := (c^o_1, (c_t)_{t=1}^\infty) \) be an \textit{allocation}. An allocation, \( c \), is \textit{feasible} if for all \( t \geq 1 \),

\[
c^y_t(s^t) + c^{o+1}_t(s^{t+1}) = e(s_t)
\]

for all \( s^t \in S^t \).

Agents derive utility from consuming in each period of life. Let \( u : \mathbb{R}_+ \to \mathbb{R} \) be an agent’s periodic utility function that is strictly concave, strictly increasing and twice-continuously differentiable. I assume that the agent’s life-time utility is additively separable with common discount factor \( \beta \in (0, 1] \). For later convenience, I define the agent’s expected life-time utility as being conditional on a history of shocks.

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8The reason why I only focus on the two-shocks case is the tractability of the model, and I do not think more fruitful results will appear even if the number of shocks increases. For the details, please see the discussion below.

9Even if an old agent’s endowment is assumed to be constant and a young agent’s endowment is fluctuating, the implications of this paper will not change.
\( s' \in S' \). For any date \( t \geq 1 \), if the generation-\( t \) agent’s allocation is \( c_t \), her expected life-time utility conditional on \( s' \in S' \) is denoted by

\[
U(c_t, s') := u(c^y_t(s')) + \beta \left[ \sum_{\tilde{s} \in S} \pi(\tilde{s}) u(c^o_{t+1}(s', \tilde{s})) \right].
\]

**2.2 Welfare concept**

In this paper, since a current shock is realized before a young agent is born, it is natural to use *interim Pareto efficiency* as social welfare concept. To put it differently, the agent is regarded as a different individual if the histories before she is born are different under interim Pareto efficiency. The formal definition is as follows:

**Definition 2.1.** An allocation, \( c \), is *interim Pareto efficient* if \( c \) is feasible and there does not exist another feasible allocation, \( \tilde{c} \), such that

\[
U(\tilde{c}_t, s') \geq U(c_t, s')
\]

for all \( t \geq 1 \) and all \( s' \in S' \), and

\[
u(\tilde{c}^o_1(s^1)) \geq u(c^o_1(s^1))
\]

for all \( s^1 \in S \), and either (2) holds with strict inequality for some \( t \geq 1 \) and some \( s' \in S' \) or (3) holds with strict inequality for some \( s^1 \in S \).

**2.3 Specification of transfer game without commitment**

In this economy, both a young agent and an old agent can voluntarily transfer part of their own endowments to the other agent who is currently alive. I assume that no young agent can commit her amount of transfer in her old age and, hence, every agent decides how much she will transfer at each age. Moreover, there is no externally enforced commitment device. An agent’s strategy determines how much of her endowment she will transfer to the other agent conditional on the histories of transfers and shocks up to this time. Let \( H' := ([0, \bar{e}]^2)^I \), where \( \bar{e} := e(2) \), \( H^0 = \emptyset \), and \( S^0 := \emptyset \). Let

\[
\sigma^y_t : H^{t-1} \times S' \to [0, \bar{e}]
\]

be a mapping from a history of transfers before date \( t \), \( h^{t-1} \in H^{t-1} \), and a history of shocks up to and including date \( t \)’s shock, \( s' \in S' \), to a real number that satisfies \( \sigma^y_t(h^{t-1}, s') \in [0, e^y(s_t)] \). Let

\[
\sigma^o_{t+1} : H' \times S^{t+1} \to [0, \bar{e}]
\]
be a mapping from a history of transfers before date \( t + 1 \), \( h' \in H' \), and a history of shocks up to and including date \( t + 1 \)’s shock, \( s_{t+1}' \in S_{t+1}' \), to a real number that satisfies \( \sigma_{t+1}(h', s_{t+1}') \in [0, e^o(s_{t+1})] \). Then, a strategy of the generation-\( t \) (\( t \geq 1 \)) agent is defined by

\[
\sigma_t := (\sigma^y_t, \sigma^o_{t+1}).
\]

A strategy of the initial old agent is

\[
\sigma_0 := \sigma^o_0 : S \rightarrow [0, e^o(s)]
\]

that satisfies \( \sigma^o_0(s) \in [0, e^o(s)] \) for all \( s \in S \). Let \( \Sigma^y_t \) and \( \Sigma^o_t \) be a set of all mappings, \( \sigma^y_t \) and \( \sigma^o_t \), respectively. Let \( \Sigma_t := \Sigma^y_t \times \Sigma^o_{t+1} \) be a set of all strategies of generation-\( t \) (\( t \geq 0 \)). When a strategy profile is \( \sigma \), the after-transfer allocations for the young and the old agents at date \( t \), given \( s' \in S' \) and \( h^{-1} \in H'^{-1} \), are

\[
\begin{align*}
\sigma^y_t(s') &= e^y(s_t) - \sigma^y_t(h^{-1}, s') + \sigma^o_t(h^{-1}, s') \\
\sigma^o_t(s') &= e^o(s_t) - \sigma^o_t(h^{-1}, s') + \sigma^y_t(h^{-1}, s').
\end{align*}
\]

### 2.4 Equilibrium concept in a transfer game

I use a subgame perfect equilibrium of this commitment structure as an equilibrium concept. In a subgame perfect equilibrium, at any point in time and history, each agent chooses the amount of transfers at that time optimally.

**Definition 2.2.** A strategy profile \( \sigma^* \) is a subgame perfect equilibrium (SPE) if for all \( t \geq 1 \), all \( s' \in S' \), and all \( h^{-1} \in H'^{-1} \),

\[
\sigma^*_t \in \arg \max_{\sigma_t \in \Sigma_t} \left\{ u \left( e^y(s_t) - \sigma^y_t(h^{-1}, s') + \sigma^o_t(h^{-1}, s') \right) + \beta \left[ \sum_{s \in S} \pi(s) u \left( e^o(s) - \sigma^o_{t+1}(h', s'), \sigma^y_t(h^{-1}, s') \right) \right] \right\},
\]  

(4)

where \( h' = (h^{-1}, (\sigma^y_t(h^{-1}, s'), \sigma^o_t(h^{-1}, s'))) \), and for all \( t \geq 1 \), all \( s' \in S' \), and all \( h^{-1} \in H'^{-1} \),

\[
\begin{align*}
u(e^y(s_t) - \sigma^o_t(h^{-1}, s') + \sigma^y_t(h^{-1}, s')) \\
\geq u(e^o(s_t) - \sigma^o_t(h^{-1}, s') + \sigma^y_t(h^{-1}, s'))
\end{align*}
\]

for all \( \sigma^o_t \in \Sigma^o_t \). An allocation, \( c \), is subgame perfect or an SPE allocation if \( c \) is induced by some SPE \( \sigma \).

In the definition, equation (4) is the incentive condition for the young agent, and equation (5) is the incentive condition for the old agent.

Now, I show that the autarky allocation is always supported by an SPE and it gives the lowest utility to the agents. This property is important and is used in later sections.

**Lemma 2.1.** The autarky allocation is always an SPE allocation and provides lower consumption to every old agent and lower conditional expected life-time utility (according to \( \beta \)) to every young agent in every history than any other SPE allocation provides.
Proof. See the Appendix.

Lemma 2.1 guarantees that the autarky-reversion trigger strategy is the best strategy when we consider SPE, like Abreu (1988), Kocherlakota (1996) and Thomas and Worrall (1988).

By using Lemma 2.1, the following result regarding the SPE allocation holds.

**Proposition 2.1.** An allocation \( c \) is an SPE allocation if and only if for all \( t \) and all \( s' \in S' \),

\[
e^o_t(s'_t) \geq e^o(s_t) \tag{6}
\]

and

\[
U(c_t, s'_t) \geq u(e^s_t) + \beta \sum_{s \in S} \pi(s) u(e^o(s)). \tag{7}
\]

Proof. See the Appendix.

Equation (6) implies that the old agent prefers an allocation \( c^o_t(s'_t) \) to her own endowment, and equation (7) implies that a young agent at date \( t \) prefers consuming her allocation \( c_t \) to consuming her endowment. If one of them is violated, an allocation \( c \) cannot be an SPE allocation. Hence, two equations are a necessary condition for \( c \) being an SPE allocation. For sufficiency, if both (6) and (7) hold, we can construct the autarky-reversion trigger strategy whose outcome is \( c \). By Lemma 2.1, the autarky allocation itself is an SPE allocation, hence, the autarky-reversion trigger strategy is an SPE.

3 Golden-Rule Type Allocation and SPE

First, to make the analysis interesting, I impose one assumption on the model hereafter.

**Assumption 3.1.** The autarkic allocation is not interim Pareto efficient.

If the autarkic allocation is interim Pareto efficient, a transfer between generations is meaningless and makes welfare worse off. Based on Aiyagari and Peled (1991) and Chattopadhyay and Gottardi (1999), this assumption is translated into the following equation:

\[
\frac{\beta \pi(1) u'(e^o(1))}{u'(e^y(1))} + \frac{\beta \pi(2) u'(e^o(2))}{u'(e^y(2))} > 1. \tag{8}
\]

One interest in a two-sided limited commitment model such as Thomas and Worrall (1988) and Kocherlakota (1996) concerns how to solve the conflict between risk sharing and incentives. To achieve efficient risk sharing, the long-run relationship is useful, while in each period each agent has an incentive to break the relationship. As shown in Thomas and Worrall (1988) and Kocherlakota (1996), when the discount factor is high enough, depending on the initial shock and the division of the surplus, an efficient risk-sharing allocation, or first-best risk-sharing allocation, is subgame perfect.
In an overlapping generations model, one difficulty with the analysis is that it is not easy to judge whether an allocation is interim Pareto efficient or not. Chattopadhyay and Gottardi (1999) provide a complete characterization of interim Pareto efficient allocations, although it is not convenient to use their condition in practice. Following studies such as Gottardi and Kubler (2011), I focus on a stationary allocation hereafter.

**Definition 3.1.** An allocation \( c \) is stationary if there is a mapping \( \tilde{c} := (\tilde{c}^y, \tilde{c}^o) \), where \( \tilde{c}^y(s) \in [0, e^y(s)] \) and \( \tilde{c}^o(s, s') \in [0, e^o(s')] \), such that for all \( t \) and all \( s \) \( t \in S 

In words, if an allocation is stationary, the young agent’s allocation just depends on the current shock and the old agent’s allocation depends on shocks while she is alive. Specifically, in this model \( c^y_t(s_t) = \tilde{c}^y(s_t) \) implies that a stationary allocation for an old agent also depends only on today’s shock because \( c^o_t(s_t) = e(s_t) - c^y_t(s_t) \). Hence hereafter, a stationary allocation \( c \) is written as \( c = (c^y, c^o) \), where \( c^y : S \rightarrow \mathbb{R}_+ \) and \( c^o : S \rightarrow \mathbb{R}_+ \) with \( e^y(s) + e^o(s) = e(s) \). Here is a necessary and sufficient condition for a stationary allocation \( c \) to be an SPE allocation.

**Corollary 3.1.** A stationary allocation \( c \) is an SPE allocation if and only if

\[
\beta \sum_{s \in S} \pi(\tilde{s}) [u(c^o(\tilde{s})) - u(e^o(\tilde{s}))] \geq \max_{s \in S} \{u(e^y(s)) - u(c^y(s))\}. \tag{9}
\]

**Proof.** By Proposition 2.1. \( Q.E.D. \)

By Chattopadhyay and Gottardi (1999), a stationary allocation \( c \) is interim Pareto efficient if and only if

\[
\frac{\beta \pi(1)u'(c^o(1))}{u'(c^y(1))} + \frac{\beta \pi(2)u'(c^o(2))}{u'(c^y(2))} \leq 1. \tag{10}
\]

Another thing worth mentioning is that an overlapping generations model does not have a corresponding notion to the first-best risk-sharing allocation in an infinitely-lived agents model. Since a young agent is born after a current state is realized, the first-best risk-sharing allocation is not achievable. However, this paper considers the following golden-rule type allocation instead of the first-best risk-sharing allocation.

**Definition 3.2.** A stationary allocation \( c \) is a golden-rule type allocation if \( c \) maximizes

\[
\{u(c^y(1)) + \beta \mathbb{E}[u(c^o(s))]) \} + \lambda \{u(c^y(2)) + \beta \mathbb{E}[u(c^o(s))]) \} \tag{11}
\]

for some \( \lambda \in (0, +\infty) \).

In words, if a stationary allocation is a golden-rule type allocation, no young agents are made better off by some stationary allocation. Notice that “young agents” here means young agents who have a
different history of shocks. Since $u$ is strictly concave, the necessary and sufficient conditions for $c$ being a solution to (11) are

$$\frac{\beta \pi(1) u'(c^o(1))}{u'(c^o(1))} = \frac{1}{1 + \lambda}$$

and

$$\frac{\beta \pi(2) u'(c^o(2))}{u'(c^o(2))} = \frac{\lambda}{1 + \lambda}.$$  

Notice that such an allocation $c$ is interim Pareto efficient, because

$$\frac{\beta \pi(1) u'(c^o(1))}{u'(c^o(1))} + \frac{\beta \pi(2) u'(c^o(2))}{u'(c^o(2))} = 1.$$  

Hereafter, I assume $\beta = 1$. This assumption is just so that the results can look clear, and even without it, the following results hold by adjusting $\pi(1)$’s value. Let

$$\pi(1) := \frac{1 - u'(c^o(2))}{u'(c^o(2))} - \frac{u'(c^o(1))}{u'(c^o(1))}.$$  

Then, (8) is rewritten as

$$\pi(1) > \pi(1).$$  

Here is the main theorem:

**Theorem 3.1.** Under Assumption 3.1, there exists a golden-rule type allocation that is subgame perfect.

To prove this theorem, let us focus on different values of $\pi(1)$.

### 3.1 Only a rich young agent transfers

First, consider a case in which only a rich young agent, i.e., a young agent with state 1, transfers some of her endowment to an old agent.

**Lemma 3.1.** For each $\pi(1) \in (\pi(1), 1)$, there exists a unique golden-rule type allocation that is achieved by transfer from only a rich young agent, i.e., $(c^y, c^o) = ((\tilde{c}, e^o(2)), (e(1) - \tilde{c}, e^o(2)))$ for some $\tilde{c} \in (\frac{e(1)}{2}, e^o(1))$. Furthermore, a rich young agent’s consumption, $\tilde{c}$, is strictly decreasing in $\pi(1)$.

**Proof.** See the Appendix. \(Q.E.D.\)

**Proposition 3.1.** There exist $\pi_\ast(1) \in (\pi(1), 1)$ and $\pi^*(1) \in [\pi_\ast(1), 1)$ such that:
(i) for all $\pi(1) \in [\pi^*(1), 1]$, the golden-rule type allocation, $(e^y, e^o) = ((\tilde{c}, e^y(2)), (e(1) - \tilde{c}, e^o(2)))$, is subgame perfect;

(ii) for all $\pi(1) \in (\underline{\pi}(1), \pi^*(1))$, it is not subgame perfect.

Proof. See the Appendix. Q.E.D.

This proposition states that for sufficiently large $\pi(1)$, a unique golden-rule type allocation that is achieved by transfer only from a rich young agent is subgame perfect, while for small $\pi(1)$, it is not. For instance, let us consider the case in which $\pi(1) \approx 1$. This case is regarded as an approximation of the deterministic overlapping generations model without discounting. In that case, consuming the same amount across two dates is the golden-rule type allocation. Since the agents do not discount the future utility, such an allocation is subgame perfect. The intuition for Proposition 3.1 is the same as in this deterministic environment.

Proposition 3.1 does not guarantee the unique cutoff of $\pi(1)$ above which the golden-rule type allocation is subgame perfect and below which it is not subgame perfect. Intuitively, the cutoff would be unique, however, it is not so obvious to prove it analytically. The golden-rule type allocation is subgame perfect if and only if

$$\pi(1) \geq D(\tilde{c}) := \frac{u(e^y(1)) - u(\tilde{c})}{u(e(1) - \tilde{c}) - u(e^o(1))},$$

where $\tilde{c}$ is the young agent’s consumption in state 1. For the golden-rule type allocation,

$$\pi(1) = 1 - \frac{\frac{u'(e^y(2))}{u'(e(2))}}{\frac{u'(e(1) - \tilde{c})}{u'(\tilde{c})} - \frac{u'(e^o(2))}{u'(e^o(2))}}.$$

Note that both $\pi(1)$ and $D$ are strictly decreasing in $\tilde{c}$. Whether there is a unique cutoff $\pi(1) \in (\underline{\pi}(1), 1)$ depends on how fast $\pi(1)$ and $D$ decreases as $\tilde{c}$ increases. To measure the speed of them, it is necessary to know the second order of them. Although the condition for the unique cutoff can be derived, it is not intuitive and instructive. Therefore, I omit the condition here.

### 3.2 Both a poor young agent and a rich young agent transfer

When $\pi(1)$ is not large enough, the golden-rule type allocation that is achieved by transfer from only a rich young agent is not subgame perfect as we saw in Proposition 3.1. Therefore, the only possibilities of the existence of a subgame-perfect golden-rule type allocation are (i) only a poor young agent transfers; (ii) both a rich young agent and a poor young agent transfer. Any allocation in the first case is not subgame perfect.

**Proposition 3.2.** Any stationary allocation that is achieved by transfer from only a poor young agent is not subgame perfect.

Proof. See the Appendix. Q.E.D.
Since in state 1 the young agent does not transfer, the incentive condition for the young agent matters only when the shock is 2. When the shock is 2, the young agent is poorer than the old agent. Since agents are risk averse, any transfer payment when an agent is poor and transfer receipt when an agent is rich hurts the lifetime payoff of the young agent. Hence, such an allocation is not subgame perfect.

**Corollary 3.2.** Any golden-rule type allocation that is achieved by transfer only from a poor young agent is not subgame perfect.

*Proof.* By Proposition 3.2. \[Q.E.D.\]

Thus, the only possibility is the case in which the young agent transfers in both \(s \in S\).

**Lemma 3.2.** For each \(\pi(1) \in (\pi(1), 1)\), there infinitely exist many golden-rule type allocations achieved by transfers from both a rich young agent and a poor young agent, \((c^y, c^o) = ((c(1), c(2)), (e(1) - c(1), e(2) - c(2)))\). Moreover, \(c(1)\) is strictly decreasing in \(c(2)\).

*Proof.* See the Appendix. \[Q.E.D.\]

For each \(\pi(1) \in (\pi(1), 1)\), there infinitely exist many golden-rule type allocations. However, not all of them are subgame perfect, while some are.

**Proposition 3.3.** For every \(\pi(1) \in (\pi(1), \pi_\gamma(1))\), there exist golden-rule type allocations that are subgame perfect.

*Proof.* See the Appendix. \[Q.E.D.\]

When \(\pi(1) < \pi_\gamma(1)\), the golden-rule type allocation achieved by transfer only from a rich young agent is not subgame perfect. This implies that the incentive constraint when the shock is 1, i.e.,

\[
\sum_{s=1}^{2} \pi(s) [u(e(s) - c(s)) - u(e^o(s))] \geq u(e^y(1)) - u(c(1)) \tag{15}
\]

is violated. This condition can be satisfied as \(c(1)\) decreases and \(c(2)\) increases. On the other hand, the incentive constraint when the shock is 2, i.e.,

\[
\sum_{s=1}^{2} \pi(s) [u(e(s) - c(s)) - u(e^o(s))] \geq u(e^\delta(2)) - u(c(2)) \tag{16}
\]

is not binding at all for the golden-rule type allocation achieved by transfer only from a rich young agent. Therefore, it is possible to make \(c(1)\) decrease and \(c(2)\) increase and satisfy the golden-rule type allocation condition \([14]\).

The formal proof in the Appendix uses the following lemmas. Since \(u\) is strictly increasing, given \(c(1) \in [0, e^\delta(1)]\), a unique \(c(2)\) is determined that satisfies \([15]\) with equality. Let \(F : [0, e^\delta(1)] \rightarrow \mathbb{R}\) be a function that determines \(c(2)\) given \(c(1)\) that satisfies \([15]\) with equality.
Lemma 3.3. F is twice-continuously differentiable and strictly concave in \( c(1) \). In addition, \( e^{y}(2) = F(e^{y}(1)) \) and \( F(0) < 0 \).

Proof. See the Appendix. \( Q.E.D. \)

Since \( u \) is strictly increasing, given \( c(2) \in [0, e^{y}(2)] \), a unique \( c(1) \) is determined that satisfies (16) with equality. Let \( G : [0, e^{y}(2)] \to \mathbb{R} \) be a function that determines \( c(1) \) given \( c(2) \) that satisfies (16) with equality.

Lemma 3.4. G is twice-continuously differentiable, strictly increasing and strictly concave. Moreover, \( e^{y}(1) = G(e^{y}(2)) \) and \( G(0) < 0 \).

Proof. See the Appendix. \( Q.E.D. \)

Lemmas 3.3 and 3.4 describe allocations that satisfy (15) and (16) with equality. Lemmas 3.3 and 3.4 imply that there is a space between \( F \) and \( G \).

Lemma 3.5. For any \( \pi(1) \in (\pi(1), \pi(\ast)(1)) \), there exists a unique \( (c^{(1)}, c^{(2)}) \in (0, e^{y}(1)) \times (0, e^{y}(2)) \) such that \( c^{(2)} = F(c^{(1)}) \) and \( c^{(1)} = G(c^{(2)}) \).

Proof. From previous lemmas. \( Q.E.D. \)

The stationary allocation defined in Lemma 3.5 is interim Pareto efficient, and at that point,

\[
\sum_{s \in S} \pi(s)\frac{u'(c(s) - c(s))}{u'(c(s))} < 1.
\]

Since the value of \( \sum_{s \in S} \pi(s)\frac{u'(c(s) - c(s))}{u'(c(s))} \) increases in \( c(s) \), it is possible to find the golden-rule type allocations that are subgame perfect by adjusting the value of \( c(s) \) for \( s \in S \). A numerical example in the next section will help us understand the results.

3.3 Numerical examples

In this subsection, I illustrate the two types of SPE with a numerical example. Let the utility function be

\[
u(c) = \frac{c^{1-\sigma}}{1-\sigma},
\]

where \( \sigma = 0.8 \). Set \( e^{y}(1) = e^{y}(2) = 0.5 \) and \( e^{y}(1) = 0.3 \) and \( e^{y}(2) = 0.7 \). Set \( \beta = 0.9 \). In the theoretical analysis, I set \( \beta = 1 \) for a better understanding. As previously mentioned, as long as \( \beta\pi(1) \) is high enough, all results so far hold. Thus, I set \( \beta = 0.9 \) instead of \( \beta = 1 \) in these numerical examples.

Figure 1 shows the case in which \( \pi(1) = 0.9 \). Figure 2 shows the case in which \( \pi(1) = 0.8 \). Figure 3 shows the case in which \( \pi(1) = 0.7 \). In all figures, the horizontal axis depicts an old agent’s consumption when the shock is 1, and the vertical axis represents for the old agent’s consumption when the shock is 2.
In all figures, a blue solid, a red dotted and a green dashed line express the same things. All points on the solid line satisfy (15) with equality and all points above it satisfy (15) with strict inequality. All points on the dotted line satisfy (16) with equality and all points below it satisfy (16) with strict inequality. All points on the dashed line express the golden-rule type allocation, and all points above it satisfy $\beta \pi(1) u'(c^{o}(1)) + \beta \pi(2) u'(c^{o}(2)) < 1$, which implies interim Pareto efficient allocations. Therefore, points surrounded by the solid line, the dotted line and the dashed line are interim Pareto efficient, subgame perfect, stationary allocation.

When $\beta \pi(1)$ is high enough, there is an interim Pareto efficient, subgame perfect allocation in which a transfer is made only if a shock is 1, i.e., a young agent has a larger endowment than an old agent. A unique golden-rule type allocation achieved by transfer only from a rich young agent is subgame perfect. This is shown by points on the horizontal axis between the solid line and the dashed line in Figure 1.

Once $\beta \pi(1)$ is getting smaller, things are changing. When $\beta \pi(1)$ is a medium value, there is no interim Pareto efficient, subgame perfect, stationary allocation in which a young agent who has a larger endowment only transfers some to an old agent, as Figure 2 shows. Although there is a stationary, subgame perfect allocation in which only a richer young agent transfers some of her endowment to a poor old agent, those allocations are not interim Pareto efficient. Still, there are interim Pareto efficient, subgame perfect allocations in this case.

When $\beta \pi(1)$ becomes much smaller, stationary allocations in which only a rich young agent transfers some of her endowment to a poor old agent are not subgame perfect, as Figure 3 shows. As in Figure 2, there are interim Pareto efficient, subgame perfect allocations even in this case.

From these three figures, we know there are always golden-rule type allocations that satisfy the limited commitment constraint.

4 Discussions

4.1 More than two shocks

In the main body, two shocks are considered. One reason why I focus on the two-shocks case is that even if the number of shocks increases, I think the model becomes more complicated without giving more insights.

Let $S := \{1, 2, \ldots, \mathcal{S}\}$ be a set of all shocks, where $\mathcal{S} > 2$. As in the model, assume $e^y(s) = e^y(s')$ for all $s \neq s' \in S$. Assume also that there exists at least one $s$ and $s'$ such that $e^y(s) > e^o(s)$ and $e^y(s') < e^o(s')$. It is clear that Lemma 2.1 and Proposition 2.1 hold. A stationary allocation $c$ is an SPE allocation if and only if

$$\beta \sum_{\hat{s} \in S} \pi(\hat{s}) [u(c^{o}(\hat{s})) - u(c^{o}(\hat{s}))] \geq u(c^{y}(s)) - u(c^{y}(s))$$  \hspace{1cm} (17)

for all $s \in S$. A stationary allocation $c$ is interim Pareto efficient if and only if

$$\sum_{s \in S} \frac{\beta \pi(s) u'(c^{o}(s))}{u'(c^{y}(s))} \leq 1.$$  \hspace{1cm} (18)
Figure 1: $\pi(1) = 0.9$
Figure 2: $\pi(1) = 0.8$
Figure 3: $\pi(1) = 0.7$
Assume

\[ \sum_{s \in S} \beta \pi(s) u'(e^s(s)) - u'(e^{\hat{s}}(s)) > 1. \] (19)

Even for a general number of shocks, at least similar results can be derived by focusing on only two shocks, \(s\) and \(s'\), \(s \neq s'\), where \(e^s(s) > e^s(\hat{s})\) and \(e^{s'}(s') < e^{s'}(s')\), and considering transfers made only when a shock is either \(s\) or \(s'\). This is in a sense an “imitation” of the main model, and under some parameter values similar results can be found. The question is what more we can find.

One interesting question would be whether there exist parameter values under which a young agent transfers some of her endowment to an old agent no matter which shock is realized and the resulting parameter values similar results can be found. The question is what more we can find.

In discussing interim Pareto efficiency, I exploit the properties \(F\) to be interim Pareto efficient is as follows: Let

\[ Q(c) := \begin{bmatrix} \frac{\beta \pi(1) u'(c^e(1))}{u'(c^e(1))} & \frac{\beta \pi(2) u'(c^e(2))}{u'(c^e(2))} \\ \frac{\beta \pi(1) u'(c^o(1))}{u'(c^o(1))} & \frac{\beta \pi(2) u'(c^o(2))}{u'(c^o(2))} \end{bmatrix} . \]

Throughout the paper, I assume that a stochastic process \(\{s_t\}\) is i.i.d. across time. Here, I assume that \(\{s_t\}\) follows a first-order Markov process, instead of an i.i.d. process. For any \(t \geq 1\), the probability of \(s' \in S_t\) is denoted by \(\pi(s') > 0\). Since the stochastic process is a first-order Markov process, \(\pi(s_t+1|s_t) = \pi(s_{t+1}|s_t) > 0\) for all \(s_{t+1} \in S\) holds.

In this setting, a necessary and sufficient condition for a stationary allocation \(c\) being subgame perfect is

\[ \forall s \in S, \quad \beta \sum_{s' \in S} \pi(s'|s) [u(c^o(s')) - u(c^o(s))] \geq u(c^o(s)) - u(c^o(s')) \]

holds. However, welfare analysis might be different.

A necessary and sufficient condition for a stationary allocation \(c\) to be interim Pareto efficient is as follows: Let
In a deterministic pure-exchange overlapping generations model, Wallace (1980) discusses the monetary
An overlapping generations model and fiat money have a deep relationship as studied in Wallace (1980).

4.3.1 Monetary equilibrium

In this section, I compare three different approaches that are common in the analysis of an overlapping
generations model: one is reputation, which is this paper’s approach; one is money; the last is political
economy.

4.3 Reputation, Money and Political Economy

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4.3.1 Monetary equilibrium

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In a deterministic pure-exchange overlapping generations model, Wallace (1980) discusses the monetary

The dominant root of this matrix, denoted by \( \eta(Q(c)) \), is the largest absolute value of the solution to

\[
\eta^2 - \left( \frac{\beta \pi(1|1)u'(c^o(1))}{u'(c^o(1))} + \frac{\beta \pi(2|2)u'(c^o(2))}{u'(c^o(2))} \right) \eta \\
+ [\pi(1|1)\pi(2|2) - \pi(2|1)\pi(1|2)] \beta^2 \frac{u'(c^o(1))}{u'(c^o(1))} \frac{u'(c^o(2))}{u'(c^o(2))} = 0
\]

with respect to \( \eta \). Let \( R(\eta; c) \) be the LHS of this equation. Note that the equation has two real number solutions, because

\[
\left( \frac{\beta \pi(1|1)u'(c^o(1))}{u'(c^o(1))} + \frac{\beta \pi(2|2)u'(c^o(2))}{u'(c^o(2))} \right)^2 \\
- 4[\pi(1|1)\pi(2|2) - \pi(2|1)\pi(1|2)] \beta^2 \frac{u'(c^o(1))}{u'(c^o(1))} \frac{u'(c^o(2))}{u'(c^o(2))} = 0
\]

If \( \pi(1|1) + \pi(2|2) = 1 \), then the similar necessary and sufficient condition will be obtained. When
\( \pi(1|1) + \pi(2|2) \neq 1 \), the following conditions characterize an interim Pareto efficient allocation.

**Proposition 4.1.** Suppose \( \pi(1|1) + \pi(2|2) \neq 1 \). Then, a stationary allocation \( c \) is interim Pareto efficient
if and only if

\[
1 \left( \frac{\beta \pi(1|1)u'(c^o(1))}{u'(c^o(1))} + \frac{\beta \pi(2|2)u'(c^o(2))}{u'(c^o(2))} \right) < 1
\]

and

\[
R(1; c) \geq 0.
\]

4.3 Reputation, Money and Political Economy

In this section, I compare three different approaches that are common in the analysis of an overlapping
generations model: one is reputation, which is this paper’s approach; one is money; the last is political
economy.

4.3.1 Monetary equilibrium

An overlapping generations model and fiat money have a deep relationship as studied in Wallace (1980).
In a deterministic pure-exchange overlapping generations model, Wallace (1980) discusses the monetary
equilibrium, and characterizes a Pareto efficient, monetary equilibrium. As for the stochastic environment, [Manuelli (1990)] and [Magill and Quinzii (2003)] discuss the same issue in a stochastic environment.

The economic environment is the same as this paper’s, except for fiat money. Let $M$ be the supply of fiat money, and it is assumed to be constant for all $t$ and all histories of shocks. A perfect foresight competitive equilibrium is defined in a standard way, and a monetary equilibrium is a perfect foresight competitive equilibrium in which the value of fiat money is strictly positive for all histories of shocks.

**Proposition 4.2.** Suppose Assumption 3.1 holds. Then, there exists a unique stationary monetary equilibrium, and its allocation is a golden-rule type allocation.

**Proof.** By Proposition 1 in [Magill and Quinzii (2003)].

From this proposition, if the money supply is independent of the shock, then an allocation achieved by transfer only from a rich young agent (an allocation discussed in Section 3.1 of this paper) is not a monetary equilibrium, because money market clearing implies that money is transferred from old to young in both states and the prices of money in both states are strictly positive.$^{10}$ It is clear that more golden-rule type allocations are supported by an equilibrium through “reputation” rather than through a constant supply of fiat money.

### 4.3.2 Political equilibrium

It is also worth mentioning the relationship between this research and the political economy approach to social security. [Boldrin and Rustichini (2000)] consider a deterministic overlapping generations model with production, and in every period the young generation decides whether or not to pay the transfers to the old generation and the setting of the tax rate that will be paid in the next period. They define a political equilibrium, in which given a sequence of taxes, the allocation constitutes a competitive equilibrium and the young generation’s behavior is optimal in the sense of subgame perfection, and characterize the equilibria. In a stochastic environment, [Demange (2009)] has a similar motivation to [Boldrin and Rustichini (2000)]. Since it is hard to trace general allocations in a stochastic environment, she focuses on a stationary allocation as in this paper.

Applying Demange’s setting to this paper’s environment, given a sequence of tax rates, $\tau = (\tau(s))_{s \in S}$ by the government (or social planner), in each period after the current shock is realized, the young generation decides whether to approve or not and if not, it chooses how much to change the tax rate by choosing the scale, $\lambda$. For instance, if the young generation prefers half of the given tax rate, $(\frac{1}{2} \tau(s))_{s \in S}$, to the current one, $\tau$, it does not approve the current tax rate and chooses half of the tax rate from the period on. Formally, for each $s \in S$, given $\tau = (\tau(s))_{s \in S}$, let

$$V(\tau, \lambda, s) = u(e^y(s) - \lambda \tau(s)) + \sum_{s' \in S} \pi(s') u(e^{o}(s') + \lambda \tau(s')).$$

$^{10}$If the money supply changes according to the shock, such an allocation can be a monetary equilibrium allocation, as shown by [Manuelli (1990)].
A contribution rule $\tau$ is sustainable if the decisive voter agrees on the scale level in the sense that $V(\tau, \lambda, s)$ is maximized at $\lambda = 1$ for each $s \in S$. (See Definition 1 in Demange (2009).)

**Proposition 4.3.** There is a sustainable rule if and only if Assumption 3.1 holds.

**Proof.** By Theorem 1 in Demange (2009). Q.E.D.

From the definition of sustainability, when $\tau$ is sustainable, the agent’s expected life-time utility should be greater than or equal to the autarkic level. If not, the decisive voter would choose $\lambda = 0$. This implies that the political equilibrium allocation satisfies the limited commitment constraint. However, the set of sustainable allocations is not as large as the set of SPE allocations. Consider an allocation in which a transfer is made only when a young agent is richer than an old agent, that is, there is no transfer from a poor young agent to a rich old agent. The contribution that achieves this allocation is not sustainable, because when the current state is 2, the decisive voter prefers a larger $\lambda > 1$. This occurs because, given that contribution rule, in state 2 the young agent does not need to pay and she will receive a positive transfer when she becomes old with some probability, and it increases the young agent’s expected lifetime utility. This is why it is not sustainable. From this argument, the set of efficient SPE allocations is at least as large as the set of sustainable allocations.

## 5 Conclusion

This paper considers an overlapping generations model with aggregate endowment shocks and a limited commitment constraint. I characterize a stationary golden-rule type allocation that satisfies a limited commitment constraint to clarify the difference from the contemporaneous, infinite-lived agents model. The implication that is common to both types of model is that history-dependent behavior improves welfare. One difference is that the golden-rule type allocation is an SPE allocation as long as the autarkic allocation is not interim Pareto efficient, even though the golden-rule type allocation is not exactly the “first-best” allocation. Another difference is that, even when a young agent is poorer than an old agent, a transfer from that young agent to an old agent is necessary for the golden-rule type allocation.

One implication of this paper is that the limited commitment constraint could be one justification as to why generational accounting data show only the positive net transfer from the young to the old and not vice versa. The reason given here is different from others such that the society cannot force people to live poorly, or that the transfer from the young to the old can improve the welfare. Another implication is that even though a young agent is facing hard times (in state 2), an appropriate intergenerational transfer improves welfare. This finding could be a justification for forcing young people to pay the social security tax even in hard times for them.

An interesting area for further research would be to characterize efficient, self-enforcing allocations completely. This paper characterizes them partially in the sense that the paper focuses on stationary

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1. Feldstein and Liebman (2002) is a good survey paper on social security. For generational accounting, see, for example, Auerbach et al. (1993) and Auerbach et al. (1999).
allocations. The beauty of contemporaneous, infinite-lived agents models such as Thomas and Worrall (1988), Kocherlakota (1996) and so forth is that they can characterize efficient, self-enforcing allocations fully. One advantage of the infinite-lived agents model is the ease in formulating the problem for the optimality. Because of this, they use dynamic programming to characterize the constrained efficient allocations. The overlapping generations model, however, cannot use a similar approach and it makes the characterization harder. It would be an interesting future research agenda to characterize constrained efficient allocations completely in an overlapping generations model.
A Appendix

A.1 Proof of Lemma 2.1

Proof. Consider a strategy profile, $\sigma$, in which no agent transfers anything to the other agent regardless of what has happened in the past. Then, $\sigma$ is an SPE, because the only deviation for any agent is to transfer a positive amount to the other agent and it decreases the agent’s utility.

Suppose there exists another SPE, $\sigma'$, that gives lower expected life-time utility to some agent at some date $t \geq 1$, some history of transfers, $h^{t-1}$, and some history of shocks, $s'$ than $\sigma$. In any SPE, no old agent transfers anything to the young agent after each history. Hence, the old agent’s consumption is her own endowment and the transfer from the young agent. Since, in autarky, no young agent transfers anything to the old agent, the autarky allocation provides the lowest consumption among the SPE. That is why I can focus on the conditional expected life-time utility of the young agent. By the supposition above,

$$u(e^y(s_t)) + \beta \sum_{s \in S} \pi(s) u(e^o(s)) > U(c_t | s', h^{t-1}, \sigma'),$$

(22)

where $U(c_t | s', h^{t-1}, \sigma')$ is the expected life-time utility under a strategy profile $\sigma'$ and a history of transfers $h^{t-1}$.

Suppose that the generation-$t$ agent deviates from $\sigma'$ when she is young by not transferring anything to the old agent when she is young and by not transferring anything to the young agent when she is old. Then, the conditional expected life-time utility of that young agent from this deviation is

$$u(e^y(s_t)) + \beta E[u(c_{t+1}^o) | \sigma'_{-t}].$$

Since this agent does not transfer anything to the young agent when she is old,

$$u(e^y(s_t)) + \beta E[u(c_{t+1}^o) | \sigma'_{-t}] \geq u(e^y(s_t)) + \beta \sum_{s \in S} \pi(s) u(e^o(s))$$

holds. By combining this equation with (22),

$$u(e^y(s_t)) + \beta E[u(c_{t+1}^o) | \sigma'_{-t}] > U(c_t | s', h^{t-1}, \sigma')$$

holds. This contradicts the fact that $\sigma'$ is an SPE, because the generation-$t$ agent has an incentive to deviate from $\sigma'_t$ when she is young.

Q.E.D.

A.2 Proof of Proposition 2.1

Proof. For necessity, suppose that for some $t$ and some $s' \in S'$, either (6) or (7) is violated. When (6) does not hold, an old agent strictly prefers doing nothing. Then, $c$ is not subgame perfect. When (7) does not hold, a young agent prefers the autarkie allocation to $c_t$. Hence, $c$ cannot be an SPE allocation.
For sufficiency, suppose that for all \( t \) and \( s' \in S' \), both \((6)\) and \((7)\) hold. Let \( \gamma_t(s') := \epsilon_t(s') - e^o(s_t) \) for all \( s' \in S' \). Notice that \( \epsilon_t'(s') = e^o(s_t) + \gamma_t(s') \). Consider the following strategy, \( \hat{\sigma}_t \); for all \( t \geq 1 \), all \( s' \in S' \) and all \( h^{-1} \in H^{-1} \),

\[
\hat{\sigma}_t^y(h^{-1}, s') = \begin{cases} 
\gamma_t(s') & \text{if no agents before date } t \text{ deviate from } \hat{\sigma}_t \text{ for } \tau < t \\
0 & \text{otherwise,}
\end{cases}
\]

and

\[
\hat{\sigma}_t^{o_1}(h', s_t, s_{t+1}) = 0
\]

for all \( h' \in H' \) and all \( s_{t+1} \in S \). For the initial old agent,

\[
\hat{\sigma}_1^o(s_1) = 0
\]

for all \( s_1 \in S \). This strategy profile, \( \hat{\sigma} \), is an SPE when \((6)\) and \((7)\) hold. The outcome of \( \hat{\sigma} \) is \( c \). Therefore, \((6)\) and \((7)\) are sufficient conditions for \( c \) being subgame perfect.

**Q.E.D.**

### A.3 Proof of Lemma 3.1

**Proof.** For any \( \pi(1) \in (\pi(1), 1) \), the allocation in the statement satisfies

\[
\pi(1) \frac{u'(e^o(1))}{u'(e^o(1))} + \pi(2) \frac{u'(e^o(2))}{u'(e^o(2))} = 1. \tag{23}
\]

Since \( e^o(2) > e^o(2) \) and \( u \) is strictly concave, \( u'(e^o(2))/u'(e^o(2)) < 1 \). Since \( e^o(1) + e^o(1) = e(1) \), \( u'(e^o(1))/u'(e^o(1)) \) is strictly decreasing in \( e^o(1) \), and since \( u \) is continuously differentiable, it is continuous in \( e^o(1) \). When \( (e^o(1), e^o(1)) = (e^o(1), e^o(1)) \), by Assumption 3.1, the value of the LHS of \((23)\) is greater than 1. When \( (e^o(1), e^o(1)) = (e^o(1), e^o(1)) \), since \( u'(e^o(1))/u'(e^o(1)) = 1 \), the LHS of \((23)\) is less than 1. By the intermediate value theorem, there exists a unique \( \hat{c} \in (e^o(1), e^o(1)) \) such that \((e^o, e^o) = ((\hat{c}, e(1) - \hat{c}), (e^o(2), e^o(2))) \) is a golden-rule type allocation that is achieved by transfer from only a rich young agent.

From \((23)\),

\[
\pi(1) = \frac{1 - \frac{u'(e^o(1))}{u'(e^o(2))}}{\frac{u'(e^o(1) - \hat{c})}{u'(\hat{c})} - \frac{u'(e^o(2))}{u'(e^o(2))}}.
\]

Notice that \( u'(e^o(1) - \hat{c})/u'(\hat{c}) \) is strictly increasing in \( \hat{c} \). Thus, if \( \pi(1) \) increases, then \( \hat{c} \) should decrease.

**Q.E.D.**
A.4 Proof of Proposition 3.1

Proof. The golden-rule type allocation is subgame perfect if and only if

\[ \pi(1) \geq \frac{u(e^y(1)) - u(\hat{c})}{u(e(1) - \hat{c}) - u(e^y(1))}. \]  

(24)

Let

\[ D(\hat{c}) := \frac{u(e^y(1)) - u(\hat{c})}{u(e(1) - \hat{c}) - u(e^y(1))}. \]

Since \( u \) is differentiable, \( D \) is also differentiable on \((0, e^y(1))\). The derivative of \( D \) is

\[
D'(\hat{c}) = \frac{-u'(\hat{c})}{u(e(1) - \hat{c}) - u(e^y(1))} + \frac{u(e^y(1)) - u(\hat{c})}{[u(e(1) - \hat{c}) - u(e^y(1))]^2} u'(e(1) - \hat{c})
\]

\[
= \left\{ -u'(\hat{c})[u(e(1) - \hat{c}) - u(e^y(1))] + u'(e(1) - \hat{c})[u(e^y(1)) - u(\hat{c})] \right\} / [u(e(1) - \hat{c}) - u(e^y(1))]^2.
\]

Since the denominator is positive and the numerator is negative because of the strict concavity of \( u \), \( D \) is strictly decreasing in \( \hat{c} \).

In addition, let

\[ \hat{D}(\hat{c}) := \frac{1 - u'(e^y(1))}{u'(e^y(1))}. \]

Then,

\[
\hat{D}'(\hat{c}) = \frac{1 - u'(e^y(1))}{u'(e^y(1))} \left\{ \frac{u''(e(1) - \hat{c})}{u'(e(1) - \hat{c})} + \frac{u'(e(1) - \hat{c})}{u'(e^y(1))} - u''(\hat{c}) \right\} < 0.
\]

Thus, \( \hat{D}(\hat{c}) \) is strictly decreasing in \( \hat{c} \).

Notice that

\[ D(\frac{e^y(1)}{2}) = \frac{u(e^y(1)) - u(e(1)/2)}{u(e(1)/2) - u(e^y(1))} < 1 \]

and

\[ D(e^y(1)) = \lim_{c \to e^y(1)} \frac{u'(c)}{u'(e(1) - c)} = \frac{u'(e^y(1))}{u'(e^y(1))} < 1. \]

Moreover, at \( \hat{c} = e(1)/2 \),

\[ \hat{D}(\frac{e^y(1)}{2}) = 1. \]
At \( \hat{c} = e^\nu(1) \),
\[
D(e^\nu(1)) - \hat{D}(e^\nu(1)) = \frac{1}{u'(e^\nu(1))} \left\{ \frac{u'(e^\nu(1))}{u'(e^\nu(2))} \left( 1 - \frac{u'(e^\nu(1))}{u'(e^\nu(2))} \right) \right\} > 0.
\]

Since \( \hat{D}(e(1)/2) > D(e(1)/2), \hat{D}(e^\nu(1)) < D(e^\nu(1)) \) and both \( D \) and \( \hat{D} \) are strictly decreasing, there exist \( \pi_* \in (\pi(1), 1) \) and \( \pi^* \in [\pi_*(1), 1] \) such that for all \( \pi(1) \in [\pi^*(1), 1] \), the golden-rule type allocation is subgame perfect and for all \( \pi(1) \in (\pi(1), \pi_*(1)) \), it is not subgame perfect. \( Q.E.D. \)

### A.5 Proof of Proposition 3.2

**Proof.** Consider an incentive condition when the current state is 2:
\[
u(c^\nu(2)) + \sum_{s=1}^{2} \pi(s)u(c^o(s)) \geq u(c^\nu(2)) + \sum_{s=1}^{2} \pi(s)u(c^o(s)).
\]
When the allocation is achieved by transfer from only a poor young agent, the above condition becomes
\[
u(c^\nu(2)) + \pi(2)u(c^o(2)) \geq u(c^\nu(2)) + \pi(2)u(c^o(2)).
\]
Since \( u \) is strictly concave, \( c^o(2) > e^o(2) > e^\nu(2) > c^\nu(2) \) and \( c^o(2) - e^o(2) = e^\nu(2) - c^\nu(2) \) in a stationary allocation,
\[
u(e^\nu(2)) - u(c^\nu(2)) > u(c^o(2)) - u(e^o(2)).
\]
Since \( \pi(2) \in (0, 1) \), this implies the incentive condition when \( s = 2 \) does not hold. Therefore, any stationary allocation that is achieved by transfer from only a poor young agent is not subgame perfect. \( Q.E.D. \)

### A.6 Proof of Lemma 3.2

**Proof.** golden-rule type allocations must satisfy
\[
\pi(1) = \frac{1 - \frac{u'(e(2) - c(2))}{u'(e(1) - c(1))}}{\frac{u'(e(2) - c(2))}{u'(e(1) - c(1))}}.
\]
Once \( \pi(1) \) is given, the relationship between \( c(1) \) and \( c(2) \) is determined. Since \( u \) is twice-continuously differentiable and strictly concave, once \( c(1) \) is given, \( c(2) \) is determined. Therefore, there are infinitely many pairs of \( (c(1), c(2)) \) that construct golden-rule type allocations. By the implicit function theorem,
\[
\frac{dc(1)}{dc(2)} = \frac{\frac{u'(e(2) - c(2)) u'(c(2)) + u'(e(2) - c(2)) u''(c(2))}{[u'(c(2))]^2} (1 - \pi(1))}{\frac{-u''(e(1) - c(1)) u'(c(1)) - u'(e(1) - c(1)) u''(c(1))}{[u'(c(1))]^2} \pi(1)} < 0.
\]
Therefore, \( c(1) \) is strictly decreasing in \( c(2) \). \( Q.E.D. \)

25
A.7 Proof of Lemma 3.3

Proof. Since $u$ is twice-continuously differentiable, so is $F$. By taking the derivative of $F$ with respect to $c(1)$,

$$\frac{dF}{dc(1)} = \frac{u'(c(1)) - \pi(1)u'(e(1) - c(1))}{\pi(2)u'(e(2) - c(2))}.$$  

The sign of $\frac{dF}{dc(1)}$ is determined by the sign of $u'(c(1)) - \pi(1)u'(e(1) - c(1))$. If $\pi(1) < \pi(1) \leq \frac{u'(e(1))}{u'(e(1))}$, $\frac{dF}{dc(1)} > 0$. When $\pi(1) > \pi(1) > \frac{u'(e(1))}{u'(e(1))}$, let $\tilde{c}(1) \in (0, e^v(1))$ be $c$ that satisfies $u'(c) - \pi(1)u'(e(1) - c) = 0$. In this case, $\frac{dF}{dc(1)} < 0$ if $c(1) > \tilde{c}(1)$, it is $0$ if $c(1) = \tilde{c}(1)$ and $\frac{dF}{dc(1)} > 0$ if $c(1) < \tilde{c}(1)$. Taking the second derivative of $F$,

$$\frac{d^2 F}{[dc(1)]^2} = \frac{u''(c(1)) + \pi(1)u''(e(1) - c(1))}{\pi(2)u'(e(2) - c(2))} + \frac{u'(c(1)) - \pi(1)u'(e(1) - c(1))}{[\pi(2)u'(e(2) - c(2))]^2} \cdot \frac{dF}{dc(1)}$$

$$< 0.$$  

In summary, when $\pi(1)$ is too low, $F$ is strictly increasing and strictly concave, while when $\pi(1)$ is not too low, $F$ is hump-shaped and strictly increasing when $c(1) < \tilde{c}(1)$ and strictly decreasing when $c(1) > \tilde{c}(1)$. Since $e^v(2) = F(e^v(1))$, when $\pi(1) > u'(e^v(1))/u'(e^v(1))$, there is a unique $c < e^v(1)$ such that $e^v(2) = F(c)$.

When $c(1) = 0$, $u(e(1)) - u(e^v(1)) < u(e^v(1)) - u(0)$ because $u$ is strictly concave. When $c(2)$ is also $0$, $u(e(2)) - u(e^v(2)) < u(e^v(1)) - u(0)$, because $u$ is strictly concave and $e^v(1) = e^v(2)$. Therefore, $(c(1), c(2)) = (0, 0)$ does not satisfy (15). When $c(1) = 0$, $F$ gives a negative number to satisfy (15) with equality. Hence, $F(0) < 0$.  

Q.E.D.

A.8 Proof of Lemma 3.4

Proof. Since $u$ is twice-continuously differentiable, $G$ is also differentiable. By the implicit function theorem,

$$\frac{dG}{dc(2)} = \frac{u'(c(2)) - \pi(2)u'(e(2) - c(2))}{\pi(1)u'(e(1) - c(1))} > 0,$$

since $c(2) \in [0, e^v(2)]$ and $e^v(2) < e^v(2)$ imply $u'(c(2)) > \pi(2)u'(e(2) - c(2))$. The second derivative of it is

$$\frac{d^2 G}{[dc(2)]^2} = \frac{u''(c(2)) + \pi(2)u''(e(2) - c(2)) \frac{dG}{dc(2)}}{\pi(1)u'(e(1) - c(1))} + \frac{u'(c(2)) - \pi(2)u'(e(2) - c(2))}{[\pi(1)u'(e(1) - c(1))]^2} \cdot \frac{dG}{dc(2)}.$$  

$$< 0.$$  

It is clear that $e^v(1) = G(e^v(2))$. When $c(2) = 0$, $u(e^v(2)) - u(0) > u(e(2)) - u(e^v(2))$ because $u$ is strictly concave. If $c(1) = 0$, $u(e(1)) - u(e^v(1)) < u(e^v(2)) - u(c(2))$ because $u$ is strictly concave and $e^v(1) = e^v(2)$. Thus, $(c(1), c(2)) = (0, 0)$ does not satisfy (16) with equality. Moreover, to satisfy (16) with equality, $c(1) < 0$. Therefore, $G(0) < 0$.  

Q.E.D.
A.9 Proof of Proposition 3.3

Proof. Let \((\bar{c}(1), \bar{c}(2))\) be a unique \((c(1), c(2))\) such that \(\bar{c}(2) = F(\bar{c}(1)) = G^{-1}(\bar{c}(1))\). Moreover, at that point,

\[
\frac{dF}{dc(1)}(\bar{c}(1)) > \frac{dG^{-1}}{dc(1)}(\bar{c}(1))
\]

must hold. This is equivalent to

\[
\frac{u'(\bar{c}(1)) - \pi(1)u'(e(1) - \bar{c}(1))}{\pi(2)u'(e(2) - \bar{c}(2))} > \frac{\pi(1)u'(e(1) - \bar{c}(1))}{u'(\bar{c}(2)) - \pi(2)u'(e(2) - \bar{c}(2))}.
\]

(25)

Here, suppose

\[
\frac{u'(\bar{c}(1)) - \pi(1)u'(e(1) - \bar{c}(1))}{\pi(2)u'(e(2) - \bar{c}(2))} < \frac{u'(\bar{c}(1))}{u'(\bar{c}(2))}.
\]

From this,

\[
\frac{\pi(1)u'(e(1) - \bar{c}(1))}{u'(\bar{c}(1))} + \frac{\pi(2)u'(e(2) - \bar{c}(2))}{u'(\bar{c}(2))} < 1.
\]

On the other hand, from (25) and the supposition,

\[
\frac{\pi(1)u'(e(1) - \bar{c}(1))}{u'(\bar{c}(1))} + \frac{\pi(2)u'(e(2) - \bar{c}(2))}{u'(\bar{c}(2))} > 1.
\]

This is a contradiction. The same argument holds when I suppose

\[
\frac{\pi(1)u'(e(1) - \bar{c}(1))}{u'(\bar{c}(2)) - \pi(2)u'(e(2) - \bar{c}(2))} \geq \frac{u'(\bar{c}(1))}{u'(\bar{c}(2))}.
\]

Therefore,

\[
\frac{u'(\bar{c}(1)) - \pi(1)u'(e(1) - \bar{c}(1))}{\pi(2)u'(e(2) - \bar{c}(2))} > \frac{u'(\bar{c}(1))}{u'(\bar{c}(2))} > \frac{\pi(1)u'(e(1) - \bar{c}(1))}{u'(\bar{c}(2)) - \pi(2)u'(e(2) - \bar{c}(2))}
\]

must hold. Then,

\[
1 > \frac{\pi(1)u'(e(1) - \bar{c}(1))}{u'(\bar{c}(1))} + \frac{\pi(2)u'(e(2) - \bar{c}(2))}{u'(\bar{c}(2))},
\]

which shows that \((\bar{c}(1), \bar{c}(2)), (e(1) - \bar{c}(1), e(2) - \bar{c}(2))\) is interim Pareto efficient. As \(c(1)\) increases from \(\bar{c}(1)\) and \(c(2)\) increases from \(\bar{c}(2)\) with satisfying incentive constraints, there is a golden-rule type allocation. Around one golden-rule type allocation, we can find other golden-rule type allocations. This completes the proof.

\(Q.E.D.\)
References


