Cyclical Behavior of Firm-level Volatility: An Explanation for the Contrast between the United States and Japan

Koki Oikawa*
Tohoku University
April 9th, 2012

Abstract

This paper presents countercyclical behavior of firm-level volatility measured by real sales growth of Japanese firms. It shows a clear contrast to the procyclicality of similarly defined volatility observed in the United States. Motivated by this fact, I build a theoretical model that is consistent with these opposite cyclical behaviors. The key is the relation between fixed and marginal cost of bankruptcy. Fixed bankruptcy cost works as an entry barrier and marginal bankruptcy cost works as an additional cost of hiring. Those distinct impacts affect compositional changes in operating firms along business cycles. The paper also examines welfare and policy implication about the optimal bankruptcy cost structure.

*I am grateful to Naohito Abe, Tatsuro Iwaisako, Tokuo Iwaisako, Boyan Jovanovic, Takao Kataoka, Akiomi Kitagawa, Minoru Kitahara, Romain Ranciere, Katsuya Takii, Yoshihiro Tamai, and seminar participants at JEA meeting, Osaka University, Waseda University, Tohoku University, New York University, Tokyo Metropolitan University, and Hitotsubashi University for their helpful comments. This work is partially supported by the Japan Society for the Promotion of Science, Grant-in-Aid for Young Scientists B (Project no. 23730183). All remaining errors are mine.
1 Introduction

The aim of this paper is to explain the contrasting cyclical behaviors of firm-level volatility, measured by real sales growth, observed in the United States and Japan. Comin and Mulani (2006) define firm-level volatility as weighted average of firms’ standard deviations of real sales growth rates. In the last part of their paper, they look into the correlation between firm-level volatility and medium-term business cycles and document procyclical behaviors of firm-level volatility. In other words, firm-level volatility decreases in a recession. As described in detail in Section 2, we similarly define firm-level volatility using firm sales data in Japan and distill medium-term cycles of Japanese real GDP time series. Interestingly, the Japanese data reveal highly significant countercyclicality during the last three decades, in other words, firm-level volatility increases in a recession.

In order to explain these opposite cyclical behaviors, I construct a simple theoretical model in Section 3, focusing on changes in firm-level volatility at the extensive margin. Suppose that entrepreneur’s ability is heterogeneous and firms operated by more competent entrepreneurs are less volatile (because they can diversify risks more). Even if each firm’s volatility is unchanged, average volatility across firms may change along business cycles, depending on a compositional change of firms in the market. If the average entrepreneur’s ability becomes higher in a recession, firm-level volatility becomes procyclical, and vice versa.

The key that alters the population of firms in the model is the bankruptcy cost structure. More concretely, it is the relative scale of the marginal cost to the fixed cost of bankruptcy, where we suppose bankruptcy costs depend positively on firm size. As Greenwald and Stiglitz (1990) argue, the coefficient of size-dependent bankruptcy cost can be interpreted as a degree of risk aversion. If a recession causes the default risk to surge, both items of the expected bankruptcy cost increase and they give birth to different impacts. An increase in expected marginal cost of bankruptcy makes entrepreneurs more risk-averse so that they employ fewer workers. Then, some people are pushed out of the labor market and open small businesses, which causes swelling

---

1Comin and Gertler (2006) introduce medium-term business cycles to capture the waves with long bandwidths that usual Hodrik-Prescott filters or "short-term" band pass filters filter out. They get medium cycles with a band pass filter with bandwidths of 2 to 200 quarters.
of the set of firms but the average efficiency of firms declines. On the other hand, an increase in expected fixed cost of bankruptcy works as a greater entry barrier, which causes shrinking of the set of firms, but the average efficiency of firms is improved. The result depends on which impact is larger. Section 3.3 shows the relative scale of fixed bankruptcy cost to marginal one settles whether firm-level volatility becomes procyclical or countercyclical. Referring to the observation mentioned above, we interpret there is relatively lower marginal bankruptcy cost in the United States and relatively higher marginal bankruptcy cost in Japan.

This story matches with the cleansing effect theory of recessions (Caballero and Hammour (1994)). If the cleansing effect of recessions works enough, average firm’s efficiency becomes higher in a recession. Caballero, Hoshi, and Kashyap (2008) argue the cleansing effect of recessions does not quite work in Japan, compared to the United States. The distinct feature of the model in this paper is that there exist less efficient entrepreneurs starting their own businesses in the Japanese case, which is worse than just without the cleansing effect.

Using the model, I investigate welfare and policy implications about optimal bankruptcy cost structure in Section 4. Section 5 discusses the consistency of the model with the cyclical behavior of idiosyncratic risks measured using financial data. As Campbell, Lettau, Malkiel, and Xu (2001) and Bekaert, Hodrick, and Zhang (2010) report, idiosyncratic risks calculated from excess returns of stocks in the United States have countercyclicality, contrary to the firm-level volatility calculated from real sales data in the United States. Even though the model in this paper does not provide stock returns, it is consistent with this reversal between real and financial data when we interpret firm-level volatility of profits as a proxy for the stock returns. Section 6 is deliberated to an extension of the model with endogenous bankruptcy probability. Section 7 is concluding remarks.

2 Cyclical Behavior of Firm-level Volatility in Japan

This section presents the cyclical behavior of firm-level volatility measured with real sales growth in Japan. The measuring methods follow Comin and Mulani (2006) to define firm-level volatility and Comin and Gertler (2006) to obtain medium term
cycles.

Data. — For firm-level data, I used Nikkei Economic Electronic Databank System (NEEDS) that contains sales data of 4962 firms (2160 manufacturing firms and 2802 non-manufacturing firms) that include listed and unlisted firms\(^2\) in all private sectors during 1963 to 2009, annually. I converted the sales data into the real term by consumer price index among all commodities. Following Comin and Mulani (2006), we calculate growth rates of real sales by each firm, and then computed the standard deviations of the growth rates of each firm within a 9 years window (from \(t-4\) to \(t+4\) for each \(t\)). Because of this window setting, the sample periods shrinks to 1967 to 2005. Finally, I define firm-level volatility in year \(t\) as the weighted average of the standard deviations across firms within a year, where weights are defined by sales shares. Table 1 summarizes the computed volatility of each firm.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>S.D.</th>
<th>Median</th>
<th>Min</th>
<th>Max</th>
<th># obs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full</td>
<td>0.17</td>
<td>1.89</td>
<td>0.097</td>
<td>0.0065</td>
<td>193.99</td>
<td>76484</td>
</tr>
<tr>
<td>Manufacturing</td>
<td>0.13</td>
<td>0.96</td>
<td>0.010</td>
<td>0.0092</td>
<td>193.99</td>
<td>41741</td>
</tr>
<tr>
<td>Non-manufacturing</td>
<td>0.21</td>
<td>2.60</td>
<td>0.093</td>
<td>0.0065</td>
<td>167.37</td>
<td>34743</td>
</tr>
</tbody>
</table>

Table 1: Summary statistics of volatility of each firm.

For business cycles, I applied a band pass filter to Japanese quarterly real GDP time series (seasonally adjusted). Following Comin and Gertler (2006), I obtained short-term components with the bandwidth of 2 to 32, and medium-term components with the bandwidth of 32 to 200. Medium-term cycles are defined by the sum of the short and medium-term components.

Observation. — First, firm-level volatility is not significantly correlated with the short-term cycle, most probably because the volatility is calculated only annually. However, it is significantly correlated with medium-term components and thus medium-term cycles. Figure 1 shows the time series behaviors of the weighted average of firm-level volatility and medium-term cycles.

\(^2\)The number of firms is changing over time. In average, each year contains 1664 listed firms and 991 unlisted firms. Davis, Haltiwanger, Jarmin, and Miranda (2006) point out that trends of firm-level volatility are significantly different between listed firms and unlisted firms. Since the dataset includes both listed and unlisted firms, the analysis evades such a sample-bias problem at least in some extent.
As seen in Figure 1, firm-level volatility in each category (manufacturing, non-
manufacturing, and full sample) and the medium-term cycles have positive correlation
until around 1975, but seemingly have negative correlation afterwards. Table 2 reports
linear regression coefficients when projecting firm-level volatility on medium-term
cycles. They are not or less significant over the whole periods but highly significantly
negative after 1975. I also show the cross-correlograms between firm-level volatility
and cycles after 1975 in Figure 2, which also indicate negative correlation in the three
categories.

There might be a structural change around 1975, otherwise, periods around 1970
could be somewhat special. Though it is also an interesting issue, we focus on the era
after 1975 because we need data dating back more years to investigate more deeply.
During the last three decades, the firm-level volatility in Japan shows countercycli-
cality to the medium-term cycles, which is a clear contrast to the case in the United States, as mentioned in Introduction.

<table>
<thead>
<tr>
<th></th>
<th>Whole periods</th>
<th>After 1975</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full</td>
<td>0.0324 (0.043)</td>
<td>-0.396** (0.062)</td>
</tr>
<tr>
<td>Manufacturing</td>
<td>0.0902* (0.048)</td>
<td>-0.519*** (0.057)</td>
</tr>
<tr>
<td>Non-manufacturing</td>
<td>-0.0257 (0.053)</td>
<td>-0.302*** (0.094)</td>
</tr>
</tbody>
</table>

Table 2: OLS coefficients on medium-term cycles. The numbers in the parentheses are standard errors. The significance levels are indicated in the standard form (*: 10%, **: 5%, and ***: 1%).

3 The Model

This section builds a theoretical model, based on Lucas (1978), to explain the opposite cyclical behaviors observed in the previous section.

3.1 Model Setting

There are \([0, 1]\) continuum of agents who can be workers or entrepreneurs. Each agent has entrepreneurial ability \(x \geq 1\) drawn from some distribution. This entrepreneurial ability indicates the maximal number of projects that an entrepreneur can manage. Since, as you will see, there is no structure that an entrepreneur does not make maximal use of his or her ability in this model, an entrepreneur with ability \(x\) manages \(x\) projects. Each project \(j\) has a revenue function such as

\[ z\epsilon_j l_j , \]

where \(z > 0\) is an exogenously predetermined common macro parameter (a decrease in \(z\) indicates a recession), \(l_j\) is project-level employment \((\alpha \in (0, 1))\), and \(\epsilon_j\) is idiosyncratic shock, uniformly distributed on \(\left[1 - \sqrt{3}\sigma, 1 + \sqrt{3}\sigma\right]\), where \(\sigma \in [0, 1/\sqrt{3}]\). I assume that projects are identical ex ante so that an entrepreneur chooses a common
\( \bar{l}_j \) across projects, say \( \bar{l} \). The total employment of an entrepreneur with ability \( x \) is \( l = x\bar{l} \). Thus, the firm’s profit is

\[
z \left[ \sum_{j=1}^{x} \epsilon_j \bar{l}^\alpha \right] - wl = \epsilon (x) x^{1-\alpha} \bar{l}^\alpha - wl,
\]

where I define

\[
\epsilon (x) \equiv \frac{1}{x} \sum_{j=1}^{x} \epsilon_j,
\]

with \( E[\epsilon (x)] = 1 \) and \( Var[\epsilon (x)] = \sigma^2/x \). Because more able entrepreneur can carry out more projects which are statistically independent thus risks can be more
diversified. The profit function for each entrepreneur with ability $x$ is

$$z \xi ( x ) x^{1-\alpha \nu} - w l.$$  

Below, to avoid complication, we deal with $x$ as if it is continuous and follows a continuous distribution $G ( x )$ defined on $[ \bar{x}, \infty )$, where $\bar{x} \geq 1$.4

**Expected bankruptcy costs.**— Now suppose that there exist some additional costs, $B ( z, l )$, that an entrepreneur has to pay other than wage payments. This additional cost is decreasing in $z$ and increasing in $l$. One natural interpretation of this cost is the *expected bankruptcy cost*, where a firm continues to be a going concern after paying the bankruptcy cost in case of bankruptcy. Specifically, we use the following function form:

$$B ( z, l ) = \delta ( z ) ( a + b l ) , \quad a \geq 0 , \ b \geq 0$$

(1)

where $\delta ( z )$ is exogenous bankruptcy probability with $\delta ' ( z ) < 0$ and $a$ is fixed bankruptcy cost and $b$ is marginal bankruptcy cost. One can also interpret this specific $B ( z, l )$ in a different way. Suppose that $\delta ( z )$ is the probability that a business partner goes bankrupt. Then the entrepreneur has to reestablish a new business line or find a new partner to continue its own business. The cost to reestablish the line is higher when the size of the firm is larger. Although $B ( z, l )$ can be interpreted in various ways,5 we specify it as the expected bankruptcy cost to make the idea and implication in this paper clear. I will discuss how the main argument works under endogenous bankruptcy probability in Section 6.

If we consider $B ( z, l )$ as the expected bankruptcy cost, is it plausible to have a

---

3The negative impact of size on volatility is observed in the dataset used in Section 2: the correlation between real sales and volatility of each firm are negative within each category. However, it is ambiguous that standard deviation is decreasing in the order of square root of the size. Hymer and Pashigian (1962) documents the firm variability is significantly decreasing in size a bit more slowly than in the order of square root. So, the assumption of independent projects may be too restrictive. The main result in this paper holds if volatility is decreasing in the size, regardless of the order of decreasing.

4To be consistent with continuous sets of projects, it is sufficient to suppose that shocks over projects follow a Brownian motion, where the standard deviation of the process should be sufficiently small to make the probability of negative production negligible.

5We can also see $B ( z, l )$ as a kind of transaction cost. Higher $z$ implies less transaction cost because they can facilitate smooth business. The total transaction cost is assumed to be increasing in firm size.
linear bankruptcy cost function as in (1)? One reason why we consider size matters is that, otherwise, bankruptcy cost becomes negligible as a firm is getting larger (Greenwald and Stiglitz (1993)). Moreover, empirical evidence of positive correlation between firm size and bankruptcy cost is reported in Bris, Welch, and Zhu (2006).

The parameter $b$ may contain firing costs and, as argued later, it can be interpreted as the degree of risk aversion of entrepreneurs. The existence of fixed cost of bankruptcy is also essential in the current model, unlike Greenwald and Stiglitz (1993). Hotchkiss, John, Mooradian, and Thorburn (2008) states that there seems to exist a fixed cost for direct bankruptcy cost, explaining why a Chapter 11 reorganization may be infeasible for some smaller firms.

### 3.2 Equilibrium and Comparative Statics

An entrepreneur maximizes the expected profit minus the expected bankruptcy cost:

$$\max_{l \geq 0} z x^{1-\alpha} l^\alpha - w l - \delta(z) (a + bl).$$

The optimal employment for an entrepreneur with ability $x$ is

$$l(x, w|z) = x \left( \frac{\alpha z}{w + b \delta(z)} \right)^{\frac{1}{\frac{1}{\alpha}}}.$$

(2)

Then the expected profit of the entrepreneur is

$$\pi^e(x, w|z) = \frac{1 - \alpha}{\alpha} \left( \frac{\alpha z}{w + b \delta(z)} \right)^{\frac{1}{\frac{1}{\alpha}}} (w + b \delta(z)) x - a \delta(z).$$

(3)

---

6Greenwald and Stiglitz (1993) use a linear bankruptcy cost function depending positively on size. Their bankruptcy cost is increasing in not employment but output.

7In the literature, bankruptcy costs are divided into direct costs and indirect costs. Direct costs contain administrative fees, compensation to lawyers, accountants, consultants and expert witnesses, and so on (Warner (1977), Altman (1984), Altman and Hotchkiss (2006), etc.). Indirect costs include unobservable opportunity costs. For examples, financially distressed firms are forced to sell assets or cut capital expenditure in an inefficient way (Altman (1984), Andrade and Kaplan (1998)); the indirect costs may be lost sales driven by firm’s deteriorating financial condition; it is also recognized as time required to complete bankruptcy procedures (Bris, Welch, and Zhu (2006)). The measurement of indirect costs depends on research, but the total of those costs have been considered more sizable than direct ones. See Hotchkiss, John, Mooradian, and Thorburn (2008) in more detail.
Free entry condition. — Because of free occupational choice, an agent prefers to be an entrepreneur if $\pi^e(x, w|z) \geq w$. Because $\pi^e(x, w|z)$ is increasing in $x$ and $\pi^e(0, w|z) < 0$, there should be a threshold $\hat{x} \in (0, \infty)$, at which the two occupations are indifferent. So

$$\text{(FE): } \frac{1 - \alpha}{\alpha} \left( \frac{\alpha z}{w + b\delta(z)} \right)^{\frac{1}{1-\alpha}} \hat{x} = \frac{w + a\delta(z)}{w + b\delta(z)}.$$  

(4)

or equivalently,

$$\hat{x} = (w + a\delta(z)) \left( w + b\delta(z) \right)^{\frac{\alpha}{1-\alpha}} \frac{\alpha}{1 - \alpha} \left( \alpha z \right)^{-\frac{1}{1-\alpha}}.$$  

(5)

Labor market clearing condition. — Equilibrium is pinned down by the labor market clearing condition:

$$\text{(LMC): } \left( \frac{\alpha z}{w + b\delta(z)} \right)^{\frac{1}{1-\alpha}} \int_{\hat{x}}^{\infty} x \, dG = G(\hat{x}).$$  

(6)

or equivalently,

$$w = \alpha z \Gamma(\hat{x})^{\alpha-1} - b\delta(z),$$  

(7)

where

$$\Gamma(\hat{x}) \equiv \frac{G(\hat{x})}{\int_{\hat{x}}^{\infty} x \, dG},$$

which is the employment for each project, $\hat{l}$, for given $\hat{x}$ because (2) implies $l = x \Gamma(\hat{x})$ (the project level employment is common across entrepreneurs).

As depicted in Figure 3, the FE condition is strictly increasing in $\hat{x}$ and the LMC condition is strictly decreasing in $\hat{x}$. Since entrepreneurs hire finite workers even with $w = 0$ when $b\delta(z) > 0$, the LMC curve touches the horizontal axis at a finite $\hat{x}$ as illustrated in the figure. Thus, there could be an equilibrium with $w = 0$ and some unemployment. To avoid this pathological case, we assume that

$$\alpha z \Gamma(\hat{x})^{\alpha-1} - b\delta(z) > 0.$$
throughout the paper, where $\hat{x}$ is the intercept of the FE curve on the horizontal axis:

$$
\hat{x} \equiv ab^{1-\alpha} \frac{\alpha}{1-\alpha} \left( \frac{\delta(z)}{\alpha z} \right)^{\frac{1}{1-\alpha}}.
$$

This assumption is satisfied when $\delta(z)/z$ is sufficiently small. Under this assumption, there exists unique equilibrium $(\hat{x}^*, w^*)$, drawn in Figure 3.

Now consider the effect of a decrease in $z$. Figure 3 also illustrates the shifts of the curves when $z$ decreases. The first intuitive but important property of the model is that $w^*$ is strictly increasing in $z$. It is because the both curves shift down in a recession. However, it is ambiguous about the change in $\hat{x}^*$. When the decline in $w^*$ is sufficiently large, or LMC shifts down sufficiently, $\hat{x}^*$ decreases in response
to a recession. In other words, more economic agents prefer to be an entrepreneur because wage is too low. The next proposition shows that the behavior of $\hat{x}^*$ depends on whether $a > b$ or not.

**Proposition 1** If $a > b$, then a decrease in $z$ leads to an increase in $\hat{x}^*$. If $a < b$, then a decrease in $z$ leads to a decrease in $\hat{x}^*$.

**Proof.** Suppose that $(\hat{x}^*(z), w^*(z))$ is equilibrium under $z$. To prepare for proof, first consider the response of the right hand side of (4) to a change in $z$.

$$\frac{d}{dz} \frac{w^*(z) + a\delta(z)}{w^*(z) + b\delta(z)} < 0 \quad \Leftrightarrow \quad (a - b) \left[ \frac{w^{*\prime}(z)}{w^*(z)} - \frac{\delta^\prime(z)}{\delta(z)} \right] > 0.$$  

Since $\delta^\prime(z) < 0$ and $w^{*\prime}(z) > 0$, this inequality holds if and only if $a > b$.

Suppose $a > b$ and $z$ decreases. (4) implies

$$\frac{1 - \alpha}{\alpha} \left( \frac{\alpha z}{w^*(z) + b\delta(z)} \right)^{\frac{1}{1-\alpha}} \hat{x}^*(z) = \frac{w^*(z) + a\delta(z)}{w^*(z) + b\delta(z)}.$$  

The value of this equation increases from the above argument. Suppose that $\hat{x}^*(z)$ decreases. Then

$$\frac{\alpha z}{w^*(z) + b\delta(z)}$$  

must increase. Now look at the LMC condition (6):

$$\left( \frac{\alpha z}{w^*(z) + b\delta(z)} \right)^{\frac{1}{1-\alpha}} \int_{\hat{x}^*(z)}^{\infty} x \, dG = G(\hat{x}^*(z)).$$  

Under the current supposition, the left hand side increases but the right hand side decreases. Contradiction. Hence, $a > b$ implies $\hat{x}^*(z)$ increases in response to a decline in $z$. The case of $a < b$ is analogous. ■

To understand this result, notice the fixed bankruptcy cost, $a$, works as the entry barrier for agents to become entrepreneurs. A recession makes any business less profitable and reduce wage. Large $a$ implies only competent agents become entrepreneurs (upward shift in $\hat{x}$).
Greenwald and Stiglitz (1990) give us interpretation on the role of \( b \). They show that risk-neutral entrepreneurs with size-dependent bankruptcy cost and risk-averse (decreasing absolute risk aversion) entrepreneurs without bankruptcy cost are identical in decision making. So we can regard \( b \) as a degree of risk aversion. A high degree of risk aversion causes large downward shifts in labor demands to avoid high cost in case of bankruptcy. Therefore, large \( b \) leads to a downward shift in \( \hat{x} \). In other words, cleansing effect of a recession works under small \( b \). Since \( a \) (entry barrier) and \( b \) (risk aversion) have opposite effects on \( \hat{x} \) and a recession raises \( a \) and \( b \) in expectation (namely, \( a \delta (z) \) and \( b \delta (z) \)) at the same time, the final consequence depends on the size relation between the two parameters.

We can see the result from another viewpoint. See the right hand side of (4),

\[
\frac{w + a \delta (z)}{w + b \delta (z)}.
\]

The shift in \( \hat{x} \) is determined by the shift in this fraction. The denominator, \( w + b \delta (z) \), is the expected unit hiring cost, or effective cost of hiring a worker (equals marginal product of labor). On the other hand, the numerator, \( w + a \delta (z) \), is wage received plus opportunity cost of entry, or effective benefit of being a worker. If \( a > b \), the effective benefit is more than the effective cost for any wage. More agents want to be workers and, at the same time, entrepreneurs can afford to hire many workers. Since a recession causes this gap expand (the fraction gets larger), the measure of entrepreneurs shrinks. Meanwhile, if \( a < b \), the effective benefit is less than the effective cost. Being workers is less attractive and entrepreneurs are not willing to hire so many workers. Hence, there are less workers and more entrepreneurs in equilibrium than under \( a > b \). Since a recession again causes this gap expand (the fraction gets smaller), the measure of entrepreneurs is stretched. If \( a = b \), the effective cost and benefit are equal and the fraction is 1, which is the same in the case without any bankruptcy cost. Therefore, a recession never causes any shift in \( \hat{x} \).
3.3 Cyclical Behavior of Firm-level Volatility

In Section 2 of this paper, I defined firm-level volatility with standard deviations of real sales growth so that increase in volatility does not capture firm size growth. Since the current model does not have any growth factor, we use the relative variance (or variability) to see firm-level volatility. The real sales are

\[ z x \Gamma (\hat{x}^*)^\alpha \epsilon (x), \]

so, the relative variance of real sales, \( v(x) \) is

\[ v(x) = \frac{\sigma^2}{x}. \]

This measurement of volatility for each firm is independent of \( z \) and the larger \( x \), the less volatile. There is no change in volatility at the intensive margin under the current setting. So, the effect on firm-level volatility stems only from a change in the composition of firms. The average firm-level volatility, \( \bar{v} \), is

\[ \bar{v}(\hat{x}^*) = \frac{\sigma^2}{1 - G(\hat{x}^*)} \int_{\hat{x}^*}^{\infty} \frac{1}{x} dG. \] (8)

Because this is decreasing in \( \hat{x}^* \) for any given distribution, the next proposition holds.

**Proposition 2** \( a > b \) implies that \( \bar{v} \) is procyclical and \( a < b \) implies that \( \bar{v} \) is countercyclical.

**Proof.** \( \bar{v}(\hat{x}) \) is strictly decreasing if

\[ \frac{\sigma^2}{\hat{x}} > \bar{v}(\hat{x}), \]

which must be satisfied since \( \sigma^2/\hat{x} \) is the maximum volatility among entrepreneurs. The remaining is straightforward from Proposition 1. ■

According to Proposition 2, firm-level volatility is *procyclical* if the fixed cost of bankruptcy is *more* than the marginal cost of bankruptcy, and *countercyclical* otherwise. Hence, the model implies \( a > b \) in the United States and \( a < b \) in
Japan. Japanese firms face more marginal bankruptcy cost (because of higher labor adjustment cost for an example), so they are more risk-averse and make decisions to avert going bankrupt and then more firms stay in the market in a recession. This is consistent with discussion in Caballero, Hoshi, and Kashyap (2008) and Hamao, Mei, and Xu (2007).

4 Welfare and Policy Implication

This section investigates welfare implication of the model. The main theme in this analysis is to propose a policy about the bankruptcy cost structure, where we suppose that the government can somewhat control fixed and marginal bankruptcy costs. Surely, if possible, \( a = b = 0 \) is the best. So I consider the situation where the bankruptcy technology is only partly manipulatable. More concretely, I look for the best relation between \( a \) and \( b \) with keeping the total bankruptcy cost constant. With this thought experiment, we can see which cost component should be given priority to be decreased when we have limited control over the bankruptcy costs.

4.1 Aggregate variables

*Aggregate income.*—The aggregate income at the equilibrium is defined by the total production minus bankruptcy costs:

\[
Y(z) = \int_{\hat{x}^*}^{\infty} z x^1 - a l (x, w^* | z)^z dG - \delta(z) \int_{\hat{x}^*}^{\infty} (a + b (x, w^* | z)) \ dG
\]

First note that the equilibrium income achieves the maximum for any given set of parameters. To see this, compute the following maximization problem:

\[
\max_{\hat{x}} z G (\hat{x}) \Gamma (\hat{x})^{z^1} - \delta(z) [a - (a - b) G (\hat{x})].
\]

The first order condition of this problem is

\[
z \Gamma (\hat{x})^{z^1} (\alpha - (1 - \alpha) \hat{x} \Gamma (\hat{x})) + \delta(z) (a - b) = 0. \tag{9}
\]
Equation (9) is equivalent to the FE condition when we solve the LMC condition, (6), about \( w \), and substitute it into (4). Hence, the equilibrium attains the maximum aggregate income.8

Note that \( Y \) is increasing in \( z \) for any set of parameters. I remark this property because, if \( a > b \) and in a recession, \( \hat{x} \) increases and each firm could produce more under lower \( z \) because of a decline in wage. So we need to make sure if a decline in \( z \) truly implies a recession.

Suppose that \( z_1 > z_2 \). To lead a contradiction, we assume \( Y (z_1) \leq Y (z_2) \). Then,

\[
\begin{align*}
  z_2 G (\hat{x}_2^*) \Gamma (\hat{x}_2^*)^{\alpha-1} &- \delta (z_2) [a - (a - b) G (\hat{x}_2^*)] \\
  \geq & \quad z_1 G (\hat{x}_1^*) \Gamma (\hat{x}_1^*)^{\alpha-1} - \delta (z_1) [a - (a - b) G (\hat{x}_1^*)] \\
  \geq & \quad z_1 G (\hat{x}_2^*) \Gamma (\hat{x}_2^*)^{\alpha-1} - \delta (z_1) [a - (a - b) G (\hat{x}_2^*)].
\end{align*}
\]

The last inequality is because the competitive equilibrium maximizes the aggregate income. Then,

\[
(z_2 - z_1) G (\hat{x}_2^*) \Gamma (\hat{x}_2^*)^{\alpha-1} \geq [\delta (z_2) - \delta (z_1)] [a - (a - b) G (\hat{x}_2^*)].
\]

However, the left hand side is negative and the right hand side is positive (\( \delta' < 0 \)). Contradiction. Hence \( Y' (z) > 0 \).

*Aggregate production.*— So far, we have confirmed that aggregate income is reduced by a decrease in \( z \). Next we consider the behavior of total production. To see this, we define the aggregate production, say \( P \), aggregate bankruptcy cost, say \( B \)

\footnote{From (6),
\[
  w = \alpha z \Gamma (\hat{x})^{\alpha-1} - b \delta.
\]

Substitute this into (4),
\[
\frac{1 - \alpha}{\alpha} \tilde{x} \Gamma (\hat{x}) = \frac{\alpha z \Gamma (\hat{x})^{\alpha-1} + (a - b) \delta}{\alpha \Gamma (\hat{x})^{\alpha-1}},
\]

which is equivalent to (9).}
Figure 4: Marginal products and marginal bankruptcy costs about $\hat{x}$

(abusing notations), and marginal values of them as follow:

\[
\begin{align*}
P (\hat{x}|z) &= zG (\hat{x}) \Gamma (\hat{x})^{\alpha - 1}, \\
MP (\hat{x}|z) &= z\Gamma (\hat{x})^{\alpha - 1} (\alpha - (1 - \alpha) \hat{x}\Gamma (\hat{x})) G' (\hat{x}), \\
B (\hat{x}|z, a, b) &= \delta (z) [a - (a - b) G (\hat{x})], \\
MB (\hat{x}|z, a, b) &= -(a - b)\delta (z) G'' (\hat{x}).
\end{align*}
\]

Notice $MP (\hat{x})/G' (\hat{x})$ is strictly decreasing. Figure 4 illustrates $MP$ and $MB$ curves (both divided by $G' (\hat{x})$) for two cases: $a > b$ and $a < b$. From this figure, one can see $\hat{x}^*|_{a > b} > \hat{x}^*|_{a < b}$, ceteris paribus. Moreover,

\[
MP (\hat{x}^*|_{a > b}) < 0 \quad \text{and} \quad MP (\hat{x}^*|_{a < b}) > 0. \quad (10)
\]
Figure 4 also illustrates the shifts of curves in response to a decline in $z$. As shown in Proposition 1, $\hat{x}^*|_{a>b}$ increases and $\hat{x}^*|_{a<b}$ decreases. Then appealing to (10), total production is reduced by a decrease in $z$.

Welfare comparison for a fixed bankruptcy cost.— At the last of this section, we consider what relation between $a$ and $b$ is desirable if the total bankruptcy cost is preserved as a constant. Define $A \equiv \{(a, b) \in \mathbb{R}^2_+ \mid B(\hat{x}^*|z, a, b) = \tilde{B} \}$, where $\tilde{B}$ is any positive real number and other parameters except $(a, b)$ are fixed. Since the total bankruptcy cost is constant for any pair $(a, b) \in A(\tilde{B}|z)$, a change in aggregate income depends only on total production, $P(\hat{x})$. From (10), $Y$ increases as $\hat{x}^*$ increases as long as $a < b$, and $Y$ increases as $\hat{x}^*$ decreases as long as $a > b$. So we have the next proposition.

**Proposition 3** Among $A(\tilde{B}|z)$ for any $\tilde{B} > 0$, the aggregate income is higher when $|a - b|$ is lower and the maximum aggregate income is attained when $a = b = \tilde{B}/\delta(z)$.

**Proof.** Suppose $a < b$. Since $MB(\hat{x}|z, a, b) / G'(\hat{x}) > 0$ and it shifts down as $b - a$ decreases, noting that $MP(\hat{x}|z) / G'(\hat{x})$ depends on neither $a$ nor $b$, $\hat{x}^*$ increases as $b - a$ decreases. Because $MP(\hat{x}|z) / G'(\hat{x}) > 0$ as long as $a < b$, $Y(z)$ increases along this change. The maximum is attained if $a = b$, where $MB(\hat{x}|z, a, b) / G'(\hat{x}) = 0$. $a = b = \tilde{B}/\delta(z)$ is from

$$
\delta(z) [a - (a - b) G(\hat{x})] = \tilde{B} \quad \Rightarrow \quad a - b = \frac{a - \tilde{B}/\delta(z)}{G(\hat{x}^*)}.
$$

If there is no bankruptcy cost ($a = b = 0$), the right hand side of (4) becomes 1, implying that wage equals marginal product, the efficient situation. Even for positive cost parameters, the efficient labor allocation continues to occur as long as $a = b$. So the gap between $a$ and $b$ gives birth to a deviation from efficient allocation.

Therefore, $a = b$ is the optimal balance between marginal bankruptcy cost and fixed bankruptcy cost if the total bankruptcy cost is given. Remember $a = b$ implies that the effective cost of hiring a worker and the effective benefit of being a worker are equal and then a change in $z$ does not shift $\hat{x}$, as argued in Section 3. In other words,
if \( \bar{x} \) shifts in response to a change in \( z \), there exists inefficiency in the bankruptcy cost structure. Hence, the government should hold down the higher cost coefficient.\(^9\)

This result is helpful in the following sense. The direct estimation of bankruptcy cost coefficients is hard because capturing indirect costs is often complicated (see, for example, Hotchkiss, John, Mooradian, and Thorburn (2008)). However, from Proposition 2 and 3, we can relate firm-level volatility and inefficiency of bankruptcy cost structure. If we observe procyclical firm-level volatility, \( a \) is more likely larger than \( b \) so the government should reduce the fixed cost of bankruptcy (e.g., simplification of bankruptcy procedure), vice versa. Proposition 3 supposes a constant bankruptcy cost, but it is just for making the argument clear. We do not have to keep the total bankruptcy cost in reality. Smaller \( \bar{B} \) leads to higher welfare in the current model.

5 Cyclical Behavior of Firm-level Volatility with Profit Measure

On firm-level volatility and business cycles, there are papers that look into cyclical behaviors of idiosyncratic risks using financial data. In this section, we discuss the relation of the model in this paper to the literature. In doing so, we regard profit volatility as a proxy of volatility of excess return of stocks because the exact variable is not defined in the current model.

Campbell, Lettau, Malkiel, and Xu (2001) report the countercyclicality of idiosyncratic risks using the data of excess returns of stocks in the United States. Bekaert, Hodrick, and Zhang (2010) point out some results of Campbell, Lettau, Malkiel, and Xu (2001) are sample-period dependent but confirm the countercyclicality. So, there is another reversal about cyclical behaviors: in the United States, firm-level volatility by real sales growth is procyclical while idiosyncratic risks from excess stock returns are countercyclical.\(^{10}\)

\(^9\)This result also implies that any cyclical behavior of firm-level volatility at the extensive margin indicates inefficient balance of the bankruptcy cost coefficients.

\(^{10}\)Hamao, Mei, and Xu (2007) report a downturn in idiosyncratic risks especially after the bubble burst in Japan using a similar method with Campbell, Lettau, Malkiel, and Xu (2001). So, it is procyclical from the bubble burst to the mid-1990s. Hamao, Mei, and Xu (2007) also report countercyclical behavior of idiosyncratic risks in the late 1990s in Japan. So, cyclicality of idiosyncratic.
To do analysis in this context, we define firm-level volatility with profit measure. So far, our definition of firm-level volatility only considers sales. Because stock return includes expectation of future profits, we need to take into account not only sales but also costs (wage payments and bankruptcy costs). Once the cost component is included into volatility measure, cyclical behavior of firm-level volatility may be reversed, depending on parameters.

The maximized profit for an entrepreneur with ability $x$ in equilibrium is

$$x \Gamma (\hat{x}^*) [z \Gamma (\hat{x}^*)^{\alpha-1} (\epsilon (x) - \alpha) + b \delta (z)] - \begin{cases} \ 0 \text{ with prob. } 1 - \delta (z), \\ a + bx \Gamma (\hat{x}^*) \text{ with prob. } \delta (z). \end{cases}$$

Hence, the expected profit and variance of profit are computed as

$$E (\pi |x, z) = (1 - \alpha) zx \Gamma (\hat{x}^*)^\alpha - a \delta (z),$$

$$Var (\pi |x, z) = (zx \Gamma (\hat{x}^*)^\alpha)^2 \frac{\sigma^2}{x} + (a + bx \Gamma (\hat{x}^*))^2 \delta (z) (1 - \delta (z)).$$

Each firm has its volatility $v_{\pi} (x|z)$ defined as the relative variance:

$$v_{\pi} (x|z) = \left( \frac{zx \Gamma (\hat{x}^*)^\alpha}{(1 - \alpha) zx \Gamma (\hat{x}^*)^\alpha - a \delta (z)} \right)^2 \frac{\sigma^2}{x}$$

$$+ \left( \frac{a + bx \Gamma (\hat{x}^*)}{(1 - \alpha) zx \Gamma (\hat{x}^*)^\alpha - a \delta (z)} \right)^2 \delta (z) (1 - \delta (z)).$$

As easily observed, $v_{\pi} (x|z)$ is decreasing in $x$, similarly to the sales volatility. However, profit volatility of each firm depends on $z$, unlike the sales volatility. Because of this dependence on $z$, we need to take into account changes in volatility at the intensive margin in analyses of firm-level volatility with profit measures. Besides, the changes at the extensive margin and intensive margin diverge when $a > b$.

**Proposition 4** Assume $\delta (z) < 1/2$. $v_{\pi} (x|z)$ is countercyclical for any $x$ if $z \Gamma (\hat{x}^*)^\alpha$ and $z \Gamma (\hat{x}^*)^{\alpha-1}$ are increasing in $z$.

---

11To obtain the results in the previous sections, $\delta (z)$ is not necessarily probability. It becomes significant here. If $\delta (z)$ is just a decreasing component in $B (z, l)$, there is no cost variability and thus no difference between sales measure and profit measure.
\textbf{Proof.} Rearrange (11) to
\[
v_{x} (x|z) = \left( 1 - \alpha - \frac{a \delta (z)}{x z \Gamma (\hat{x}^{*})^\alpha} \right)^{-2} \frac{\sigma^{2}}{x} + \frac{a}{(1 - \alpha) x z \Gamma (\hat{x}^{*})^\alpha - a \delta (z)} + \frac{b x}{z \Gamma (\hat{x}^{*})_{-1} (1 - \alpha) x - \frac{a \delta (z)}{z \Gamma (\hat{x}^{*})^\alpha}} \right)^{2} \times \delta (z) (1 - \delta (z)),
\]
which is decreasing in $z$. \Halmos

The condition that $z \Gamma (\hat{x}^{*})^\alpha$ and $z \Gamma (\hat{x}^{*})_{-1}$ are procyclical is sufficient for countercyclicality of $v_{x} (x|z)$. Below, I consider only the cases where these conditions hold because they are plausible in the following sense. First, procyclicality of $z \Gamma (\hat{x}^{*})^\alpha$ implies that all entrepreneurs obtain less profits under a smaller $z$. Notice that the expected profit minus the expected bankruptcy cost is
\[
\pi^{e} (x|z) = (1 - \alpha) x z \Gamma (\hat{x}^{*})^\alpha - a \delta (z).
\]
If $z \Gamma (\hat{x}^{*})^\alpha$ increases under $a > b$ in a recession, then $\pi^{e} (x|z)$ increases in a recession for entrepreneurs with sufficiently large $x$, implying that a decline in $z$ is a boom for highly able people. I omit this possibility. If $a < b$, $z \Gamma (\hat{x}^{*})^\alpha$ is automatically procyclical.

Second, procyclicality of $z \Gamma (\hat{x}^{*})_{-1}$ implies that marginal expected cost of hiring is procyclical because $z \Gamma (\hat{x}^{*})_{-1} \propto w^{*} + b \delta (z)$ from (7). In other words, procyclicality of $z \Gamma (\hat{x}^{*})_{-1}$ implies that an increment in $b \delta (z)$ is smaller than a drop in the wage in response to a decline in $z$. If $a > b$, $z \Gamma (\hat{x}^{*})_{-1}$ is automatically procyclical.

Put differently, procyclicality of $z \Gamma (\hat{x}^{*})^\alpha$ and $z \Gamma (\hat{x}^{*})_{-1}$ constrains the response of $\hat{x}^{*}$ to a shift in $z$. From (5) and (7), $\hat{x}^{*}$ in equilibrium satisfies
\[
(1 - \alpha) z \hat{x}^{*} \Gamma (\hat{x}^{*})^\alpha = \alpha z \Gamma (\hat{x}^{*})_{-1} + (a - b) \delta (z).
\] (12)
Total differentiation of this function implies

\[ \frac{d\hat{x}^*}{dz} = \frac{1}{z} \left( \frac{\hat{x}^* \Gamma(\hat{x}^*) - \frac{\alpha}{1-\alpha}}{1 + \alpha (1 + \hat{x}^* \Gamma(\hat{x}^*))} \right) \left( \frac{\frac{\delta'(z)}{\delta(z)} - 1}{\frac{\Gamma'(\hat{x}^*)}{\Gamma(\hat{x}^*)}} \right). \]

Therefore, both \( z \Gamma(\hat{x}^*)^\alpha \) and \( z \Gamma(\hat{x}^*)^{\alpha-1} \) are procyclical if and only if

\[ -\frac{1}{\alpha} < \frac{\hat{x}^* \Gamma(\hat{x}^*) - \frac{\alpha}{1-\alpha}}{\hat{x}^* \Gamma(\hat{x}^*) + 1} \frac{\frac{\delta'(z)}{\delta(z)} - 1}{\frac{\Gamma'(\hat{x}^*)}{\Gamma(\hat{x}^*)}} + \alpha < \frac{1}{1 - \alpha}, \tag{13} \]

where I used

\[ \Gamma'(\hat{x}) = \Gamma(\hat{x}) \frac{G'(\hat{x})}{G(\hat{x})} (1 + \hat{x} \Gamma(\hat{x})). \]

Noting, from (12),

\[ a \left\{ \begin{array}{c} > \\ \leq \end{array} \right\} b \iff \hat{x}^* \Gamma(\hat{x}^*) \left\{ \begin{array}{c} > \\ \leq \end{array} \right\} \frac{\alpha}{1 - \alpha}, \]

the first fraction in the middle term in (13) is less than \( 1/\alpha \) if \( a > b \) and it is more than \(-1/(1 - \alpha)\) if \( a < b \). So, whether (13) holds purely depends on the elasticity of \( \delta(z) \) and \( G'/G \) (it holds regardless \( \delta \) and \( G \) when \( a = b \)). So procyclicality of \( z \Gamma(\hat{x}^*)^\alpha \) and \( z \Gamma(\hat{x}^*)^{\alpha-1} \) requires bankruptcy probability changes moderately along business cycles.

Now define the firm-level volatility of profit as the average of \( v_\pi(x|z) \) over entrepreneurs:

\[ \bar{v}_\pi(z) = \frac{1}{1 - G(\hat{x}^*)} \int_{\hat{x}^*}^{\infty} v_\pi(x|z) \, dG. \]

We can show the following proposition.

**Proposition 5** Assume \( \delta(z) < 1/2 \). Suppose \( z \Gamma(\hat{x}^*)^\alpha \) and \( z \Gamma(\hat{x}^*)^{\alpha-1} \) are increasing in \( z \). \( \bar{v}_\pi(z) \) is countercyclical if \( a \leq b \).
Proof.

\[
\frac{d}{dz} \tilde{v}_\pi (z) = \frac{1}{1 - G(\hat{x}^*)} \left[ \int_{\hat{x}^*}^{\infty} \frac{\partial v(x|z)}{\partial z} dG - G'(\hat{x}^*) (v(x|z) - \tilde{v}_\pi (z)) \frac{d\hat{x}^*}{dz} \right]
\]  

(14)

From Proposition 4 and the fact that \( v(x|z) \geq \tilde{v}_\pi (z) \), this is strictly negative if \( d\hat{x}^*/dz \geq 0 \), which corresponds to \( a \leq b \) from Proposition 1.

In (14), the first term in the big parenthesis stands for the intensive margin, which is negative as long as \( z \Gamma(\hat{x}^*)^{\alpha} \) and \( z \Gamma(\hat{x}^*)^{\alpha - 1} \) are procyclical, and the second term (including the sign) is the extensive margin, which could be positive (if \( a > b \)) or negative (if \( a < b \)). Therefore, when \( a > b \), the total effect depends on which margin is dominant. When we consider firm-level volatility with sales measure, there is no intensive margin so that procyclicality or countercyclicality purely determined by the relation between \( a \) and \( b \). Profit volatility introduces the intensive margin and it is countercyclical, the cyclical behavior of volatility can be reversed when \( a > b \), which we interpret as the situation in the United States. Because the source of intensive margins is \( \delta (z) (v(x|z) = (1 - \alpha)^{-2} \sigma^2/x \) if \( \delta (z) = 0 \), the observed countercyclicity of idiosyncratic risk implies the probability of bankruptcy is not negligible and cancels out the procyclicality of extensive margin.

Figures 5 is a numerical example that shows procyclicality of sales volatility and countercyclicality of profit volatility under \( a > b \).

6 Endogenous Bankruptcy Probability

One shortcoming in the above model is that the bankruptcy probability is constant among firms. I use such a setting because of tractability but there is a natural

\[\text{In this numerical example, } x \text{ follows a Pareto distribution (} x = 1): \]

\[G(x) = 1 - x^{-\rho} \quad (\rho > 1)\]

and the probability of bankruptcy is

\[\delta (z) = \bar{\delta} e^{-\beta z}.
\]

The parameters are set at: \( \alpha = 0.6, \rho = 2, \bar{\delta} = 0.1, \beta = 0.1, \sigma^2 = 1/3 \). Surely, \( z \Gamma(\hat{x}^*)^{\alpha} \) and \( z \Gamma(\hat{x}^*)^{\alpha - 1} \) are increasing in \( z \) in this example.
question: what happens if probability of bankruptcy is different among firms? Here, I extend the model in Section 3 with endogenous bankruptcy probability and then present the bankruptcy cost structure is still significant.

Suppose that firms commit to the level of employment and pay wages after shocks are revealed. A firm goes bankrupt if $\epsilon(x) z x^{1-\alpha} l^a < w l$. Let $\delta_x$ be the distribution function of $\epsilon(x)$. I slightly modify the setting of distribution of idiosyncratic shocks for tractability. $\delta_x$ is continuous, defined on a positive support and satisfies $E(\epsilon(x)) = 1$ and $Var(\epsilon(x))$ is strictly decreasing in $x$. So the probability of bankruptcy is $\delta_x(\omega)$ where

$$\omega(1|x, w, z) \equiv \frac{wl^{1-a}}{z x^{1-\alpha}}.$$ 

**Assumption 1** For all $\omega \in [0, \alpha]$ and for all $x$,

$$\frac{\omega \delta_x''(\omega)}{\delta_x'(\omega)} > \frac{\alpha}{1-\alpha},$$
and, for any $x$

$$
\lim_{l \to 0} \frac{\omega (l|x, w, z) \delta'_x (\omega (l|x, w, z))}{l} < \infty.
$$

The other settings are identical to the basic model. The entrepreneur’s maximization problem is

$$
\max_{l} xz^{1-\alpha} l^\alpha - wl - \delta_x (\omega) (a + bl) \quad \text{subject to } \omega = \frac{wl^{1-\alpha}}{xz^{1-\alpha}}.
$$

The first order condition is

$$
\alpha z x^{1-\alpha} l^{\alpha-1} = w + \psi_x (\omega, l),
$$

where

$$
\psi_x (\omega, l) \equiv b \delta_x (\omega) + (1 - \alpha) \frac{\omega \delta'_x (\omega)}{l} (a + bl).
$$

By definition, $\psi_x (\omega, l) \geq 0$ for any $x$ and $l$. Since the marginal expected bankruptcy cost is nonnegative, $l$ should be chosen to satisfy $\omega (l|x, w, z) < \alpha$ from the second line of (15). In other words, $\omega = \alpha$ is the optimal decision when there is no risk of bankruptcy.

The second order condition is not necessarily satisfied because the expected bankruptcy cost, $\delta_x (\omega) (a + bl)$, is not convex in general. However, Assumption 1 guarantees the uniqueness and existence of optimal labor demand.

**Proposition 6** Assume Assumption 1. There uniquely exists $l$ to satisfy (15) The optimal $l(x, w, z)$ is decreasing in $w$ and increasing in $z$. Also, the optimal labor demand, $l(x, w, z)$, is increasing in $x$ if $\psi_x (\omega, l)$ is nonincreasing in $x$ for given $l$ and $\omega$ satisfying $\omega < \alpha$.

**Proof.** To show the existence and uniqueness of demand, it suffices to show that $\lim_{l \to 0} \psi_x (\omega, l) < \infty$ and $\frac{d}{dl} \psi_x (\omega, l) > 0$ when $\omega < \alpha$. The first condition is assumed by the latter half of Assumption 1. The second condition holds true for any given $x$.
because
\[ \frac{d}{dt} \psi_x (\omega, l) = \left( 1 - \alpha \right) \frac{\omega \delta_x' (\omega)}{l} \left[ 2b + \left( 1 - \alpha \right) \frac{a + bl}{l} \left[ \frac{\omega \delta_x'' (\omega)}{\delta_x' (\omega)} - \frac{\alpha}{1 - \alpha} \right] \right] > 0 \]
under Assumption 1.

Next, an increase in \( w \) increases \( \psi_x \) because \( \omega \) is increasing in \( w \), \( \delta_x \) is increasing in \( \omega \), and \( \delta_x' \) is also increasing in \( \omega \) \( (\delta_x'' > 0 \text{ by Assumption 1}) \), while the marginal revenue does not change. Hence, \( l (x, w, z) \) is decreasing in \( w \). On the other hand, an increase in \( z \) decreases \( \omega \) and thus \( \psi_x \), and shifts the marginal revenue upwardly. Hence, \( l (x, w, z) \) is increasing in \( z \).

Finally, an increase in \( x \) increases the left hand side of (15). About the right hand side of (15), for a given \( l \),
\[ \frac{d \psi_x}{dx} = \frac{\partial \psi_x}{\partial \omega} \frac{\partial \omega}{\partial x} + \frac{\partial \psi_x}{\partial x} < 0 \]
since the first term is negative and the second term is also nonpositive by assumption. Hence, \( l (x, w, z) \) is increasing in \( x \). \( \blacksquare \)

From this proposition, one can easily show that the expected profit for a given \( x \) is increasing in \( x \). So, there exist a threshold \( \hat{x} \) above which economic agents become entrepreneurs. Now I write the FE condition:
\[ z \hat{x}^{1 - \alpha} l (\hat{x}, w, z)^{\alpha} - w l (\hat{x}, w, z) - \delta_x (\hat{\omega}) (a + bl (\hat{x}, w, z)) = w, \quad (16) \]
where
\[ \hat{\omega} \equiv \omega (l (\hat{x}, w, z) | \hat{x}, w, z). \]
Note that (16) is upward sloping on the \( \hat{x}-w \) diagram.

The labor market clearing condition in this version of the model is simply
\[ G (\hat{x}) = \int_{\hat{x}}^{\infty} l (x, w, z) dG, \quad (17) \]
which is downward sloping on the \( \hat{x}-w \) diagram. Therefore, as long as \( \hat{x} \) (the intercept of (16)) is sufficiently small, there exists unique equilibrium \( (\hat{x}^*, w^*) \). Similarly to the
basic model, the FE condition shifts leftward in response to an increase in \( z \) because higher \( z \) implies higher expected profits for each \( x \) so less able entrepreneurs can start their business. The LMC condition shifts rightward in response to an increase in \( z \) because higher \( z \) implies more labor demands from each entrepreneur, making \( \hat{x} \) lower to clear the labor market.

In a recession, \( w^* \) definitely decreases but it is ambiguous about \( \hat{x}^* \). To see how does the direction of the shift in \( \hat{x}^* \), rewrite the FE condition in equilibrium as

\[
\frac{1 - \alpha}{\alpha} l (\hat{x}^*, w^*, z) = \frac{w^* + a \delta_{\hat{x}^*} (\hat{\omega}^*)}{w^* + b \delta_{\hat{x}^*} (\hat{\omega}^*) + \frac{\hat{\omega}^* \delta'_{\hat{x}^*} (\hat{\omega}^*)}{l^*} (a + b \hat{\omega}^*)},
\]  

(18)

where \( \hat{l}^* \equiv l (\hat{x}^*, w^*, z), \hat{\omega}^* \equiv \omega (\hat{l}^* | \hat{x}^*, w^*, z) \). The following Proposition 7 shows that the behavior of \( \hat{l}^* \) determines the behavior of \( \hat{x}^* \).

**Proposition 7** If \( \hat{l}^* \) increases in response to an increase in \( z \), \( \hat{x}^* \) increases, vice versa.

**Proof.** First note that a rise in \( z \) implies an upward shift of labor demand schedule for each \( x \). Now suppose an increase in \( z \) leads to a reduction in \( \hat{l}^* \). Then \( \hat{x}^* \) should decrease obviously. Next, suppose an increase in \( z \) leads to an increase in \( \hat{l}^* \). To lead contradiction, suppose that \( \hat{x}^* \) declines. Since labor demands from any entrepreneurs are larger than those before the change, there are greater labor demands after change while labor supply declines by the decline in \( \hat{x}^* \). This never holds in equilibrium, so \( \hat{x}^* \) increases if an increase in \( z \) leads to a rise in \( \hat{l}^* \). □

From Proposition 7 and the supposition that volatility of each firm is decreasing in \( x \), firm-level volatility is countercyclical if the right hand side of (18) is increasing in \( z \), vice versa. In the rest of this section, I focus on behaviors of the right hand side of (18).

Equation (18) is similar to the FE condition in the basic model, (4), except the last term in the denominator,

\[
\frac{\hat{\omega}^* \delta'_{\hat{x}^*} (\hat{\omega}^*)}{l^*} (a + b \hat{\omega}^*),
\]  

(19)
which stands for marginal increase in bankruptcy probability times the current bankruptcy cost. If the impact of this term is negligible, the right hand side of (18) is increasing in $z$ if

$$(a - b) \left( \frac{\delta_{\hat{x}^*} (\hat{\omega}^*)}{dz} - w^* \frac{d\delta_{\hat{x}^*} (\hat{\omega}^*)}{dz} \right) < 0$$

$$\Rightarrow (a - b) \left( \frac{dw^*}{dz} \frac{z}{w^*} - \frac{d\delta_{\hat{x}^*} (\hat{\omega}^*)}{dz} \frac{z}{\delta_{\hat{x}^*} (\hat{\omega}^*)} \right) < 0.$$ 

So, as long as the elasticity of $w^*$ to $z$ is sufficiently greater than the elasticity of $\delta_{\hat{x}^*} (\hat{\omega}^*)$ to $z$ ($d\delta_{\hat{x}^*} (\hat{\omega}^*)/dz \leq 0$ is sufficient), $a < b$ implies that $\hat{l}^*$ and $\hat{x}^*$ are increasing in $z$ so that firm-level volatility is countercyclical like in the basic model.

Since (19) is not negligible in general, the result is much more complicated and the results depend on the distribution $\delta_{\hat{x}}$. In any case, however, the results in the basic model holds true in some extent.

7 Concluding Remarks

This paper has three contributions. First, it documents a contrasting observation of cyclical behaviors of firm-level volatility between Japan and the United States. In the recent three decades, the Japanese firms show countercyclical volatility along medium-term business cycles while firm-level volatility in the United States shows procyclicality (Comin and Mulani (2006)). Second, it presents a simple model to explain the opposite cyclical behaviors. We focused on the extensive margin for the shift in firm-level volatility. Since the bankruptcy cost affects what types of entrepreneurs run businesses and the averaged measurement of firm-level volatility depends on the population of firms, firm-level volatility depends on the bankruptcy cost structure. Third, using the model, this paper provides a policy implication about how bankruptcy cost should be intervened. The model indicates which component of bankruptcy costs is more harmful to the aggregate economy, fixed cost or variable cost of bankruptcy. This is useful when direct estimation of bankruptcy cost is difficult.

As the final note of this paper, I mention about future research in this context. First is dynamics. Since entrepreneurs make decisions about entry and exit, they should also decide those timing considering the current and future economic environ-
ments. So the population of firms must be affected by their dynamic optimization. Second is introduction of endogenous mass of agents. Because the total measure of workers and entrepreneurs is exogenously given in the current setting, individual labor demand and aggregate employment is countercyclical if \( a > b \). Therefore, the model should be extended to have unemployment or elastic mass of agents to fit the model quantitatively. This extension also affects the measure of firms. The present model implies there are more firms in a recession in Japan. If the size of pool of workers/entrepreneurs shrinks in a recession, then the measure of firms would also shrink in a recession even in \( a < b \) and the above results continue to hold.

**References**


