Cointegrated Commodity Pricing Model

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Abstract

In this paper, we propose a commodity pricing model that extends Gibson-Schwartz two-factor model to incorporate the effect of linear relations among commodity prices, which include co-integration under certain conditions. We derive futures and call option pricing formulae, and show that unlike Duan and Pliska (2004), the linear relations among commodity prices, or the error correction term, should affect the commodity derivative prices. Using crude oil and heating oil market data, we estimate the proposed model. The result suggests that there is co-integration among these commodity prices, and that its effect on derivative prices should not be ignored empirically.

Keywords: co-integration, commodity prices, convenience yield, energy.

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1 Introduction

Economy is full of equilibrium relations and co-movements. They are, for example, purchasing power parity, covered or uncovered interest rate parity, spot-forward relations, money demand equation, consumption spending model, relations among commodity prices and so on. Although these relations are widely known, it does not seem that they are sufficiently utilized in finance, especially in the area of derivative valuations.

These relations are modeled as co-integration which was first implicitly used by Davidson, Hendry, Srba, and Yeo (1978) and established by Engle and Granger (1987). Co-integration refers to property that holds among two or more non-stationary time series variables. That is, if certain linear combinations of several non-stationary variables become stationary, these variables are said to be co-integrated. It is interpreted as long term relationship or equilibrium among variables. This is because co-integrated variables are tied to each other to keep certain linear combinations stationary, and hence tend to move together. Thus, it is natural to consider whether and how such co-movements among co-integrated variables affect prices of their derivatives.

While academic papers that analyze co-integration relationship among economic variables are abound, researches on derivative pricing with co-integration are limited. In our best knowledge, Duan and Pliska (2004) was the first to incorporate co-integration effect in derivative pricing. They focused on stocks and priced their options under the assumption termed the local risk-neutral valuation relationship, which by definition implies that the drift terms of stock returns are equal to the risk-free rate under the risk-neutral probability. In this setting, they concluded that co-integration affects option prices only when volatilities are stochastic.

Commodity prices, however, behave differently from stock prices. They are strongly affected by production and inventory conditions, and tend to deviate temporarily from the prices that would hold without those effects. These characteristics are recognized from the theory of storage by Kaldor (1939) and Working (1948). To incorporate such temporary deviations, convenience yield is introduced, which is one of the crucial ingredients in commodity pricing models. For example, Gibson and Schwartz (1990, 1993) proposed a two-factor model with spot commodity prices and mean reverting convenience yields, and priced commodity futures and options. Schwartz (1997) investigated three different (one-, two-, and three-factor) models including the Gibson-Schwartz two-factor model using the data of crude oil,
gold, and copper prices, and analyzed their long term hedging strategies. Smith and Schwartz (2000) have modeled commodity dynamics in a different setting using long and short term factors and finds that the model is equivalent with the Gibson-Schwartz model. There are many other models that generalize the above including Miltersen and Schwartz (1998), Nielsen and Schwartz (2004), Casassus and Collin-Dufresne (2005).

When convenience yield exists, the drift of commodity prices may deviate from the risk-free rate even under the risk-neutral probability. Thus, in the standard commodity pricing models, Duan and Pliska (2004)’s risk-neutral valuation framework does not hold, and their results cannot be directly applied to commodity derivative pricing. This is the reason why we need to extend Duan and Pliska (2004)’s framework and investigate commodity pricing with co-integration.

For this purpose, we generalize the Gibson-Schwartz two-factor model by explicitly incorporating linear relations among commodity prices, which include co-integration under certain conditions. More specifically, we formulate a commodity price model in which the temporary deviation of drift terms from the risk-free rate under the risk-neutral probability is described by convenience yield and linear relations among commodity prices, or error correction terms under appropriate conditions. Since in the preceding papers, such temporary deviation has been modeled as a whole by convenience yield, this paper can also be regarded as a proposal of a new model for specifying a part of the temporary deviation in terms of linear relations among prices that include co-integration.

It is important to notice that few preceding papers on commodities have incorporated co-integration in their derivative pricing models. One exception is the research by Dempster, et al. (2008) who analyzed spread options on two commodity prices where their spread was assumed to be stationary. However, they did not explicitly model the spot prices, and directly modeled the spread instead. This approach made their model simple, but did not enable us to price futures and options on each commodities, not their spread, whose prices were co-integrated.

Therefore, this paper is the first to investigate the effect of linear relations among prices that include co-integration on derivative pricing of commodities for which Duan and Pliska (2004)’s risk-neutral valuation does not hold, and to evaluate derivatives on commodities prices, not their spreads. More precisely, we formulate the Gibson-Schwartz two-factor model with linear relations among commodity prices, or co-integration under certain conditions,
and obtain the analytic formula for commodity futures and options. We also investigate empirically the effect of such linear relations on derivatives prices using the data of crude oil and heating oil both traded in NYMEX.

The rest of this paper is organized as follows: In section 2, we model commodity spot prices and convenience yields with co-integration using error correction term in the drift of spot prices. We also investigate the relation between our model and Gibson-Schwartz model, and derive the closed-form pricing formulae of futures and call options. In section 3, we show the state equation and observation equation for the Kalman filter, and conduct empirical analysis using crude oil and heating oil. Section 4 gives discussions, and section 5 concludes.

2 The Model

2.1 Gibson-Schwartz two-factor with Co-integration (GSC) Model

We propose a model that extends Gibson-Schwartz two-factor model, hereafter the GS model, (Gibson and Schwartz (1990), Schwartz (1997)) to explicitly incorporate linear relations among commodity prices, or co-integration under certain conditions. We adopt the continuous-time specification of co-integrated systems shown by Duan and Pliska (2004). As usual in commodity price models, we start with describing the behavior of spot prices and convenience yields under the risk-neutral probability.

Assume that there are \(n\) commodities whose spot prices and convenience yields follow

\[
\begin{align*}
\text{d}S_i(t) &= S_i(t) \left( r - \delta_i(t) + b_iz(t) \right) dt + S_i(t) \sigma_i \text{d}W_S(t), \quad i = 1, \ldots, n \tag{1} \\
\text{d}\delta_i(t) &= \kappa_i (\hat{\alpha}_i - \delta_i(t)) dt + \sigma_{\delta_i} \text{d}W_{\delta_i}(t), \quad i = 1, \ldots, n. \tag{2}
\end{align*}
\]

under the risk-neutral probability. Here, \(r\) is risk-free interest rate which is assumed to be constant. \(b_i, \sigma_{S_i}, \kappa_i, \hat{\alpha}_i,\) and \(\sigma_{\delta_i}\) are constant coefficients. \(W(t) = [W_{S_1}(t) \ldots W_{S_n}(t) W_{\delta_1}(t) \ldots W_{\delta_n}(t)]^\top\) is the \(2n\) dimensional Brownian motion under the risk-neutral probability with

\[
\text{d}W_{S_i}(t) \text{d}W_{S_j}(t) = \rho_{S_iS_j} dt, \text{d}W_{S_i}(t) \text{d}W_{\delta_j}(t) = \rho_{S_i\delta_j} dt, \text{d}W_{\delta_i}(t) \text{d}W_{\delta_j}(t) = \rho_{\delta_i\delta_j} dt \quad i, j = 1, \ldots, n.
\]
We assume that the commodity prices are related linearly through

\[ z(t) = \mu_z + a_0 t + \sum_{i=1}^{n} a_i \ln S_i(t) \]  (3)

where \( \mu_z, a_0, \) and \( a_i \)'s are constant.\(^1\) Assume that \( \ln S_i \) are co-integrated, then by rearranging the equation as \( \ln S_1(t) = (-\mu_z - a_0 t - \sum_{i=2}^{n} a_i \ln S_i(t) + z(t))/a_1, \) if \( a_1 \neq 0, \) \( z(t) \) can be interpreted as an error correction term, \( a_i \) as co-integration vectors, and \( b_i \) as adjustment speed of the error correction term. Using Ito's lemma, the dynamics of \( z(t) \) is

\[ dz(t) = -b(m - z(t))dt + \sum_{i=1}^{n} a_i \delta_i(t)dt + \sum_{i=1}^{n} a_i \sigma_i dW_{S_i} (t) \]  (4)

\[ b = \sum_{i=1}^{n} b_i a_i \]

\[ m = \frac{-a_0 - \sum_{i=1}^{n} a_i r + \frac{1}{2} \sum_{i=1}^{n} a_i \sigma_i^2}{b} \]  (5)

The set of equations (1), (2), and (3) corresponds to the GS model with one exception that a linear relation \( z(t) \) among prices affects the drift terms. \( z(t) \) represents the error correction term of co-integration among commodity prices when they are co-integrated. Thus, we call this model the Gibson-Schwartz 2-factor with co-integration model, hereafter the GSC model.

It is worth mentioning that while the GSC model bases its specification of co-integration on that of Duan and Pliska (2004), their price dynamics, especially the drift terms, under the risk-neutral probabilities are different. This difference comes from the characteristics of underlying assets. For stocks, which Duan and Pliska (2004) focused on, it is natural to assume that the drift terms of returns should be equal to the risk-free rate under the risk-neutral probability. On the other hand, for commodities, it is standard to assume that the drift terms may deviate temporarily from the risk-free rate.

\(^1\)Although we treat the case where there is only one linear relation among prices i.e., the case with one dimensional \( z(t) \), we can extend the model to have several linear relations, or multi-dimensional \( z(t) \).
even under the risk-neutral probability by reflecting inventory and production conditions. The GSC model assumes that such deviation is described by convenience yield and the term \( z(t) \). Consequently, in the GSC model, Duan and Pliska (2004)'s risk-neutralization method does not hold, and linear relations of prices, or co-integration under certain conditions, affect derivative prices even though volatilities \( \sigma_{S_i} \) are constant.

2.2 Relation between GSC and GS models

Since the GSC model specifies a part of temporary deviation of drifts from the risk-free rate by the linear relation \( z(t) \) and the rest by convenience yield, while the GS model (and other usual commodity price models) specifies the whole temporary deviation by convenience yield, it is interesting to investigate the relation between the GSC and GS models.

To see this, let us fix \( i = 1 \). To distinguish the GSC and GS models, we assume the GS model described by

\[
\begin{align*}
    dS_1(t) &= S_1(t)(r - \delta'_1(t))dt + S_1(t)\sigma_{S_1}dW_{S_1}(t) \\
    d\delta'_1(t) &= \kappa'_1(\alpha'_1 - \delta'_1(t))dt + \sigma'_{S_1}dW_{\delta_1}(t).
\end{align*}
\]

where \( \delta'_1 \) is the convenience yield in the GS model.

Now, if \( b_1 = 0 \) in the GSC model, then \( \delta'_1 = \delta_1 \), and the GSC model clearly coincides with the GS model. On the other hand, if \( b_1 \neq 0 \), then for both models to describe exactly the same price process, we must have \( \delta'_1(t) = \delta_1(t) - b_1z(t) \). Its dynamics is given by

\[
\begin{align*}
    d\delta'_1(t) &= d\delta_1(t) - b_1dz(t) \\
    &= \{ \kappa_1\hat{\alpha}_1 + b_1bm - b_1bz(t) + (b_1a_1 - \kappa_1)\delta_1(t) + b_1\sum_{i=2}^{n} a_i\delta_i(t) \}dt \\
    &\quad + (\sigma_{\delta_1} - b_1a_1\sigma_{S_1})dW_{\delta_1}(t) - b_1\sum_{i=2}^{n} a_i\sigma_{S_i}dW_{S_i}(t)
\end{align*}
\]

By the uniqueness of drift and volatility terms, we have

\[
\begin{align*}
    \kappa'_1\alpha'_1 - \kappa'_1\delta'_1(t) &= \kappa_1\hat{\alpha}_1 + b_1bm - b_1bz(t) + (b_1a_1 - \kappa_1)\delta_1(t) + b_1\sum_{i=2}^{n} a_i\delta_i(t) \\
    \sigma_{\delta'_1} &= \sigma_{\delta_1} - b_1a_1\sigma_{S_1} \\
    0 &= -b_1a_i\sigma_{S_i} \quad i = 2, \ldots, n
\end{align*}
\]
Focusing on the drift term and the corresponding coefficients of \( z(t), \delta_i(t) \) and the constant term, we have

\[
\begin{align*}
\kappa' \alpha' &= \kappa_1 \alpha_1 + b_1 m \\
-\kappa' &= b_1 a_1 - \kappa_1 \\
\kappa' b_1 &= -b_1 b \\
b_1 a_i &= 0.
\end{align*}
\]

The forth equation leads to \( a_i = 0 (i = 2, \ldots, n) \) and the second and third equations imply

\[
\begin{align*}
\kappa' &= -b = b_1 a_1 + \kappa_1 \\
\Rightarrow \kappa_1 &= 0 \\
\alpha' &= -b_1 m.
\end{align*}
\]

Hence, the long term mean of convenience yield \( \delta'_1 \) should be \( -b_1 m \) and the adjustment speed of convenience yield is \( b_1 a_1 \).

Thus, we obtain that if \( b_1 \neq 0 \), for the GSC and GS models describe exactly the same price process, it should hold that \( a_i = 0 (i = 2, \ldots, n) \) and

\[
\begin{align*}
d\delta_1(t) &= \sigma_{\delta_1} dW_{\delta_1}(t) \\
z(t) &= \mu_z + a_0 t + a_1 \ln S_1(t) \\
dz(t) &= \kappa' \delta_1(t) + \sigma' \delta_1(t) dt + a_1 \sigma_{S_1} dW_{S_1}(t) \\
b &= b_1 a_1 \\
m &= \frac{-a_0 - a_1 r + \frac{1}{2} a_1 \sigma_{S_1}^2}{b}.
\end{align*}
\]

We can now summarize the relation between the GSC and GS models as follows:

1. If \( b_1 = 0 \), then the GSC and GS models coincide.

2. If \( b_1 \neq 0 \) and \( a_i = 0 (i = 2, \ldots, n) \), then for the GSC and GS models to describe exactly the same price process, the following should hold.

\[
\begin{align*}
d\delta'_1(t) &= \kappa'_1 (\alpha'_1 - \delta'_1(t)) dt + \sigma' \delta'_1 dt \\
\kappa'_1 &= -b_1 a_1 \\
\alpha'_1 &= -b_1 m \\
\sigma' &= \sigma_{\delta_1} - b_1 a_1 \sigma_{S_1}.
\end{align*}
\]
and the GSC convenience yield $\delta_1$ is

$$d\delta_1(t) = \sigma_{\delta_1} dW_{\delta_1}(t)$$

with error term $z(t)$ as

$$z(t) = \mu_z + a_0 t + a_1 \ln S_1(t)$$

3. If $b_1 \neq 0$ and at least one of $a_i$ ($i = 2, \ldots, n$) is not 0, then the GSC and GS models cannot describe the same price process.

The GSC model includes the GS model as a special case, and in the third case above, utilizing the GS model might cause misspecification if the GSC model is correct.

### 2.3 Futures and Option Prices for GSC model

We can derive the futures and European call option prices on commodity $i$ in the closed forms. Note that under the assumptions above, the spot price
of commodity $i$ is calculated as

$$S_i(T) = S_i(t) \exp \{X_i(t, T)\}$$

$$X_i(t, T) = \left( r + b_i m - \frac{\sigma_i^2}{2} - \hat{\alpha}_i + \sum_{j=1}^{n} b_i a_j \hat{\delta}_j \right) (T - t)
+ \frac{b_i (m - z(t))}{b} (1 - e^{b(T-t)}) - \sum_{j=1}^{n} \frac{b_i a_j \hat{\alpha}_j}{b(b + \kappa_j)} (e^{b(T-t)} - e^{-\kappa_j(T-t)})
- \sum_{j=1}^{n} \frac{b_i a_j \hat{\alpha}_j}{b^2} (e^{b(T-t)} - 1) + \sum_{j=1}^{n} \frac{b_i a_j \hat{\alpha}_j}{b(b + \kappa_i)} (e^{b(T-t)} - e^{-\kappa_i(T-t)})
+ \frac{(\hat{\alpha}_i - \delta_i(t))}{\kappa_i} (1 - e^{-\kappa_i(T-t)}) - \sum_{j=1}^{n} \frac{b_i a_j (\hat{\alpha}_j - \delta_j(t))}{b\kappa_j} (1 - e^{-\kappa_j(T-t)})
+ \sigma_i \{W_{S_i}(T) - W_{S_i}(t)\} - \frac{1}{\kappa_i} \sigma_i \{\dot{W}_{\delta_i}(T) - \dot{W}_{\delta_i}(t)\}
- \sum_{j=1}^{n} \frac{b_i a_j}{b} \sigma_j \{W_{S_j}(T) - W_{S_j}(t)\} + \sum_{j=1}^{n} \frac{b_i a_j}{b\kappa_j} \sigma_j \{W_{S_j}(T) - W_{S_j}(t)\}
+ \sum_{j=1}^{n} \frac{b_i a_j}{b} \int_{t}^{T} e^{b(T-s)} \sigma_j dW_{S_j}(s)
+ \int_{t}^{T} \frac{e^{-\kappa_i(T-s)}}{\kappa_i} \sigma_i dW_{\delta_i}(s) - \sum_{j=1}^{n} \int_{t}^{T} \frac{b_i a_j e^{-\kappa_j(T-s)}}{b\kappa_j} \sigma_j dW_{\delta_j}(s)
- \sum_{j=1}^{n} \int_{t}^{T} \frac{b_i a_j}{b(b + \kappa_j)} (e^{b(T-s)} - e^{-\kappa_j(T-s)}) \sigma_j dW_{\delta_j}(s).$$

Denote by $E[\cdot]$ the expectation under the risk-neutral probability. Using risk neutrality and property of moment generating function, we obtain the futures price of commodity $i$ as follows (cf. Cox, et al., 1981).

**Proposition 2.1.** Assuming (1), (2), and (3), the futures price of commodity $i$ with maturity $t$ is given by

$$G_i(t, T) = E_t[S_i(T)] = S_i(t) \exp \left\{ \mu_{X_i}(t, T) + \frac{\sigma_{X_i}^2(t, T)}{2} \right\}$$
where

\[ \mu_{X_i}(t, T) = \frac{E_t[X_i(t, T)]}{T - t} \]

\[ = \left( r + b_i m - \frac{\sigma_i^2}{2} - \delta_i + \sum_{j=1}^{n} \frac{b_i a_j \alpha_j}{b} \right) (T - t) \]

\[ + \frac{b_i (m - z(t))}{b} \left( 1 - e^{b(T-t)} \right) - \sum_{j=1}^{n} \frac{b_i a_j \delta_j(t)}{b(b + \kappa_j)} \left( e^{b(T-t)} - e^{-\kappa_j(T-t)} \right) \]

\[ - \sum_{j=1}^{n} \frac{b_i a_j \alpha_j}{b^2} (e^{b(T-t)} - 1) + \sum_{j=1}^{n} \frac{b_i a_j \hat{\alpha}_j}{b(b + \kappa_i)} \left( e^{b(T-t)} - e^{-\kappa_j(T-t)} \right) \]

\[ + \frac{(\hat{\alpha}_i - \delta_i(t))}{\kappa_i} \left( 1 - e^{-\kappa_i(T-t)} \right) - \sum_{j=1}^{n} \frac{b_i a_j (\hat{\alpha}_j - \delta_j(t))}{b \kappa_j} \left( 1 - e^{-\kappa_j(T-t)} \right) \]

and
\[
\sigma^2_{X_i}(t, T) = E_t[(X_i(t, T) - \mu_{X_i}(t, T))^2] \\
= \left( \frac{\sigma_{S_i}^2}{\kappa_i^2} + \frac{\sigma_{S_i}^2}{\kappa_i^2} - \frac{2\sigma_{S_i}\delta_i}{\kappa_i} - \sum_{j=1}^{n} \frac{2b_ja_j\sigma_j\delta_j}{b\kappa_j\kappa_j} \right) \\
+ \sum_{j=1}^{n} \frac{2b_ja_j\sigma_j\delta_j}{b\kappa_j} + \sum_{j,k=1}^{n} \frac{b_j^2a_jak\delta_j\delta_k}{b\kappa_j\kappa_k} + \sum_{j=1}^{n} \frac{2b_ja_j\sigma_j\delta_j}{b\kappa_j} - \sum_{j,k=1}^{n} \frac{2b_j^2a_jak\sigma_j\delta_k}{b^2\kappa_j} \\
- \sum_{j=1}^{n} \frac{2b_ja_j\sigma_j\delta_j}{b} + \sum_{j,k=1}^{n} \frac{b_j^2a_jak\sigma_j\delta_k}{b^2} (T - t) \\
+ \frac{\sigma_{S_i}^2}{2\kappa_i^2}(1 - e^{-2\kappa_i(T-t)}) - \sum_{j=1}^{n} \frac{2b_ja_j\sigma_j\delta_j}{b\kappa_j} (1 - e^{-(\kappa_i + \kappa_j)(T-t)}) \\
+ \sum_{j,k=1}^{n} \frac{b_j^2a_jak\sigma_j\delta_k}{b^2\kappa_j\kappa_k}(1 - e^{-(\kappa_j + \kappa_k)(T-t)}) \\
+ \sum_{j,k=1}^{n} \frac{b_j^2a_jak\sigma_j\delta_k}{b^2(b + \kappa_j)(b + \kappa_k)} \left\{ \frac{-1}{2b} (1 - e^{2b(T-t)}) - \frac{1}{\kappa_j - b} (1 - e^{-(\kappa_j - b)(T-t)}) \right\} \\
- \frac{1}{\kappa_j - b} (1 - e^{-(\kappa_k - b)(T-t)}) + \frac{1}{\kappa_j + \kappa_k} (1 - e^{-(\kappa_j + \kappa_k)(T-t)}) \right\} \\
- \sum_{j,k=1}^{n} \frac{b_j^2a_jak\sigma_j\delta_k}{b^2(2b(T-t))} \\
- \sum_{j,k=1}^{n} \frac{2b_j^2a_jak\sigma_j\delta_k}{b^2(b + \kappa_j)} \left\{ \frac{-1}{2b} (1 - e^{2b(T-t)}) - \frac{1}{\kappa_j - b} (1 - e^{-(\kappa_j - b)(T-t)}) \right\} \\
+ 2 \left\{ \left( \frac{\sigma_{S_i}^2}{\kappa_i^2} + \frac{\sigma_{S_i}^2}{\kappa_i^2} - \sum_{j=1}^{n} \frac{b_ja_j\sigma_j\delta_j}{b\kappa_j^2\kappa_j} - \sum_{j=1}^{n} \frac{b_ja_j\sigma_j\delta_j}{b\kappa_j^2\kappa_j} \right) (1 - e^{-\kappa_i(T-t)}) \right\} \\
+ \sum_{j=1}^{n} \frac{2(b_ja_j\sigma_j\delta_j)}{b\kappa_j^2\kappa_j} - \sum_{j=1}^{n} \frac{b_j^2a_jak\sigma_j\delta_k}{b^2\kappa_j^2\kappa_k} + \sum_{k=1}^{n} \frac{b_j^2a_jak\sigma_k\delta_k}{b^2\kappa_k^2\kappa_k} \right\} \\
\times (1 - e^{-\kappa_i(T-t)})
Assuming (1), (2), and (3), the European call option price of commodity Proposition 2.2.

and Pliska (1981), we now obtain the European call option price as follows:

\[
C_i(t, T) = e^{-r(T-t)} E_t[(S_i(T) - K)^+] \\
= S_i(t) e^{-r(T-t) + \mu X_i(t, T) + \frac{1}{2} \sigma^2_{X_i}(t, T)} \Phi(d_{i1}) - K e^{-r(T-t)} \Phi(d_{i2}) \\
d_{i1} = \frac{\ln(S_i(t)/K) + \mu X_i(t, T) + \sigma^2_{X_i}(t, T)}{\sigma_{X_i}(t, T)} \\
d_{i2} = d_{i1} - \sigma_{X_i}(t, T).
\]

Applying the usual method by Harrison and Kreps (1979) or Harrison and Pliska (1981), we now obtain the European call option price as follows:

**Proposition 2.2.** Assuming (1), (2), and (3), the European call option price of commodity $i$ with maturity $t$ is given by

\[
C_i(t, T) = e^{-r(T-t)} E_t[(S_i(T) - K)^+] \\
= S_i(t) e^{-r(T-t) + \mu X_i(t, T) + \frac{1}{2} \sigma^2_{X_i}(t, T)} \Phi(d_{i1}) - K e^{-r(T-t)} \Phi(d_{i2}) \\
d_{i1} = \frac{\ln(S_i(t)/K) + \mu X_i(t, T) + \sigma^2_{X_i}(t, T)}{\sigma_{X_i}(t, T)} \\
d_{i2} = d_{i1} - \sigma_{X_i}(t, T).
\]

Notice that in both futures and call option pricing formulae, $a_i$ and $b_i$, the coefficients for the error term $z(t)$ appear through $\mu X_i$ and $\sigma_{X_i}$. This implies the futures and option prices of commodities depend on co-integration. Recall also that the volatilities $\sigma_{S_i}$ of underlying commodity prices are assumed to
be constant. Thus, as long as the assumptions made on the GSC model are correct, co-integration should affect derivative prices even though the volatilities of underlying commodity prices are constant.

The next question is how much co-integration affects derivative prices. In the following section, we empirically investigate this point.

3 Empirical Analysis

3.1 The Dynamics of Commodity Spot Prices, Convenience Yields, and Error Term under Natural Probability

Since neither commodity spot prices nor convenience yields are observable, we have to estimate the parameters using the Kalman filter with their futures prices. Since we have modeled commodity spot prices and convenience yields under the risk-neutral probability, we need to specify the sdes of commodity spot prices, convenience yields, and the linear relation term \( z(t) \) under the natural probability to estimate the model. For this purpose, we have to formulate the market price of risk which transforms the risk-neutral probability to the natural probability.

Let us assume that Brownian motions under the risk-neutral probability \( W(t) \) and Brownian motions under the natural probability \( W^P(t) \) satisfies

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Footnote:

2For the Kalman Filter, see Hamilton (1994).
\[ W(t) = W^P(t) + \int_0^t \theta(s, \delta(s), z(s)) ds \]
\[ W(t) = [ W_{S_1}(t) \cdots W_{S_n}(t) \ W_{\delta_1}(t) \cdots W_{\delta_n}(t) ]^\top \]
\[ W^P(t) = [ W_{S_1}^P(t) \cdots W_{S_n}^P(t) \ W_{\delta_1}^P(t) \cdots W_{\delta_n}^P(t) ]^\top \]
\[ \theta(s, \delta(s), z(s)) = \hat{\beta}_0 + \sum_{i=1}^n \hat{\beta}_{\delta_i} \delta_i(t) + \hat{\beta}_z z(t) \]

\[ \hat{\beta}_0 = \begin{bmatrix} \hat{\beta}_{S_0} \\ \vdots \\ \hat{\beta}_{\delta_0} \\ \vdots \\ \hat{\beta}_{S_n} \end{bmatrix}, \quad \hat{\beta}_{\delta_i} = \begin{bmatrix} \hat{\beta}_{S_i, \delta_i} \\ \vdots \\ \hat{\beta}_{S_i, \delta_0} \\ \vdots \\ \hat{\beta}_{\delta_i, \delta_0} \end{bmatrix}, \quad \hat{\beta}_z = \begin{bmatrix} \hat{\beta}_{S_z} \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \]

\( \theta \) is the market price of risk which is assumed to be linear in \( \delta_1(t), \ldots, \delta_n(t), \) and \( z(t) \). Since this assumption includes the case where market price of risk to be constant, it is a generalized form of the GS model. The consequence of this assumption can be seen in the following sdes.

\[ d\ln S_i(t) = (\beta_{S_i,0} + \beta_{S_i,\delta_i} \delta_i(t) + \beta_{S_i,z} z(t)) dt + \sigma_{S_i} dW^P_{S_i}(t) \] (9)
\[ \beta_{S_i,0} = r - \frac{\sigma^2_{S_i}}{2} + \sigma_{S_i} \hat{\beta}_{S_i,0}, \beta_{S_i,\delta_i} = -1 + \sigma_{S_i} \hat{\beta}_{\delta_i}, \beta_{S_i,z} = b_i + \sigma_{S_i} \hat{\beta}_{S_i,z} \]
\[ d\delta_i(t) = (\beta_{\delta_i,0} + \beta_{\delta_i,\delta_i} \delta_i(t)) dt + \sigma_{\delta_i} dW^P_{\delta_i}(t) \] (10)
\[ \beta_{\delta_i,0} = \kappa_i \hat{\alpha}_i + \sigma_{\delta_i} \hat{\beta}_{\delta_i,0}, \beta_{\delta_i,\delta_i} = -\kappa_i + \sigma_{\delta_i} \hat{\beta}_{\delta_i}, \]
\[ dz(t) = (\beta_{z,0} + \sum_{j=1}^n \beta_{z,\delta_j} \delta_j(t) + \beta_{z,z} z(t)) dt + \sum_{i=1}^n a_i \sigma_{S_i} dW^P_{S_i}(t) \] (11)
\[ \beta_{z,0} = -bm + \sum_{i=1}^n a_i \sigma_{S_i} \hat{\beta}_{S_i,0}, \beta_{z,\delta_i} = -a_i + a_i \sigma_{S_i} \hat{\beta}_{S_i,\delta_i}, \beta_{z,z} = b + \sum_{i=1}^n a_i \sigma_{S_i} \hat{\beta}_{S_i,z} \]

Note that convenience yields do not depend on commodity prices or the term \( z(t) \). On the other hand, the drift term of \( z(t) \) depends not only on \( z(t) \) but also on convenience yields.

--Note that this assumption satisfies the condition for Gisanov theorem. See Liptser and Shiryaev (2000), section 6.2, example 3 (b).
Each log commodity price depends on its corresponding convenience yield and the linear relation term $z(t)$. Most of the existing models assume that the drift terms of commodity spot prices may deviate from the risk-free rate, and that the convenience yields represent such deviation. While convenience yields are usually explained by the theory of storage, there may be other causes, such as transaction costs or impacts from other commodity prices, that make the drift terms to deviate temporarily from the risk-free rate. In the GSC model, we use the concept of co-integration, and enhance the GS model to incorporate by the term $z(t)$ these other elements for deviation. In other words, this model can be interpreted as formulating the drift term of a commodity spot price in terms of three components; the risk-free rate, convenience yield, and the linear relation term $z(t)$. Furthermore, the impact from other commodity prices and convenience yields are passed through the term $z(t)$ to each commodity prices.

In appendices, we derive the solution for sdes (9), (10), and (11) and show the state equation, observation equation, the Kalman filter, forecasts, and the maximum likelihood for this model. We also provide a generalized state equation which assumes that the market price of risk is linear with $\ln S_1(t), \ldots, \ln S_n(t), \delta_1(t), \ldots, \delta_n(t)$, and $z(t)$.

### 3.2 Data

We use WTI and heating oil daily closing prices traded at NYMEX from January 2, 1990 to February 27, 2009. Five futures contracts labeled Maturity 1, Maturity 3, Maturity 5, Maturity 7, Maturity 9, are used in the estimation. Maturity 1 stands for the contract closest to maturity, Maturity 3 stands for the third closest maturity, and so on. Time to maturity corresponding to these prices are also used. We fixed the risk-free rate to be 4%.

The basic statistics for these data are described in Table 1. Since the maturity dates are fixed, the time to maturity changes as time progresses. Comparing WTI crude oil with heating oil, we can see that the standard deviation of heating oil is higher because the average price of heating oil is higher than that of crude oil. The mean maturity and its standard deviation are quite close to each other. Also, note that the correlation between WTI futures price and heating oil are 0.996.
Figure 1: WTI and heating oil daily closing price from January 2, 1990 to February 27, 2009. Solid line and dashed line are the price of crude oil and heating oil, respectively.
Table 1: Statistics of Data.

<table>
<thead>
<tr>
<th>Futures Contract</th>
<th>Mean price</th>
<th>Mean maturity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(Standard deviation)</td>
<td>(Standard deviation)</td>
</tr>
<tr>
<td>WTI crude oil</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Maturity 1</td>
<td>34.14 (23.89)</td>
<td>0.10 (0.04)</td>
</tr>
<tr>
<td>Maturity 3</td>
<td>34.10 (24.15)</td>
<td>0.35 (0.04)</td>
</tr>
<tr>
<td>Maturity 5</td>
<td>33.91 (24.30)</td>
<td>0.59 (0.04)</td>
</tr>
<tr>
<td>Maturity 7</td>
<td>33.71 (24.40)</td>
<td>0.83 (0.04)</td>
</tr>
<tr>
<td>Maturity 9</td>
<td>33.51 (24.45)</td>
<td>1.08 (0.04)</td>
</tr>
<tr>
<td>Heating oil</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Maturity 1</td>
<td>95.36 (68.24)</td>
<td>0.09 (0.04)</td>
</tr>
<tr>
<td>Maturity 3</td>
<td>95.59 (69.33)</td>
<td>0.34 (0.04)</td>
</tr>
<tr>
<td>Maturity 5</td>
<td>95.41 (70.06)</td>
<td>0.58 (0.04)</td>
</tr>
<tr>
<td>Maturity 7</td>
<td>95.04 (70.45)</td>
<td>0.83 (0.04)</td>
</tr>
<tr>
<td>Maturity 9</td>
<td>94.56 (70.40)</td>
<td>1.07 (0.04)</td>
</tr>
</tbody>
</table>

3.3 Estimation Results

Now, we estimate the model using the Kalman filter. In table 2, we report the estimated parameters with standard errors. As we can see, the estimated co-integration vectors are (4.90, -7.65) and the adjustment speed are 0.18 and 0.30 respectively. These values are significant, which suggests that there exists co-integration and hence that the GS model is misspecified. The values of $b_1, b_2$ measures to what degree co-integration affects the spot prices. It is suggested that heating oil price is affected by co-integration much more than crude oil price. Note that $a_0$ is significant, which means that the term $z(t)$ includes time drift.

Both $\alpha$ in GSC model are estimated as 0. Which signifies that there were no long term means of convenience yields. However, in the GS model $\alpha$ are significantly positive. Also, notice that $\kappa_s$ in the GS and the GSC model are different. These are caused by the term $z(t)$, which suggest that the long term mean have been replaced by $z(t)$.

Let us turn to the volatility parameters. Crude oil and heating oil spot prices have positive correlation. The corresponding spot price and conve-
Table 2: Parameters estimates and standard errors in paranthesis. Data are WTI and heating oil daily closing prices traded at NYMEX from January 2, 1990 to February 27, 2009.

<table>
<thead>
<tr>
<th></th>
<th>GS crude oil</th>
<th>GS heating oil</th>
<th>GSC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility Parameters</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_{S_1}$</td>
<td>0.350524</td>
<td>0.672612</td>
<td></td>
</tr>
<tr>
<td>$\sigma_{S_2}$</td>
<td>0.335416</td>
<td>0.711369</td>
<td></td>
</tr>
<tr>
<td>$\delta_{S_1}$</td>
<td>0.102625</td>
<td>0.739491</td>
<td></td>
</tr>
<tr>
<td>$\delta_{S_2}$</td>
<td>0.107671</td>
<td>0.516627</td>
<td></td>
</tr>
<tr>
<td>$\rho_{S_1,S_2}$</td>
<td>0.998027</td>
<td>0.531324</td>
<td></td>
</tr>
<tr>
<td>$\rho_{S_1,\delta_1}$</td>
<td>0.102625</td>
<td>0.739491</td>
<td></td>
</tr>
<tr>
<td>$\rho_{S_2,\delta_1}$</td>
<td>0.107671</td>
<td>0.516627</td>
<td></td>
</tr>
<tr>
<td>$\rho_{S_1,\delta_2}$</td>
<td>0.843748</td>
<td>0.000003</td>
<td></td>
</tr>
<tr>
<td>$\rho_{S_2,\delta_2}$</td>
<td>0.000003</td>
<td>0.278442</td>
<td></td>
</tr>
<tr>
<td>Convenience yield parameters</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\kappa_1$</td>
<td>1.116475</td>
<td>2.683139</td>
<td></td>
</tr>
<tr>
<td>$\kappa_2$</td>
<td>0.614441</td>
<td>2.109540</td>
<td></td>
</tr>
<tr>
<td>$\tilde{\alpha}_1$</td>
<td>0.040246</td>
<td>-0.003669</td>
<td></td>
</tr>
<tr>
<td>$\tilde{\alpha}_2$</td>
<td>0.046778</td>
<td>0.032036</td>
<td></td>
</tr>
<tr>
<td>Cointegration parameters</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b_1$</td>
<td>0.178504</td>
<td>0.018504</td>
<td></td>
</tr>
<tr>
<td>$b_2$</td>
<td>0.297776</td>
<td>0.003617</td>
<td></td>
</tr>
<tr>
<td>$a_0$</td>
<td>-0.069018</td>
<td>0.000954</td>
<td></td>
</tr>
<tr>
<td>$a_1$</td>
<td>4.895701</td>
<td>0.030117</td>
<td></td>
</tr>
<tr>
<td>$a_2$</td>
<td>-7.649909</td>
<td>0.046642</td>
<td></td>
</tr>
<tr>
<td>Market price of risk parameters</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_{S_0}$</td>
<td>0.366217</td>
<td>0.005565</td>
<td></td>
</tr>
<tr>
<td>$\beta_{S_1}$</td>
<td>0.10984 (0.237399)</td>
<td>-0.301429 (0.361120)</td>
<td></td>
</tr>
<tr>
<td>$\beta_{S_2}$</td>
<td>0.012156</td>
<td>0.108242</td>
<td></td>
</tr>
<tr>
<td>$\beta_{S_3}$</td>
<td>-0.014736 0.014736 (0.113800)</td>
<td>0.014181 (0.289836)</td>
<td></td>
</tr>
<tr>
<td>$R(1,1)$</td>
<td>0.000551</td>
<td>0.000252</td>
<td></td>
</tr>
<tr>
<td>$R(2,2)$</td>
<td>0.000001</td>
<td>0.000000</td>
<td></td>
</tr>
<tr>
<td>$R(3,3)$</td>
<td>0.000009</td>
<td>0.000000</td>
<td></td>
</tr>
<tr>
<td>$R(4,4)$</td>
<td>0.000000</td>
<td>0.000000</td>
<td></td>
</tr>
<tr>
<td>$R(5,5)$</td>
<td>0.000023</td>
<td>0.000000</td>
<td></td>
</tr>
<tr>
<td>$R(6,6)$</td>
<td>0.002771</td>
<td>0.004669</td>
<td></td>
</tr>
<tr>
<td>$R(7,7)$</td>
<td>0.0000010</td>
<td>0.000000</td>
<td></td>
</tr>
<tr>
<td>$R(8,8)$</td>
<td>0.00290</td>
<td>0.000006</td>
<td></td>
</tr>
<tr>
<td>$R(9,9)$</td>
<td>0.000000</td>
<td>0.000000</td>
<td></td>
</tr>
<tr>
<td>$R(10,10)$</td>
<td>0.001324</td>
<td>0.000102</td>
<td></td>
</tr>
<tr>
<td>Log-likelihood</td>
<td>81600.23</td>
<td>57518.53</td>
<td>151614.81</td>
</tr>
<tr>
<td>sample size</td>
<td>4801</td>
<td>4801</td>
<td>9602</td>
</tr>
</tbody>
</table>
nience yield have relatively high positive correlation, which is consistent with
the GS model, however their correlations are more lower. Moreover, Crude
oil spot price and heating oil convenience yield have no correlation. We can
also see that the correlation for heating oil spot price and crude oil convenience
yield are relatively low. It is intuitive that spot price and convenience yield
among different commodities should not be strongly correlated. Volatility
of spot price and convenience yield are different with the GS model and the
GSC model have larger values. The larger value of volatility of convenience
yield is caused by $a_i$, $b_i$ and $\rho$.

Table 3: RMSE (root mean square error) and ME (mean error) for each
futures.

<table>
<thead>
<tr>
<th>Contracts</th>
<th>RMSE</th>
<th>ME</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>GS</td>
<td>GSC</td>
</tr>
<tr>
<td>Crude oil</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Maturity 1</td>
<td>0.033298</td>
<td>0.029263</td>
</tr>
<tr>
<td>Maturity 3</td>
<td>0.020038</td>
<td>0.020033</td>
</tr>
<tr>
<td>Maturity 5</td>
<td>0.018406</td>
<td>0.018152</td>
</tr>
<tr>
<td>Maturity 7</td>
<td>0.017075</td>
<td>0.017052</td>
</tr>
<tr>
<td>Maturity 9</td>
<td>0.016965</td>
<td>0.016228</td>
</tr>
<tr>
<td>Heating oil</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Maturity 1</td>
<td>0.056661</td>
<td>0.070991</td>
</tr>
<tr>
<td>Maturity 3</td>
<td>0.020187</td>
<td>0.020138</td>
</tr>
<tr>
<td>Maturity 5</td>
<td>0.025290</td>
<td>0.023325</td>
</tr>
<tr>
<td>Maturity 7</td>
<td>0.017833</td>
<td>0.020902</td>
</tr>
<tr>
<td>Maturity 9</td>
<td>0.040683</td>
<td>0.039045</td>
</tr>
</tbody>
</table>

Table 3 shows root mean square error (RMSE) and mean error (ME) of
the model. Both model have small values which means that the models are
well fitted and it is hard to decide which have better performance.
4 Hedging Futures

In this section, we implement the GSC and GS model for hedging long term futures\(^4\) contracts, where we call it as target futures, using short term futures.

Since the GS model have two stochastic variables, we need two futures which have different maturities to hedge and the weight can be calculated by solving the following system of equations\(^5\).

\[
\Phi w = \varphi \\
\Phi = \begin{bmatrix}
\frac{\partial G_i(t,T_1)}{\partial S_i} & \frac{\partial G_i(t,T_2)}{\partial S_i} \\
\frac{\partial G_i(t,T_1)}{\partial \delta_i} & \frac{\partial G_i(t,T_2)}{\partial \delta_i}
\end{bmatrix}
\]

\[
w = [w_1, w_2]^T \\
\varphi = [\frac{\partial G_i(t,T)}{\partial S_i}, \frac{\partial G_i(t,T)}{\partial \delta_i}]^T
\]

where \(w_i\) are weights for futures with maturity \(T_i\) and \(T\) is the maturity of target futures. For the GSC model which has \(S_i, \delta_i, \text{ and } z\) as the stochastic variables, we need three futures to hedge and now the system of equations for (12) are

\[
\Phi = \begin{bmatrix}
\frac{\partial G_i(t,T_1)}{\partial S_i} & \frac{\partial G_i(t,T_2)}{\partial S_i} & \frac{\partial G_i(t,T_3)}{\partial S_i} \\
\frac{\partial G_i(t,T_1)}{\partial \delta_i} & \frac{\partial G_i(t,T_2)}{\partial \delta_i} & \frac{\partial G_i(t,T_3)}{\partial \delta_i} \\
\frac{\partial G_i(t,T_1)}{\partial z} & \frac{\partial G_i(t,T_2)}{\partial z} & \frac{\partial G_i(t,T_3)}{\partial z}
\end{bmatrix}
\]

\[
w = [w_1, w_2, w_3]^T \\
\varphi = [\frac{\partial G_i(t,T)}{\partial S_i}, \frac{\partial G_i(t,T)}{\partial \delta_i}, \frac{\partial G_i(t,T)}{\partial z}]^T.
\]

To calculate the hedging portfolio, we need the values of state variables \(S_i(t)\), \(\delta_i(t)\), and \(z(t)\). There are two methods to calculate state variables. One way is to use the Kalman filters, which we call Kalman filter method. Another way is to calculate state variables by solving the observation equation which only needs futures price and estimated parameters. This method will be called as simultaneous equation method. We implemented both method. Hedging error ratio is calculated by dividing hedging error value by the target futures

---
\(^4\)Recall that we are assuming risk-free rate as constant which implies that futures and forwards should be equally valued.

price of each hedging start period. The hedging error value is the total of difference between target futures price and the value of hedge portfolio.

We hedge futures which matures in 9 months and 10 years with futures which matures in 1, 3, 5 months. Since we can not calculate the hedging error precisely with long term futures that are not traded in the market such as 10-year futures, we also hedge 9-month futures for precise evaluation of the hedging error. For 10-years futures, we calculate the hedging error by using the theoretical price of 10-year futures. The total hedging period is from January 2, 1990 to February 27, 2009. We roll the futures 3 business days before it matures and each hedging period is roughly 1 month. The hedging weight and the hedging error are calculated daily.

The performance of hedging simulation for 9-month futures are indicated in table 4 and figure 2. While the hedging error ratio for crude oil are relatively small, the performance for heating oil are not good. This is true for both GSC and GS model. We also see that the Kalman filter method is rather better than simultaneous equation method. Figure 3 shows the weights of futures in the hedging portfolio which state variables are calculated by Kalman filters. It indicates that long term futures are hold positively and short term futures are hold negatively or rather moderately than longer terms. Since the target futures are long term, this result is consistent.

Table 4: Performance of hedging 9-month futures. ’Kalman filter’ indicates that the state variables are calculated by using Kalman filters. ’Simultaneous’ indicates that the state variables are calculated by solving the observation equation.

<table>
<thead>
<tr>
<th>Contracts</th>
<th>Mean of hedging error ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Method</td>
</tr>
<tr>
<td>Crude oil</td>
<td>Kalman filter</td>
</tr>
<tr>
<td></td>
<td>Simultaneous</td>
</tr>
<tr>
<td>Heating oil</td>
<td>Kalman filter</td>
</tr>
<tr>
<td></td>
<td>Simultaneous</td>
</tr>
</tbody>
</table>

In table 5 and figure 4, we show the performance of hedging simulation for 10-year futures. Obviously, the hedging error ratio is poor than the 9-month
Figure 2: Performance of hedging 9-month futures. Solid line and dashed line indicates hedge performance of WTI crude oil and heating oil, respectively.
Figure 3: Weights of futures for hedging 9-month futures. For GS model, solid line and dashed line indicates hedging weights of 1-month futures and 3-month futures, respectively. For GSC model, solid line, dashed line, and dotted line indicates hedging weights of 1-month futures, 3-month futures, and 5-month futures, respectively.

*Graphs showing trend lines for Delta Weight for Commodity 1 and Commodity 2 over time between Dec-91 and Dec-07.*
futures. However, note that this hedging error ratio is calculated by using theoretical price. The hedging weight which state variables are calculated by Kalman filters for this simulation are shown in figure 5. Again we can see that the longer term futures are positive and shorter term futures are negative or relatively small.

Table 5: Performance of hedging 10-year futures. 'Kalman filter' indicates that the state variables are calculated by using Kalman filters. 'Simultaneous' indicates that the state variables are calculated by solving the observation equation.

<table>
<thead>
<tr>
<th>Contracts</th>
<th></th>
<th>Mean of hedging error ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Method</td>
<td>GS</td>
</tr>
<tr>
<td>Crude oil</td>
<td>Kalman filter</td>
<td>-0.233085</td>
</tr>
<tr>
<td></td>
<td>Simultaneous</td>
<td>0.022894</td>
</tr>
<tr>
<td>Heating oil</td>
<td>Kalman filter</td>
<td>0.369785</td>
</tr>
<tr>
<td></td>
<td>Simultaneous</td>
<td>0.014367</td>
</tr>
</tbody>
</table>

5 Discussion

5.1 Relations among futures prices

It should be noted that the linear relations among commodity spot prices cannot be replaced by those among their futures prices. Namely, there is no straightforward way to transform the linear relations among unobservable spot prices into the linear relations among observable futures prices. Indeed, from (3) and the pricing formula for futures, we have

\[ z(t) = \mu_z + a_0 t + \sum_{i=1}^{n} a_i \ln G_i(t, T) - a_i \left( \mu_X_i(t, T) + \frac{\sigma_X^2(t, T)}{2} \right) \]

Since there are \( \mu_X_i(t, T) \) and \( \sigma_X^2(t, T) \) which contains \( a_i \) in a non-linear fashion in the equation, we can not apply linear estimation for \( a_i \) such as (3).
Figure 4: Performance of hedging 10-year futures. Solid line and dashed line indicates hedge performance of WTI crude oil and heating oil, respectively.
Figure 5: Weights of futures for hedging 10-year futures. For GS model, solid line and dashed line indicates hedging weights of 1-month futures and 3-month futures, respectively. For GSC model, solid line, dashed line, and dotted line indicates hedging weights of 1-month futures, 3-month futures, and 5-month futures, respectively.
Things are more complicated with \( b_i \). Let us see the dynamics of futures price.

\[
dG_i(t, T) = \left\{ \frac{\partial G_i}{\partial t} + (r - \delta_i(t) + b_i z(t)) S_i(t) \frac{\partial G_i}{\partial S_i} + \kappa_i(\hat{\alpha}_i - \delta_i(t)) \frac{\partial G_i}{\partial \delta_i} \right. \\
+ \left. (-b(m - z(t))) \sum_{j=1}^{n} a_j \delta_j(t) \right) \frac{\partial G_i}{\partial z} \\
+ \frac{1}{2} \sigma_{S_i}^2 S_i(t) \frac{\partial^2 G_i}{\partial S_i^2} + \frac{1}{2} \sigma_{\delta_i}^2 \frac{\partial^2 G_i}{\partial \delta_i^2} + \frac{1}{2} \sum_{j,k=1}^{n} a_j a_k \sigma_{S_i S_k} \frac{\partial^2 G_i}{\partial z \partial \delta_i} \\
+ \sigma_{S_i \delta_i} S_i(t) \frac{\partial^2 G_i}{\partial S_i \partial \delta_i} + \sum_{j=1}^{n} a_j \sigma_{S_i S_j} S_i(t) \frac{\partial^2 G_i}{\partial S_i \partial z} \\
+ \sum_{j=1}^{n} a_j \sigma_{S_i S_j} \frac{\partial^2 G_i}{\partial \delta_i \partial z} \right\} dt \\
+ \sigma_{S_i} S_i(t) \frac{\partial G_i}{\partial S_i} dW_S(t) + \sigma_{\delta_i} \frac{\partial G_i}{\partial \delta_i} dW_{\delta_i}(t) + \sum_{j=1}^{n} a_j \sigma_{S_i} \frac{\partial G_i}{\partial z} dW_{S_j}(t) \\
= \sigma_{S_i} S_i(t) (\hat{\beta}_{S_{i0}} + \hat{\beta}_{S_{i\delta}} \delta_i(t) + \hat{\beta}_{S_{i z}} z(t)) \frac{\partial G_i}{\partial S_i} dt \\
+ \sigma_{\delta_i} (\hat{\beta}_{\delta_{i0}} + \hat{\beta}_{\delta_{i \delta}} \delta_i(t)) \frac{\partial G_i}{\partial \delta_i} dt \\
+ \sum_{j=1}^{n} a_j \sigma_{S_j} (\hat{\beta}_{S_{j0}} + \hat{\beta}_{S_{j \delta}} \delta_j(t) + \hat{\beta}_{S_{j z}} z(t)) \frac{\partial G_i}{\partial z} dt \\
+ \sigma_{S_i} S_i(t) \frac{\partial G_i}{\partial S_i} dW_P^S(t) + \sigma_{\delta_i} \frac{\partial G_i}{\partial \delta_i} dW_P^\delta(t) + \sum_{j=1}^{n} a_j \sigma_{S_j} \frac{\partial G_i}{\partial z} dW_P^{S_j}(t) \\
\]

where we used martingale property for \( G_i(t, T) = E_t[S_i(T)] \) in the second equality and changed the measure to natural probability in the last equality. Comparing this equation to (1), the adjustment coefficients is 0 under risk-neutral probability and \( \sigma_{S_i} \hat{\beta}_{S_{i z}} \frac{\partial G_i}{\partial S_i} + \sum_{j=1}^{n} a_j \sigma_{S_j} \hat{\beta}_{S_{j z}} \frac{\partial G_i}{\partial S_j(t)} \frac{\partial G_i}{\partial z} \). This means that the adjustment coefficients for futures have nothing to do with \( b_i \) which are
adjustment coefficients for spot prices.

Furthermore, we emphasize that the linear relation is not observable in
the GSC model. There are two aspect of this unobservability. First, it is
modeled as spot prices which is not observable. If we model the linear relation
using futures price, the advantage of the model will be the observability of
the price which allows us to use easy regression analysis and avoid using
technical Kalman filter. Second, we modeled the linear relation under risk-
neutral probability which is not observable from historical data that should
move under natural probability. While $a_i$ does not change with probabilities,
$b_i$ does as we have seen in the above equation. The adjustment coefficients are
changed by the market price of risks and it implies that if co-integration exist,
the effects of error term on spot prices under natural probability and under
risk-neutral probability will be different. Thus, it may be interesting to model
the linear relations among observable futures prices instead of unobservable
spot prices under natural probability, and analyze the effects on spot prices
and other derivatives.

5.2 Conditions for co-integration

Although the GSC model was conceived by the concept of co-integration or
VECM model, the model need not to be co-integrated. The only conditions
needed is the existence and uniqueness of strong solution of sdes (1) and
(2). However, if we assume the GSC model to be co-integrated, we can use
cointegration tests that are widely used. Let us discretize the sdes (1), (2),
and (4).

$$y(t) = c + \Phi_0 y(t - 1) + \varepsilon(t)$$  \hspace{1cm} (13)

where

$$y(t) = \begin{bmatrix}
\ln S_1(t) - \ln S_1(t - 1) \\
\vdots \\
\ln S_n(t) - \ln S_n(t - 1) \\
d_1(t) \\
\vdots \\
d_n(t) \\
z(t)
\end{bmatrix}$$
and other coefficients are given in the appendix. If we assume that

\[ |I_n - \Phi_0 z| = 0 \text{ then } |z| > 1, \]

then \( y \) is stationary\(^6\) which implies co-integration in discrete time. However, note that this condition is strong. We do not need that convenience yields to be stationary, but if we exclude convenience yields from the condition, \( \ln S_i(t) \) may not be I(1) since there are convenience yields in the drift terms. The difficulty stems from the correlation of convenience yield with other variables. Another condition may be utilized which is proposed by Comte (1999) for continuous autoregressions but it is also strong because of the existence of convenience yield\(^7\).

### 5.3 Multi-dimensional \( z(t) \)

In this paper we have assumed that there are only one linear relation which is represented by the term \( z(t) \). This can be relaxed to \( h(<n) \) different linear relations \([z_1(t) \ldots z_h(t)]^\top\) that can be formalized as,

\[
\begin{align*}
dS_i(t) &= S_i(t)(r - \delta_i(t)) + \sum_{j=1}^{h} b_{ij}z_j(t)dt + S_i(t)\sigma_i dW_{S_i}(t), \quad i = 1, \ldots, n \\
d\delta_i(t) &= \kappa_i(\hat{\alpha}_i - \delta_i(t))dt + \sigma_i dW_{\delta_i}(t), \quad i = 1, \ldots, n \\
z_j(t) &= \mu_z + a_{0j}t + \sum_{i=1}^{n} a_{ij} \ln S_i(t) \quad j = 1, \ldots, h.
\end{align*}
\]

With a straight forward argument we can derive the futures and call option formulae. We can also extend the assumption on market price of risk and formalize the state and observation equation for Kalman filters. The difficulty of this model is that there are many parameters to consider when implementing the model to historical data. The parameters needed to be estimated would be \( n(1 + 2n) \) parameters for volatilities and correlations, \( 2n \) parameters for convenience yields \((\hat{\alpha}, \kappa)\), \( h(2n + 1) \) parameters for linear relations \((a_{0j}, a_{ij}, b_{ij})\), \( 2n(1 + n) + nh \) parameters for market price of risks \((\hat{\beta})\), and other parameters which depends on the number of commodities and futures maturity data used for covariance matrix \( R \) in the observation

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\(^6\)See Hamilton (1994) for stationarity of VAR processes.

\(^7\)See also Bergstrom (1990) for stationary condition of continuous autoregressions.
equation. If we assume 3 commodities and two linear relations for the model using 3 maturities of futures for each commodity, the number of parameters will be 80 parameters! In order to conduct a realistic empirical analysis, the number of commodities and linear relations used have to be very small.

6 Conclusion

In this paper, we formulated a commodity pricing model that incorporated the effect of linear relations among commodity prices, which included co-integration under certain conditions. We derived futures and call option pricing formulae, and showed that unlike Duan and Pliska (2004), the linear relations among commodity prices, or the error correction term under appropriate conditions, should affect those derivative prices in the standard setup of commodity pricing.

We emphasize that the proposed model can be interpreted as a generalization of standard commodity models, especially the GS model. This is because we decompose the deviation of commodity return from the risk-free rate under the risk-neutral probability into two components; convenience yield and the linear relation term $z(t)$. The proposed model can thus describe not only the usual storage effects captured by convenience yield, but also other causes such as impacts from other commodity prices and transaction costs. Comparing the GS with the GSC model, it should be noted that if price of a commodity is affected by convenience yields of other commodities through the term $z(t)$, the GS model is misspecified as long as the GSC model is correct.

In the empirical analysis, we assumed that the market price of risk is linear in convenience yield and the term $z(t)$, and utilized the Kalman filter technique. Using crude oil and heating oil market data, we estimated the proposed model. The result suggested that there are co-integration among these commodity prices, and that its effect on derivative prices should not be ignored empirically. We also implement the model to hedging long term futures.

Finally, it should be noted that while the linear relations among spot prices play an important role, such spot prices are assumed to be unobservable in the standard commodity pricing models including ours. Thus, it would be interesting to model the linear relations among observable futures prices instead of unobservable spot prices, and analyze the effects of the lin-
ear relation, or co-integration under certain conditions, on derivatives. We lay this topic for future study.
Appendices

Appendix 1  Derivation of equation (8)

In this section, we derive (8). This is done by calculating the term $z(T)$. First, note that

$$\int_t^T z(s)ds = \frac{1}{b}(z(T) - z(t)) + m(T - t) + \sum_{j=1}^n \int_t^T \frac{a_j}{b} \delta_j(s)ds$$

$$- \sum_{j=1}^n \frac{a_j}{b} \sigma_j(W_{S_i}(T) - W_{S_i}(t))$$

Since $z(t)$ is a linear sde it can be derived as

$$z(T) = e^{b(T-t)}z(t) + \int_t^T e^{b(T-s)}(-bm - \sum_{i=1}^n a_i \delta_i(s))ds$$

$$+ \sum_{i=1}^n \int_t^T e^{b(T-s)}a_j \sigma_{S_j}dW_{S_i}(s)$$

$$= e^{b(T-t)}z(t) + m(1 - e^{b(T-t)})$$

$$- \int_t^T e^{b(T-s)}\sum_{i=1}^n a_i \delta_i(s))ds + \sum_{j=1}^n \int_t^T e^{b(T-s)}a_j \sigma_{S_j}dW_{S_i}(s)$$

$$= e^{b(T-t)}z(t) + m(1 - e^{b(T-t)})$$

$$- \sum_{i=1}^n \frac{a_i}{b + \kappa_i} (e^{b(T-t)} - e^{-\kappa_i(T-t)})$$

$$- \sum_{i=1}^n \frac{a_i \hat{\delta}_i}{b + \kappa_i} (e^{b(T-t)} - 1) + \sum_{i=1}^n \frac{a_i \hat{\delta}_i}{b + \kappa_i} (e^{b(T-t)} - e^{-\kappa_i(T-t)})$$

$$- \sum_{i=1}^n \int_t^T \frac{a_i}{b + \kappa_i} (e^{b(T-s)} - e^{-\kappa_i(T-s)})\sigma_{S_i}dW_{S_i}(s)$$

$$+ \sum_{i=1}^n \int_t^T e^{b(T-s)}a_j \sigma_{S_j}dW_{S_i}(s)$$
where we used

\[
\int_t^T e^{b(T-s)} \delta_i(s) ds = \int_t^T e^{b(T-s)-\kappa_i(s-t)} \delta_i(t) + e^{b(T-s)-\kappa_i(s-t)} \hat{\alpha}_i \, ds \\
+ \int_t^T \int_t^s e^{b(T-s)-\kappa_i(s-u)} \sigma_{\delta_i} dW_{\delta_i}(u) ds \\
= e^{bT+\kappa_i t} \frac{\delta_i(t)}{b + \kappa_i} \left( e^{-\kappa_i(t)} - e^{-(b+\kappa_i)T} \right) \\
+ \frac{\hat{\alpha}_i}{b} \left( e^{b(T-t)} - 1 \right) - e^{bT+\kappa_i t} \frac{\hat{\alpha}_i}{b + \kappa_i} \left( e^{-(b+\kappa_i)t} - e^{-(b+\kappa_i)T} \right) \\
+ \int_t^T e^{bT+\kappa_i u} \frac{1}{b + \kappa_i} \left( e^{-\kappa_i u} - e^{-(b+\kappa_i)T} \right) \sigma_{\delta_i} dW_{\delta_i}(u).
\]
The last equation is a consequence of Fubini’s theorem for stochastic integrals. Collecting terms, we have

\[
X_i(t, T) \triangleq \int_t^T \left( r - \frac{\sigma_i^2}{2} - \delta_i(s) + b_i z(s) \right) \, ds + \int_t^T \sigma_i \, dW_{S_i}(s)
\]

\[
= \left( r + b_i m - \frac{\sigma_i^2}{2} - \hat{\alpha}_i + \sum_{j=1}^n \frac{b_i a_j \hat{\alpha}_j}{b} \right) (T - t)
\]

\[
+ \frac{b_i (m - z(t))}{b} \left( 1 - e^{b(T-t)} \right) - \sum_{j=1}^n \frac{b_i a_j \delta_j(t)}{b(b + \kappa_j)} (e^{b(T-t)} - e^{-\kappa_j(T-t)})
\]

\[
- \sum_{j=1}^n \frac{b_i a_j \hat{\alpha}_j}{b^2} (e^{b(T-t)} - 1) + \sum_{j=1}^n \frac{b_i a_j \hat{\alpha}_j}{b(b + \kappa_j)} (e^{b(T-t)} - e^{-\kappa_j(T-t)})
\]

\[
+ \frac{(\hat{\alpha}_i - \delta_i(t))}{\kappa_i} (1 - e^{-\kappa_i(T-t)}) - \sum_{j=1}^n \frac{b_i a_j (\hat{\alpha}_j - \delta_j(t))}{b\kappa_j} (1 - e^{-\kappa_j(T-t)})
\]

\[
+ \sigma_i (W_{S_i}(T) - W_{S_i}(t)) - \frac{1}{\kappa_i} \sigma_i (W_{\hat{\delta}_i}(T) - W_{\hat{\delta}_i}(t))
\]

\[
- \sum_{j=1}^n \frac{b_i a_j}{b} \sigma_j (W_{S_j}(T) - W_{S_j}(t)) + \sum_{j=1}^n \frac{b_i a_j}{b\kappa_j} \sigma_j (W_{S_i}(T) - W_{S_i}(t))
\]

\[
+ \sum_{j=1}^n \frac{b_i a_j}{b} \int_t^T e^{b(T-s)} \sigma_j \, dW_{S_j}(s)
\]

\[
+ \int_t^T \frac{e^{-\kappa_i(T-s)}}{\kappa_i} \sigma_i \, dW_{\hat{\delta}_i}(s) - \sum_{j=1}^n \int_t^T \frac{b_i a_j e^{-\kappa_j(T-s)}}{b\kappa_j} \sigma_j \, dW_{\hat{\delta}_j}(s)
\]

\[
- \sum_{j=1}^n \int_t^T \frac{b_i a_j}{b(b + \kappa_j)} (e^{b(T-s)} - e^{-\kappa_j(T-s)}) \sigma_j \, dW_{\hat{\delta}_j}(s).
\]

Using properties of stochastic integrals, we have \( \mu_{X_i}(t, T) \) and \( \sigma_{X_i}(t, T) \) as in the paper.

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8See Ikeda and Watanabe (1989), chapter III, lemma 4.1 or Heath, Jarrow, and Morton (1992), appendix lemma 0.1, and its corollaries.
Appendix 2  Proof of Proposition 2.2

We prove the call option pricing formula. From Harrison and Kreps (1979) or Harrison and Pliska (1981), we have

\[ C_i(t, T) = e^{-r(T-t)}E_i[(S_i(T) - K)^+] \]

\[ = e^{-r(T-t)} \int_D (S_i(t)e^{x_i} - K)n(x_i|\mu_{X_i}(t, T), \sigma^2_{X_i}(t, T))dx_i \]

where \( n(x|\mu, \sigma^2) \) is density function of normal distribution with mean \( \mu \) and variance \( \sigma^2 \) and

\[ D = \left\{ x_i \mid x_i \geq \ln \left( \frac{K}{S_i(t)} \right) \right\} \]

The integral can be calculated as

\[
\int_D e^{x_i}n(x|\mu_{X_i}, \sigma^2_{X_i})dx = \int_D \frac{1}{\sqrt{2\pi}\sigma_{X_i}} e^{-\frac{(x_i-\mu_{X_i})^2}{2\sigma^2_{X_i}}} dx \\
= \frac{1}{\sqrt{2\pi}\sigma_{X_i}} \int_D e^{\mu_{X_i} + \frac{\sigma^2_{X_i}}{2} - \frac{(x_i-\mu_{X_i})^2}{2\sigma^2_{X_i}}} dx \\
= e^{\mu_{X_i} + \frac{\sigma^2_{X_i}}{2}} \int_D e^{-\frac{(x_i-\mu_{X_i})^2}{2\sigma^2_{X_i}}} dx \\
= e^{\mu_{X_i} + \frac{\sigma^2_{X_i}}{2}} \int_{-\infty}^{d_{i1}} e^{-\frac{y^2}{2}} dy
\]

where

\[ d_{i1} = \frac{\ln \left( \frac{S_i(t)}{K} \right) + \mu_{X_i} + \sigma^2_{X_i}}{\sigma_{X_i}} \]

and \( \mu_{X_i} = \mu_{X_i}(t, T), \sigma_{X_i} = \sigma_{X_i}(t, T) \) for notational convenience. Also,

\[
\int_D \frac{1}{\sqrt{2\pi}\sigma_{X_i}} e^{-\frac{(x_i-\mu_{X_i})^2}{2\sigma^2_{X_i}}} dx_i = \int_{-\infty}^{d_{i2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy
\]
where
\[ d_{i2} = \frac{\ln \left( \frac{S_i(t)}{K} \right) + \mu X_i}{\sigma X_i} \]
Collecting all terms,
\[ C_i(t, T) = S_i(t)e^{-r(T-t) + \mu X_i(t,T) + \frac{r^2 \sigma_i^2(t,T)}{2}}\Phi(d_{i1}) - Ke^{-r(T-t)}\Phi(d_{i2}) \]

Appendix 3  Solutions for equations (9), (10), and (11)

\[ \ln S_i(t) = \ln S_i(0) + \frac{\beta S_i \delta_i(0)}{\beta \delta_i} (e^{\beta_i t} - 1) + \frac{\beta S_i \delta_i^2}{\beta \delta_i^2} (e^{\beta_i t} - 1) - \frac{\beta S_i \delta_i}{\beta \delta_i} \frac{\beta S_i \delta_i}{\beta \delta_i} (e^{\beta_i t} - 1) + \sum_{j=1}^{n} \int_0^t \frac{\beta S_i \delta_j \sigma_j}{\beta \delta_j} (e^{\beta_j t} - 1) dW^P_{S_i}(s) + \sigma S_i W^P_{S_i}(t) \]
\[ \delta_i(t) = e^{\beta_{\delta_i}t} \delta_i(0) + \frac{\beta_{\delta_i,0}}{\beta_{\delta_i}} (e^{\beta_{\delta_i}t} - 1) + \int_0^t e^{\beta_{\delta_i}(t-s)} \sigma_{\delta_i} dW_{\delta_i}(s) \]

\[ z(t) = e^{\beta_{zz}t} z(0) + \frac{\beta_{zz,0}}{\beta_{zz}} (e^{\beta_{zz}t} - 1) \]

\[ + \sum_{i=1}^n \left\{ \frac{\beta_{\delta_i} \beta_{\delta_i,0} \delta_i(0) + \beta_{\delta_i} \beta_{\delta_i,0}}{\beta_{\delta_i} (\beta_{\delta_i} - \beta_{zz})} (e^{\beta_{\delta_i}t} - e^{\beta_{zz}t}) \right\} \]

\[ + \sum_{i=1}^n \int_0^t \frac{\beta_{zz} \sigma_{\delta_i}}{\beta_{\delta_i} - \beta_{zz}} (e^{\beta_{\delta_i}(t-s)} - e^{\beta_{zz}(t-s)}) dW_{\delta_i}(s) \]

\[ + \sum_{i=1}^n \int_0^t e^{\beta_{zz}(t-s)} a_i \sigma_s dW_{S_i}(s) \]
Appendix 4  State equation, observation equation, Kalman filters, forecasts, and maximum likelihood

\[ x_t = Fx_{t-1} + C^x + v_t \]

where

\[ x_t = \begin{bmatrix} \ln S_1(t) & \cdots & \ln S_n(t) & \delta_1(t) & \cdots & \delta_n(t) & z(t) \end{bmatrix}^\top \]

\[ F = \begin{bmatrix} F_{S_1S_1}(\Delta t) & \cdots & F_{S_1S_n}(\Delta t) & F_{S_1\delta_1}(\Delta t) & \cdots & F_{S_1\delta_n}(\Delta t) & F_{S_1z}(\Delta t) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{S_nS_1}(\Delta t) & \cdots & F_{S_nS_n}(\Delta t) & F_{S_n\delta_1}(\Delta t) & \cdots & F_{S_n\delta_n}(\Delta t) & F_{S_nz}(\Delta t) \\ F_{\delta_1S_1}(\Delta t) & \cdots & F_{\delta_1S_n}(\Delta t) & F_{\delta_1\delta_1}(\Delta t) & \cdots & F_{\delta_1\delta_n}(\Delta t) & F_{\delta_1z}(\Delta t) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{\delta_nS_1}(\Delta t) & \cdots & F_{\delta_nS_n}(\Delta t) & F_{\delta_n\delta_1}(\Delta t) & \cdots & F_{\delta_n\delta_n}(\Delta t) & F_{\delta_nz}(\Delta t) \\ F_{zS_1}(\Delta t) & \cdots & F_{zS_n}(\Delta t) & F_{z\delta_1}(\Delta t) & \cdots & F_{z\delta_n}(\Delta t) & F_{zz}(\Delta t) \end{bmatrix} \]

\[ C^x = \begin{bmatrix} C_1^x(\Delta t) \\ \vdots \\ C_n^x(\Delta t) \\ C_1(\Delta t) \\ \vdots \\ C_n(\Delta t) \\ C_1(\Delta t) \\ \vdots \\ C_n(\Delta t) \end{bmatrix} \]

\[ Q = \text{Cov}(v_t) = \begin{bmatrix} \sigma_{S_1S_1}(\Delta t) & \cdots & \sigma_{S_1S_n}(\Delta t) & \sigma_{S_1\delta_1}(\Delta t) & \cdots & \sigma_{S_1\delta_n}(\Delta t) & \sigma_{S_1z}(\Delta t) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{S_nS_1}(\Delta t) & \cdots & \sigma_{S_nS_n}(\Delta t) & \sigma_{S_n\delta_1}(\Delta t) & \cdots & \sigma_{S_n\delta_n}(\Delta t) & \sigma_{S_nz}(\Delta t) \\ \sigma_{\delta_1S_1}(\Delta t) & \cdots & \sigma_{\delta_1S_n}(\Delta t) & \sigma_{\delta_1\delta_1}(\Delta t) & \cdots & \sigma_{\delta_1\delta_n}(\Delta t) & \sigma_{\delta_1z}(\Delta t) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{\delta_nS_1}(\Delta t) & \cdots & \sigma_{\delta_nS_n}(\Delta t) & \sigma_{\delta_n\delta_1}(\Delta t) & \cdots & \sigma_{\delta_n\delta_n}(\Delta t) & \sigma_{\delta_nz}(\Delta t) \\ \sigma_{zS_1}(\Delta t) & \cdots & \sigma_{zS_n}(\Delta t) & \sigma_{z\delta_1}(\Delta t) & \cdots & \sigma_{z\delta_n}(\Delta t) & \sigma_{zz}(\Delta t) \end{bmatrix} \]
\[ F_{S_iS_j}(t) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases} \]

\[ F_{S_i\delta_j}(t) = \begin{cases} \frac{\beta_{S_iS_j}}{\beta_{S_iS_j}} (e^{\beta_{S_iS_j} t} - 1) + \frac{\beta_{S_i\delta_j}}{\beta_{S_i\delta_j} - \beta_{S_iS_j}} \left\{ \frac{1}{\beta_{S_iS_j}} (e^{\beta_{S_iS_j} t} - 1) - \frac{1}{\beta_{S_i\delta_j}} (e^{\beta_{S_i\delta_j} t} - 1) \right\}, & i = j \\ \frac{\beta_{S_i\delta_j}}{\beta_{S_i\delta_j} - \beta_{S_iS_j}} \left\{ \frac{1}{\beta_{S_i\delta_j}} (e^{\beta_{S_i\delta_j} t} - 1) - \frac{1}{\beta_{S_i\delta_j}} (e^{\beta_{S_i\delta_j} t} - 1) \right\}, & \text{otherwise} \end{cases} \]

\[ F_{S_i\delta}(t) = \frac{\beta_{S_i\delta}}{\beta_{\delta\delta}} (e^{\beta_{S_i\delta} t} - 1) \]

\[ F_{\delta\delta}(t) = 0 \]

\[ F_{z\delta}(t) = \begin{cases} e^{\beta_{z\delta} t}, & i = j \\ 0, & \text{otherwise} \end{cases} \]

\[ F_{zz}(t) = 0 \]

\[ C_{S_i}^x(t) = \beta_{S_00} t + \frac{\beta_{S_z0}}{\beta_{S_z0}} (e^{\beta_{S_z0} t} - 1) - \frac{\beta_{S_z0}}{\beta_{S_z0}} t + \frac{\beta_{S_z0}}{\beta_{S_z0}} (e^{\beta_{S_z0} t} - 1) - \frac{\beta_{S_z0}}{\beta_{S_z0}} t \]

\[ + \sum_{j=1}^{\infty} \frac{\beta_{S_z\delta_j} \beta_{S_00}}{\beta_{S_z\delta_j} (\beta_{S_\delta\delta} - \beta_{S_zS_z})} \left\{ \frac{1}{\beta_{S_z\delta_j}} (e^{\beta_{S_z\delta_j} t} - 1) - \frac{1}{\beta_{S_z\delta_j}} (e^{\beta_{S_z\delta_j} t} - 1) \right\} \]

\[ + \sum_{j=1}^{\infty} \frac{\beta_{S_z\delta_j} \beta_{S_00}}{\beta_{S_z\delta_j} (\beta_{S_\delta\delta} - \beta_{S_zS_z})} \left\{ \frac{1}{\beta_{S_z\delta_j}} (e^{\beta_{S_z\delta_j} t} - 1) - \frac{1}{\beta_{S_z\delta_j}} (e^{\beta_{S_z\delta_j} t} - 1) \right\} \]

\[ + \frac{\beta_{z0}}{\beta_{z\delta}} \beta_{S_z0} (e^{\beta_{z0} t} - 1) \]

\[ C_{\delta_i}^x(t) = \frac{\beta_{\delta_00}}{\beta_{\delta_i}} (e^{\beta_{\delta_00} t} - 1) \]

\[ C_{z}^x(t) = \frac{\beta_{z0}}{\beta_{zz}} (e^{\beta_{z0} t} - 1) - \sum_{j=1}^{\infty} \frac{\beta_{z\delta_j} \beta_{z0}}{\beta_{\delta_j} (\beta_{\delta_j\delta_j} - \beta_{zz})} (e^{\beta_{z\delta_j} t} - e^{\beta_{z0} t}) \]

\[ + \frac{\beta_{z0}}{\beta_{zz}} (1 - e^{\beta_{z0} t}) \]
\[
\sigma_{r,s}(t) = \frac{\beta S_{i,t} \beta S_{j,t} \rho \delta_{i,j} \sigma \delta_{i,j} - 1}{\beta \beta \beta \beta} (1 - e^{(\beta \beta \beta \beta + \beta \beta \beta \beta) t}) + \frac{1}{\beta \beta \beta \beta} (1 - e^{\beta \beta \beta \beta t})
\]

\[
+ \frac{1}{\beta \beta \beta \beta} (1 - e^{\beta \beta \beta \beta t}) + t \}
\]

\[
+ \sum_{k=1}^{n} \frac{\beta S_{i,t} \beta S_{j,t} \beta \delta_{i,t} \rho \delta_{i,t} \sigma \delta_{i,t} \sigma \delta_{i,t}}{\beta \beta \beta \beta} (1 - e^{(\beta \beta \beta \beta + \beta \beta \beta \beta) t})
\]

\[
+ \frac{1}{\beta \beta \beta \beta} (1 - e^{\beta \beta \beta \beta t}) + \frac{1}{\beta \beta \beta \beta} (1 - e^{\beta \beta \beta \beta t}) + \frac{t}{\beta \beta \beta \beta}
\]

\[
+ \frac{1}{\beta \beta \beta \beta} (1 - e^{\beta \beta \beta \beta t}) - \frac{1}{\beta \beta \beta \beta} (1 - e^{\beta \beta \beta \beta t})
\]

\[
+ \frac{1}{\beta \beta \beta \beta} (1 - e^{\beta \beta \beta \beta t}) + \frac{t}{\beta \beta \beta \beta}
\]

\[
+ \frac{1}{\beta \beta \beta \beta} (1 - e^{\beta \beta \beta \beta t}) + \frac{1}{\beta \beta \beta \beta} (1 - e^{\beta \beta \beta \beta t}) + \frac{t}{\beta \beta \beta \beta}
\]

\[
+ \frac{1}{\beta \beta \beta \beta} (1 - e^{\beta \beta \beta \beta t}) - \frac{1}{\beta \beta \beta \beta} (1 - e^{\beta \beta \beta \beta t})
\]
\begin{align*}
- \frac{1}{\beta_k \delta_k \beta_{zz}^2} (1 - e^{\beta_{zz} t}) - \frac{t}{\beta_k \delta_k \beta_{zz}} &+ \frac{1}{\beta_{zz} \beta_k \delta_k (\beta_{zz} + \beta_k \delta_k)} (1 - e^{(\beta_{zz} + \beta_k \delta_k) t}) \\
- \frac{1}{\beta_{zz} \beta_k \delta_k} (1 - e^{\beta_{zz} t}) - \frac{t}{\beta_{zz} \beta_k \delta_k} &- \frac{1}{\beta_{zz} \beta_k \delta_k} (1 - e^{\beta_k \delta_k t}) - \frac{t}{\beta_{zz} \beta_k \delta_k} \\
- \frac{1}{2 \beta_{zz}^2} (1 - e^{2 \beta_{zz} t}) + \frac{2}{\beta_{zz}^2} (1 - e^{\beta_{zz} t}) + \frac{t}{\beta_{zz}^2} &\right) \\
+ \sum_{k,l=1}^{n} \beta_{S_i} \beta_{z_k} \rho_{S_k} \sigma_{S_k} \sigma_{S_l} &\left\{ - \frac{1}{\beta_k \delta_k} \left( 1 - e^{\beta_k \delta_k t} \right) - \frac{t}{\beta_k \delta_k} + \frac{1}{\beta_{zz}^2} (1 - e^{\beta_{zz} t}) + \frac{t}{\beta_{zz}} \right\} \\
+ \frac{1}{\beta_k \delta_k} (1 - e^{\beta_k \delta_k t}) + \frac{1}{\beta_k \delta_k} (1 - e^{\beta_{zz} t}) + \frac{t}{\beta_k \delta_k} &\left\{ - \frac{1}{\beta_{zz}} \left( 1 - e^{(\beta_{zz} + \beta_k \delta_k) t} \right) + \frac{1}{\beta_{zz}} (1 - e^{\beta_{zz} t}) \right\} \\
+ \frac{1}{\beta_k \delta_k} (1 - e^{\beta_k \delta_k t}) + \frac{t}{\beta_k \delta_k} &\right}\end{align*}
\[ \sigma_{s_{zz}}(t) = \sum_{k=1}^{n} \frac{\beta_{s_{z_{k}}\beta_{s_{z_{k}}}}}{\beta_{s_{z_{k}}}} \left( \frac{-1}{\beta_{s_{z_{k}}}} (1 - e^{(\beta_{s_{z_{k}}})^{t}}) \right) \]

\[ + \frac{1}{\beta_{s_{z_{k}}}} (1 - e^{(\beta_{s_{z_{k}}})^{t}}) + \frac{1}{\beta_{s_{z_{k}}} + \beta_{s_{z_{k}}}} (1 - e^{(\beta_{s_{z_{k}}})^{t}}) - \frac{1}{\beta_{s_{z_{k}}} + \beta_{s_{z_{k}}}} (1 - e^{(\beta_{s_{z_{k}}})^{t}}) \]

\[ + \sum_{k=1}^{n} \frac{\beta_{s_{z_{k}}}}{\beta_{s_{z_{k}}}} \frac{a_{k} \rho_{s_{z_{k}}} \sigma_{s_{z_{k}}} \sigma_{s_{z_{k}}}}{\beta_{s_{z_{k}}}} \left( \frac{-1}{\beta_{s_{z_{k}}}} (1 - e^{(\beta_{s_{z_{k}}})^{t}}) + \frac{1}{\beta_{s_{z_{k}}} + \beta_{s_{z_{k}}}} (1 - e^{(\beta_{s_{z_{k}}})^{t}}) \right) \]

\[ + \sum_{k,l=1}^{n} \frac{\beta_{s_{z_{k}}}}{\beta_{s_{z_{k}}}} \frac{a_{k} \rho_{s_{z_{k}}} \sigma_{s_{z_{k}}} \sigma_{s_{z_{k}}}}{\beta_{s_{z_{k}}}} \left( \frac{-1}{\beta_{s_{z_{k}}}} (1 - e^{(\beta_{s_{z_{k}}})^{t}}) + \frac{1}{\beta_{s_{z_{k}}}} (1 - e^{(\beta_{s_{z_{k}}})^{t}}) \right) \]

\[ + \sum_{k,l=1}^{n} \frac{\beta_{s_{z_{k}}}}{\beta_{s_{z_{k}}}} \frac{a_{k} \rho_{s_{z_{k}}} \sigma_{s_{z_{k}}} \sigma_{s_{z_{k}}}}{\beta_{s_{z_{k}}}} \left( \frac{-1}{\beta_{s_{z_{k}}}} (1 - e^{(\beta_{s_{z_{k}}})^{t}}) + \frac{1}{\beta_{s_{z_{k}}}} (1 - e^{(\beta_{s_{z_{k}}})^{t}}) \right) \]

\[ + \sum_{k,l=1}^{n} \frac{\beta_{s_{z_{k}}}}{\beta_{s_{z_{k}}}} \frac{a_{k} \rho_{s_{z_{k}}} \sigma_{s_{z_{k}}} \sigma_{s_{z_{k}}}}{\beta_{s_{z_{k}}}} \left( \frac{-1}{\beta_{s_{z_{k}}}} (1 - e^{(\beta_{s_{z_{k}}})^{t}}) + \frac{1}{\beta_{s_{z_{k}}}} (1 - e^{(\beta_{s_{z_{k}}})^{t}}) \right) \]

\[ + \sum_{k,l=1}^{n} \frac{\beta_{s_{z_{k}}}}{\beta_{s_{z_{k}}}} \frac{a_{k} \rho_{s_{z_{k}}} \sigma_{s_{z_{k}}} \sigma_{s_{z_{k}}}}{\beta_{s_{z_{k}}}} \left( \frac{-1}{\beta_{s_{z_{k}}}} (1 - e^{(\beta_{s_{z_{k}}})^{t}}) + \frac{1}{\beta_{s_{z_{k}}}} (1 - e^{(\beta_{s_{z_{k}}})^{t}}) \right) \]

\[ + \sum_{k=1}^{n} \frac{\beta_{s_{z_{k}}}}{\beta_{s_{z_{k}}}} \frac{a_{k} \rho_{s_{z_{k}}} \sigma_{s_{z_{k}}} \sigma_{s_{z_{k}}}}{\beta_{s_{z_{k}}}} \left( \frac{-1}{\beta_{s_{z_{k}}}} (1 - e^{(\beta_{s_{z_{k}}})^{t}}) + \frac{1}{\beta_{s_{z_{k}}}} (1 - e^{(\beta_{s_{z_{k}}})^{t}}) \right) \]

\[ - \sum_{k=1}^{n} \frac{a_{k} \rho_{s_{z_{k}}} \sigma_{s_{z_{k}}} \sigma_{s_{z_{k}}}}{\beta_{s_{z_{k}}}} (1 - e^{(\beta_{s_{z_{k}}})^{t}}) \]
\[ \sigma_{s_i,j}(t) = \frac{\beta_{s_i,j} \rho_{s_i,j} \sigma_{s_i,j}}{\beta_{s_i,j}} \left\{ -1 + \frac{1}{\beta_{s_i,j} + \beta_{s_i,j}} \left( 1 - e^{(\beta_{s_i,j} + \beta_{s_i,j})t} \right) \right\} \]

\[ + \sum_{k=1}^n \frac{\beta_{s_i,j} \beta_{s_k,j} \rho_{s_k,j} \sigma_{s_k,j}}{\beta_{s_i,j}} \left\{ -1 + \frac{1}{\beta_{s_k,j} + \beta_{s_i,j}} \left( 1 - e^{(\beta_{s_k,j} + \beta_{s_i,j})t} \right) \right\} \]

\[ + \frac{1}{\beta_{s_i,j}} \left( 1 - e^{\beta_{s_i,j}t} \right) - \frac{1}{\beta_{s_i,j}} \left( 1 - e^{\beta_{s_i,j}t} \right) \]

\[ \sigma_{s_i,j}(t) = \frac{-\rho_{s_i,j} \sigma_{s_i,j}}{\beta_{s_i,j} + \beta_{s_i,j}} \left( 1 - e^{(\beta_{s_i,j} + \beta_{s_i,j})t} \right) \]

\[ \sigma_{s_i,z}(t) = \sum_{l=1}^n \frac{\beta_{s_i,j} \rho_{s_i,j} \sigma_{s_i,j}}{\beta_{s_i,j} - \beta_{s_i,j}} \left\{ -1 + \frac{1}{\beta_{s_i,j} + \beta_{s_i,j}} \left( 1 - e^{(\beta_{s_i,j} + \beta_{s_i,j})t} \right) \right\} \]

\[ + \frac{1}{\beta_{s_i,j} + \beta_{s_i,j}} \left( 1 - e^{(\beta_{s_i,j} + \beta_{s_i,j})t} \right) \]

\[ - \sum_{l=1}^n \frac{a_k \rho_{s_k,j} \sigma_{s_k,j}}{\beta_{s_k,j} + \beta_{s_k,j}} \left( 1 - e^{(\beta_{s_k,j} + \beta_{s_k,j})t} \right) \]

\[ \sigma_{z,z}(t) = \sum_{k,l=1}^n \frac{\beta_{s_k,j} \beta_{s_l,j} \rho_{s_k,j} \sigma_{s_k,j} \sigma_{s_l,j}}{(\beta_{s_k,j} - \beta_{s_l,j}) (\beta_{s_l,j} - \beta_{s_k,j})} \left\{ -1 + \frac{1}{\beta_{s_k,j} + \beta_{s_l,j}} \left( 1 - e^{(\beta_{s_k,j} + \beta_{s_l,j})t} \right) \right\} \]

\[ + \frac{1}{\beta_{s_k,j} + \beta_{s_l,j}} \left( 1 - e^{(\beta_{s_k,j} + \beta_{s_l,j})t} \right) \]

\[ - \frac{1}{2\beta_{s_l,j}} \left( 1 - e^{2\beta_{s_l,j}t} \right) \]

\[ + \sum_{k,l=1}^n \frac{\beta_{s_k,j} a_k \rho_{s_k,j} \sigma_{s_k,j} \sigma_{s_l,j}}{\beta_{s_k,j} - \beta_{s_l,j}} \left\{ -1 + \frac{1}{\beta_{s_k,j} + \beta_{s_l,j}} \left( 1 - e^{(\beta_{s_k,j} + \beta_{s_l,j})t} \right) \right\} \]

\[ + \frac{1}{2\beta_{s_k,j}} \left( 1 - e^{2\beta_{s_k,j}t} \right) \]

\[ - \sum_{k,l=1}^n \frac{a_k a_l \rho_{s_k,j} \sigma_{s_k,j} \sigma_{s_l,j}}{2\beta_{s_k,j}} \left( 1 - e^{2\beta_{s_k,j}t} \right) \]
The observation equation is

\[ y_t = H_t x_t + C_t^y + w_t \]

where

\[
y_t = \begin{bmatrix}
\ln G_1(t, T_1, 1) & \cdots & \ln G_1(t, T_1, k) & \ln G_2(t, T_2, 1) & \cdots & \ln G_n(t, T_n, k)
\end{bmatrix}^T
\]

\[
H_t = \begin{bmatrix}
H_{G_1 T_1} S_i & \cdots & H_{G_1 T_1} S_n & H_{G_1 T_1} \delta_1 & \cdots & H_{G_1 T_1} \delta_n & H_{G_1 T_1} z \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
H_{G_2 T_2} S_i & \cdots & H_{G_2 T_2} S_n & H_{G_2 T_2} \delta_1 & \cdots & H_{G_2 T_2} \delta_n & H_{G_2 T_2} z \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
H_{G_n T_n} S_i & \cdots & H_{G_n T_n} S_n & H_{G_n T_n} \delta_1 & \cdots & H_{G_n T_n} \delta_n & H_{G_n T_n} z
\end{bmatrix}
\]

\[
C_t^y = \begin{bmatrix}
C_1(t, T_1, 1) + \frac{\sigma_{G_1 T_1}^2(t, T_1, 1)}{2} \\
\vdots \\
C_1(t, T_1, k) + \frac{\sigma_{G_1 T_1}^2(t, T_1, k)}{2} \\
C_2(t, T_2, 1) + \frac{\sigma_{G_2 T_2}^2(t, T_2, 1)}{2} \\
\vdots \\
C_n(t, T_n, 1) + \frac{\sigma_{G_n T_n}^2(t, T_n, 1)}{2}
\end{bmatrix}
\]

\[
R = \text{Cov}(w_t) = \begin{bmatrix}
R_{1,1} & 0 \\
0 & \ddots \\
& & R_{nk, nk}
\end{bmatrix}
\]

\[
H_{G_{Ti} S_j} = \begin{cases}
1, & i = k \\
0, & \text{otherwise}
\end{cases}
\]

\[
H_{G_{Ti} \delta_j} = \begin{cases}
-\frac{1-e^{-\kappa_i T_i k}}{\kappa_i} - \frac{b_{a_i}(e^{b_{Ti} k} - e^{-\kappa_i T_i k})}{b(b+\kappa_i)} + \frac{b_{a_i}(1-e^{-\kappa_i T_i k})}{b \kappa_i}, & i = j \\
-\frac{b_{a_i}(e^{b_{Ti} k} - e^{-\kappa_j T_i k})}{b(b+\kappa_i)} + \frac{b_{a_i}(1-e^{-\kappa_j T_i k})}{b \kappa_j}, & \text{otherwise}
\end{cases}
\]

\[
H_{G_{Ti} z} = \frac{b_i}{b}(e^{b_{Ti} k} - 1)
\]
\[ C_i(t, T) = \left( r + b_i m - \frac{\sigma_i^2}{2} - \hat{\alpha}_i + \sum_{j=1}^n b_i a_j \hat{\alpha}_j \right) (T - t) \]
\[ + \frac{b_i m}{b} (1 - e^{b(T-t)}) - \sum_{j=1}^n \frac{b_i a_j \hat{\alpha}_j}{b} (e^{b(T-t)} - 1) \]
\[ + \sum_{j=1}^n \frac{b_i a_j \hat{\alpha}_j}{b(b + \kappa_j)} (e^{b(T-t)} - e^{-\kappa_j(T-t)}) + \frac{\hat{\alpha}_i}{\kappa_i} (1 - e^{-\kappa_i(T-t)}) \]
\[ - \sum_{j=1}^n \frac{b_i a_j \hat{\alpha}_j}{b\kappa_j} (1 - e^{-\kappa_j(T-t)}). \]

Here we assumed that there are \( k \) maturities for each commodity futures which can be generalized to different maturities for each commodity and used the notation \( T_{ij} \) for the maturity of commodity \( i \) futures contract \( j \) th closest to maturity.

The forecasts and filters are

\[ x_{t|t-1} = F_t x_{t-1|t-1} + C_t^x \]
\[ P_{t|t-1} = F_t P_{t-1|t-1} F_t^\top + Q_t \]
\[ \Sigma_{t|t-1} = H_t P_{t|t-1} H_t^\top + R_t \]
\[ K_t = P_{t|t-1} H_t^\top \Sigma_{t|t-1}^{-1} \]
\[ x_t = x_{t|t-1} + K_t (y_t - H_t x_{t|t-1} - C_t^y) \]
\[ P_t = (I - K_t H_t) P_{t|t-1} \]

and the maximum log likelihood is

\[ \mathcal{L}(\vartheta) = -\frac{1}{2} \left\{ nm T \ln(2\pi) + \sum_{t=1}^T \ln |\Sigma_{t|t-1}| + \sum_{t=1}^T u_t^\top \Sigma_{t|t-1}^{-1} u_t \right\} \]
\[ u_t = y_t - H_t x_{t|t-1} - C_t^y. \]
Appendix 5  The Generalized state equation

We generalize the model by assuming that market price of risk is linear with \( \ln S_1(t), \ldots, \ln S_n(t), \delta_1(t), \ldots, \delta_n(t), \) and \( z(t) \). For ease of notation, we use the following sdes instead of sdes (1), (2), (4).

\[
\begin{align*}
    dS_i(t) &= S_i(t)(r - \delta_i(t) + b_i z(t))dt + S_i(t)\sigma_{S_i}^\top dW(t), \quad i = 1, \ldots, n \\
    d\delta_i(t) &= \kappa_i(\hat{\alpha}_i - \delta_i(t))dt + \sigma_{\delta_i}^\top dW(t), \quad i = 1, \ldots, n \\
    dz(t) &= -b(m - z(t))dt + \sum_{i=1}^n a_i \delta_i(t)dt + \sigma_z^\top dW(t) \quad (16)
\end{align*}
\]

\( \sigma_z^\top = \sum_{i=1}^n a_i \sigma_{S_i} \).

where \( W(t) \) are \( 2n \) dimensional standard Brownian motion and

\[
\sigma_{S_i}^\top \sigma_{S_j} = \rho_{S_i S_j} \sigma_{S_i} \sigma_{S_j} \\
\sigma_{S_i}^\top \sigma_{\delta_i} = \rho_{S_i \delta_i} \sigma_{S_i} \sigma_{\delta_i}.
\]

The difference between equations (1), (2), (4) and the above are that the Brownian motions are correlated or not, however the distribution are the same which makes no difference when modeling with either equations. We just need to apply Cholesky decomposition to the covariance matrix in order to generate \( \sigma_{S_i} \) and \( \sigma_{\delta_i} \). These sdes (14), (15), (16) are more easier to derive the solutions in matrix form.

Let us assume that Brownian motions under equivalent martingale measure \( W(t) \) and Brownian motions under natural probability \( W^P(t) \) satisfies

\[
W(t) = W^P(t) + \int_0^t \theta(s, S(s), \delta(s), z(s))ds
\]

\[
\theta(s, S(s), \delta(s), z(s)) = \hat{\beta}_0 + \sum_{i=1}^n \hat{\beta}_{S_i} \ln S_i(t) + \sum_{i=1}^n \hat{\beta}_{\delta_i} \delta_i(t) + \hat{\beta}_z z(t)
\]

\[
\hat{\beta}_0 = \begin{bmatrix}
    \hat{\beta}_{S_1,0} \\
    \vdots \\
    \hat{\beta}_{S_n,0} \\
    \hat{\beta}_{\delta_0} \\
    \vdots \\
    \hat{\beta}_{\delta_n} \\
\end{bmatrix}, \hat{\beta}_{S_i} = \begin{bmatrix}
    \hat{\beta}_{S_1,1} \\
    \vdots \\
    \hat{\beta}_{S_n,1} \\
\end{bmatrix}, \hat{\beta}_{\delta_i} = \begin{bmatrix}
    \hat{\beta}_{\delta_1,1} \\
    \vdots \\
    \hat{\beta}_{\delta_n,1} \\
\end{bmatrix}, \hat{\beta}_z = \begin{bmatrix}
    \hat{\beta}_{S_1 z} \\
    \vdots \\
    \hat{\beta}_{S_n z} \\
    \hat{\beta}_{\delta_1 z} \\
    \vdots \\
    \hat{\beta}_{\delta_n z} \\
\end{bmatrix}
\]
\( \theta \) is market price of risk which is assumed that it is linear to \( S_1(t), \ldots, S_n(t), \delta_1(t), \ldots, \delta_n(t), \) and \( z(t) \). Now we can derive the sdes under natural probability.

\[
d\ln S_i(t) = \left( \beta_{S,0} + \sum_{j=1}^{n} \beta_{S,S_j} \ln S_j(t) + \sum_{j=1}^{n} \beta_{S,\delta_j} \delta_j(t) + \beta_{S,z} z(t) \right) dt + \sigma_{S_i}^T dW^P(t)
\]

\[
\beta_{S,0} = r - \frac{\sigma_{S_i}^2}{2} + \sigma_{S_i}^T \beta_0, \quad \beta_{S,S_j} = \sigma_{S_i}^T \beta_{S_j}, \quad \beta_{S,\delta_j} = -1 + \sigma_{S_i}^T \beta_{\delta_j},
\]

\[
\beta_{S,z} = b_i + \sigma_{S_i}^T \beta_z
\]

\[
d\delta_i(t) = \left( \beta_{\delta,0} + \sum_{j=1}^{n} \beta_{\delta,S_j} \ln S_j(t) + \sum_{j=1}^{n} \beta_{\delta,\delta_j} \delta_j(t) + \beta_{\delta,z} z(t) \right) dt + \sigma_{\delta_i}^T dW^P(t)
\]

\[
\beta_{\delta,0} = \kappa_i \hat{\alpha}_i + \sigma_{\delta_i}^T \beta_0, \quad \beta_{\delta,S_j} = \sigma_{\delta_i}^T \beta_{S_j}, \quad \beta_{\delta,\delta_j} = -\kappa_i + \sigma_{\delta_i}^T \beta_{\delta_j},
\]

\[
\beta_{\delta,z} = \sigma_{\delta_i}^T \beta_z
\]

\[
dz(t) = \left( \beta_{z,0} + \sum_{j=1}^{n} \beta_{z,S_j} \ln S_j(t) + \sum_{j=1}^{n} \beta_{z,\delta_j} \delta_j(t) + \beta_{zz} z(t) \right) dt + \sigma_{z}^T \cdot dW^P(t)
\]

\[
\beta_{z,0} = bm + \sigma_{z}^T \beta_0, \quad \beta_{z,S_j} = \sigma_{z}^T \beta_{S_j}, \quad \beta_{z,\delta_j} = -a_j + \sigma_{z}^T \beta_{\delta_j},
\]

\[
\beta_{zz} = b + \sigma_{z}^T \beta_z
\]
In matrix form, the solution to this equation is

\[
X(t) = e^{t\beta} \left\{ X(0) + \int_{0}^{t} e^{-s\beta} \Sigma dW^p(s) + \int_{0}^{t} e^{-s\beta} \beta_0 ds \right\}
\]

\[
X(t) = \begin{bmatrix}
\ln S_1(t) & \cdots & \ln S_n(t) & \delta_1(t) & \cdots & \delta_n(t) & z(t)
\end{bmatrix}^T
\]

\[
\beta_0 = \begin{bmatrix}
\beta_{S_10} & \cdots & \beta_{S_n0} & \beta_{\delta_10} & \cdots & \beta_{\delta_n0} & \beta_{z0}
\end{bmatrix}^T
\]

\[
\beta = \begin{bmatrix}
\beta_{S_1s_1} & \cdots & \beta_{S_1s_n} & \beta_{S_1\delta_1} & \cdots & \beta_{S_1\delta_n} & \beta_{S_1z}
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots
\beta_{z_s_s_1} & \cdots & \beta_{z_s_s_n} & \beta_{z_s_\delta_1} & \cdots & \beta_{z_s_\delta_n} & \beta_{z_s_z}
\end{bmatrix}
\]

\[
\Sigma = \begin{bmatrix}
\sigma_{s_1} & \cdots & \sigma_{s_n} & \sigma_{\delta_1} & \cdots & \sigma_{\delta_n} & \sigma_{z}
\end{bmatrix}^T
\]

This equation can be used as the state equation for Kalman filters. With this generalization, we can handle market price of risk as linear with log price of assets, convenience yields, or may be stochastic interest rates which we did not consider in this paper. However, the drawback of the generalized model is that there are too many parameters need to be estimated.

**Appendix 6  Coefficients for equation (13)**

We show the coefficients for equation (13) in this subsection.

\[
c = \begin{bmatrix}
c_{s_1}(\Delta t) & \cdots & c_{s_n}(\Delta t) & c_{\delta_1}(\Delta t) & \cdots & c_{\delta_n}(\Delta t) & c_z(\Delta t)
\end{bmatrix}^T
\]

\[
\Phi_0 = \begin{bmatrix}
\phi_{S_1s_1}(\Delta t) & \cdots & \phi_{S_1s_n}(\Delta t) & \phi_{S_1\delta_1}(\Delta t) & \cdots & \phi_{S_1\delta_n}(\Delta t) & \phi_{S_1z}(\Delta t)
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots
\phi_{s_n s_1}(\Delta t) & \cdots & \phi_{s_n s_n}(\Delta t) & \phi_{s_n \delta_1}(\Delta t) & \cdots & \phi_{s_n \delta_n}(\Delta t) & \phi_{s_n z}(\Delta t)
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots
\phi_{\delta_1 s_1}(\Delta t) & \cdots & \phi_{\delta_1 s_n}(\Delta t) & \phi_{\delta_1 \delta_1}(\Delta t) & \cdots & \phi_{\delta_1 \delta_n}(\Delta t) & \phi_{\delta_1 z}(\Delta t)
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots
\phi_{\delta_n s_1}(\Delta t) & \cdots & \phi_{\delta_n s_n}(\Delta t) & \phi_{\delta_n \delta_1}(\Delta t) & \cdots & \phi_{\delta_n \delta_n}(\Delta t) & \phi_{\delta_n z}(\Delta t)
\phi_{z_{s_1}}(\Delta t) & \cdots & \phi_{z_{s_n}}(\Delta t) & \phi_{z_{\delta_1}}(\Delta t) & \cdots & \phi_{z_{\delta_n}}(\Delta t) & \phi_{z_{z}}(\Delta t)
\end{bmatrix}
\]

\(^{9}\text{Cf. Karatzas and Shreve (1991), section 5.6 or Ikeda and Watanabe (1981), p. 232, Ex. 8.1.}\)
\[ c_i(\Delta t) = r + b_i m - \frac{\sigma_i^2}{2} - \hat{\alpha}_i + \sum_{j=1}^{n} \frac{b_i a_j \hat{\alpha}_j}{b} + \frac{b_i m}{b}(1 - e^{b \Delta t}) \]

\[ - \sum_{j=1}^{n} \frac{b_i a_j \hat{\alpha}_j}{b^2}(e^{b \Delta t} - 1) + \sum_{j=1}^{m} \frac{b_i a_j \hat{\alpha}_j}{b(b + \kappa_j)}(e^{b \Delta t} - e^{-\kappa_j \Delta t}) \]

\[ + \frac{\hat{\alpha}_i}{\kappa_i}(1 - e^{-\kappa_i \Delta t}) - \sum_{j=1}^{n} \frac{b_i a_j \hat{\alpha}_j}{b \kappa_j}(1 - e^{-\kappa_j \Delta t}) \]

\[ c_i(\Delta t) = \hat{\alpha}_i(1 - e^{-\kappa_i \Delta t}) \]

\[ c_z(\Delta t) = m(1 - e^{b \Delta t}) - \sum_{i=1}^{n} \frac{a_i \hat{\alpha}_i}{b}(e^{b \Delta t} - e^{-\kappa_i \Delta t}) \]

\[ \phi_{S_i S_j}(\Delta t) = 0 \quad \forall i, j \]

\[ \phi_{S_i \delta_j}(\Delta t) = - \frac{b_i a_j}{b(b + \kappa_i)}(e^{b \Delta t} - e^{-\kappa_j \Delta t}) - \frac{1}{\kappa_i}(1 - e^{-\kappa_i \Delta t}) \]

\[ + \frac{b_i a_j}{b \kappa_i}(1 - e^{-\kappa_i \Delta t}) \]

\[ \phi_{S_i \delta_j} = - \frac{b_i a_j}{b(b + \kappa_j)}(e^{b \Delta t} - e^{-\kappa_i \Delta t}) + \frac{b_i a_j}{b \kappa_j}(1 - e^{-\kappa_j}) \quad i \neq j \]

\[ \phi_{S_i z} = - \frac{b_i}{b}(1 - e^{b \Delta t}) \]

\[ \phi_{\delta_i S_j} = 0 \]

\[ \phi_{\delta_i \delta_j} = e^{-\kappa_i \Delta t} \]

\[ \phi_{\delta_i \delta_j} = 0 \quad i \neq j \]

\[ \phi_{\delta_i z} = 0 \]

\[ \phi_{z S_i} = 0 \]

\[ \phi_{z \delta_i} = - \frac{a_i}{b + \kappa_i}(e^{b \Delta t} - e^{-\kappa_i \Delta t}) \]

\[ \phi_{zz} = 0 \]
References


