On the Existence of Bayesian Cournot Equilibrium

By

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Presented by:

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- **Topological (Nash) approach:** When the inverse demand function is concave and decreasing, and costs are convex, the payoff function of each firm is concave in its output. Existence follows from Nash's theorem.

- **Lattice approach:** Non-concave demand and non-convex costs can be allowed: Novshek (1985) (condition on the inverse demand that has a flavor of concavity but more general), Amir (1996) (log-concavity of the inverse demand).

Both Novshek and Amir conditions imply the property of *strategic substitutes*: each firm's reaction function is decreasing.

With incomplete information, there is a major obstacle to using the Nash and the lattice approaches, for reasons to be explained.
What is known about the existence of Bayesian equilibrium in the incomplete information Cournot oligopolies?

**Literature on oligopolies with incomplete information:**

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![Diagram of linear price function](image)

- Prices can be negative

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But prices in reality are non-negative, and so in a model they need to be truncated:

This truncation (insisting on always non-negative prices) has no impact on equilibrium existence in the complete information case, but makes existence a much scarcer phenomenon under incomplete information.
**Main finding:** The existence of Bayesian equilibrium in Cournot oligopolies with incomplete information (and always non-negative prices) is a much more limited phenomenon compared to the existence of Nash equilibrium in Cournot oligopolies with complete information.

Bayesian equilibrium fails to exist even in very simple, linear, Cournot duopolies with incomplete information.

What causes the non-existence?
Why, under incomplete information, always non-negative prices can have such a detrimental effect on equilibrium existence, even in linear duopolies?
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Had possibly negative prices been allowed, the demand function would be linear for all possible outputs.

Payoff function of each firm, in each state of nature, is concave in that firm’s strategy:

The expected payoff functions inherit concavity, and the existence of equilibrium follows from Nash theorem.
Why, under incomplete information, always non-negative prices can have such a detrimental effect on equilibrium existence, even in linear duopolies?

But with always non-negative prices the situation is different. The inverse demand is truncated:

Payoffs are not concave, but merely quasi-concave:

But *expected* payoff functions may not be quasi-concave, which often results in equilibrium non-existence.
Example of non-existence of Bayesian Cournot equilibrium

Two states of nature:

\[ \begin{align*}
\text{state } \omega_1, \ p(\omega_1) &= \frac{1}{13} \\
\text{state } \omega_2, \ p(\omega_2) &= \frac{12}{13}
\end{align*} \]

Two firms: firm 1 is completely informed about the state of nature, firm 2 has no information about it. Marginal costs:

\[
\begin{align*}
c^1(\omega_1) &= 5 \\
c^2(\omega_1) &= 0.001
\end{align*} \quad \Rightarrow \quad \begin{align*}
c^1(\omega_2) &= 0.001 \\
c^2(\omega_2) &= 0.001
\end{align*}
\]

Note that \( c^1(\omega_1) > \bar{p} \), and thus firm 1 will not produce at \( \omega_1 \) in any best response. Therefore firm 1’s strategy can w.l.o.g. be assumed to be a scalar \( q^1 = q^1(\omega_2) \)

Firm 2’s strategy is a scalar \( q^2 = q^2(\omega_1) = q^2(\omega_2) \)
For simplicity of presentation, approximate $c'(\omega) = c^2 = 0.001 = 0$

Since Firm 1 doesn't produce at $\omega$, it only faces $P(\omega_2, Q) = \max(1 - Q, 0)$.

Therefore Firm 1's reaction function is given by

$$R^1(q^2) = \max\left(\frac{1 - q^2}{2}, 0\right).$$
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\[ R^1(q^2) = \max\left(\frac{1 - q^2}{2}, 0\right). \]
With possibly negative prices, Firm 2's reaction function is:

\[ R_2^2(q^1) = \max\left(\frac{8}{13} - \frac{6}{13}q^1, 0\right) \]

The reaction functions intersect at \( q^1 = \frac{1}{4}, q^2 = \frac{1}{2} \).

This is the unique Bayesian Cournot equilibrium when possibly negative prices are allowed.
But with always non-negative prices, the reaction function of Firm 2 is neither decreasing nor continuous! It has the following form:

\[
R^2(q^1) = \begin{cases} 
\frac{8}{13} - \frac{6}{13} q^1 & \text{if } q^1 < 0.13148 \\
0.5547,2 & \text{if } q^1 = 0.13148 \\
2 & \text{if } q^1 > 0.13148 
\end{cases}
\]

There is no Bayesian Cournot equilibrium with always non-negative prices, because the graphs of the reaction functions never cross each other.
Part I

With possibly negative prices:

\[ \pi_-^2 \left( q_1 = \frac{1}{4}, q_2 \right) = \]
\[ = \frac{1}{13} \cdot (4 - 0 - q_2) \cdot q_2 + \frac{12}{13} \cdot (1 - 1 - q_2) \cdot q_2 \]
Part II

With always non-negative prices:

\[ \pi^2 \left( q_1 = \frac{1}{4}, q_2 \right) = \]

\[ = \frac{1}{13} \cdot \max(4 - 0 - q_2, 0) \cdot q_2 + \frac{12}{13} \cdot \max(1 - \frac{1}{4} - q_2, 0) \cdot q_2 \]

Part III
Conditions under which a Bayesian equilibrium exists

For simplicity of this presentation, assume that the space of states of nature $\Omega$ is finite, and that the information endowment of each firm $i$ is given by a partition $\pi^i$ of $\Omega$.

At every state of nature $\omega$ we make the following assumptions:

1. $P(\omega, Q)$ is decreasing in $Q$
2. $P(\omega, Q)$ is non-negative, and reaches zero at a state-dependent (horizontal) intercept $Q(\omega)$
3. $c^i(\omega, q^i)$ is continuous and increasing in $q^i$
4. $P(\omega, Q)$ is twice continuously differentiable and satisfies the Novshek condition below $Q$:

$$\forall Q \in [0, Q(\omega)]: QP''(\omega, Q) + P'(\omega, Q) \leq 0$$
But conditions (1)—(4) do not suffice to establish Bayesian equilibrium existence, as the previous example showed. Thus, an *additional condition* will be needed.

In our example of non-existence, the primitives of the model (inverse demand and costs) are well behaved, which indicates that it is hard to find a simple condition on the primitives of the model that would guarantee equilibrium existence.

The next condition (5) is assumed on the payoff functions:

(5) for each firm $i$ there exists a state-dependent threshold $q^i(\omega)$ measurable w.r.t. $\pi^i$, such that:

a) no firm will exceed its threshold in (some) best response,

\[ \sum_i q^i(\omega) \leq Q(\omega). \]

If marginal costs increase sufficiently fast, such thresholds are likely to be found.
**Theorem 1**: Suppose (1), (2), (3), (4), and (5) holds. Then there exists a Bayesian equilibrium if either:

(a) there are only two firms; or

(b) cost functions of all firms are convex.
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It is interesting to compare this with what happens when negative prices are allowed (that is, the inverse demand is not required to be zero when \( Q > Q(\omega) \), and (4) holds on the entire \( \mathbb{R}_+ \)). Then (5) is not needed:

**Theorem 2**: Suppose \( P \) is allowed to obtain negative values but, except for that, (1)-(4) hold and condition (4) holds on the entire \( \mathbb{R}_+ \). Then there exists a Bayesian equilibrium if either:

(a) there are only two firms; or

(b) cost functions of all firms are convex.
Uniqueness of Bayesian equilibrium

Even if the equilibrium exists in an oligopoly, it may not be unique:

There is a simple example of a duopoly with a *state-independent* demand $P = \min(1-Q^2,0)$, and state dependent linear costs, in which there are two Bayesian Cournot equilibria.

However:

**Theorem:**

In a duopoly with always non-negative prices and at least one Bayesian Cournot equilibrium, suppose that:

1) one firm better informed than the other;

2) there is a complete information on $Q$;

3) condition (4) holds;

4) costs are strictly increasing, convex, and twice continuously differentiable;

Then the Bayesian equilibrium is *unique*. Moreover, this is also the case in oligopoly, provided that there are only two types of firms, and firms of one type are better informed.
Conclusions:

1. Bayesian Cournot equilibrium fails to exist quite often in oligopolies with always non-negative prices. It is difficult to find simple and general conditions on primitives that would guarantee equilibrium existence, although we have made some progress.

2. However, when negative prices are allowed, Bayesian Cournot equilibrium exists under quite general conditions.

3. All this is unlike what happens under complete information, where the cases of always non-negative (truncated) and possibly negative (non-truncated) prices are equivalent in terms of equilibrium existence.

4. An equilibrium with possibly negative prices, even if prices in it are positive in every state of nature, may fail to be an equilibrium with always non-negative (truncated) prices.

5. Uniqueness of Bayesian Cournot equilibrium in oligopolies with always non-negative prices is problematic too. Uniqueness is guaranteed only under strong conditions on information.