Dual representations of cardinal preferences

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Outline

1. Objectives
2. Formalism
3. Results
   - General dualities
   - Duality between utilities and Arrow-Pratt functions
   - Extension
4. Summary
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Objective 1

- Generate **dual representations of cardinal preferences**

  Given the set of lotteries \( \Delta(O) \) over a set of outcomes \( O \), a **cardinal preference** is a binary relation \( \succeq \) over \( \Delta(O) \) that has a von Neumann-Morgenstern representation \( u : O \rightarrow \mathbb{R} \), i.e., for lotteries \( \mu, \lambda \in \Delta(O) \)

  \[
  \mu \succeq \lambda \iff \int_O \mu(dx) u(x) \geq \int_O \lambda(dx) u(x)
  \]

- Do the same for risk averse preferences

- This will enable precise and flexible specification of preferences in applications, just as duality theory has done in the case of ordinal preferences
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- Generate these dualities for vector outcomes, not merely scalar outcomes
  - Enables applications with vector outcomes
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What is duality theory in the context of cardinal preferences?

- What are the ‘natural’ dual objects?
  - von Neumann-Morgenstern utility functions
  - Certainty equivalent mappings
  - Risk premia mappings
  - Acceptance set mappings
  - Generalized Arrow-Pratt functions

- What is duality?
  - Bijections between specified sets of the above-mentioned objects
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The setting

- $(X, \succeq)$ is a partially ordered topological vector space,
  - with very general and standard linear-topological-ordering structure
- $O \subset X_+$ is the convex and compact outcome set
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- \(O \subset X_+\) is the convex and compact outcome set
von Neumann-Morgenstern utility functions

- \( \mathcal{U} \) consists of (von Neumann-Morgenstern utility) functions \( u : O \rightarrow \mathbb{R} \) such that
  - \( u(0) = 0 \)
  - \( u \) increasing with respect to \( \geq \)
  - \( u \) continuous

- \( \mathcal{U}_a \) is the set of risk averse \( u \in \mathcal{U} \), i.e., for every \( \mu \in \Delta(O) \)

\[
u \left( \int_O \mu(dz) \, z \right) \geq \int_O \mu(dz) \, u(z)
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- \( u \in \mathcal{U} \) is identified with the equivalence class \([u]\) of functions \( v : O \rightarrow \mathbb{R} \) that are increasing affine transformations of \( u \)
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Certainty equivalent mappings

\( \mathcal{F} \) is a set of mappings \( F : \Delta(O) \mapsto O \)

- Interpretation: \( F(\mu) \) is the set of certainty equivalents corresponding to lottery \( \mu \)
- For \( u \in U \), define \( \phi(u) : \Delta(O) \mapsto O \) by

\[
\phi(u)(\mu) = \{ x \in O \mid u(x) = U(\mu) \}
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where \( U(\mu) = \int_O \mu(dz) u(z) \) is the expected utility from lottery \( \mu \)

- \( \mathcal{F}_a \subset \mathcal{F} \) is the collection of “risk averse” certainty equivalent mappings
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Risk premia mappings

\( \mathcal{P} \) is a set of mappings \( P : \Delta(O) \rightarrow X \)

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- For \( F \in \mathcal{F} \), define \( \psi(F) : \Delta(O) \rightarrow X \) by

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\psi(F)(\mu) = \{ x \in X \mid m_\mu - x \in F(\mu) \}
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- \( \mathcal{P}_a \subset \mathcal{P} \) is the collection of “risk averse” risk premia mappings
Acceptance set mappings

$A$ is a set of mappings $A : O \mapsto \Delta(O)$

- Interpretation: $A(x)$ is the acceptance set corresponding to outcome $x$
- For $u \in \mathcal{U}$, define $\xi(u) : O \mapsto \Delta(O)$ by

$$\xi(u)(x) = \{\mu \in \Delta(O) \mid u(x) \leq U(\mu)\}$$

- $A_a \subset A$ is the collection of “risk averse” acceptance set mappings
Acceptance set mappings

\( \mathcal{A} \) is a set of mappings \( \mathcal{A} : O \Rightarrow \Delta(O) \)

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Dual representations of cardinal preferences
General dualities

Theorem

Given appropriate definitions of $\mathcal{F}$, $\mathcal{P}$ and $\mathcal{A}$

$$
\phi : \mathcal{U} \rightarrow \mathcal{F} \quad \psi : \mathcal{F} \rightarrow \mathcal{P} \quad \xi : \mathcal{U} \rightarrow \mathcal{A}
$$

are bijections.

Corollary

$$
\psi \circ \phi : \mathcal{U} \rightarrow \mathcal{P} \quad \xi \circ \phi^{-1} : \mathcal{F} \rightarrow \mathcal{A} \quad \xi \circ \phi^{-1} \circ \psi^{-1} : \mathcal{P} \rightarrow \mathcal{A}
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General dualities for risk averse preferences

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4. Summary
Let \( O \subset \mathbb{R}^n \) be a “nice” outcome space with boundary \( \partial O \).

For \( u : O \to \mathbb{R} \), the (generalized) Arrow-Pratt function \( \Gamma_1(u) : O \to \mathbb{R}^n \) is

\[
\Gamma_1(u)(x) = \frac{-D^2 u(x)Du(x)}{\|Du(x)\|^2}
\]

if \( u \) is twice differentiable at \( x \) and \( \|Du(x)\| > 0 \); otherwise, \( \Gamma_1(u)(x) = 0 \).

- Reduces to scalar Arrow-Pratt coefficient for \( n = 1 \).
- Has economic interpretation that it yields same ordering of risk averse preferences as other criteria.
Let $O \subset \mathbb{R}^n$ be a “nice” outcome space with boundary $\partial O$.

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First (simple-minded) conjecture

Conjecture. \( \Gamma_1 : \mathcal{U}^{nd} \rightarrow \mathcal{R}^n \) is a bijection given appropriate sets

- \( \mathcal{U}^{nd} \) of utility functions \( u : O \rightarrow \mathcal{R} \)
- \( \mathcal{R}^n \) of Arrow-Pratt functions \( a : O \rightarrow \mathcal{R}^n \)

The conjecture is false as “initial value” auxiliary conditions, of the kind used in the scalar case, are inadequate for making \( \Gamma_1 \) injective...this can be shown very generally

So, we need an extra dual object to supplement the Arrow-Pratt function and identify distinct preferences in terms of the pair of dual objects
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Second (less simple-minded) conjecture

- The appropriate supplementary object is boundary data
- Let $\Gamma_2(u) = u_{\partial O}$ denote the restriction of $u$ to $\partial O$
- $\mathcal{G}$ is a set of boundary data $g : \partial O \rightarrow \mathbb{R}$
- Conjecture. $\Gamma : \mathcal{U}^{nd} \rightarrow \mathbb{R}^n \times \mathcal{G}$ is a bijection
- The key is to solve the master problem: for every $(a, g) \in \mathbb{R}^n \times \mathcal{G}$, there exists a unique $u \in \mathcal{U}^{nd}$ such that

\[ \Gamma_1(u)(.) = \frac{-D^2 u(.) Du(.)}{\|Du(.)\|^2} = a(.) \]

and

\[ \Gamma_2(u) = u_{\partial O} = g \]

- Master problem involves a system of non-linear PDEs.
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- It is easily checked that

**Lemma**

If \( u \in \mathcal{U}^{nd} \), then \( \Gamma(u) \in \mathcal{R}^n \times \mathcal{G} \).

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For every \((a, g) \in \mathcal{R}^n \times \mathcal{G}\), there exists \( u : O \rightarrow \mathcal{R} \) such that \( [u] \cap \mathcal{U}^{nd} \neq \emptyset \) and \( \Gamma(u) = (a, g) \).
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Outline

1. Objectives
2. Formalism
3. Results
   - General dualities
   - Duality between utilities and Arrow-Pratt functions
   - Extension
4. Summary

Sudhir A. Shah

Dual representations of cardinal preferences
Consider the risk averse preferences represented by $A_1, A_2 \in \mathcal{A}_a$

Is it legitimate to say “the preference represented by $A_1$ is more risk averse than that represented by $A_2$” if $A_1(.) \subset A_2(.)$?

Yes, because the duality result implies

1. $\xi^{-1}(A_1) \in \mathcal{U}_a$ and $\xi^{-1}(A_2) \in \mathcal{U}_a$, and
2. $\xi \circ \xi^{-1}(A_1)(.) = A_1(.) \subset A_2(.) = \xi \circ \xi^{-1}(A_2)(.)$

which is the Yaari criterion for comparative risk aversion extended to the vector outcome context.
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