Dynamic Campaign Spending*

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Abstract

We build a model of electoral campaigning in which two candidates allocate money over time to control the movement of relative popularity. If any gain in a candidate's popularity tends to decay over time (making it harder for the candidate to maintain or increase her lead when she is already ahead) then the candidates increase their spending over time ahead of the election. Relative popularity follows a modified Brownian motion whose long-run mean depends only on the ratio of starting budgets.

We also explore an extension of the model in which the candidates' budgets evolve over time in response to movements in relative popularity. If building an early lead in the polls helps candidates raise more money that they can put to use in the later stages of the campaign, then candidates have a stronger incentive to spend early, and resources are allocated more evenly over time.

Key words: campaigns, dynamic allocation problems, contests
JEL codes: C72

preliminary work; comments welcome

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1 Introduction

Electoral campaigns are dynamic contests in which the candidates allocate their resources over time to increase their relative popularity prior to the election.

In the 2012 U.S. presidential race, for example, the candidates of the two major political parties raised approximately $1 billion each and spent close to their entire war chests by the election on November 6th. The incumbent, Barack Obama, had spent $986 million and had only $28 million cash on hand by the end of November while the challenger, Mitt Romney, spent $992 million and had $29 million cash on hand (after most, but not all, payments to campaign staff were made). Figure 1, which shows the pattern of spending for each of the two candidates, reveals how both candidates increased their spending over time, ramping it up in the final months. This pattern is the same across other elections as well, not just presidential races. And, in this particular case, spending in each month is also roughly equal for the two candidates.

These patterns are the consequence of deliberate and strategic choices made by the campaigns—a fact that raises the question of what the underlying calculations are that drive these decisions. We answer this question through a simple dynamic allocation model in which two candidates allocate their stock of available resources across a finite number of periods to influence the movement of their relative popularity.

The game begins with one of the two candidates being possibly more popular than the other. At each moment in time, relative popularity may go up, meaning that candidate 1 increases his lead in the polls; or it may go down, meaning that candidate 2 increases her lead. Relative popularity evolves between periods according to a modified Brownian motion so that the next period's starting level of relative popularity is normally distributed with a fixed variance and a mean that depends only on the current level of relative popularity and the ratio of candidate 1's spending to candidate 2's spending that period. At the final date, an election takes place and the more popular candidate wins office. Money left over has no value, so the game is zero-sum.[1]

Dynamic contests are notoriously difficult to solve. We make progress by establishing a key result: at every history, the equilibrium ratio of candidate 1's spending to candidate 2's spending equals the ratio of their available resources. This result enables us to provide a complete characterization of the equilibrium path of spending over time for various

[1]The candidates are, therefore, purely office-motivated— an assumption that is suited for elections to offices with highly value (e.g., the U.S. presidency) or in which candidates are constrained in their use of leftover campaign funds.

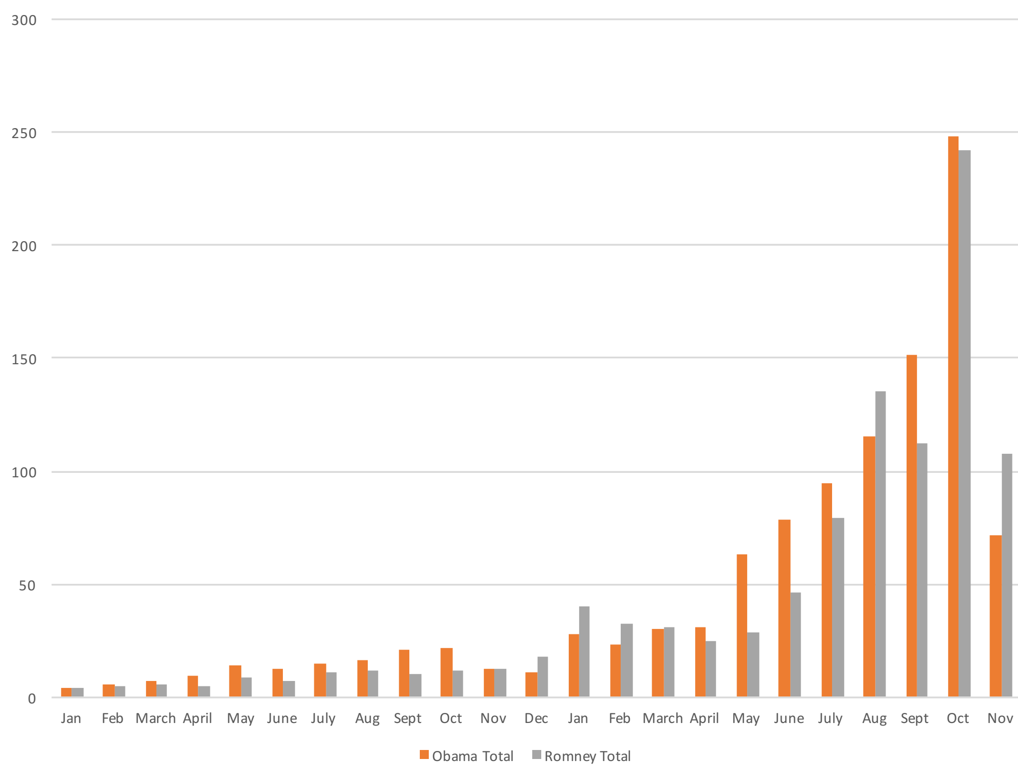

Figure 1. Monthly spending by Obama and Romney in millions of dollars for the 2012 presidential race. Federal Election Commission data reported by the *New York Times*.

special cases of our model. For example, in the case where the drift of the popularity process is affected only by relative spending, the two players spread their resources evenly over periods independent of the current level of relative popularity. Therefore, along the equilibrium path relative popularity follows a constant-drift Brownian motion, where the drift depends only on the ratio of starting budgets.

Alternatively, if any gains in a candidate's relative popularity tends to decay over time, then maintaining an advantage in popularity is harder. As a result, the players increase their spending over time, and relative popularity follows a mean-reverting Ornstein-Uhlenbeck process, with long-term mean determined by the ratio of starting budgets. Moreover, the rate at which spending increases over time is higher when the speed of reversion of the stochastic process is greater. This case is salient because it rationalizes the spending pattern in Figure 1.

We also study an extension of the model in which budgets evolve over time in response to movements in popularity. The extension accommodates both the case in which donors

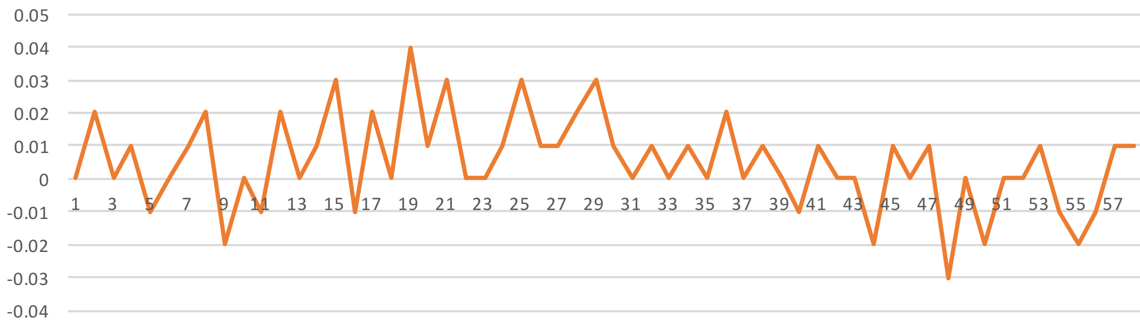

Figure 2. Obama's weekly average support in polls minus Romney's weekly average for the 58 weeks prior to the election. Data are from Pollster (see footnote 2).

flock to the candidate that is leading, as well as the case in which they channel their support to the underdog. For the former case, we show that there is a new tradeoff between spending early versus saving for later. Spending early now has greater benefit even though its effects on popularity are short-lived because short-term increases in popularity can attract more money that can be spent later.

Finally, we calibrate our model to spending data from the Obama-Romney race depicted in Figure 1, and data on relative popularity measured as the difference in the candidates' polling numbers. These data, depicted in Figure 2, are from Pollster.[2] The parameters that best fit the model feature both a substantial level of mean-reversion in the popularity process and a long run mean that slightly favors Obama.

Related Literature. We build on a long tradition of using contest theory to model electoral campaigning. Prior work in this literature has studied static models (e.g. Meirowitz, 2008, Erikson and Palfrey, 2000, Skaperdas and Grofman, 1995). More recently, de Roos and Sarafidis (2018) study a dynamic model to explain how candidates that have won past races may enjoy "momentum."[3] In their model, momentum results from a complementarity between prior electoral success and current spending.[4] Our paper differs from this and other prior work in that we consider a dynamic game in which

[2]See `https://elections.huffingtonpost.com/pollster/`.

[3]A model features "momentum" if a candidate that has won a previous races, or who is leading in the polls, has an advantage in subsequent races or polls as well.

[4]Other explanations for momentum include social learning in the presence of incomplete information about the candidates' types (Callander, 2007, Knight and Schiff, 2010, Ali and Kartik, 2012) or through the informational content conveyed by polls about future competition (Denter and Sisak, 2015).

candidates strategically decide how to allocate their campaign resources across multiple periods in the face of an inter-temporal budget constraint.

Our paper also relates to prior empirical work studying the effects of campaigns on the election outcome (see Jacobson, 2015, for a review of the literature). As Jacobson (1978) noted several decades ago, the main challenge for this literature is how to tackle simultaneity bias: campaign spending decisions affect the outcome of the election, but these decisions are themselves affected by expectations about how the campaign will unfold, who will win, and with what margin. Erikson and Wlezien (2012) address these concerns with a time-series analysis, and produce evidence that campaign spending in U.S. presidential elections influences polling numbers, which measure popularity.

Two recent papers have taken a structural approach to address these and various other econometric challenges.[5] First, Kawai and Sunada (2015) build on Erikson and Palfrey (2000) and estimate a dynamic model of fund-raising and campaigning in which the inter-temporal resource allocation decisions that candidates make are across different elections rather than across periods in the run-up to a particular election. Second, Iaryczower et al. (2017) estimate a dynamic model in which campaign spending weakens electoral accountability, enabling politicians to run on platforms that are closer to their own ideological preferences.[6] Their paper focuses on the strategic interaction between voters and politicians, while we study competition between candidates.

On the theory side, our paper relates to the literature on dynamic contests (see Konrad et al., 2009, and Vojnović, 2016, for reviews of this literature). In this literature, Gross and Wagner (1950) study a continuous Blotto game; Harris and Vickers (1985, 1987), Klumpp and Polborn (2006) and Konrad and Kovenock (2009) study models of races; and Glazer and Hassin (2000) and Hinnosaar (2018) study sequential contests. Ours is the first paper, to our knowledge, that studies a strategic allocation problem in which players allocate their campaign funds across time.

2 Model

Consider the following complete information *dynamic campaigning game* between two candidates, $i = 1, 2$, ahead of an election. Time runs continuously from 0 to T and

[5] Others, such as Green and Krasno (1988), Gerber (1998), and Cox and Thies (2000) have taken a more reduced-form instrumental variables approach.

[6] Their work connects in this way to the accountability literature— e.g., work by Ashworth (2006) who develops a model in which the need to raise money from lobbies to finance the campaign affects the preferences of voters over candidates.

the candidates take actions at times in $\mathcal{T} := \{0, \Delta, 2\Delta, ..., (N-1)\Delta\}$, with $\Delta := T/N$. We identify these times with N discrete periods indexed by $n \in \{0, ..., N-1\}$. For all $t \in [0, T]$, we use $\underline{t} := \max\{\tau \in \mathcal{T} : \tau \leq t\}$ to denote the last time that the players took actions.

At the start of the game the candidates are endowed with positive resource stocks, $X_0 \geq 0$ and $Y_0 \geq 0$ respectively for candidates 1 and 2.[7] They allocate their resources across periods to influence changes in their relative *popularity.* Relative popularity at time t is measured by a continuous random variable $Z_t \in \mathbb{R}$ whose realization at time t is denoted by z_t. We will interpret this as a measure of candidate 1's lead in the polls. If $z_t > 0$, then candidate 1 is ahead of candidate 2. If $z_t < 0$, then candidate 2 is ahead; and if $z_t = 0$, it is a dead heat. We assume that at the beginning of the game, relative popularity is equal to $z_0 \in \mathbb{R}$.

At any time $t \in \mathcal{T}$, the candidates simultaneously decide how much of their resource stock to invest in influencing their future relative popularity. Candidate 1's investment is denoted x_t while candidate 2's is denoted y_t. The size of the resource stock that is available to candidate 1 at time $t \in \mathcal{T}$ is denoted $X_t = X_0 - \sum_{s \in \{\tau \in \mathcal{T} : \tau < t\}} x_s$ and that available to candidate 2 is $Y_t = Y_0 - \sum_{s \in \{\tau \in \mathcal{T} : \tau < t\}} y_s$.

Throughout, we will maintain the assumption that for all times t, the evolution of popularity is governed by the following modified Brownian motion:

$$dZ_t = \left(q\left(x_{\underline{t}}/y_{\underline{t}}\right) - \lambda Z_t\right) dt + \sigma dW_t \tag{1}$$

where $\lambda \geq 0$ and $\sigma > 0$ are parameters and $q(\cdot)$ is a strictly increasing, strictly concave function on $[0, \infty)$. Thus, the drift of popularity depends on the ratio of investments through the function $q(\cdot)$, and it may be mean-reverting if $\lambda > 0$.[8]

Finally, we assume that the *winner* of the election collects a payoff of 1 while the *loser* collects a payoff of 0. For analytical convenience, we make the assumption that if either candidate $i = 1, 2$ invests an amount equal to 0 at any time in $\mathcal{T}$, then the game ends immediately. If $j \neq i$ invested a positive amount at that time, then j is the winner

[7] Although candidates raise funds over time, our assumption that they start with a fixed stock is tantamount to assuming that they can forecast how much will be available to them. In fact, some large donors make pledges early on and disburse their funds as it is needed in the campaign.

[8] If $\lambda = 0$ the process governing the evolution of popularity in the interval between two consecutive times in $\mathcal{T}$ is a standard Brownian motion— the continuous time limit of the random walk in which popularity goes up with probability probability $\frac{1}{2} + q(x_{\underline{t}}/y_{\underline{t}})\sqrt{\Delta}$ and goes down with complementary probability. If $\lambda > 0$, instead, popularity evolves in this interval according to the Ornstein-Uhlenbeck process, under which the leading player's lead has a tendency to decay.

while if j also invested 0 at that time, then each candidate wins with equal probability.[9] If both players invest a positive amount at every time $t \in \mathcal{T}$, then the game only ends at time T, with candidate 1 winning if $z_T > 0$, losing if $z_T < 0$, and both candidates winning with equal probability if $z_T = 0$. In other words, if the game does not end before time T, then the winner is the candidate that is more popular at time T, and if they are equally popular they win with equal probability.

3 Analysis

Since the game is in continuous time, strategies must be measurable with respect to the filtration generated by W_t. However, since candidates take actions only at discrete times, we will forgo this additional formalism and treat the game as a game in discrete time. To that end, let us define the function

$$p\left(x/y\right) = \begin{cases} q(x/y) & \text{if } \lambda = 0 \\ q(x/y)/\lambda & \text{if } \lambda > 0 \end{cases}$$

Then by our assumption about the popularity process in (1), the distribution of $Z_{t+\Delta}$ at any time $t \in \mathcal{T}$, conditional on (x_t, y_t, z_t), is normal with constant variance and a mean that is a weighted sum of $p(x_t/y_t)$ and z_t; specifically,

$$Z_{t+\Delta} \mid (x_t, y_t, z_t) \sim \begin{cases} \mathcal{N}\left(p(x_t/y_t)\Delta + z_t, \sigma^2\Delta\right) & \text{if } \lambda = 0 \\ \mathcal{N}\left(p(x_t/y_t)(1 - e^{-\lambda\Delta}) + z_t e^{-\lambda\Delta}, \sigma^2(1 - e^{-2\lambda\Delta})/2\lambda\right) & \text{if } \lambda > 0 \end{cases}$$

where $\mathcal{N}(\cdot, \cdot)$ denotes the normal distribution whose first component is mean and second is variance. Note that the mean and variance of $Z_{t+\Delta}$ in the $\lambda = 0$ case correspond to the limits as $\lambda \to 0$ of the mean and variance in the $\lambda > 0$ case.

Therefore, our model is strategically equivalent to a discrete time model in which relative popularity is a state variable that transitions over discrete periods such that popularity in each period is normally distributed with a constant variance and a mean that depends on the popularity in the last period and on the ratio of candidates' spending.

[9]These assumptions help close the model since the function q, which depends on spending *ratios*, is undefined if the denominator in the ratio is 0. The assumptions also guarantee that relative popularity, Z_t, follows an Itô process at every history. This model can be considered the limiting case of two different models. One is a model in which the marginal return to investing an ϵ amount of resources starting at 0 goes to infinity. The other is a model in which candidates must spend a minimum amount ϵ in each period to sustain the campaign, and ϵ goes to 0.

With this, our equilibrium concept is subgame perfect Nash equilibrium (SPE) in pure strategies. We will refer to this concept succinctly as "equilibrium."

In the remainder of this section, we establish results on the paths of spending and popularity over time. We begin with a key observation, established in Section 3.1 below, that facilitates the analysis: on the equilibrium path of play, the ratio of the candidates' spending, x_t/y_t, is constant across all periods $t \in \mathcal{T}$.

3.1 Equal Spending Ratios

We refer to the ratio of a candidate's current spending to current budget as that candidate's *spending ratio*. For candidate 1 this is x_t/X_t and for candidate 2 it is y_t/Y_t. We will show that on the equilibrium path, these two ratios equal each other at every time t that the candidates make spending decisions.

Consider any time $t \in \mathcal{T}$ at which the game has not ended and candidates have to make their investment decisions. If $t = (N-1)\Delta$, then both candidates will spend their remaining budgets, i.e. $x_{(N-1)\Delta} = X_{(N-1)\Delta}$ and $y_{(N-1)\Delta} = Y_{(N-1)\Delta}$. Therefore, both candidates' spending ratios equal 1.

Now suppose that $t < (N-1)\Delta$ and assume that the stock of resources available to the two candidates are $X_t, Y_t > 0$.[10] Also, suppose that after the players choose their spending levels x_t and y_t, the probability that candidate 1 will win the election at time T when evaluated at time $t+\Delta$ depends on $X_{t+\Delta} = X_t - x_t$ and $Y_{t+\Delta} = Y_t - y_t$ only through the ratio $(X_t - x_t)/(Y_t - y_t)$. Denote this probability by $\pi_t\left((X_t - x_t)/(Y_t - y_t), z_{t+\Delta}\right)$. Further, let $F(z_{t+\Delta}|x_t, y_t, z_t)$ denote the c.d.f. of $Z_{t+\Delta}$ conditional on (x_t, y_t, z_t), and let $f(z_{t+\Delta}|x_t, y_t, z_t)$ denote the associated p.d.f. As noted above, these are normal distributions that depend on x_t and y_t only through the ratio x_t/y_t.

If both candidates spend a positive amount in every period, candidate 1's expected payoff at time t is given by

$$\Pi_t(x_t, y_t|X_t, Y_t, z_t) = \int \pi_t\left(\frac{X_t - x_t}{Y_t - y_t}, z_{t+\Delta}\right) dF\left(z_{t+\Delta}|x_t/y_t, z_t\right)$$

[10] Recall that if either X_t or Y_t equal 0, the game will end at time t: either both candidates have no money to spend, or the one with a positive budget will spend any positive amount and win.

and candidate 2's expected payoff is $1 - \Pi_t(x_t, y_t | X_t, Y_t, z_t)$. The pair of necessary first order conditions for interior equilibrium values of x_t and y_t are

$$\frac{1}{y_t} \int \pi_t \left(\frac{X_t - x_t}{Y_t - y_t}, z_{t+\Delta} \right) \frac{\partial f\left(z_{t+\Delta} | x_t/y_t, z_t\right)}{\partial (x_t/y_t)} dz_{t+\Delta} = \\ = \frac{1}{Y_t - y_t} \int \frac{\partial \pi_t(\frac{X_t - x_t}{Y_t - y_t}, z_{t+\Delta})}{\partial (\frac{X_t - x_t}{Y_t - y_t})} dF\left(z_{t+\Delta} | x_t/y_t, z_t\right); \tag{2}$$

$$\frac{x_t}{(y_t)^2} \int \pi_t \left(\frac{X_t - x_t}{Y_t - y_t}, z_{t+\Delta} \right) \frac{\partial f\left(z_{t+\Delta} | x_t/y_t, z_t\right)}{\partial (x_t/y_t)} dz_{t+\Delta} = \\ = \frac{X_t - x_t}{(Y_t - y_t)^2} \int \frac{\partial \pi_t(\frac{X_t - x_t}{Y_t - y_t}, z_{t+\Delta})}{\partial (\frac{X_t - x_t}{Y_t - y_t})} dF\left(z_{t+\Delta} | x_t/y_t, z_t\right). \tag{3}$$

Taking the ratios of the respective left and right hand sides of these equations implies that $x_t/y_t = (X_t - x_t)/(Y_t - y_t)$, or $x_t/y_t = X_t/Y_t$. This observation suggests that our supposition that the remaining budgets $X_t - x_t$ and $Y_t - y_t$ affect continuation payoffs only through their ratio can be established by induction provided that the second order conditions are satisfied. The main steps in the proof of the following proposition involve establishing these facts. This and all other proofs appear in the Appendix.

Proposition 1. *There exists an essentially unique equilibrium. If $X_t, Y_t > 0$ are the remaining budgets of candidates 1 and 2 at any time $t \in \mathcal{T}$, then in all equilibria,*

$$x_t/X_t = y_t/Y_t.$$

The word "essentially" appears in the proposition above only because the equilibrium is not unique at histories at which either $X_t = 0 < Y_t$ or $X_t > 0 = Y_t$ — histories that do not arise on the path of play. In these cases, the candidate with a positive resource stock may spend any amount in period t and win. Apart from this trivial source of multiplicity, the equilibrium is unique.

3.2 Equilibrium Spending and Popularity Paths

An immediate corollary of Proposition 1 is a characterization of the process governing the evolution of relative popularity on the equilibrium path.

Corollary 1. *On the equilibrium path, relative popularity follows the process*

$$dZ_t = \left(q(X_0/Y_0) - \lambda Z_t\right) dt + \sigma dW_t \tag{4}$$

If $\lambda = 0$, this is a Brownian motion with constant drift $p(X_0/Y_0)$. If $\lambda > 0$, it is the Ornstein-Uhlenbeck process with long-term mean $p(X_0/Y_0)$ and speed of reversion λ.

Therefore, when $\lambda > 0$ popularity leads have a tendency to decay towards zero. The instantaneous volatility of the process is σ and the stationary variance is $\sigma^2/2\lambda$. Figure 5 in Section 5 shows some sample paths for calibrated values of the parameters.

Proposition 1 also enables us to solve, in closed form, for the equilibrium spending ratio at each history.

Proposition 2. *Let $t \in \mathcal{T}$ be a time at which $X_t, Y_t > 0$. Then, in equilibrium, spending ratios depend only on calendar time and on the speed of reversion λ. In particular,*

$$\frac{x_t}{X_t} = \frac{y_t}{Y_t} = \begin{cases} \frac{e^{-\lambda(T-t-\Delta)} - e^{-\lambda(T-t)}}{1-e^{-\lambda(T-t)}} & \text{if } \lambda > 0 \\ \Delta/(T-t) & \text{if } \lambda = 0 \end{cases}$$

which is continuous at $\lambda = 0$.

Recall that $T = N\Delta$ and actions are taken at times $t = n\Delta$ with $n \in \{0, 1, ..., N-1\}$. Therefore, if $\lambda = 0$, the candidates will, in equilibrium, spend a fraction $1/(N-n)$ of their available resources in each period. This implies that on the equilibrium path, candidates spread their initial resources X_0 and Y_0 exactly evenly across periods: $x_t = X_0/N$ and $y_t = Y_0/N$ for every time $t \in \mathcal{T}$.

For the $\lambda > 0$ case, recursively define for all $n \in \{1, 2, ..., N-1\}$,

$$\gamma_\lambda(0) = \frac{e^{-\lambda(T-\Delta)} - e^{-\lambda T}}{1 - e^{-\lambda T}}$$

$$\gamma_\lambda(n) = \frac{e^{-\lambda(T-(n+1)\Delta)} - e^{-\lambda(T-n\Delta)}}{1 - e^{-\lambda(T-n\Delta)}} \prod_{m=0}^{n-1} (1 - \gamma_\lambda(m))$$

Then, the candidates' spending at each time $n\Delta \in \mathcal{T}$ is given by

$$x_{n\Delta} = \gamma_\lambda(n) X_0 \qquad \text{and} \qquad y_{n\Delta} = \gamma_\lambda(n) Y_0,$$

Hence, $\gamma_\lambda(n)$ is the fraction of initial budget that each player spends on the equilibrium path at time $n\Delta$, and ratio of spending in consecutive periods n and $n+1$ is

$$r_n(\lambda) = \frac{\gamma_\lambda(n+1)}{\gamma_\lambda(n)}.$$

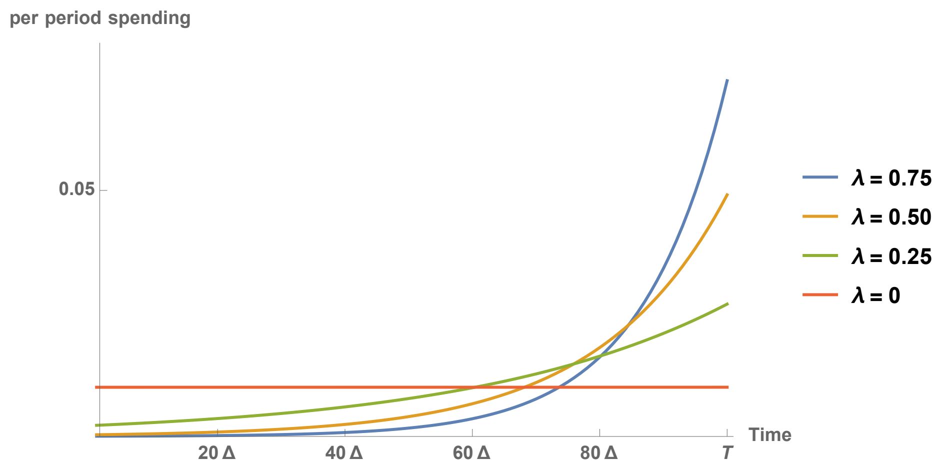

Figure 3. If $X_0 = Y_0$ then per period spending on the equilibrium path is the same for both players (i.e., $x_{n\Delta} = y_{n\Delta}$ for all $n\Delta \in \mathcal{T}$ when $X_0 = Y_0$). Here, we plot per period spending for $X_0 = Y_0 = 1$, $N = 100$ and various values of λ.

Figure 3 depicts spending patterns for different values of λ and shows that spending increases over time if $\lambda > 0$. The figure also shows that the ratio of spending at time $(n+1)\Delta$ to spending at time $n\Delta$ is uniformly increasing in λ. These are general properties that we summarize in the following proposition.

Proposition 3. *Fix the number of periods N, total time T, and consider the case in which $\lambda > 0$. Then, for all $n \in \{0, 1, ..., N-2\}$, the ratio $r_n(\lambda)$ of spending in consecutive periods n and $n+1$ is*

(i) greater than 1, and

(ii) increasing in the speed of reversion, λ.

To understand Proposition 3, consider the case where there is no mean-reversion, $\lambda = 0$. We have already argued that in this case both candidates spread their investments evenly across periods. Now, suppose there is some mean-reversion, $\lambda > 0$. Although early investments may produce a lead in popularity, the effect of the investment is only short-term since the lead tends to decay over time. This makes the lead difficult to grow or even maintain. This calculation results in candidates investing higher shares of the initial budget in later periods compared to previous ones, as depicted in Figure 1. Moreover, the same calculation favors spending in later periods even more as the speed of reversion λ increases.

4 Endogenous Budgets

Our baseline model captures the idea that when there is mean-reversion in the relative popularity process, then building up an early lead is less advantageous because the lead is harder to maintain. However, a key idea coming out of prior work (reviewed in the introduction) that is not captured in the baseline model is that candidates who have built up an early lead may enjoy the benefits of "momentum." One important source of momentum is discussed by Aldrich (1980), who postulates that candidates who outperform expectations early on may attract more resources from campaign donors. In this section, we explore an extension that incorporates this possibility.

4.1 A Model with Evolving Budgets

Instead of assuming that candidates are endowed with a fixed budget at the start of the game that they must allocate over time, we assume that the size of the resources stock also evolves in a way that depends on the evolution of popularity. In particular, we retain all the features of the baseline model except the ones we describe as follows.

Candidates start with exogenous budgets X_0 and Y_0 as in the baseline model. However, the budgets now evolve according to the following geometric Brownian motions

$$\frac{dX_t}{X_t} = az_t dt + \sigma_X dW_t^X$$
$$\frac{dY_t}{Y_t} = bz_t dt + \sigma_Y dW_t^Y$$

where a, b, σ_X and σ_Y are constants, and W_t^X and W_t^Y are Wiener processes. Under this assumption, the evolution of the budget is stochastic because it depends on the evolution of relative popularity, which is itself stochastic, and because there also an exogenous source of randomness. In addition, even though none of our results hinge on it, we make the assumption for simplicity that dW_t is independent of dW_t^X and of dW_t^Y, while dW_t^X and dW_t^Y have covariance $\rho \geq 0$.

This setting is general enough to allow for several possibilities. For example, donors may support the candidate that is leading in the polls and withdraw support from the one that is trailing. This is the case where $b < 0 < a$. Alternatively, donors may channel their resources to the underdog, which is the case where $a < 0 < b$. Popularity therefore feeds back into the budget process. If the difference $a - b$ is positive, then the feedback is positive and if it is negative then the feedback is negative. We refer to a and b as the feedback parameters.[11]

All other features of the model are exactly the same as in the baseline model, including the process (1) governing the evolution of popularity, though we now assume for analytical tractability that

$$q(x/y) = \log(x/y).$$

4.2 Analysis

Our analysis of the baseline model suggests that the ratio of budgets, X_t/Y_t, plays an important role in characterizing the equilibrium. Applying Itô's lemma, we can write the process governing the evolution of this ratio for the extended model as:

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = \mu_{XY}(z_t)dt + \sigma_X dW_t^X - \sigma_Y dW_t^Y, \tag{5}$$

where

$$\mu_{XY}(z_t) = (a-b)z_t + \sigma_Y^2 - \rho\sigma_X\sigma_Y.$$

Hence, the instantaneous volatility of this process is simply $\sigma_{XY} = \sqrt{\sigma_X^2 + \sigma_Y^2 - \rho\sigma_X\sigma_Y}$. Therefore, if at time $t \in \mathcal{T}$ the candidates have an amount of available resources equal to X_t and Y_t and spend x_t and y_t, then the log of the next period's ratio of budgets conditional on all information, $\mathcal{I}_t$, available at time time t is a normal random variable:

$$\log\left(\frac{X_{t+\Delta}}{Y_{t+\Delta}}\right) \Big| \, \mathcal{I}_t \sim \mathcal{N}\left(\log\left(\frac{X_t - x_t}{Y_t - y_t}\right) + \mu_{XY}(z_t)\Delta, \sigma_{XY}^2\Delta\right).$$

Our main finding for this extension is that even though candidates' budgets now evolve stochastically over time, the main feature of equilibrium in the baseline model still holds here: the equilibrium spending ratios, x_t/X_t and y_t/Y_t at time t are equal; thus x_t/y_t is pinned down by the ratio of available budgets, X_t/Y_t, at time t.

Proposition 4. *In the model with evolving budgets, for every N, T, and $\lambda > 0$, there exists $-\eta < 0$ such that whenever $a - b \geq -\eta$, there is an essentially unique equilibrium.*

[11] Also, note that dX_t and dY_t may be negative. One interpretation is that X_t and Y_t are expected total budgets available for the remainder of the campaign, where the expectation is formed at time t. Depending on the level of relative popularity, the candidates revise their expected future inflow of funds and adjust their spending choices accordingly.

For all $t \in \mathcal{T}$, if $X_t, Y_t > 0$, then in all equilibria,

$$x_t/X_t = y_t/Y_t.$$

To understand the condition $a-b \geq -\eta$, note that when $a < 0 < b$, there is a negative feedback between popularity and the budget flow: a candidate's budget increases when she is less popular than her opponent. The condition $a - b \geq -\eta$ puts a bound on how negative this feedback can be. If this condition is not satisfied, candidates may want to *reduce* their popularity as much as they can in the early stages of the campaign to accumulate a larger war chest to use in the later stages. This could undermine the existence of an SPE in pure strategies.

As in the case of the baseline model, we can characterize the stochastic process of relative campaign spending for this model with evolving budgets as well. To that end, fix the number of periods N that the candidates take actions. Let $g_1(0) = 1$ and $g_2(0) = 0$, and define recursively for every $m \in \{1, ..., N-1\}$,

$$\begin{pmatrix} g_1(m) \\ g_2(m) \end{pmatrix} = \begin{pmatrix} e^{-\lambda\Delta} & a-b \\ \frac{1-e^{-\lambda\Delta}}{\lambda} & 1 \end{pmatrix} \begin{pmatrix} g_1(m-1) \\ g_2(m-1) \end{pmatrix}$$

Proposition 5. *Let $t = (N-m)\Delta \in \mathcal{T}$ be a time at which $X_t, Y_t > 0$. Then, in the essentially unique equilibrium, spending ratios are equal to*

$$\frac{x_t}{X_t} = \frac{y_t}{Y_t} = \frac{g_1(m-1)}{g_1(m-1) + g_2(m-1)\frac{\lambda}{1-e^{-\lambda\Delta}}}. \tag{6}$$

Moreover, in equilibrium, $(\log(x_{t+n\Delta}/y_{t+n\Delta}), z_{t+n\Delta}) \mid \mathcal{I}_t$ follows a bivariate normal distribution with mean

$$\begin{pmatrix} 1 & (a-b)\,\Delta \\ \frac{1-e^{-\lambda\Delta}}{\lambda} & e^{-\lambda\Delta} \end{pmatrix}^n \begin{pmatrix} \log\left(\frac{X_t}{Y_t}\right) - \frac{\lambda\Delta(\sigma_Y^2-\rho\sigma_X\sigma_Y)}{a-b}\frac{e^{-\lambda\Delta}}{1-e^{-\lambda\Delta}} \\ z_t + \frac{(\sigma_Y^2-\rho\sigma_X\sigma_Y)}{a-b} \end{pmatrix} + \begin{pmatrix} \frac{\lambda(\sigma_Y^2-\rho\sigma_X\sigma_Y)}{a-b}\frac{e^{-\lambda\Delta}}{1-e^{-\lambda\Delta}} \\ \frac{\rho\sigma_X\sigma_Y-\sigma_Y^2}{a-b} \end{pmatrix}$$

and variance

$$\begin{pmatrix} 1 & (a-b)\,\Delta \\ \frac{1-e^{-\lambda\Delta}}{\lambda} & e^{-\lambda\Delta} \end{pmatrix}^n \begin{pmatrix} \sigma_{XY}^2\Delta & 0 \\ 0 & \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda} \end{pmatrix} \begin{pmatrix} 1 & \frac{1-e^{-\lambda\Delta}}{\lambda} \\ (a-b)\,\Delta & e^{-\lambda\Delta} \end{pmatrix}^n.$$

4.3 Spending Early or Late?

In the baseline model, when $\lambda > 0$ the difficulty in maintaining an early lead means that there is a disincentive to spend resources early on. This produces the result, reported in Proposition 3 and depicted in Figure 3, that spending is increasing over time. However, in the model of this section, if $b < 0 < a$, there is a force working in the other direction: spending money to build early leads may advantageous because it results in faster growth of the war chest, which is valuable for the future. The disincentive to spend early is mitigated by this opposing force, and may even be overturned if a is much larger than b, i.e., if donors have a greater tendency to flock to the leading candidate.

We now establish this intuition more formally. Recall that $r_n(\lambda)$ gave the ratio of equilibrium spending in consecutive periods, n and $n+1$. For the model with evolving budgets, we define the analogous ratio, $\tilde{r}_n$, which we note depends on a and b only through the difference $a - b$ (see the proof of Proposition 6 below in the Appendix) and is the same for both players in equilibrium, given Proposition 5. Specifically,

$$\tilde{r}_n(\lambda, a-b) = \frac{x_{(n+1)\Delta}/X_{(n+1)\Delta}}{x_{n\Delta}/X_{n\Delta}} = \frac{y_{(n+1)\Delta}/Y_{(n+1)\Delta}}{y_{n\Delta}/Y_{n\Delta}}$$

The following proposition establishes the key properties of this ratio, particularly its dependence on the feedback parameters, a and b.

Proposition 6. *Fix the number of periods N, total time T, and consider the case in which $\lambda > 0$. Then, for all $n \in \{0, 1, ..., N-2\}$, if $a - b$ is sufficiently small then the ratio $\tilde{r}_n(\lambda, a-b)$ of spending in consecutive periods n and $n+1$ conditional on the history up to period n is*

(i) greater than 1,

(ii) increasing in λ, and

(iii) decreasing in $a - b$.

The baseline model is the special case of the model with evolving budgets in which there is no budget volatility: $a = b = \sigma_X = \sigma_Y = 0$. Starting with this special case, as we increase the difference $a - b$, spending plans becomes more balanced over time. What Proposition 6 establishes is that when $a - b$ grows from zero, there is a greater incentive to spend in earlier periods of the race than there is if $a = b$.

Finally, it is worth noting that the comparative statics result reported in Proposition 6 does not hold when $a - b$ is very large. Indeed, we have examples in which $\tilde{r}_n(\lambda, a-b)$ is *increasing* in $a - b$ for large λ, n, and $a - b$.[12] The intuition behind these examples rests on the fact that when the degree of mean reversion is high, then it is important for candidates to build up a large war chest that they can deploy in the final stages of the race. If the election date is distant and $a - b$ is large, then early spending is mostly for the purpose of building up these resources. But how should the candidates allocate their spending across the very early periods? Spending too much in any one period is risky: if the resource stock does not grow (or even if it grows but insufficiently) then there is less money, and hence less opportunity, to grow it in the subsequent periods. Since q is concave, the candidates would like to have many attempts to grow the war chest early on, and this is even more the case as the importance of the feedback $a - b$ gets large.

5 Calibration

In this section, we calibrate our baseline model to Federal Election Commission (FEC) spending data for the 2012 presidential contest between Obama (candidate 1) and Romney (candidate 2), and polling data for these candidates as reported by Pollster (see footnote 2).[13] We begin by establishing an identification result that shows that we can empirically identify λ and $q(X_0/Y_0)$ without having to fix Δ. In other words, we can calibrate λ and $q(X_0/Y_0)$ using data collected with arbitrary frequency over time.

Proposition 7. *Fix any integer $\kappa \geq 1$, and for $t \in \mathcal{T}$ let $\bar{x}_t = \sum_{j=0}^{\kappa-1} x_{t+\Delta j}$ be aggregate equilibrium spending over a unit of time $\bar{\Delta} := \kappa\Delta$. Then, given the remaining budget X_t at time t,*

$$\frac{\bar{x}_t}{X_t} = \begin{cases} \frac{e^{-\lambda(T-t-\bar{\Delta})} - e^{-\lambda(T-t)}}{1-e^{-\lambda(T-t)}} & \text{if } \lambda > 0 \\ \bar{\Delta}/(T-t) & \text{if } \lambda = 0 \end{cases} \tag{7}$$

The key implication of this proposition is that λ and Δ cannot be separately identified from the data; only their product $\lambda\Delta$ can be identified.

To illustrate how we use this identification result, suppose that we have aggregated spending data in periods of length $\bar{\Delta} = \kappa\Delta$ for some κ. Normalizing $T = 1$, if the data is collected k times, and assuming that $\lambda > 0$, we can calibrate λ to minimize the sum

[12] One example is $\lambda = 0.8$, $\Delta = 0.9$, and $n = a - b = 10$.

[13] Since (a subset of) $(x_{j\Delta}/y_{j\Delta}, z_{j\Delta})_{j=1}^{N-1}$ is observable from the data, we could also calibrate the relevant parameters of the model with evolving budgets.

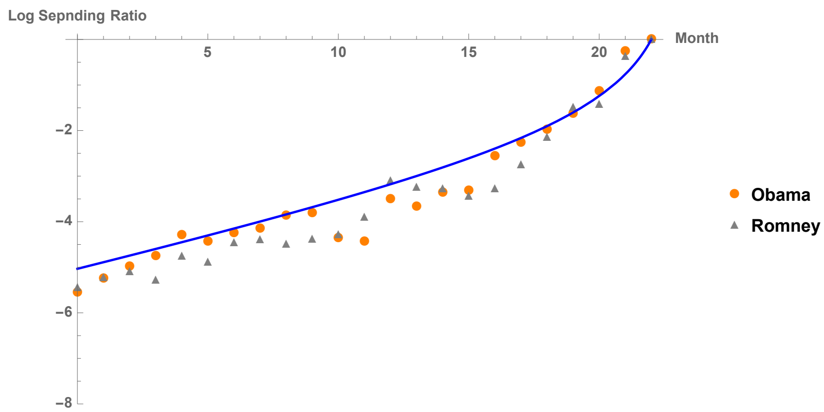

Figure 4. Logs of Obama's and Romney's spending ratios ($\log(x_t/X_t)$ and $\log(y_t/Y_t)$, respectively) over the twenty-three month period prior to the election. The blue line is the fit of (7) for $T = 1$, $\bar{\Delta} = 1/23$, and the calibrated value of $\lambda = 3.162$.

of squared deviations from the ratio of $\bar{x}_t/X_t$; i.e., to solve

$$\min_{\lambda} \sum_{j=0}^{k-1} \left| \frac{\bar{x}_j}{X_j} - \frac{e^{-\lambda(1-\frac{j}{k}-\frac{1}{k})} - e^{-\lambda(1-\frac{j}{k})}}{1 - e^{-\lambda(1-\frac{j}{k})}} \right|^2 \tag{8}$$

We numerically compute the minimizer by taking monthly aggregates of the spending data over a twenty-three month period prior to the election.[14] We computed $\lambda = 3.162$ as the solution to the minimization of (8) in the data.[15] Figure 4 shows the fit. It also shows that the spending ratios for the two candidates, while not exactly equal, are roughly similar over time.

Next, suppose that relative popularity data is collected ℓ times at fixed intervals over a period $\tilde{T} \leq T$. Then compounding the normal distribution, for every $j = 0, ..., \ell - 1$,

$$Z_{\frac{j+1}{\ell}} \sim \mathcal{N}\left((1 - e^{-\frac{\lambda}{\ell}\tilde{T}})q(X_0/Y_0)/\lambda + e^{-\frac{\lambda}{\ell}\tilde{T}} z_{\frac{j}{\ell}}, \sigma^2(1 - e^{-\frac{2\lambda}{\ell}\tilde{T}})/2\lambda\right).$$

[14] Finer levels of aggregation are possible but there is likely to be greater error in interpreting whether spending that is *reported* in a certain period was actually put to effective *use* in the electoral contest in that period. We therefore use monthly aggregates rather than weekly or daily measures. Fortunately, the result in Proposition 7 implies that the choice of the period of aggregation, Δ, does not matter.

[15] Our calibration problem assumes $\lambda > 0$. Since both candidates' rate of spending is increasing over time, it is obvious that $\lambda = 0$ would fit the data worse. See Figure 4.

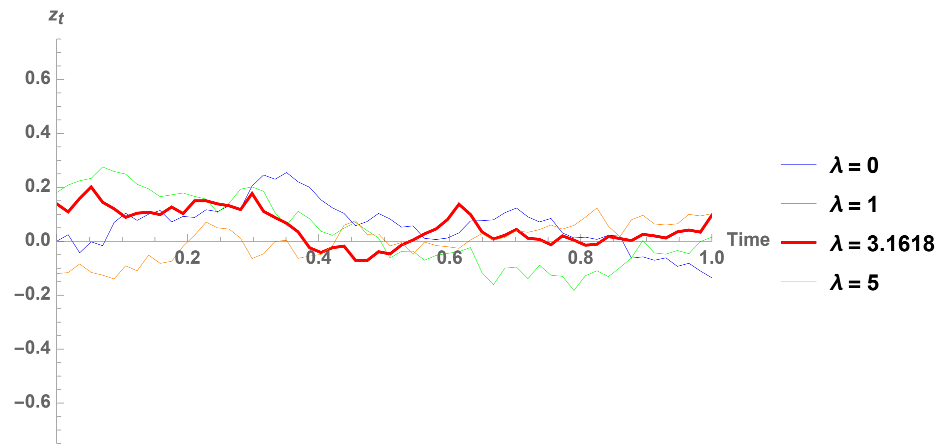

Figure 5. Equilibrium sample paths for relative popularity z_t for four values of λ, including the calibrated value, when $z_0 = 0$ and σ and $q(X_0/Y_0)$ take their calibrated values of 0.258 and 0.0234, respectively.

since with our normalization of $T = 1$, we have $\tilde{T}/T = \tilde{T}$. Therefore, given λ calibrated from above, we can calibrate $q(X_0/Y_0)$ to minimize the sum of squared deviations of z_{j+1} from its mean; i.e., to solve

$$\min_{q(X_0/Y_0)} \sum_{j=0}^{\ell-1} \left| z_{\frac{j+1}{\ell}} - \left[(1 - e^{-\frac{\lambda}{\ell}\tilde{T}}) q(X_0/Y_0)/\lambda + e^{-\frac{\lambda}{\ell}\tilde{T}} z_{\frac{j}{\ell}} \right] \right|^2, \tag{9}$$

which gives

$$q(X_0/Y_0) = \frac{\lambda}{\ell} \cdot \frac{\sum_{j=0}^{\ell-1} z_{\frac{j+1}{\ell}} - e^{-\frac{\lambda}{\ell}\tilde{T}} z_{\frac{j}{\ell}}}{1 - e^{-\frac{\lambda}{\ell}\tilde{T}}}.$$

After computing $q(X_0/Y_0)$, we then calibrate σ such that σ^2 is the normalized average of the squared residuals from (9) (taking into account that we freely chose $q(X_0/Y_0)$ in the minimization problem):

$$\sigma^2 = \frac{\frac{1}{\ell-1} \sum_{j=0}^{\ell-1} \left| z_{\frac{j+1}{\ell}} - \left[(1 - e^{-\frac{\lambda}{\ell}\tilde{T}}) q(X_0/Y_0)/\lambda + e^{-\frac{\lambda}{\ell}\tilde{T}} z_{\frac{j}{\ell}} \right] \right|^2}{(1 - e^{-\frac{2\lambda}{\ell}\tilde{T}})/2\lambda}$$

For the calibration, we defined the realized values of relative popularity Z_t to be the difference in the weekly average of support for Obama and support for Romney (in percentage points) from Pollster data over a 57 week period prior to the election.[16] This means that $\tilde{T} = 0.58$ and $\ell = 57$. Combining this and the data with the calibrated value of $\lambda = 3.162$ from above, we computed $q(X_0/Y_0) = 0.0234$ and $\sigma = 0.258$. For these calibrated values, we can generate sample paths for the equilibrium process for Z_t reported in (4) in Corollary 1. Figure 5 shows one such sample path. It also depicts sample paths for other values of λ.

Finally, to examine the fit of the model we estimate the correlations of the residuals from the minimization problem in (9) with the realized values of z_t and the candidates' spending ratios x_t/X_t (Obama) and y_t/Y_t (Romney). The following table shows that there are almost no correlations.

z_t	x_t/X_t	y_t/Y_t
-0.0315	0.0362	0.0353
(0.817)	(0.783)	(0.789)

Table 1. Correlation coefficients of the residuals from the minimization problem in (9) with z_t, x_t/X_t and y_t/Y_t. p values in parentheses are for the null hypothesis that the correlations are zero.

6 Final Remarks

Since our paper focused on the strategic choices made by the campaigns, we abstracted away from some important considerations. For example, we left unmodeled the behavior of the voters that generates over-time fluctuation in relative popularity. And, in the extension on endogenous budgets, we abstracted away from the motivations and choices of the donors, and the effort decisions of the candidates in how much time to allocate to campaigning versus fundraising. These abstractions leave open questions about how to micro-found the behavior of voters and donors, and effort allocation decision for the candidates. We leave these questions to future work.[17]

Finally, although we calibrated λ and other parameters from the data, we did not estimate the shape of the function q. In real life, campaigns may not know what the return to spending is at the various stages of the campaign, as this may be specific to

[16] Polling data get increasingly sparse more than a year out from the date of the election, with many missing weeks. So we start at 57 weeks from the election.

[17] A recent paper by Bouton et al. (2018) addresses some of these questions in a static model. The paper studies the strategic choices of donors who try to affect the electoral outcome and shows that their behavior is affected by the competitiveness of the election.

the personal characteristics of their respective candidates, and changes in the political environment, including the "mood" of voters. Real-life campaigns face an optimal experimentation problem whereby they try to learn about their environment through early spending. Our model also abstracted away from this important question of how early spending may benefit campaigns by providing them with information about what kinds of campaign strategies seem to work well for their candidate. This is a considerably difficult problem, especially in the face of a fixed election deadline, and the endogeneity of donor interest and available resources. But there is no doubt that well-run campaigns spend to acquire valuable information about how voters are engaging with and responding to the candidates over the course of the campaign. These are interesting and important questions that ought to be the subject of future work.

Appendix

A Proofs

Proof of Proposition 1

We consider the case of $\lambda > 0$. The $\lambda = 0$ must be handled separately, but is very similar, so we omit the details.[18]

We prove by induction that, in any equilibrium, if $X_t > 0, Y_t > 0$, then for all $t \in \mathcal{T}$,

(i) $x_\tau / y_\tau = X_t / Y_t$ at all times $\tau \geq t$ at which the players take actions;

(ii) if $t < (N-1)\Delta$, then the distribution of Z_T computed at time $t + \Delta \in \mathcal{T}$ given $z_{t+\Delta}$ is

$$\mathcal{N}\left(p\left(\frac{X_t - x_t}{Y_t - y_t}\right)(1 - e^{-\lambda(T-t-\Delta)}) + z_{t+\Delta}e^{-\lambda(T-t-\Delta)}, \frac{\sigma^2(1 - e^{-2\lambda(T-t-\Delta)})}{2\lambda}\right)$$

The claim is obviously true at $t = (N-1)\Delta$, since in any equilibrium the candidates' payoffs depend only on z_T and in the final period they must spend the remainder of their budget.

Suppose, for the inductive step, that for all $\tau \geq t + \Delta$, both statements (i) and (ii) above hold. The distribution of $Z_{t+\Delta}$ at time $t \in \mathcal{T}$ given (x_t, y_t, z_t) is

$$\mathcal{N}\left(p\left(\frac{x_t}{y_t}\right)(1 - e^{-\lambda\Delta}) + z_t e^{-\lambda\Delta}, \frac{\sigma^2(1 - e^{-2\lambda\Delta})}{2\lambda}\right)$$

By this hypothesis, the distribution of Z_T computed at time $t + \Delta \in \mathcal{T}$ given $z_{t+\Delta}$ is

$$\mathcal{N}\left(p\left(\frac{X_t - x_t}{Y_t - y_t}\right)(1 - e^{-\lambda(T-t-\Delta)}) + z_{t+\Delta}e^{-\lambda(T-t-\Delta)}, \frac{\sigma^2(1 - e^{-2\lambda(T-t-\Delta)})}{2\lambda}\right)$$

[18] It is worth mentioning that we have continuity at the limit: all of the results for the $\lambda = 0$ case hold as the limits of the $\lambda > 0$ case as $\lambda \to 0$.

The compound of normal distributions is also a normal distribution. Therefore, the distribution of Z_T at time t, given (x_t, y_t, z_t) is normal with mean and variance:

$$\mu_{Z_T|t} = p\left(\frac{X_t - x_t}{Y_t - y_t}\right)(1 - e^{-\lambda(T-t-\Delta)}) + p\left(\frac{x_t}{y_t}\right)(e^{-\lambda(T-t-\Delta)} - e^{-\lambda(T-t)}) + z_t e^{-\lambda(T-t)}$$
$$\sigma^2_{Z_T|t} = \frac{\sigma^2(1 - e^{-2\lambda(T-t)})}{2\lambda}.$$

These expressions follow from the law of iterated expectation, $\mu_{Z_T|t} = E_t[E_{t+1}[Z_T]]$, and the law of iterated variance, $\sigma^2_{Z_T|t} = E_t[Var_{t+1}[Z_T]] + Var_t[E_{t+1}[Z_T]]$.

Now, define the standardized random variable

$$\tilde{Z}_T = \frac{Z_T - \mu_{Z_T|t}}{\sigma_{Z_T|t}}.$$

Candidate 1 wins if $Z_T > 0$ or, equivalently, if

$$\tilde{Z}_T > -\frac{\mu_{Z_T|t}}{\sigma_{Z_T|t}} =: \tilde{z}^*_T$$

Therefore, taking y_t as given, candidate 1's objective is to maximizes his probability of winning, which is given by

$$\pi_t(x_t, y_t | X_t, Y_t, z_t) := \int_{\tilde{z}^*_T}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-s/2} ds.$$

Factoring common constants, the first order condition for this optimization problem is satisfied if and only if $0 = \partial\mu_{Z_T|t}/\partial x_t$, i.e.,

$$0 = p'\left(\frac{x_t}{y_t}\right)\frac{e^{-\lambda(T-t-\Delta)} - e^{-\lambda(T-t)}}{y_t} - p'\left(\frac{X_t - x_t}{Y_t - y_t}\right)\cdot\frac{1 - e^{-\lambda(T-t-\Delta)}}{Y_t - y_t} \tag{10}$$

Moreover, substituting the first order condition in the second order condition and rearranging terms, we get that the second order expression is given by a positive constant that multiplies

$$\frac{\partial^2\mu_{Z_T|t}}{\partial(x_t)^2} = p''\left(\frac{x_t}{y_t}\right)\frac{(e^{-\lambda(T-t-\Delta)} - e^{-\lambda(T-t)})}{(y_t)^2} + p''\left(\frac{X_t - x_t}{Y_t - y_t}\right)\cdot\frac{1 - e^{-\lambda(T-t-\Delta)}}{(Y_t - y_t)^2}$$

Because the function q is strictly concave, p is strictly concave as well. Hence, the second order condition is always satisfied and the objective function is strictly quasi-concave in x_t. By an analogous argument, we can show that candidate 2's problem is strictly quasi-concave in y_t.

Therefore, the first order approach in the main text of Section 3.1 is valid, and we have $x_t/y_t = X_t/Y_t$. Therefore, we have $x_\tau/y_\tau = X_t/Y_t$ for all $\tau \geq t$. This implies $(X_t - x_t)/(Y_t - y_t) = X_t/Y_t$. Therefore, we can conclude that the distribution of Z_T computed at time t is given by a normal distribution with mean and variance:

$$\mu_{Z_T|t} = p\left(\frac{x_t}{y_t}\right)(1 - e^{-\lambda(T-t)}) + z_t e^{-\lambda(T-t)},$$
$$\sigma^2_{Z_T|t} = \frac{\sigma^2(1 - e^{-2\lambda(T-t)})}{2\lambda}.$$

□

Proof of Proposition 2

Suppose that $\lambda > 0$. Then, the first order condition for x_t from (10) is

$$p'\left(\frac{x_t}{y_t}\right)\frac{e^{-\lambda(T-t-\Delta)} - e^{-\lambda(T-t)}}{y_t} = p'\left(\frac{X_t - x_t}{Y_t - y_t}\right)\cdot\frac{1 - e^{-\lambda(T-t-\Delta)}}{Y_t - y_t}$$

The analogous first order condition for y_t is

$$p'\left(\frac{x_t}{y_t}\right)\frac{x_t\left(e^{-\lambda(T-t-\Delta)} - e^{-\lambda(T-t)}\right)}{(y_t)^2} = p'\left(\frac{X_t - x_t}{Y_t - y_t}\right)\frac{(X_t - x_t)\left(1 - e^{-\lambda(T-t-\Delta)}\right)}{(Y_t - y_t)^2}$$

These two equations (together with the fact that $x_t/y_t = (X_t - x_t)/(Y_t - y_t)$) imply

$$\frac{x_t}{X_t} = \frac{y_t}{Y_t} = \frac{e^{-\lambda(T-t-\Delta)} - e^{-\lambda(T-t)}}{1 - e^{-\lambda(T-t)}}.$$

Now consider the $\lambda = 0$ case. The corresponding first order conditions for x_t and y_t are, respectively,

$$p'\left(\frac{x_t}{y_t}\right)\frac{\Delta}{y_t} = p'\left(\frac{X_t - x_t}{Y_t - y_t}\right)\cdot\frac{T-t}{Y_t - y_t},$$
$$p'\left(\frac{x_t}{y_t}\right)\frac{x_t\Delta}{(y_t)^2} = p'\left(\frac{X_t - x_t}{Y_t - y_t}\right)\frac{(X_t - x_t)(T-t)}{(Y_t - y_t)^2}.$$

Therefore, we have $x_t/X_t = y_t/Y_t = \Delta/(T-t)$. □

Proof of Proposition 3

Pick any $n \in \{0, 1, ..., N-2\}$. To prove (i), observe that $\gamma_\lambda(n+1) > \gamma_\lambda(n)$ if and only if

$$\frac{e^{-\lambda(N\Delta-(n+2)\Delta)} - e^{-\lambda(N\Delta-(n+1)\Delta)}}{1-e^{-\lambda(N\Delta-(n+1)\Delta)}}\left(1-\frac{e^{-\lambda(N\Delta-(n+1)\Delta)} - e^{-\lambda(N\Delta-n\Delta)}}{1-e^{-\lambda(N\Delta-n\Delta)}}\right) > \frac{e^{-\lambda(N\Delta-(n+1)\Delta)} - e^{-\lambda(N\Delta-n\Delta)}}{1-e^{-\lambda(N\Delta-n\Delta)}}$$

Simplifying, this inequality we get

$$\left(e^{-\lambda(N\Delta-(n+2)\Delta)} - e^{-\lambda(N\Delta-(n+1)\Delta)}\right) > \left(e^{-\lambda(N\Delta-(n+1)\Delta)} - e^{-\lambda(N\Delta-n\Delta)}\right),$$

which must hold because $e^{-\lambda x}$ is decreasing and convex.

To prove (ii), note that

$$\frac{\gamma_\lambda(n+1)}{\gamma_\lambda(n)} = \frac{e^{-\lambda(N\Delta-(n+2)\Delta)} - e^{-\lambda(N\Delta-(n+1)\Delta)}}{e^{-\lambda(N\Delta-(n+1)\Delta)} - e^{-\lambda(N\Delta-n\Delta)}}$$

Taking the derivative of the right side with respect to λ, we get

$$\frac{\Delta[(e^{-\lambda(2N\Delta-2n\Delta-2\Delta)} - e^{-\lambda(2N\Delta-2n\Delta-\Delta)}) - (e^{-\lambda(2N\Delta-2n\Delta-\Delta)} - e^{-\lambda(2N\Delta-2n\Delta)})]}{(e^{-\lambda(N\Delta-(n+1)\Delta))} - e^{-\lambda(N\Delta-n\Delta)})^2} > 0,$$

where the inequality holds again because the function $e^{-\lambda x}$ is convex and decreasing. $\square$

Proof of Proposition 4

Consider time $t = n\Delta \in \mathcal{T}$ and suppose that the game has arrived at time t with both candidates having a positive amount of resources still available, $X_t, Y_t > 0$. Let x_t and y_t be the amounts spent by the two candidates in period t and denote with $\mathcal{I}_t$ all the information available at time t. Then, we have

$$Z_{t+\Delta} \mid \mathcal{I}_t \sim \mathcal{N}\left(\log\left(\frac{x_t}{y_t}\right)\frac{1-e^{-\lambda\Delta}}{\lambda} + z_t e^{-\lambda\Delta}, \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda}\right),$$

and Itô's lemma implies that

$$\log\left(\frac{X_{t+\Delta}}{Y_{t+\Delta}}\right) \mid \mathcal{I}_t \sim \mathcal{N}\left(\log\left(\frac{X_t - x_t}{Y_t - y_t}\right) + \mu_{XY}(z_t)\Delta, \sigma^2_{XY}\Delta\right).$$

We will prove the proposition by induction on the times at which candidates take actions, $t = (N-m)\Delta \in \mathcal{T}$, $m = 1, 2, ..., N$. To simplify notation, it is convenient to define recursively the following expressions. Let

$$g_1(1) = e^{-\lambda\Delta}, \quad g_2(1) = \frac{1-e^{-\lambda\Delta}}{\lambda}, \quad g_3(1) = 0, \quad g_4(1) = \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda}.$$

and define recursively for any $m > 1$

$$\begin{aligned}
g_1(m) &= e^{-\lambda\Delta} g_1(m-1) + g_2(m-1)(a-b); \\
g_2(m) &= g_1(m-1)\frac{1-e^{-\lambda\Delta}}{\lambda} + g_2(m-1); \\
g_3(m) &= g_2(m-1) + g_3(m-1). \\
g_4(m) &= (g_1(m-1))^2 \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda} + (g_2(m-1))^2 \sigma_{XY}^2 \Delta + g_4(m-1)
\end{aligned}$$

Since we can write

$$\begin{pmatrix} g_1(m) \\ g_2(m) \end{pmatrix} = \begin{pmatrix} e^{-\lambda\Delta} & a-b \\ \frac{1-e^{-\lambda\Delta}}{\lambda} & 1 \end{pmatrix} \begin{pmatrix} g_1(m-1) \\ g_2(m-1) \end{pmatrix}, \tag{11}$$

by diagonalizing the matrix

$$\begin{pmatrix} e^{-\lambda\Delta} & a-b \\ \frac{1-e^{-\lambda\Delta}}{\lambda} & 1 \end{pmatrix}$$

and solving for $(g_1(m), g_2(m))'$ with initial conditions $g_0(1) = 1$ and $g_2(0) = 0$, we can conclude that, for each $N \in \mathbb{N}$ and $\lambda, \Delta > 0$, there exists $-\eta < 0$ such that, for each $a - b \geq -\eta$, both $g_1(m)$ and $g_2(m)$ are non-negative for each $m = 1, ..., N$. For the rest of the proof, we assume $g_1(m) \geq 0$ and $g_2(m) \geq 0$.

Consider the following inductive hypothesis: for every $s = (N-m)\Delta \in \mathcal{T}$, $m \in \{1, ..., N\}$, if $X_s, Y_s > 0$, then

(i) the continuation payoff of each candidate is a function of current popularity z_s, current budget ratio X_s/Y_s and calendar time s;

(ii) the distribution of Z_T given z_s and X_s/Y_s is $\mathcal{N}\left(\hat{\mu}_{(N-m)\Delta}(z_s), \hat{\sigma}^2_{(N-m)\Delta}\right)$, where

$$\hat{\mu}_{(N-m)\Delta}(z_{(N-m)\Delta}) = g_1(m) z_{(N-m)\Delta} + g_2(m) \log\left(\frac{X_{(N-m)\Delta}}{Y_{(N-m)\Delta}}\right) + g_3(m)(\sigma_Y^2 - \rho\sigma_X\sigma_Y),$$

and $\hat{\sigma}^2_{(N-m)\Delta} = g_4(m)$.

Suppose the game reaches period $t = (N-1)\Delta$ and both players still have a positive amount of resources, $X_{(N-1)\Delta}, Y_{(N-1)\Delta} > 0$. Because money left at time T is useless, candidates spend all their remaining resources at t, $x_{(N-1)\Delta} = X_{(N-1)\Delta}$ and $y_{(N-1)\Delta} = Y_{(N-1)\Delta}$. Hence, $x_{(N-1)\Delta}/y_{(N-1)\Delta} = X_{(N-1)\Delta}/Y_{(N-1)\Delta}$ and

$$Z_T \mid \mathcal{I}_{(N-1)\Delta} \sim \mathcal{N}\left(\log\left(\frac{X_{(N-1)\Delta}}{Y_{(N-1)\Delta}}\right)\frac{1-e^{-\lambda\Delta}}{\lambda} + z_{(N-1)\Delta}e^{-\lambda\Delta}, \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda}\right).$$

Because Z_T fully determines candidates' payoffs, the continuation payoff of candidates is a function of current popularity $z_{(N-1)\Delta}$, the ratio $X_{(N-1)\Delta}/Y_{(N-1)\Delta}$ and calendar time. Furthermore, at $t = (N-1)\Delta$ the distribution of Z_T given $z_{t+\Delta}$ is a degenerate distribution on $z_{t+\Delta}$. Given the recursive expressions defined above, we can conclude that the second part of the inductive hypothesis also holds at $t = (N-1)\Delta$. This concludes the base step.

Suppose the inductive hypothesis holds true at any time $s = (N-m)\Delta \in \mathcal{T}$ with $m \in \{1, 2, ..., n^* - 1\}$, $m^* < N$. We want to show that at time $t = (N-m^*)\Delta \in \mathcal{T}$, if $X_t, Y_t > 0$, then (i) an equilibrium exists, and in all equilibria, $x_t/y_t = X_t/Y_t$ and the continuation payoffs of both candidates are functions of relative popularity z_t, the ratio X_t/Y_t, and calendar time t, and (ii) Z_T given period t information is distributed according to $\mathcal{N}\left(\hat{\mu}_{(N-m^*)\Delta}(z_t), \hat{\sigma}^2_{(N-m^*)\Delta}\right)$.

Consider period $N - m^*$ and let $x, y > 0$ be the candidate's spending levels in this period. Exploiting the inductive hypothesis, we can compound the normal distributions and conclude that $Z_T \mid \mathcal{I}_t \sim \mathcal{N}(\tilde{\mu}, \tilde{\sigma}^2)$ where

$$\begin{aligned}\tilde{\mu} = {} & g_1(m^*-1)\left[\log\left(\frac{x}{y}\right)\frac{1-e^{-\lambda\Delta}}{\lambda} + z_t e^{-\lambda\Delta}\right] \\ & + g_2(m^*-1)\left[\log\left(\frac{X_{(N-m^*)\Delta}-x}{Y_{(N-m^*)\Delta}-y}\right) + \mu_{XY}(z_t)\Delta\right] + g_3(m^*-1)(\sigma_Y^2 - \rho\sigma_X\sigma_Y) \\ \tilde{\sigma}^2 = {} & (g_1(m^*-1))^2\frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda} + (g_2(m^*-1))^2\sigma_{XY}^2\Delta + g_4(m^*-1)\end{aligned}$$

Then, note that we can write

$$\tilde{\mu} = \hat{\mu}_t(x,y) := G_1\log\left(\frac{x}{y}\right) + G_2\log\left(\frac{X_{(N-m^*)\Delta}-x}{Y_{(N-m^*)\Delta}-y}\right) + G_3,$$

where

$$\begin{aligned} G_1 &= g_1(m^* - 1)\frac{1 - e^{-\lambda\Delta}}{\lambda} \\ G_2 &= g_2(m^* - 1) \\ \text{and } G_3 &= g_1(m^* - 1)z_t e^{-\lambda\Delta} + g_2(m^* - 1)\mu_{XY}(z_t)\Delta + g_3(m^* - 1)(\sigma_Y^2 - \rho\sigma_X\sigma_Y) \end{aligned} \tag{12}$$

Furthermore $\tilde{\sigma}^2$ is independent of x and y.

Candidate 1 wins the election if $Z_T > 0$. Thus, in equilibrium he chooses x to maximize his winning probability

$$\int_{\frac{-\hat{\mu}_t(x,y)}{\tilde{\sigma}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-s/2} ds.$$

The first order necessary condition for x is given by

$$\frac{1}{\sqrt{2\pi}} e^{\frac{\hat{\mu}_t(x,y)}{2\hat{\sigma}_t}} \frac{\hat{\mu}_t'(x,y)}{\tilde{\sigma}} = \frac{1}{\sqrt{2\pi}\tilde{\sigma}} e^{\frac{\hat{\mu}_t(x,y)}{2\tilde{\sigma}}} \left[\frac{G_1(X_t - x) - G_2 x}{x(X_t - x)}\right].$$

Furthermore, when the first order necessary condition holds, the second order condition is given by

$$\frac{1}{\sqrt{2\pi}} e^{\frac{\hat{\mu}_t(x,y)}{2\hat{\sigma}_t}} \frac{\hat{\mu''}_t(x,y)}{\tilde{\sigma}} = \frac{-1}{\sqrt{2\pi}} e^{\frac{\hat{\mu}_t(x,y)}{2\tilde{\sigma}}} \left[\frac{G_1(X_t - x)^2 + G_2 x^2}{x^2(X_t - x)^2}\right] < 0.$$

Hence, the problem is strictly quasi-concave for candidate 1 for each y. A symmetric argument shows that the corresponding problem for candidate 2 is strictly quasi-concave for each x. Hence an equilibrium exists and the optimal investment of the two candidates is pinned down by the first order necessary condition. Solving the first order condition, we get

$$\frac{x_t}{X_t} = \frac{G_1}{G_1 + G_2}, \qquad \frac{y_t}{Y_t} = \frac{G_1}{G_1 + G_2}. \tag{13}$$

Therefore, we have

$$\frac{x_t}{y_t} = \frac{X_t - x_t}{Y_t - y_t} = \frac{X_t}{Y_t}.$$

Exploiting this equilibrium condition, we conclude that the distribution of

$$Z_T \mid \mathcal{I}_t,$$

and hence the continuation payoffs of candidates at times $t = (N - m)\Delta$, are functions only of current popularity z_t, the current budget ratio X_t/Y_t, and calendar time t. Furthermore, substituting $\mu_{XY}(z_t) = (a - b)z_t + \sigma_Y^2 - \rho\sigma_X\sigma_Y$ in the distribution of

$$Z_T \mid \mathcal{I}_{(N-m^*)\Delta},$$

the equilibrium condition also implies that the mean of the distribution is

$$\begin{aligned}\tilde{\mu} = [g_1(m^* - 1)e^{-\lambda\Delta} + g_2(m^* - 1)(a - b)]z_t \\ + [g_1(m^* - 1)\frac{1 - e^{-\lambda\Delta}}{\lambda} + g_2(m^* - 1)] \log\left(\frac{X_{(N-m^*)\Delta}}{Y_{(N-m^*)\Delta}}\right) \\ + [g_2(m^* - 1) + g_3(m^* - 1)](\sigma_Y^2 - \rho\sigma_X\sigma_Y).\end{aligned}$$

Given the expressions recursively defined above, we conclude that

$$Z_T \mid \mathcal{I}_{(N-m^*)\Delta} \sim \mathcal{N}(\hat{\mu}_{(N-m^*)\Delta}, \hat{\sigma}^2_{(N-m^*)\Delta})$$

where

$$\begin{aligned}\hat{\mu}_{(N-m^*)\Delta}(z_{(N-m^*)\Delta}) &= g_1(m^*)z_{(N-m^*)\Delta} + g_2(m^*) \log\left(\frac{X_{(N-m)\Delta}}{Y_{(N-m)\Delta}}\right) + g_3(m^*)(\sigma_Y^2 - \rho\sigma_X\sigma_Y), \\ \hat{\sigma}^2_{(N-m^*)\Delta} &= g_4(m^*).\end{aligned}$$

□

Proof of Proposition 5

The expression of x_t/X_t and y_t/Y_t follows from (11), (12), and (13). To derive the distribution, we start by using the findings from our proof of Proposition 4 above to derive the distribution of $x_{t+j\Delta}/y_{t+j\Delta}$ and $z_{t+j\Delta}$ given x_t/y_t and z_t. Note that, since $X_t/Y_t = x_t/y_t$ for each t, we can recursively write that

$$\left.\begin{pmatrix} \log\left(\frac{x_{t+n\Delta}}{y_{t+n\Delta}}\right) \\ z_{t+n\Delta} \end{pmatrix}\right| \begin{pmatrix} \frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}} \\ z_{t+(n-1)\Delta} \end{pmatrix}$$

follows the multivariate normal distribution

$$\mathcal{N}\left(\left(\begin{array}{c}\log\left(\frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}}\right)+\mu_{XY}(z_{t+(n-1)\Delta})\Delta\\ \log\left(\frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}}\right)\frac{1-e^{-\lambda\Delta}}{\lambda}+z_{t+(n-1)\Delta}e^{-\lambda\Delta}\end{array}\right),\left(\begin{array}{cc}\sigma_{XY}^2\Delta & 0\\ 0 & \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda}\end{array}\right)\right)$$

Therefore,

$$\left(\begin{array}{c}\log\left(\frac{x_{t+n\Delta}}{y_{t+n\Delta}}\right)-\frac{\lambda(\sigma_Y^2-\rho\sigma_X\sigma_Y)}{a-b}\frac{e^{-\lambda\Delta}}{1-e^{-\lambda\Delta}}\\ z_{t+n\Delta}+\frac{(\sigma_Y^2-\rho\sigma_X\sigma_Y)}{a-b}\end{array}\right)\Bigg|\left(\begin{array}{c}\log\left(\frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}}\right)-\frac{\lambda(\sigma_Y^2-\rho\sigma_X\sigma_Y)}{a-b}\frac{e^{-\Delta\lambda}}{1-e^{-\lambda\Delta}}\\ z_{t+(n-1)\Delta}+\frac{(\sigma_Y^2-\rho\sigma_X\sigma_Y)}{a-b}\end{array}\right)$$

follows the multivariate normal distribution

$$\mathcal{N}\left(\left(\begin{array}{cc}1 & (a-b)\,\Delta\\ \frac{1-e^{-\lambda\Delta}}{\lambda} & e^{-\lambda\Delta}\end{array}\right)\left(\begin{array}{c}\log\left(\frac{x_{t+(n-1)\Delta}}{y_{t+(n-1)\Delta}}\right)-\frac{\lambda\Delta(\sigma_Y^2-\rho\sigma_X\sigma_Y)}{a-b}\frac{e^{-\lambda\Delta}}{1-e^{-\lambda\Delta}}\\ z_{t+(n-1)\Delta}+\frac{(\sigma_Y^2-\rho\sigma_X\sigma_Y)}{a-b}\end{array}\right),\left(\begin{array}{cc}\sigma_{XY}^2\Delta & 0\\ 0 & \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda}\end{array}\right)\right).$$

Therefore,

$$\left(\begin{array}{c}\log\left(\frac{x_{t+n\Delta}}{y_{t+n\Delta}}\right)-\frac{\lambda(\sigma_Y^2-\rho\sigma_X\sigma_Y)}{a-b}\frac{e^{-\lambda\Delta}}{1-e^{-\lambda\Delta}}\\ z_{t+n\Delta}+\frac{(\sigma_Y^2-\rho\sigma_X\sigma_Y)}{a-b}\end{array}\right)\Bigg|\left(\begin{array}{c}\log\left(\frac{x_t}{y_t}\right)-\frac{\lambda\Delta(\sigma_Y^2-\rho\sigma_X\sigma_Y)}{a-b}\frac{e^{-\lambda\Delta}}{1-e^{-\lambda\Delta}}\\ z_t+\frac{(\sigma_Y^2-\rho\sigma_X\sigma_Y)}{a-b}\end{array}\right)$$

follows the multivariate normal distribution with mean

$$\left(\begin{array}{cc}1 & (a-b)\,\Delta\\ \frac{1-e^{-\lambda\Delta}}{\lambda} & e^{-\lambda\Delta}\end{array}\right)^n\left(\begin{array}{c}\log\left(\frac{X_t}{Y_t}\right)-\frac{\lambda\Delta(\sigma_Y^2-\rho\sigma_X\sigma_Y)}{a-b}\frac{e^{-\lambda\Delta}}{1-e^{-\lambda\Delta}}\\ z_t+\frac{(\sigma_Y^2-\rho\sigma_X\sigma_Y)}{a-b}\end{array}\right)$$

and variance

$$\left(\begin{array}{cc}1 & (a-b)\,\Delta\\ \frac{1-e^{-\lambda\Delta}}{\lambda} & e^{-\lambda\Delta}\end{array}\right)^n\left(\begin{array}{cc}\sigma_{XY}^2\Delta & 0\\ 0 & \frac{\sigma^2(1-e^{-2\lambda\Delta})}{2\lambda}\end{array}\right)\left(\begin{array}{cc}1 & \frac{1-e^{-\lambda\Delta}}{\lambda}\\ (a-b)\,\Delta & e^{-\lambda\Delta}\end{array}\right)^n.$$

□

Proof of Proposition 6

Fix λ and Δ and let $N-m=n$. We must show that for all $m\in\{1,...,N-1\}$,

$$\hat{r}_m(a-b)=\frac{x_{m\Delta}}{X_{m\Delta}}/\frac{x_{(m+1)\Delta}}{X_{(m+1)\Delta}}$$

is decreasing in $\alpha := a - b$ around $\alpha = 0$. (Note that $\hat{r}_m$ is the same as $\tilde{r}_{N-m}$ but $\hat{r}_m$ counts time backwards.)

Since we can write

$$\hat{r}_m(\alpha) = \frac{\dfrac{g_1(m-1)}{g_1(m-1) + g_2(m-1)\frac{\lambda}{1-e^{-\lambda\Delta}}}}{\dfrac{g_1(m)}{g_1(m) + g_2(m)\frac{\lambda}{1-e^{-\lambda\Delta}}}} = \frac{g_1(m-1)}{g_1(m)}\frac{g_2(m+1)}{g_2(m)},$$

we first relate $(g_1(m), g_2(m+1))$ to $(g_1(m-1), g_2(m))$. From

$$\begin{pmatrix} g_1(m) \\ g_2(m) \end{pmatrix} = \begin{pmatrix} e^{-\lambda\Delta} g_1(m-1) + \alpha g_2(m-1) \\ \frac{1-e^{-\lambda\Delta}}{\lambda} g_1(m-1) + g_2(m-1) \end{pmatrix},$$

we can derive

$$g_1(m) = \frac{(\lambda+\alpha)e^{-\lambda\Delta} - \alpha}{\lambda} g_1(m-1) + \alpha g_2(m)$$

and

$$g_2(m+1) = \frac{1-e^{-\lambda\Delta}}{\lambda}\frac{(\lambda+\alpha)e^{-\lambda\Delta} - \alpha}{\lambda} g_1(m-1) + \frac{\alpha - \alpha e^{-\lambda\Delta} + \lambda}{\lambda} g_2(m).$$

Hence,

$$\begin{pmatrix} g_1(m) \\ g_2(m+1) \end{pmatrix} = \begin{pmatrix} \frac{(\lambda+\alpha)e^{-\lambda\Delta}-\alpha}{\lambda} g_1(m-1) + \alpha g_2(m) \\ \frac{1-e^{-\lambda\Delta}}{\lambda}\frac{(\lambda+\alpha)e^{-\lambda\Delta}-\alpha}{\lambda} g_1(m-1) + \frac{\alpha-\alpha e^{-\lambda\Delta}+\lambda}{\lambda} g_2(m) \end{pmatrix}. \tag{14}$$

Substituting in the expression for $\hat{r}_m(\alpha)$ and simplifying, we get

$$\hat{r}_m(\alpha) = \frac{1}{\frac{(\lambda+\alpha)e^{-\lambda\Delta}-\alpha}{\lambda} + \alpha g_m}\left(\frac{1-e^{-\lambda\Delta}}{\lambda}\frac{(\lambda+\alpha)e^{-\lambda\Delta} - \alpha}{\lambda}\frac{1}{g_m} + \frac{\alpha + \lambda - \alpha e^{-\lambda\Delta}}{\lambda}\right) \tag{15}$$

where $g_m := g_2(m)/g_1(m-1)$. From (14), we further have

$$g_{m+1} = \frac{\frac{1-e^{-\lambda\Delta}}{\lambda}\frac{(\lambda+\alpha)e^{-\lambda\Delta}-\alpha}{\lambda} + \frac{\alpha+\lambda-\alpha e^{-\lambda\Delta}}{\lambda} g_m}{\frac{(\lambda+\alpha)e^{-\lambda\Delta}-\alpha}{\lambda} + \alpha g_m}. \tag{16}$$

Computing (15) one step forward and substituting for g_{m+1} as obtained from (16) and, subsequently, for g_m as obtained from (15), we get $\hat{r}_{m+1}$ as a function of α and $\hat{r}_m$:

$$\hat{r}_{m+1}\left(\alpha,\hat{r}_m\right) = \frac{\left(\begin{array}{c}\alpha\left(-2e^{\lambda\Delta}+\left(\hat{r}_m-3\right)\left(-e^{2\Delta\lambda}\right)+\hat{r}_m-1\right)+\lambda\left(e^{\lambda\Delta}\left(e^{\lambda\Delta}+\hat{r}_m-1\right)+\hat{r}_m\right)+\\ +(1+e^{\lambda\Delta})\sqrt{\left(\alpha(\hat{r}_m+1)\left(e^{\Delta\lambda}-1\right)+\lambda\left(e^{\lambda\Delta}-\hat{r}_m\right)\right)^2-4\alpha\hat{r}_m\left(e^{\lambda\Delta}-1\right)\left(\alpha\left(e^{\lambda\Delta}-1\right)-\lambda\right)}\end{array}\right)}{\left(\begin{array}{c}\alpha\left(e^{\lambda\Delta}-1\right)\left((\hat{r}_m+1)e^{\Delta\lambda}-\hat{r}_m+1\right)+\lambda\left(e^{\lambda\Delta}\left(e^{\lambda\Delta}-\hat{r}_m+1\right)+\hat{r}_m\right)+\\ +(1+e^{\lambda\Delta})\sqrt{\left(\alpha(\hat{r}_m+1)\left(e^{\Delta\lambda}-1\right)+\lambda\left(e^{\lambda\Delta}-\hat{r}_m\right)\right)^2-4\alpha\hat{r}_m\left(e^{\lambda\Delta}-1\right)\left(\alpha\left(e^{\lambda\Delta}-1\right)-\lambda\right)}\end{array}\right)}.$$

We first show that $\hat{r}_m > e^{\lambda\Delta} > 1$ for each m around $\alpha = 0$. For $m = 1$, since we have $\frac{x_{(N-1)\Delta}}{X_{(N-1)\Delta}} = 1$ and $\frac{x_{(N-2)\Delta}}{X_{(N-2)\Delta}} = \dfrac{g_1(1) + g_2(1)\frac{\lambda}{1-e^{-\lambda\Delta}}}{g_1(1)}$, we have

$$\hat{r}_1 - e^{\lambda\Delta} = 1 + \frac{\frac{1-e^{-\lambda\Delta}}{\lambda}}{e^{-\lambda\Delta}}\frac{\lambda}{1-e^{-\lambda\Delta}} - e^{\lambda\Delta} = 1,$$

so $\hat{r}_1 > 1$. Given $\hat{r}_m > e^{\lambda\Delta}$, subtracting $e^{\lambda\Delta}$ from the right hand side of the expression of $\hat{r}_{m+1}$ and setting $\alpha = 0$, we get

$$\hat{r}_{m+1} - e^{\lambda\Delta} = 1 - \frac{e^{\lambda\Delta}}{\hat{r}_m} > 0.$$

Hence, if $\hat{r}_m > e^{\lambda\Delta}$, then $\hat{r}_{m+1} > e^{\lambda\Delta}$. Therefore, we have $\hat{r}_m > e^{\lambda\Delta}$ for each m in a neighborhood of $\alpha = 0$.

Now we prove that $\hat{r}'_m(\alpha) < 0$ around $\alpha = 0$. To this end, observe that $\hat{r}_{m+1}(\alpha, \hat{r}_m)$ is decreasing in α and increasing in $\hat{r}_m$:

$$\left.\frac{\partial\hat{r}_{m+1}\left(\alpha,\hat{r}_m\right)}{\partial\alpha}\right|_{\alpha=0} = -\frac{\left(\hat{r}_m-1\right)e^{\lambda\Delta}\left(e^{2\lambda\Delta}-1\right)}{\hat{r}_m\left(\hat{r}_m-e^{\lambda\Delta}\right)} < 0;$$
$$\left.\frac{\partial\hat{r}_{m+1}\left(\alpha,\hat{r}_m\right)}{\partial\hat{r}_m}\right|_{\alpha=0} = \frac{e^{\lambda\Delta}}{\left(\hat{r}_m\right)^2} > 0.$$

Hence, for each m, $\hat{r}_m(\alpha)$ is decreasing in α inductively.

Finally, now viewing $\hat{r}_m$ as a function of λ as well, we have

$$\left.\frac{\partial\hat{r}_{m+1}\left(\alpha,\hat{r}_m,\lambda\right)}{\partial\lambda}\right|_{\alpha=0} = \frac{e^{\lambda\Delta}\left(\hat{r}_m-1\right)\Delta}{\hat{r}_m} > 0 \text{ for each } \lambda > 0.$$

Hence, for each m, $\hat{r}_m$ is increasing in λ inductively near $\alpha = 0$. $\square$

Proof of Proposition 7

For $k = 1$ the result reduces to the formula reported in Proposition 2. Suppose $k > 0$ and let $\lambda > 0$. Then

$$\begin{aligned}
\frac{\bar{x}_t}{X_t} &= \frac{x_t + x_{t+\Delta} + \cdots + x_{t+(k-1)\Delta}}{X_t} \\
&= \frac{x_t}{X_t} + \frac{x_{t+\Delta}}{X_{t+\Delta}}\frac{X_{t+\Delta}}{X_t} + \cdots + \frac{x_{t+(k-1)\Delta}}{X_{t+(k-1)\Delta}}\frac{X_{t+(k-1)\Delta}}{X_{t+(k-2)\Delta}} \cdots \frac{X_{t+\Delta}}{X_t} \\
&= \frac{e^{-\lambda(T-t-\Delta)} - e^{-\lambda(T-t)}}{1 - e^{-\lambda(T-t)}} + \frac{e^{-\lambda(T-t-2\Delta)} - e^{-\lambda(T-t-\Delta)}}{1 - e^{-\lambda(T-t-\Delta)}}\left(1 - \frac{e^{-\lambda(T-t-\Delta)} - e^{-\lambda(T-t)}}{1 - e^{-\lambda(T-t)}}\right) + \cdots \\
&= e^{-\lambda(T-t)}\frac{e^{\lambda k\Delta} - 1}{1 - e^{-\lambda(T-t)}} = \frac{e^{-\lambda(T-t-\overline{\Delta})} - e^{-\lambda(T-t)}}{1 - e^{-\lambda(T-t)}}
\end{aligned}$$

which is the formula stated in the proposition. The proof of the case in which $\lambda = 0$ is similar and omitted. $\square$

References

Aldrich, J. H. (1980): "A Dynamic Model of Presidential Nomination Campaigns," *American Political Science Review*, 74, 651669.

Ali, S. and N. Kartik (2012): "Herding with collective preferences," *Economic Theory*, 51, 601–626.

Ashworth, S. (2006): "Campaign finance and voter welfare with entrenched incumbents," *American Political Science Review*, 100, 55–68.

Bouton, L., M. Castanheira, and A. Drazen (2018): "A Theory of Small Campaign Contributions," *NBER Working Paper No. 24413.*

Callander, S. (2007): "Bandwagons and Momentum in Sequential Voting," *The Review of Economic Studies*, 74, 653–684.

Cox, G. W. and M. F. Thies (2000): "How Much Does Money Matter? Buying Votes in Japan, 1967-1990," *Comparative Political Studies*, 33, 37–57.

de Roos, N. and Y. Sarafidis (2018): "Momentum in dynamic contests," *Economic Modelling*, 70, 401–416.

Denter, P. and D. Sisak (2015): "Do polls create momentum in political competition?" *Journal of Public Economics*, 130, 1 – 14.

Erikson, R. S. and T. R. Palfrey (2000): "Equilibria in campaign spending games: Theory and data," *American Political Science Review*, 94, 595–609.

Erikson, R. S. and C. Wlezien (2012): *The timeline of presidential elections: How campaigns do (and do not) matter*, University of Chicago Press.

Gerber, A. (1998): "Estimating the effect of campaign spending on senate election outcomes using instrumental variables," *American Political Science Review*, 92, 401–411.

Glazer, A. and R. Hassin (2000): "Sequential rent seeking," *Public Choice*, 102, 219–228.

Green, D. P. and J. Krasno (1988): "Salvation for the spendthrift incumbent: Reestimating the effects of campaign spending in House elections," *American Journal of Political Science*, 884–907.

Gross, O. and R. Wagner (1950): "A Continuous Colonel Blotto game," *Manuscript*.

Harris, C. and J. Vickers (1985): "Perfect Equilibrium in a Model of a Race," *The Review of Economic Studies*, 52, 193–209.

——— (1987): "Racing with uncertainty," *The Review of Economic Studies*, 54, 1–21.

Hinnosaar, T. (2018): "Optimal sequential contests," *Manuscript*.

Iaryczower, M., G. L. Moctezuma, and A. Meirowitz (2017): "Career Concerns and the Dynamics of Electoral Accountability," *Manuscript*.

Jacobson, G. (1978): "The effects of campaign spending in congressional elections," *American Political Science Review*, 72, 469–491.

Jacobson, G. C. (2015): "How Do Campaigns Matter?" *Annual Review of Political Science*, 18, 31–47.

Kawai, K. and T. Sunada (2015): "Campaign finance in us house elections," *Manuscript*.

Klumpp, T. and M. K. Polborn (2006): "Primaries and the New Hampshire effect," *Journal of Public Economics*, 90, 1073–1114.

Knight, B. and N. Schiff (2010): "Momentum and Social Learning in Presidential Primaries," *Journal of Political Economy*, 118, 1110–1150.

Konrad, K. A. and D. Kovenock (2009): "Multi-battle contests," *Games and Economic Behavior*, 66, 256–274.

Konrad, K. A. et al. (2009): *Strategy and dynamics in contests*, Oxford University Press.

Meirowitz, A. (2008): "Electoral contests, incumbency advantages, and campaign finance," *The Journal of Politics*, 70, 681–699.

Skaperdas, S. and B. Grofman (1995): "Modeling Negative Campaigning," *The American Political Science Review*, 89, 49–61.

Vojnović, M. (2016): *Contest theory: Incentive mechanisms and ranking methods*, Cambridge University Press.