Optimal Reference-Point Adaptation

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Introduction

This talk is about a model of **reference-dependent** preferences with **loss aversion** and **endogenously** formed reference points.

- **Reference-dependent utility/preferences:** when utility from an outcome depends on comparisons to relevant “reference levels” or “reference points.”
- **Loss aversion:** people dislike losses relative to the reference point more than they like same-sized gains.

Introduction—An Example of Reference-Dependency

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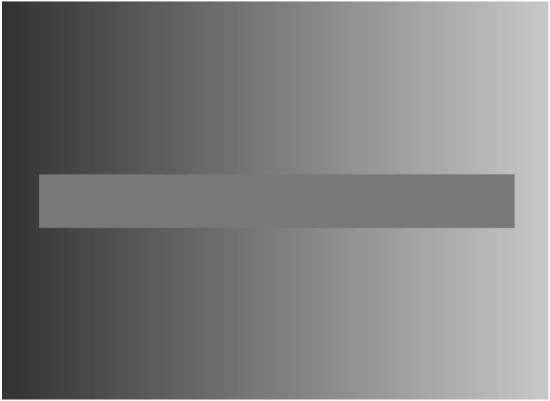


Introduction

My take on reference-dependent preference:

- Reference dependence should not be viewed as a mistake on part of the decision maker, or DM.

Rayo and Becker (2007)

It is an optimal feature of biological measurement instruments designed to guide the DM's decisions to maximize fitness.

- Analogous with how the eyes work in the above example.
- Can be grounded in a restriction on how evolution can shape preferences:
  - There is a limit on the DM's perception sensitivity.

Introduction

Why is reference dependence and loss aversion important?

1. The most cited paper in Econometrica to date is Kahneman and Tversky's Prospect Theory paper .
   - "An essential feature of the present theory is that the carriers of value are changes in wealth or welfare, rather than final states."
2. The Rabin's (2000) Paradox:
   - People are generally risk averse over small-stakes gambles but moderately risk averse over high-stakes gambles.
3. Reference-dependent preferences can explain a number of puzzling phenomena incompatible with neoclassic theory.
   - For example the endowment effect, labor supply, the disposition effect, and the equity premium puzzle.

Introduction

Crucial question: **what is the reference point**?

Kahneman and Tversky's answer: most of the time **status quo**.

However, the status quo is often make unintuitive predictions when applying prospect theory in economics.

- For example, an employee who is very confident of getting a 10% pay raise is likely to view a 5% raise as a loss.

Introduction

I model reference point formation for time-consistent preferences.

- This results in reference point that is "as if" the DM is able to choose the reference point freely.

An obvious **problem**:

- It seems like there can be no drawback by setting a low reference point as possible.

Solution:

- Any realistic model of reference point formation should equate the reference point with the outcome when there is no uncertainty.

Introduction

Approach: I model the formation of the reference point in two stages.

1. **Ex ante:** the reference point is determined by the probability distribution over outcomes in the ex post stage.
2. **Ex post:** expected value of gain-loss utility taking the reference point as given.

I connecting ex ante and ex post preferences by assuming time-consistent preferences.

For the resulting utility function, it is as if the DM faces a trade-off between anticipatory utility with the risk of being disappointed.

Adaptive Reference Points

I extend the model to an infinite-horizon setting where the DM strives to maximize the sum of discounted utility in each period.

Per-period utility is given by the same utility function as above, but I introduce a friction in reference point formation.

In this model, the reference point is partly determined by the history of previous reference points.

- Thus, the reference point has both a **backward-looking** and a **forward-looking** component.

Outline of Talk

1. I derive ex ante preferences over lotteries that the DM has had time to get "adjusted" to.
2. I extend the model to an dynamic setting with adaptive reference points.
3. I apply the model to an infinite-horizon consumptions-savings problem.

Related Literature

Closest papers to mine are **Kőszegi and Rabin (2006, 2007, 2009)**. They model reference point formation as determined by the DM's expectations from the recent past.

Here, the reference point is stochastic and the DM evaluates an outcome against every possible outcome, determined by her expectations.

The model does not satisfy first order stochastic dominance, features multiplicity of optimal plans and time-inconsistent preferences.

My results can be viewed from a long the lines of robustness of the predictions of Kőszegi and Rabin's model.

Related Literature

- **Optimal Reference Points:** Ben-Tal and Teboulle (1986, 2007), Gollier and Muermann (2010), and Sarver (2017).
- **Reference Points as Expectations:** Bell (1985), Loomes and Sugden (1986), and Gul (1991).
- **Habit Formation:** Becker and Murphy (1988), Constantinides (1990), Bowman et al. (1999) and Campbell and Cochrane (1999).

Baseline Model

I develop a utility function for situations when the DM has had the time to form her reference point **given** the risk she is facing.

Ex ante: In this stage the DM **forms** her reference point given the uncertainty she faces.

Ex post: In this stage all uncertainty is resolved and the DM takes the reference point as **given**.

Preview—Ex Ante Preferences

I **derive** the following utility function as a function of lotteries μ over outcomes in the ex post stage:

The Optimized Reference-Dependent, ORD, Utility Function

$$V(\mu) = \underbrace{u(r^\mu)}_{\text{anticipatory utility}} + \underbrace{\int \phi(u(x) - u(r^\mu))d\mu(x)}_{\text{gain-loss utility}} \tag{1}$$

1. **Ex ante:** $V(\mu)$ with $r^\mu \in \mathbb{R}$ given by

$$r^\mu \in \arg\max_{r \in \mathbb{R}} \; u(r) + \int \phi(u(x) - u(r))d\mu(x).$$

2. **Ex post:** gain-loss utility given a reference point, r^μ.

Model: Ex-Post Utility

- Given some reference level, or point, $r \in \mathbb{R}$, when the outcome x is drawn according to a probability measure μ, **gain-loss utility** is given by a Bernoulli utility function

$$U_r(\mu) = \int \phi(u(x) - u(r)) d\mu(x).$$

  - I assume that $\phi : \mathbb{R} \to \mathbb{R}$ is continuous, differentiable except possibly at zero, and strictly increasing. I normalize $\phi(0) = 0$.
  - The difference between x and r is measured by an **underlying** function $u : \mathbb{R} \to \mathbb{R}$. It is assumed to be continuous, differentiable, and strictly increasing.
  - For simplicity, U_r is linear in probabilities (could allow for nonlinear probability weighting as in Kahneman and Tversky (1979)).

Model: Ex-Post Utility with Loss Aversion

Loss aversion for small stakes is captured by the following property:

Loss Aversion: $\lim_{y \to 0} \phi'(|y|) \equiv \eta$ and $\lim_{y \to 0} \phi'(-|y|) \equiv \lambda\eta$

Here, $\lambda \geq 1$ is referred to as the *loss aversion parameter*.

- When $\lambda = 1$, ϕ is an everywhere differentiable function.

Model: Ex Ante Utility

I now consider the ex ante utility function.

Arguably no standard way of modeling this. I assume the following:

- The utility from an outcome, $x \in \mathbb{R}$, and a reference point, $r \in \mathbb{R}$, is given by a function $w : \mathbb{R}^2 \to \mathbb{R}$.
- Let
$$W(\mu, r) = \int w(x, r) d\mu(x)$$
and assume that w is continuous and differentiable in both arguments, except possibly at $x = r$ (when ϕ has a kink at zero).
- The reference point $r^\mu \in \mathbb{R}$ (given the choice of lottery μ) is
$$r^\mu \in \arg\sup_{r \in \mathbb{R}} W(\mu, r)$$
.

Model: Ex Ante Utility

To get some structure, I make three assumptions on W.

A.1 **Consumption Utility:** For every $x \in \mathbb{R}$, $\sup_{r \in \mathbb{R}} W(\delta_x, r) = u(x)$.

A.2 **Time Consistency:** Fix a reference point $r \in \mathbb{R}$, then for every pair of lotteries μ, ν the following holds:

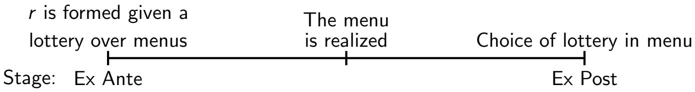
$$W(\mu, r) \geq W(\nu, r) \iff U_r(\mu) \geq U_r(\nu).$$

A.3 **Acclimation:** For every $x \in \mathbb{R}$, $W(\delta_x, r) \geq W(\delta_x, r')$ for $r' \leq r < x$ and $W(\delta_x, r) \geq W(\delta_x, r')$ for $r' \geq r > x$.

Remark: Is the time-consistency assumption too strong?

Not in an enriched framework, then this assumption is both sufficient and necessary.

- Consider the following setting:



- With prob. p the menu is degenerate $\{x\}$ and with prob $1-p$ the menu is $\{\mu, \theta, \ldots\}$. Consider the case when $p \to 1$ then $r = x$ (assumption A.3).

- Since x can be arbitrarily chosen, $\mu \succeq_{exa} \theta$ is equivalent to $\mu \succeq_{exp} \theta$ if only if A.2 holds.

Model: Ex Ante Utility

Proposition

Suppose $\lambda = 1$ and η is normalized to unity. W satisfies assumption A.1 to A.3 if and only if

$$W(\mu, r^{\mu}) = \underbrace{u(r^{\mu})}_{\text{anticipatory utility}} + \underbrace{\int \phi(u(x) - u(r^{\mu}))d\mu(x)}_{\text{gain-loss utility}}$$

where $\frac{\partial \phi(y)}{\partial y} \leq 1$ for $y > 0$ and $\frac{\partial \phi(y)}{\partial y} \geq 1$ for $y < 0$.

For any $\lambda\eta \geq 1 \geq \eta$, $W(\mu, r^{\mu})$ satisfies assumption A.1-A.3. The reversed implication is not true.

Optimized Reference-Dependent Utility

For lotteries μ with compact support in $\mathbb{R}$, it is possible to define the utility function V.

The Optimized Reference-Dependent, ORD, Utility Function

$$V(\mu) = \max_{r \in \mathbb{R}} \left[u(r) + \int \phi(u(x) - u(r)) d\mu(x) \right]$$

An Example—Expectations Determines Reference Point

1. Ex Ante: W.p. p the DM gets a 10% raise (10 utils) and w.p. $(1-p)$ she gets a 5% raise (5 utils).

2. Assume for simplicity that ϕ is piecewise linear:

$$\phi(y) = \begin{cases} \lambda\eta y & \text{for } y \leq 0, \\ \eta y & \text{for } y > 0, \end{cases}$$

.

3. The optimal reference point, r^*, is for any $\varepsilon > 0$ given by

$$p\phi'(10 - u(r^* + \varepsilon)) + (1-p)\phi'(5 - u(r^* + \varepsilon)) > 1 \\ > p\phi'(10 - u(r^* - \varepsilon)) + (1-p)\phi'(5 - u(r^* - \varepsilon))$$

.

4. By piecewise linearity of ϕ, $r^* = 10$ if $p > \underline{p}$ (otherwise $r^* = 5$) for $\underline{p} \in (0,1)$ that solves

$$5 + \underline{p}\eta 5 = 10 - (1 - \underline{p})\lambda\eta 5.$$

Three Example of ORD Indirect Utility Functions

Three Examples from Ben-Tal and Teboulle (2007), Gollier and Muermann (2010), and Sarver (2017):

1. $$V(\mu) = \int u(x)\mu(x) \Leftrightarrow \phi(x) = x$$

2. Given a piecewise linear ϕ then $V(\mu) = \int u(x)d(w \circ F_\mu)(x)$ with

$$w(p) = \begin{cases} \lambda\eta p & \text{for} \quad p \leq \frac{\eta^{-1}-1}{(\lambda-1)}, \\ \eta p + 1 - \eta & \text{for} \quad p > \frac{\eta^{-1}-1}{(\lambda-1)}. \end{cases}$$

3. When $U_r(x) = 1 - exp(-[u(x) - u(r)])$ (i.e. $\lambda = \eta = 1$), then

$$V(\mu) = -log\left(\int exp(-u(x))d\mu(x)\right).$$

Some Properties

Proposition

For any ORD utility function V the following holds:

1. *V satisfy first order stochastic dominance.*

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3. *For any lottery μ with maximal and minimal outcomes $\bar{x}$ and $\underline{x}$ respectively, the optimal reference point, r^{μ}, is such that $\underline{x} \leq r^{\mu} \leq \bar{x}$*

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4. *V is convex in probabilities, that is,* $\gamma V(\mu) + (1-\gamma)V(\nu) \geq V(\gamma\mu + (1-\gamma)\nu)$ *for all* $\gamma \in [0,1]$.

Some Properties

Proposition

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3. *For any lottery μ with maximal and minimal outcomes $\bar{x}$ and $\underline{x}$ respectively, the optimal reference point, r^{μ}, is such that $\underline{x} \leq r^{\mu} \leq \bar{x}$*
4. *V is convex in probabilities, that is, $\gamma V(\mu) + (1-\gamma)V(\nu) \geq V(\gamma\mu + (1-\gamma)\nu)$ for all $\gamma \in [0,1]$.*
5. *V exhibits first-order risk-aversion at all wealth levels if and only if $\lambda > 1$.*

Infinite-Horizon Extension

- In some situations it is reasonable to believe that the DM does not have time to prepare herself to the risk she is facing, depending on her previous reference level.
- This can give rise to history-dependent risk-aversion.
- I extend the model to allow for partial reference point adjustment.
- The notion of an adaptive and slow-adjusting reference point is supported by a body of empirical evidence
  - See Lant (1992), Mas (2006), Arkes et al. (2008), Post et al. (2008), Baucells et al. (2011), Card and Dahl (2011), DellaVigna et al. (2017) and Thakral and Tô (2017).

Infinite-Horizon Extension

Here the DM has preferences over the timing of resolution of uncertainty, therefore the model's uncertainty is dated by the time of its resolution.

The preferences presented here are defined recursively over infinite-horizon **temporal lotteries**, $\mu_t(x_t, \mu_{t+1})$.

- The model has an infinite amount of discrete time periods, where period 1 refers to the first time the DM pays attention to the problem.
- In period $t-1$, μ_{t-1} is a measure not only of consumption x_{t-1} in period $t-1$ but also of temporal lotteries μ_t for period t.
- Epstein and Zin (1989) show that such lotteries are well-defined if their support is compact

Infinite-Horizon Extension

The anticipation in a period t is now represented by $a_t \in \mathbb{R}$ which **partly** determines the reference point r_t.

Contemporaneous utility in period $t > 1$:

$$W(x_t, a_t; r_{t-1}) = u(a_t) + U_{r_t(a_t, r_{t-1})}(x_t).$$

a_t maximizes discounted utility from period t and onwards as before.

$r_t : \mathbb{R}^2 \to \mathbb{R}$ is an **adaptive** reference point in period t. It is a function of today's anticipation level, a_t, and yesterday's reference level, r_{t-1}.

Infinite-Horizon Extension—Adaptive Reference Points

For simplicity:

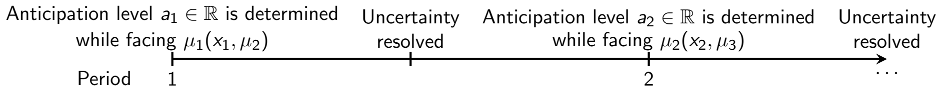
- r_t is given by an autoregressive law of motion

$$r_t = \delta a_t + (1 - \delta) r_{t-1}$$

  with $\delta \in [0, 1]$.
- In period 1, r_0 is given by $a_1 \in \mathbb{R}$.
  - This model choice is in line with the psychological literature on cognitive reference points (Rosch, 1975).

Infinite-Horizon Extension —Timeline

Given a temporal lottery μ_1:



Infinite-Horizon Extension—Total indirect utility

Total indirect utility, standing in period $t > 1$, facing μ_t given r_{t-1}, can be expressed recursively as

$$V(\mu_t; r_{t-1}) = \max_{a_t \in \mathbb{R}} \Big[u(a_t) + \int \left[U_{r_t(a_t, r_{t-1})}(x_t) + \beta V(\mu_{t+1}; r_t(a_t, r_{t-1})) \right] d\mu_t(x_t, \mu_{t+1}) \Big]$$

for $\beta \in (0, 1)$.

Finally, when $t = 1$ total indirect utility is given by

$$V_1(\mu_1) = \max_{a_1 \in \mathbb{R}} \Big[\delta^{-1} u(a_1) + \int \left[U_{r_1(a_1, a_1)}(x_t) + \beta V(\mu_2; r_2(a_2, a_1)) \right] d\mu_2(x_2, \mu_3) \Big].$$

Infinite-Horizon Extension—Basic Properties

- The DM has time-consistent preferences.
- In a richer framework where the DM is allowed to make choices, any *optimal plan*, that is, pairs of consumption and anticipation levels in each period, induces a temporal lottery that maximizes V_1.
- For the rest of this talk assume that u is of the CRRA class. The results holds for a much larger class but with some technical assumptions.

Deterministic Consumption-Savings Model

Assume that the DM inherits a wealth level $\omega \in \mathbb{R}$ that she is to distribute over all periods. An optimal plan in this context is the division of ω into consumption in each period, given by $x_1, x_2, \ldots$, such that the DM maximizes utility given optimal reference points.

Proposition

There exists $\beta^* \in (0,1)$ such that for any wealth level $\omega > 0$, in any optimal plan, $x_t = r_t$ for all t whenever $\beta \in (\beta^*, 1)$.

Moreover, the optimal consumption path satisfies

$$u'\left(x_{t-1} - \frac{x_{t-1} - x_t}{\delta}\right) = \beta u'\left(x_t - \frac{x_t - x_{t+1}}{\delta}\right), \quad (2)$$

for all $t > 1$ with $u'(x_1) = \beta u'(x_1 - \frac{x_1 - x_2}{\delta})$.

Deterministic Consumption-Savings Model—Intuition

When the reference point is at the consumption level in each period, the DM derives utility from anticipation only.

- The DM smooths anticipatory utility according to the standard Euler-equation $u'(a_t) = \beta u'(a_{t+1})$ for all $t > 1$.
  - Since $r_t = \delta a_t + (1 - \delta) r_{t-1} = x_t$ and $r_t = x_t$ in each period, $\Rightarrow$

$$u'(x_{t-1} - \frac{x_{t-1} - x_t}{\delta}) = \beta u'(x_t - \frac{x_t - x_{t+1}}{\delta}).$$

- Note that $x_t > x_{t+1}$ for all t since $a_t > a_{t+1}$.
- The above result can equivalently be expressed by fixing $\beta \in (0, 1)$ and adjusting $\delta \in (0, 1)$.

Wealth Shock in Consumption-Savings Model

A puzzle regarding the adjustment speed of the reference point:

- A number of field studies seem to imply that the reference point **adjusts slowly** (see Mas (2006), Post et al. (2008), Card and Dahl (2011), DellaVigna et al. (2017), and Thakral and Tô (2017))
- Experimental evidence, using small monetary incentives, indicate that the reference point **adjusts very fast** (see Buffat and Senn (2015), Song (2016), and Smith (2012))

Wealth Shock in Consumption-Savings Model

Consider a wealth shock occurring in period t such that the DM has wealth $\omega_t > 0$ given some inherited reference point r_{t-1}.

Proposition

Let $\beta \in (\beta^*, 1)$ as in the above Proposition. There exist $r_{t-1} > 0$, ($\eta^* \in (0,1),$), and $\omega_t^* > 0$ such that, in any optimal plan, $x_t < r_t$ ($x_t > r_t$) whenever ($\eta \in (\eta^*, 1),$) $\omega_t < \omega_t^*$ ($\omega_t > \omega_t^*$). Moreover, $x_{t'} < r_{t'}$ ($x_{t'} > r_{t'}$) for $t' \leq \tau < \infty$ and $x_{t'} = r_{t'}$ for $t' > \tau$.

There is an asymmetry between adjustment to gains and loses.

Wealth Shock in Consumption-Savings Model—Intuition

Assume that $x_{t'} = r_{t'}$ for all $t' \geq t$ given any $\omega_t > 0$ and $r_{t-1} > 0$. For a negative wealth shock:

The optimal reference point is given by

$$\delta\lambda\eta u'(r_t) + \underbrace{\beta(1-\delta)u'(a_{t+1})}_{\text{since } \phi \text{ is kinked}} \geq u'(a_t) \geq \eta\delta u'(r_t) + \underbrace{\beta(1-\delta)u'(a_{t+1})}_{\text{since } \phi \text{ is kinked}}$$

$$\Leftrightarrow \lambda\eta u'(r_t) \geq u'(a_t) \geq \eta u'(r_t)$$

- Since c_t is increasing in ω_t, c_t can be small relative to r_{t-1}.
- Since marginal utility is diminishing, there exists a $r_{t-1} > 0$ such that $\lambda\eta u'(\delta a_t + (1-\delta)r_{t-1}) < u'(a_t)$ when $r_t > c_t$.
- Heuristically, the DM cares about smoothing anticipatory utility as u is concave.
- Doesn't work for gains if δ is large relative to η.

Wealth Shocks: A Preference for Increasing Consumption Streams

Two influential surveys by Loewenstein and Sicherman (1991) and Loewenstein and Prelec (1993) has found that people can exhibit an ex ante preference for increasing consumption profiles.

The ORD utility model with sticky reference points can help rationalize this behavior.

The argument builds on the idea of diminishing sensitivity of gain-loss utility as proposed by Kahneman and Tversky (1979).

This is true for the ORD utility function if:

A.4 **Reflection:** ϕ is twice differentiable with $\phi''(y) < 0$ for $y > 0$ and $\phi''(y) > 0$ for $y < 0$.

Wealth Shock in Consumption-Savings Model—A Preference for Increasing Consumption Streams

Corollary

Given a wealth shock as in the Proposition above. Then consumption in any period $t \leq t' \leq \tau - 1$ *is given by*

$$u'(x_{t'})\phi'(u(x_{t'}) - u(r_{t'})) = \beta u'(x_{t'+1})\phi'(u(x_{t'+1}) - u(r_{t'+1})).$$

Moreover, there exists $\beta^* \in (0, 1)$ *such that if* ϕ *satisfies assumption A.4 then every consumption level* $x_{t'}$ *is increasing in* t' *for* $t \leq t' \leq \tau$ *whenever* $\beta \in (\beta^*, 1)$.

Conclusion

1. I suggest a model featuring endogenous reference points that seems to capture salient features of reference point formation.
2. I show in a dynamic setting how the model is compatible with partly backward-looking reference points.
3. What is left is to apply the model to a stochastic problem.

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Ex Post Decisions

Consider now the situation when the DM has chosen her reference point ex ante and takes it as given in an ex post decision between lotteries.

Informally, the DM faces a lottery almost surely but also faces a vanishingly small probability of a choice between lotteries, and the choice situation happens to occur.

That is, the DM evaluates a lottery by an expected utility function U_r given a reference point r.

Endowment Effect for Risk

In line with experimental findings (e.g. Isoni et al. (2011) and Sprenger (2015)) the model can generate an endowment effect for risk.

The DM is no more willing to accept a lottery μ on top of some wealth level y when her reference point is y, then she is to accept μ when she is already facing the degenerate lottery y given the optimal reference point for μ given by $r^{\mu} \in X$.

Proposition

Suppose ϕ is piecewise linear and u is linear. For any lotteries μ and ν and any $x, y \in \mathbb{R}$, if $U_y(y + \mu) \geq U_y(y)$ then $U_x(\mu + \nu) \geq U_x(\nu)$.