Optimal Reference-Point Adaption*

Peter Wikman†

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Abstract

This paper develops a dynamic model of reference-dependent preferences and loss aversion. The decision maker's adaptive reference point is determined endogenously in each period by balancing the utility from anticipation of future consumption with the risk of disappointment. In the baseline model, reference point formation is guided by behavioral assumptions taking standard reference-dependent preferences as given. I show that the decision maker's preferences always respect first order stochastic dominance and exhibit first-order risk-aversion at all wealth levels. Absent risk in an infinite-horizon setting, the adaptive reference point distorts the optimal consumption path compared to the standard model, while consumption coincides with the reference point in each period. I also show that it might be optimal to consume above or below the reference level in the presences of wealth shocks and how this can rationalize a preference for increasing consumption streams.

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†Email: peter.wikman@tse-fr.eu.

1. Introduction

Although prospect theory (Kahneman and Tversky, 1979) is still the leading descriptive decision criterion under risk, applications in economics are surprisingly limited. This is often attributed to that people in the model derive utility from gains and losses measured relative to some reference level. This reference level is most of the time hard to identify and has often been left as a free variable. Pinning down this variable in a disciplined way poses a major challenge to any researcher attempting to apply any form of model featuring reference-dependent preferences.

This paper proposes a solution to this problem by providing a model of dynamic reference-dependent preferences featuring endogenously determined reference points. Its main feature is that today's anticipation of tomorrow's consumption determines tomorrow's reference point. More specifically, it is 'as if' the decision maker, or DM, ex ante chooses an anticipation level of consumption from which she derives utility according to some underlying function. This consumption level anchors her expectations of consumption ex post which she evaluates against the reference level. Thus, when determining the reference point, the DM trades off anticipatory utility ex ante with the risk of being disappointed ex post. As the DM is rational, she chooses an anticipation level so as to maximize the expected sum of these two utilities.

By introspection, it is not hard to gauge the importance of expectation management to avoid future disappointment. For example, a Google search on the phrase "manage your expectations" gives around half a million results. This seems to imply that people, indeed, take into account the trade-off between anticipation and disappointment. This idea is also corroborated by experimental evidence. For example, Mellers et al. (1997) find that decision under risk can be described as a maximization of expected emotional experiences (this finding is support by Shepperd and McNulty (2002)). Moreover, Sweeny and Shepperd (2010) examine upstream and downstream costs and benefits of optimism and pessimism. They conclude that their subjects were aware of the downstream costs of optimistic expectations.

There exists a large body of evidence indicating that peoples' preferences depend on reference levels, or reference points (see, e.g., Kahneman and Tversky (1979)). For example, the fact that people are generally risk averse over small-stakes gambles but moderately risk averse over high-stakes gambles is hard to reconcile with utility functions that are reference independent.[1] Moreover, reference dependence is a feature of most sensory and perceptual dimensions and the evolutionary advantages of this feature can be derived from physical constraints on the nervous system (Rayo and Becker, 2007). Thus, instead of viewing reference dependence as a mistake by the DM, it should be viewed as an optimal feature of biological measurement instruments designed to guide her decisions.[2]

In the original formulation of prospect theory, the reference point is taken to be the current asset position of the DM. However, Kahneman and Tversky acknowledges that "[a] discrepancy between the reference point and the current asset position may also arise because of recent changes in wealth to which one ha not yet adopted." (Kahneman and Tversky, 1979, p.287) However, they provide no guidelines as how to model reference point adjustment.

Kőszegi and Rabin (2006) is the first paper I know of that explicitly models reference point formation (but see also Bell (1985), Loomes and Sugden (1986), and Gul (1991)). The authors argue that the reference point should be determined by a person's expectations formed in recent times of the event at hand. The model allows the reference point to be stochastic in that the DM evaluates an outcome against every possible outcome determined by her expectations.[3] These expectations are pinned down by assuming that the DM has rational expectations.

A key implication of the above modeling strategy is that the DM has time-inconsistent preferences. That is, she might benefit from committing herself to an action ex post while ex ante forming her expectations. To see this, note that, typically, the DM's expectations depend on her future behavior. The issue is that, since she takes the reference point as given ex post, she does not internalize the fact that her behavior must be consistent with the expectations generating the reference point. She might, thus, be

[1] For example, if an expected utility maximizer turns down a 50-50 gamble losing \$10 or gaining \$11 for any wealth level, she will turn down any 50-50 gamble where the loss is \$100, no matter how large the upside of the gamble is. This is called Rabin's (2000) paradox and it has shown to extend to virtually any reference independent utility function (see the discussion in Bleichrodt et al. (2017)).

[2] I work under the assumption that preferences are optimized to maximize the expected fitness of our ancestors in their natural environment, and that this environment might share limited characteristics with that of the modern world.

[3] An alternative, as in Gul (1991), is specifying that the reference point is the certain equivalent of the expected outcome.

unable to credibly commit to a plan she ex ante prefers. This is true of any model in which the reference point is explicitly determined by the DM's future behavior. Moreover, it seems to imply that forward-looking reference points are unlikely to be apart of a fitness-maximizing solution to an evolutionary problem. This is unfortunate as forward-looking reference points are able to explain seemingly disparate phenomena such as the endowment effect (Ericson and Fuster, 2011, Sprenger, 2015) and the apparent negative relationship between labor supply and daily earnings for New York City cab drivers (Camerer et al., 1997, Thakral and Tô, 2017).

The class of utility functions developed in this paper, called the optimized reference-dependent, or ORD, resolves this tension by severing the explicit connection between expectations and reference point formation. Instead I model reference point formation by letting the DM to 'choose' her reference point in the ex ante stage. This can be interpreted as if the DM can—given sufficient time—mentally prepare herself for the risk at hand. I then show that is possible to regain the connection between expectations and reference point formation by some 'calibrational' assumptions described below.

To stay close to the previous literature, I assume that ex post utility from an outcome —given some reference point—is a function of the difference between utility from actual and reference-point consumption, measured by an underlying function (this is in line with much of the previous literature, see, e.g., Loomes and Sugden (1986) and Kőszegi and Rabin (2006)). I focus on cases where the ex post utility function is kinked at the origin with marginal utility being larger on the loss side than on the gain side.

When deriving the ORD class of ex ante utility functions, I consider a general class of preferences that can be represented by an utility function that depends continuously on ex post preferences and the reference point. I then impose some structure on this utility function by three assumptions. The first assumption is that the DM's preferences over outcomes are time-consistent. This is a standard simplifying assumption motivated by the above discussion. The second assumptions states that, when there is no uncertainty, the DM's indirect utility function, i.e. utility as a function of a lottery assuming that she chooses the best possible reference point given the lottery she is facing, coincides with the underlying function. This assumption justifies the use of the latter function as the measures of the distance between the actual outcome and the reference level. Finally, the third assumptions states that, if the DM is facing a degenerate lottery, she maximize ex ante utility by equating the reference point with the actual outcome. This is a calibrational assumption that links the DM's expectations with the formation of the reference point.

I show that the only class of utility functions that satisfies these assumptions, when ex post utility is everywhere differentiable (i.e. not kinked), is what I call the ORD class. It can be described, as above, as if the DM strives to balance utility from anticipation of future consumption with the risk of being disappointed. Furthermore, I show that the ex ante preferences can be viewed as a subclass of the preferences axiomatized by Sarver (2017) which have been shown to be surprisingly tractable. For example, an ORD utility function is monotone with respect to first order stochastic dominance and exhibits first-order risk-aversion at all wealth levels if the ex post utility function is kinked. These two features are not necessarily true for Kőszegi and Rabin (2006)'s model for empirically identified parameter values.

I extend the model to a dynamic setting where I allow the reference point to be adaptive. Here, the initial reference point is determined as in the baseline model whereas, for all later periods, the reference point is not only determined by the anticipation level in that period but also by previous periods' anticipation levels. The notion of an adaptive and slow-adjusting reference point is supported by a body of empirical evidence (see Lant (1992), Mas (2006), Arkes et al. (2008), Post et al. (2008), Baucells et al. (2011), Card and Dahl (2011), DellaVigna et al. (2017) and Thakral and Tô (2017)). Due to the nature of this reference point, the DM's preferences are nonstationary and both marginal utility and risk preferences, such as potential first-order risk-aversion, may be affect by previous periods' reference level.

I show that, in a deterministic environment, the optimal consumption path is distorted compared to standard model, while consumption coincide with the reference point in each period. This is in contrast with the literature on habit formation where consumption is not at the reference point in most periods (see, e.g., Constantinides (1990) and Campbell and Cochrane (1999)). Moreover, I show that given a sufficiently large wealth shock, the DM finds it optimal to adjust her reference point over time to the new consumption level. This implies that she might consume above or below her references point for a finite number of periods before she has acclimated to her new wealth level. Moreover, I show how this can give rise to a preference for increasing consumption streams, as found in surveys by Loewenstein and Sicherman (1991) and Loewenstein and Prelec (1993).

The structure of the paper is as follows. In section 2, I develop the baseline model. Thereafter in section 3, I introduce the infinite-horizon model with adaptive reference points. Behavioral implications of the model are derived in section 4. In Section 5, I show that the model can generate an endowment effect for risk. Finally, in section 6 I compare the model to already establish models in the literature.

2. Baseline Model

The purpose of this section is to develop an utility function that represents reference-dependent preferences of a decision maker, or DM, that has had the time to form her reference level *given* the risk she is facing. Such as situation is referred to as the *ex ante* stage. By contrast, the *ex post* stage is a situation in which the reference level has been determined earlier, and the DM has preferences as described by a Bernoulli function determined by this referent.

The ex ante utility function is derived from a set of behavioral assumptions together with the assumed relationship between the reference level and the ex post utility function. The reason for basing the ex ante utility function on explicitly modeled ex post preferences is that the latter has a relatively standard formulation that is well understood and performs well in experimental settings (see, e.g., Kahneman and Tversky (1979), Bell (1985), Loomes and Sugden (1986), Bleichrodt et al. (2001), Köbberling and Wakker (2005), and Kőszegi and Rabin (2006)). The primitive of this section is the ex ante choice set of lotteries (with compact support in $\mathbb{R}$) the DM focuses on in isolation from other choices and risks (the analysis of ex post preferences is postponed until Section 5). Any uncertainty is here, for simplicity, assumed to be resolved in the ex post stage.

2.1. Ex Post Utility

I will start by presenting the ex post utility function. Given some reference level, or point, $r \in \mathbb{R}$, when the outcome $x \in \mathbb{R}$ is drawn according to a probability measure, μ, utility is given by a Bernoulli function[4]

$$U_r(\mu) = \int \phi(u(x) - u(r))d\mu(x). \tag{1}$$

The difference between the outcome and the reference level is measured by an *underlying* function u.[5] It should be interpreted as intrinsic utility from consumption and is assumed to be continuous, differentiable, and strictly increasing. The function $\phi : \mathbb{R} \to \mathbb{R}$ is called the *gain-loss* function. It is assumed to be continuous, differentiable except possibly at zero, and strictly increasing. It is normalized such that $\phi(0) = 0$.

As ϕ may be kinked at zero, *loss aversion for small stakes* is captured by the following property:

$$\lim_{y\to 0} \phi'(|y|) \equiv \eta \text{ and } \lim_{y\to 0} \phi'(-|y|) \equiv \lambda\eta \text{ with } \lambda > 1.$$

Here, $\lambda \geq 1$ is referred to as the *loss aversion parameter* and η as the *salience parameter*. When $\lambda = 1$, the gain-loss function, ϕ, is an everywhere differentiable function.

2.2. Ex Ante Utility

I now consider a class of ex ante preferences over lotteries. These preferences are represented by a utility function as follows:

$$V(\mu) = \max_{r\in\mathbb{R}} \left[u(r) + \int \phi[u(x) - u(r)]d\mu(x) \right], \tag{2}$$

$\frac{\partial\phi(y)}{\partial y} \leq 1$ for $y > 0$ and $\frac{\partial\phi(y)}{\partial y} \geq 1$ for $y < 0$. I will refer to this class of utility functions as *optimized reference-dependent*, or ORD. This indirect utility function (with respect to r) is well-defined due to Berge's Maximum principle since the supports of the lotteries I consider are compact subsets of $\mathbb{R}$.

The utility function given by equation (2) has a didactic interpretation. Heuristically, it as if the DM determines the reference point ex ante by imagining a consumption level for which she derives utility according to her underlying function. This consumption level determines her reference

[4]Letting U_r be linear in probabilities is for simplicity. As discussed in Section 6, the baseline model is compatible with nonlinear probability weighting as in Kahneman and Tversky (1979) and Tversky and Kahneman (1992).

[5] For experimental evidence supporting the view that much of the behavior captured by reference-dependent preferences stems from differences between anticipated and actual experienced utility, see Knutson and Peterson (2005) and Kermer et al. (2006).

level which is evaluated against her actual consumption level. Thus, the rule determining the reference point trades off anticipated utility ex ante with gain-loss utility ex post.

To justify the restriction to the above class of preferences, I will derive it from a relatively large class of ex ante utility functions. The utility functions I consider are given by

$$W(\mu, r) = \int w(x, r) d\mu(x),$$

where, notably, W is linear in probabilities holding r fixed.[6] Let $w : \mathbb{R}^2 \to \mathbb{R}$, a function of outcomes $x \in \mathbb{R}$ and reference points $r \in \mathbb{R}$, be continuous and differentiable in both arguments except possibly when $x = r$ (when ϕ has a kink at zero). The reference point $r \in \mathbb{R}$ is determined by a function $r : \Delta(X) \to \mathbb{R}$ where

To allow for precise predictions, it is necessary to put some additional structure on the ex ante utility. I will make the following three assumptions on W.

A.1 Consumption Utility: For every $x \in \mathbb{R}$, $\sup_{r \in \mathbb{R}} W(\delta_x, r) = u(x)$, where δ_x denotes the Dirac probability concentrated at x.

A.2 Time Consistency: Fix a reference point $r \in \mathbb{R}$, then for every pair of lotteries μ, ν the following holds:

$$W(\mu, r) \geq W(\nu, r) \iff U_r(\mu) \geq U_r(\nu).$$

A.3 Acclimation: For every $x \in \mathbb{R}$, $W(\delta_x, r) \geq W(\delta_x, r')$ for $r' \leq r < x$ and $W(\delta_x, r) \geq W(\delta_x, r')$ for $r' \geq r > x$.

The first assumption implies that deterministic consumption is evaluated according to intrinsic utility only. It justifies the use of the underlying function u as the relevant unit of comparison in ϕ. The second assumption implies that the DM's incentives are aligned in both stages. It is a simplifying assumption that I view as appropriate as there seems to be no inherent connection between reference-dependent preferences and time inconsistency. See Remark 1 below for a justification of its formulation. The acclimation assumption was introduce by Bowman et al. (1999). In the present context, it implies that the DM prefers the reference level to equate the consumption

[6] Again, this is done for simplicity. See footnote 4.

level when there is no uncertainty. Thus, the DM dislikes being elated or disappointed by outcomes that are certain to occur.

Remark 1 *At a first glance, the time-consistent assumption given by A.2 seems too restrictive. However, the restriction that ex ante and ex post preferences are aligned for any reference point—as opposed to any optimal reference point—is, in fact, a necessary condition for the DM to not benefit from a commitment device when the model is slightly enriched (as in Section 5).*

Consider the setting in which uncertainty is partly resolved between the ex ante and the ex post stage and the DM can make decisions in both stages. For example, in the limiting case when the DM has no decision to make ex ante and is almost certain to face an outcome x in the ex post stage, the optimal reference point is (for any fixed alternative situation that could occur) setting r equal x. In the unlikely case in the DM, instead of the outcome x, ends up facing an arbitrary choice between lotteries ex post. For the DM to have time-consistent preferences, she must have the same preference over these lotteries ex ante and ex post. Since x can be arbitrarily chosen, the preferences over the lotteries is the same ex ante and ex post if only if assumption A.2 holds. The above formulation is purely for ease of presentation since it avoids specifying this enriched environment.

Together, these assumptions imply that it is almost without loss of generality to focus on a special class of the function W that is equivalent to the ORD class.

Proposition 1 *Suppose $\lambda = 1$ and η is normalized to unity. Then W satisfies assumption A.1 to A.3 if and only if*

$$W(\mu, r) = u(r) + \int \phi(u(x) - u(r)) d\mu(x) \tag{3}$$

where $\frac{\partial \phi(y)}{\partial y} \leq 1$ for $y > 0$ and $\frac{\partial \phi(y)}{\partial y} \geq 1$ for $y < 0$.

It is straightforward to show that the utility function given by equation (3) satisfy assumption $A.1$ to $A.3$ for all pairs of loss aversion and salience parameters $\lambda\eta \geq 1 \geq \eta$. The other direction of the argument does not hold in the presence of loss aversion. This is, arguably, an 'artifact' of the kink in ϕ.[7] For simplicity, I will only consider ex ante utility functions given by equation (3) for $\lambda\eta \geq 1 \geq \eta$.

It is without loss of generality to restrict the DM's choice of reference point to be deterministic rather than a distribution of reference points that is independent of the distribution of outcomes, as in Kőszegi and Rabin (2006).[8] The reason is that the DM weakly prefers a deterministic reference point over any stochastic one, since the latter is a convex combination of deterministic references points and W is linear in probabilities.

A couple of fundamental properties of ORD utility functions can be obtained from the analyses of similar functions by Ben-Tal and Teboulle (1986, 2007) and Sarver (2017). To this aim, it is necessary to introduce some notation. For μ and ν, let $\mu + \nu$ be the lottery defined by $(\mu + \nu)(A) = \mu(A) + \nu(A)$ for any set $A \subset \mathbb{R}$.

Proposition 2 *For any ORD utility function V the following holds:*

1. *V satisfy first order stochastic dominance.*
2. *V satisfy second order stochastic dominance if and only if u and ϕ are concave.*
3. *For any lottery μ with maximal and minimal outcomes $\bar{x}$ and $\underline{x}$ respectively, the optimal reference point,*

$$r^{\mu} \in \arg\max_{r\in\mathbb{R}} u(r) + \int \phi(u(x) - u(r))d\mu(x),$$

is such that $\underline{x} \leq r^{\mu} \leq \bar{x}$.

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[7] For example, let $k : \mathbb{R} \to [0, 1]$ be any continuous function with $0 \leq \partial k(r)/\partial r \leq 1$, then the following function satisfy satisfies assumption $A.1$ to $A.3$:

$$W(F, r) = u(r) + [k(r) \cdot (\lambda - 1) + 1] \int \phi(u(x) - u(r))dF(x).$$

[8] Of course, the space of lotteries the DM can choose from has to be well-defined. A possible construction is to restrict attention to some compact interval, $[w, b] = X$, on the real line and let $\Delta(X)$ denote the set of all lotteries over X (countably-additive Borel probability measures), endowed with the topology of weak convergence. Thus, the ex ante utility function could be defined by

$$V(\mu) = \sup_{\nu\in\Delta(X)} \left[\int u(r)d\nu(r) + \int\int \phi(u(x) - u(r))d\mu(x)\nu(r)\right]. \quad (4)$$

4. *V is convex in probabilities, that is, $\gamma V(\mu) + (1-\gamma)V(\nu) \geq V(\gamma\mu + (1-\gamma)\nu)$ for all $\gamma \in [0,1]$.*

The first result in Proposition 2 is reassuring. The bulk of experimental evidence indicates that people adhere to first order stochastic dominance (see, e.g., Starmer (2000) for a review). Statement 2 shows that for the model to exhibit aversion to mean-preserving spreads, similar to any Bernoulli function, both u and ϕ need to be concave given any reference point. Statement 3 implies that the DM prefers a reference point that is within the range of the support of the lottery she is facing. Finally, the fourth statement says that if the DM is indifferent between two lotteries, she weakly prefers any of the two lotteries over any mixture of them. This shows that, only considering ex ante decisions, the model is similar to a special case of the class utility functions axiomatized by Sarver (2017) in an infinite-horizon setting.

Given an utility function V, the certainty equivalent of the lottery μ is defined implicitly as the outcome $x \in \mathbb{R}$ such that $V(\delta_x) = V(\mu)$. The difference between the expected value and the certain equivalent of a lottery μ is called the risk premium and is denoted $\pi(\mu)$.

Definition 1 A DM exhibits first-order risk aversion at a wealth level $x \in \mathbb{R}$ if for all $\mu \neq \delta_0$ with $\int x d\mu(x) = 0$, $\frac{\partial \pi(\delta_x + \varepsilon\mu)}{\partial \varepsilon}|_{\varepsilon=0^+} < 0$.

Proposition 3 *Suppose the DM's preferences can be represented by an ORD utility function. Then the DM is first-order risk-averse at all wealth levels if and only if $\lambda > 1$.*

It is possible to analyze large-stake risk aversion by using local expected utility analysis in the sense of Machina (1982). Notice that for any lottery μ with compact support in $\mathbb{R}$, the ORD utility function $V(\mu)$ can be expressed as $W(\mu, r^*(\mu))$ where $r^*(\mu) \in \arg\max_{r\in\mathbb{R}} u(r) + \int \phi(u(x) - u(r)) d\mu(x)$. Then the Arrow-Pratt measure of risk-aversion at the local utility function $W(\delta_x, r^*(\mu))$ is given by

$$-\frac{u''(x)}{u'(x)} - \frac{u'(x)\phi''(u(x) - u(r^*(\mu)))}{\phi'(u(x) - u(r^*(\mu)))}, \tag{5}$$

given that both u'' and ϕ'' exist. This measure is also valid for small-stake risks when $\lambda = 1$. This is also a good measure of small-stake risk aversion when $\lambda > 1$ if the wealth level is not at $r^*(\mu)$ as can be the case for the model presented in Section 3.

If the DM is averse to mean-preserving spreads (that is, $u'' \leq 0$ and $\phi'' \leq 0$) and $\frac{\phi''(y)}{\phi'(y)}$ is increasing in y (for example, when $\phi''' > 0$), then the local Arrow-Pratt measure is increasing in the optimal reference point $r^*(\mu)$. From the analysis by Gollier and Muermann (2010), the optimal reference point is decreasing in any change that the deteriorates the risk in the first order stochastic dominance sense, and likewise for a deterioration in the second order stochastic dominance sense if $\phi''' > 0$. Thus, for such changes the DM becomes locally less risk-averse. As noted by Gollier and Muermann, this can explain the Allais paradox as the indifference curves of V 'fans out' in the Marschak-Machina triangle.

A standard measure for comparative risk aversion for large stage risk is defined as follows. The first DM with ORD utility function V_1 is *more risk averse than* the second DM with ORD utility function V_2 if, for any μ (with compact support) and any $x \in \mathbb{R}$, $V_1(\mu) \geq V_1(\delta_x) \Rightarrow V_2(\mu) \geq V_2(\delta_x)$. It is possible to show that V_1 is more risk averse than V_2 if

$$V_1(\mu) = \max_{r \in \mathbb{R}} u_2(r) + \int f\Big(\phi_2(u_2(x) - u_2(r))\Big)\mu(x)$$

where f is strictly increasing and concave with $f(0) = 0$ and $\lim_{y \to 0} f'(-|y|) \geq (\lambda\eta)^{-1}$.

A particularly nice parametric example is when $\phi(y)$ is equal to $\eta v(y)$ for $y \geq 0$ and $\lambda\eta v(y)$ for $y < 0$(where v is differentiable everywhere with $v(0) = 0$ and $v'(y) \leq \eta^{-1}$ for $y > 0$ and $v'(y) \geq (\lambda\eta)^{-1}$ for $y \leq 0$. If both have the same underlying function u and the same v then DM 1 is more risk averse than DM 2 if DM 1's loss aversion parameter, λ_1, is weakly greater than DM 2's loss aversion parameter, λ_2.[9]

I finish this section by introducing three characterizations of special cases of ORD utility functions. Firstly, the ORD utility function is a Bernoulli function if and only if ϕ is the identity function (Gollier and Muermann, 2010, Proposition 4 p.1277). Secondly, if ϕ is piecewise linear

$$\phi(y) = \begin{cases} \lambda\eta y & \text{for } y \leq 0, \\ \eta y & \text{for } y > 0, \end{cases} \tag{6}$$

the ORD function is a special case of rank-dependent utility function (see, e.g., Abdellaoui (2002) for a recent axiomatization). The proposition be-

[9]See Sarver (2017, p.17-18) for a more comprehensive treatment of measure of risk aversion.

low is a minor generalization of a result in Sarver (2017, Proposition S.1 Supplementary Appendix p.5).

Proposition 4 *Suppose that V is an ORD utility function and ϕ is given by (6) with $\lambda\eta > 1 > \eta$, then the optimal reference point, r^μ, when facing a lottery μ, is given by*

$$r^\mu = \inf\left\{x \in X : F_\mu(x) > \frac{1-\eta}{(\lambda-1)\eta}\right\}, \tag{7}$$

where F_μ is the c.d.f. induced by μ. Moreover, $V(\mu) = \int u(x)d(w \circ F_\mu)(x)$ with

$$w(x) = \begin{cases} \lambda\eta x & \textit{for} \quad x \leq \frac{1-\eta}{(\lambda-1)\eta}, \\ \eta x + 1 - \eta & \textit{for} \quad x > \frac{1-\eta}{(\lambda-1)\eta}. \end{cases} \tag{8}$$

Lastly, as shown by Ben-Tal and Teboulle (2007, Example 2.1) another special case is when $U_r(x) = 1 - exp(-[u(x) - u(r)])$, then $V(\mu) = -log\left(\int exp(-u(x))d\mu(x)\right)$.

3. The Dynamic Model

In this section, I extend the baseline model to an infinite-horizon setting. The ORD utility model is generalized by introducing adaptive reference points, that is, the reference point in any period t does not only depend on this period's anticipation level $a_t \in \mathbb{R}$ but also on the anticipation level of any period $1 \leq t' < t$. Thus, the preference described here are generally nonstationary.

As the DM has preferences over the timing of resolution of uncertainty, the present model's uncertainty is dated by the time of its resolution. The preferences presented here are defined over infinite-horizon *temporal lotteries*. In any period t, a temporal lottery μ_t is a joint measure not only of consumption x_t (with compact support in $\mathbb{R}$) in period t but also of consumption lotteries μ_{t+1} for period $t+1$, and so on for each period $1, 2, \ldots$. Epstein and Zin (1989), among others, have shown that it is possible to construct a well-defined space of lotteries given these restrictions. The timeline of this model is given in Figure 1.

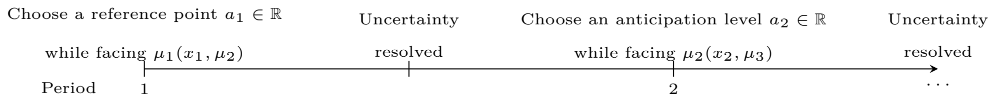

Figure 1: Timeline of the infinite-horizon ORD utility model.

In any period $t > 0$, the DM's utility from consumption is given by the same reference-dependent utility function, $U_{r_t(\mathbf{a}_t)}$, as in the baseline model. However, $r_t(\mathbf{a}_t)$ may now be an *adaptive* reference point possibly depending on the history of *anticipation levels*, $\mathbf{a}_t = (a_1, \dots, a_t)$. I assume for simplicity that r_t is given by an autoregressive law of motion

$$r_t = \delta a_t + (1 - \delta) r_{t-1}, \tag{9}$$

for $\delta \in (0, 1]$. Given that r_t is determined by equation (9), in period t the previous period's reference point, r_{t-1}, is a sufficient description of the DM's preferences. Thus, I write $r_t(a_t, r_{t-1})$.

In period 1, I assume, in line with the psychological literature on cognitive reference points (Rosch, 1975), that the entire reference point is determined by that period's anticipation level. That is, $r_1 = a_1$. Since anticipation not only completely determine the reference level in the current period but also affects the reference level in future periods, anticipatory utility has to be weighted by δ^{-1}.[10] However, in some contexts, it makes more sense to ignore the first period and let the initial reference point be exogenously given.

Contemporaneous utility in period $t > 1$ is similar to the baseline model, i.e., $u(a_t) + U_{r_t(a_t, r_{t-1})}(x_t)$. The DM strives to maximize the expected value of the sum of discounted contemporaneous utility. I assume that the DM chooses a_t in period t optimally as before. Therefore, *total indirect utility*, standing in period $t > 1$, facing μ_t given r_{t-1}, can be expressed recursively as

$$V(\mu_t; r_{t-1}) = \max_{a_t \in \mathbb{R}} \left[u(a_t) + \int \left[U_{r_t(a_t, r_{t-1})}(x_t) + \beta V(\mu_{t+1}; r_t(a_t, r_{t-1})) \right] d\mu_t(x_t, \mu_{t+1}) \right] \tag{10}$$

for $\beta \in (0, 1)$.

[10] The reason for this is best seen by considering the marginal utility from changing the reference point in period 1 as supposed to any other period. Assume that the reference level and consumption level is the same in each period, $\lambda = 1$ so that $\phi'(0) = 1$, and ignore discounting. Then, changing the anticipation level in period 1 gives $\delta^{-1} u' - u' - (1 - \delta) u' - \delta (1 - \delta)^2 u' - \ldots = 0$. By the same token, changing anticipation in any period $t > 1$ gives $u' - \delta u' - \delta (1 - \delta) u' - \delta (1 - \delta)^2 u' - \ldots = 0$.

In period 1, $u(a_1)$ in equation (10) is weighted by δ^{-1} and V_1 is a function of μ_1 only. Thus,

$$V_1(\mu_1) = \max_{a_t \in \mathbb{R}} \left[\delta^{-1} u(a_t) + \int \left[U_{r_t(a_t, r_{t-1})}(x_t) + \beta V(\mu_{t+1}; r_t(a_t, r_{t-1})) \right] d\mu_t(x_t, \mu_{t+1}) \right].$$

When the DM is allowed to make decisions regarding consumption, it is possible to solve for the *optimal plan* as the pairs of consumption and anticipation levels for each period such that the DM maximizes V_1 given the decision problem she is facing.

I now present a couple of basic properties that are straightforward consequences of the analysis of the baseline model. Firstly, it is readily verifiable that the DM has time-consistent preferences. Secondly, using results from Sarver (2017), it is possible to show that the DM's preference respects first order stochastic dominance, and respects second order stochastic dominance if and only if both u and ϕ are concave. Thirdly, the model implies an aversion to late resolution of uncertainty (see Kreps and Porteus (1978)). The reason is that the earlier the DM knows about her future consumption the earlier she can start to align her anticipation levels with these consumption levels. Finally, and importantly, the above model differ from most prominent dynamic reference-dependent utility formulations in that the DM is not averse to information (see the discussion in Gollier and Muermann (2010)).

4. Behavioral Implications in a Consumption-Savings Model

An ORD utility function with adaptive reference points can be fully described by specifying u, ϕ, δ, and β. The results in this section hinges on that loss aversion for small stakes is present. Therefore, $\lambda\eta > 1 > \eta$ for all ϕ considered here and δ is strictly lower than unity. Moreover, besides u being strictly concave (for obvious reasons), a technical assumption on u that will be used throughout this section is that intertemporal elasticity of substitution lie in the interval $[a, b] \subset (0, \infty)$ when the consumption level tends to infinity and is strictly larger than zero when consumption tends to zero.[11] Finally, I assume that u is sufficiently concave relative to the loss aversion parameter, that is $\lim_{y\to 0} u'(y) > \lambda \lim_{x\to\infty} u'(x)$. These assumptions are sufficient for the behavioral results below.

I will focus on analyzing a simple infinite-horizon consumption-savings model to highlight some of the behavioral implications of introducing an adaptive reference point. The reason for focusing on an infinite-horizon model is to avoid problems with the DM having incentives to increase the reference level over time as she nears the end period.

4.1. Behavior Under Certainty

I first show that a DM—with preferences as described by an ORD utility function with adaptive reference points—that is sufficiently patient, consumes at her reference level in each period when there is no uncertainty. I consider a setting in which the DM inherits a wealth level $\omega > 0$ that she is to distribute as consumption over all periods.

A preliminary observation is that when the DM is impatient ($\beta < 1$), consuming at the reference level is incompatible with an adaptive reference point when she is not loss averse for small stakes. The reason is that the incentives to adjust the reference point is not (necessary) in line with the incentives to smooth discounted consumption. However, when the gain-loss function is kinked, if the DM is patient enough she always benefits from consuming at the reference level.

Proposition 5 *There exists $\beta^* \in (0,1)$ such that for any wealth level $\omega > 0$, in any optimal plan, $x_t = r_t$ for all t whenever $\beta \in (\beta^*, 1)$. Moreover, the optimal consumption path satisfies*

$$u'\left(x_{t-1} - \frac{x_{t-1} - x_t}{\delta}\right) = \beta u'\left(x_t - \frac{x_t - x_{t+1}}{\delta}\right), \tag{11}$$

for all $t > 1$ with $u'(x_1) = \beta u'(x_1 - \frac{x_1 - x_2}{\delta})$.

To get some intuition regarding Proposition 5, note that when the reference point is at the consumption level in each period, the DM derives utility from anticipatory only (since gain-loss utility is nil in each period). Thus,

[11] This assumption rules out that u is of the CARA class but allows for that u is of, e.g., the CRRA class. It, moreover, allows for boundary solutions when consumption is low, that is, $u'(x)$ may be finite when $x \to 0$.

when U_r is kinked at r and β is sufficiently large, the DM smooths anticipatory utility according to the standard Euler-equation $u'(a_t) = \beta u'(a_{t+1})$ for all $t > 1$. Since $r_t = \delta a_t + (1-\delta)r_{t-1} = x_t$ in each period, $a_t = x_{t-1} - \frac{x_{t-1}-x_t}{\delta}$ implies that actual consumption is given by $u'(x_{t-1} - \frac{x_{t-1}-x_t}{\delta}) = \beta u'(x_t - \frac{x_t - x_{t+1}}{\delta})$. Note that $x_t > x_{t+1}$ for all t since $a_t > a_{t+1}$.

Although consumption is typically not in accordance with the standard Euler-equation, it has to be the case that, on average, actual consumption in two adjacent periods almost satisfies $u'(x_t) = \beta u'(x_{t+1})$. Assume that this is not the case and that there exists a constant $k > 0$ such that $u'(x_t) \geq \beta(u'(x_{t+1}) - k)$ for all $t > 1$ (the argument when $k < 0$ is symmetric). Such a plan cannot be optimal since, iterating forward, the distortion between period t and $t + \tau$ grows unbounded when $\tau \to \infty$. Thus, it is better to shift consumption from period t to period $t + \tau$, consuming above the reference point in period $t + \tau$ and below it in period t.

The above result can equivalently be expressed by fixing $\beta \in (0, 1)$ and adjusting $\delta \in (0, 1)$. Formally, there exists $\delta^* \in (0, 1)$ such that for any wealth level $\omega > 0$, in any optimal plan, $x_t = r_t$ for all t whenever $\delta \in (\delta^*, 1)$. This implies that the result holds independent of the time between the periods. That is, increasing the time in between two adjacent periods implies a lower β, but at the same time the DM has more time to adjust her reference point which should increase δ.

It is reasonable to question just how close to 1 the discount factor has to be for the above result to hold. Assume that u is of the CRRA class, then the above result holds if $\beta\lambda\eta \geq 1$ independent of the value of $\delta \in (0, 1]$, and $u'(x_t) > \beta u'(x_{t+1})$ for all t, with $u'(x_t) = \beta u'(x_{t+1})$ as t tend to infinity. Since, the standard estimate of the loss aversion parameter is $\lambda \geq 2$, the above Proposition holds if $\beta > 3/4$ when $\eta = 2/3$ (which is the value of η that implies that the optimal reference point is equal to the median in the baseline model when ϕ is piecewise linear).

Proposition 5 shows how the model contrasts with existing dynamic models featuring reference-dependent utility and habit formation, such as Becker and Murphy (1988), Constantinides (1990), and Campbell and Cochrane (1999). In those models, the consumption level is typically not at the reference point in most periods. Loss aversion for small stakes plays a limited role in such models as for it to affect behavior, the reference point has to be close to the planned consumption level.

As noted by Kőszegi and Rabin (2009), the key difference between habit formation models and models where reference point formation is forward looking is that, in the former model, increasing consumption today increases the reference point tomorrow. By contrast, when reference-point adaption is forward looking, an increase in consumption today will lower the reference point tomorrow as the DM anticipates that she will have less to consume tomorrow.

4.2. Reference Point Adaption

The main reason for developing the ORD model with adaptive reference points is to capture behavior in situations where the DM has had limited time to adjust the reference point to news about consumption. In the baseline model, the DM is either able to fully adjust to news about consumption or is unable to affect the reference level at all. I show that the dynamic model is able to capture partial reference point adjustment in which it is optimal to not fully adjust the reference level with the new consumption level.

Consider the setting in which the DM receives an unexpected wealth shock in period t (formally, consider the situation where the probability of incurring the wealth shock tends to zero) such that her new wealth level is $\omega_t > 0$. Furthermore, assume that ω_t is sufficiently small (large) relative to the expected wealth level ω'_t, for which the previous reference points was determined according to, such that r_{t-1}, is large (small) relative to the optimal consumption level in period t. Proposition 6 below then states that it is optimal for the DM to set the reference level above (below) the actual consumption level in the same period. Moreover, absent new shocks, the DM is going to continue consuming below (above) the reference point until some finite period τ after which she will consume at the reference level in all remaining periods.

Proposition 6 *Let $\beta \in (\beta^*, 1)$ as in Proposition 5. There exist ($\eta^* \in (0,1)$,) $\omega_t^* > 0$ and $r_{t-1}^* > 0$ such that, in any optimal plan, $x_t < r_t$ ($x_t > r_t$) whenever ($\eta \in (\eta^*, 1)$,) $\omega_t < \omega_t^*$ ($\omega_t > \omega_t^*$) and $r_{t-1} > r_{t-1}^*$ ($r_{t-1} < r_{t-1}^*$). Moreover, $x_{t'} < r_{t'}$ for $t' \leq \tau < \infty$ and $x_{t'} = r_{t'}$ for $t' > \tau$.*

To get the intuition behind the result, assume that the reference level is at the consumption level in each period given any wealth level, ω_t, and any inherited reference level, r_{t-1}. Moreover, assume for simplicity that ϕ is piecewise linear. I consider the case of a negative wealth shock (that is, the

DM finds herself poorer than expected). Then the optimal reference point in period t is given by

$$\delta\lambda\eta u'(r_t) + \beta(1-\delta)u'(a_{t+1}) \geq u'(a_t) \geq \delta\eta u'(r_t) + \beta(1-\delta)u'(a_{t+1})$$
$$\Leftrightarrow \lambda\eta u'(r_t) \geq u'(a_t) \geq \eta u'(r_t) \quad (12)$$

according to Proposition 5 since $u'(a_t) = \beta u'(a_{t+1})$. Notice that, due to the kink in ϕ, it is optimal for the DM to adjust a_{t+1} to any change in a_t so that consumption is at the reference level. Hence the term $\beta(1-\delta)u'(a_{t+1})$.

As a higher wealth level implies a higher consumption level in any period $t' \geq t$, if the wealth shock is large r_{t-1} can be large relative to the optimal c_t given ω_t. Since marginal utility is diminishing and the assumption on the curvature of u, there exists a $r_{t-1} > 0$ such that $\lambda\eta u'(\delta a_t + (1-\delta)r_{t-1}) < u'(a_t)$ when $r_t > c_t$ (since we can make the optimal c_t arbitrary low relative to r_{t-1} by decreasing ω_t). Thus, condition (12) does not hold for $r_t = c_t$. Heuristically, the DM cares about smoothing anticipatory utility as u is concave, therefore she finds it optimal to consume below her reference level to even out her anticipation levels over time.

As the gain-loss function is kinked at the origin, reference point adjustment is bounded above zero. By contrast, consumption smoothing implies that the DM tends to even out her consumption level over time. Therefore, the reference level is bound to catch up with the consumption level within a finite period of time.

There are two caveats regarding Proposition 6. First, if the initial reference level is low, there might not exist wealth shocks such that the DM has to set the consumption level below the reference level. This happens when the reference level is close to zero and marginal utility is bounded as the consumption level tends to zero. The reason is that there might not exist large enough wealth shocks leaving the DM with a positive net wealth. Secondly, for positive wealth shocks, the result depends on that the salience parameter, η, is small enough relative to δ.

To illustrate the second caveat, I will give a parametric example. Let $u(x) = ln(x)$, ϕ be piecewise linear, and $\delta \in (\eta, 1)$. I will show that, no matter the size of the positive wealth shock, the always prefers equating the reference point with the consumption level in any optimal plan when there is no remaining uncertainty. First, notice that condition (12) boils down to $r_t \geq \eta a_t$. However, this implies that $\delta a_t + (1-\delta)r_{t-1} \geq \eta a_t$ which is always true.

What happens here is that the marginal cost of a higher reference point, which depends on η, is always lower than the marginal gain from increasing the anticipation level when the speed of adjustment, δ, is large.

4.3. Preferences for Increasing Consumption Streams

Two influential surveys by Loewenstein and Sicherman (1991) and Loewenstein and Prelec (1993) (later corroborated by a number of studies, see, e.g. Duffy and Smith (2013) and the references therein) has found that people can exhibit an ex ante preference for increasing consumption profiles. I here show that the ORD model with adaptive reference points can help rationalize this behavior. The model can also help predict when one should expect it to occur.

The argument builds on the idea by Kahneman and Tversky (1979) that the gain-loss function exhibits diminishing sensitivity. In their theory, the gain-loss function is concave over gains and convex over losses. This implies that the DM gets increasingly insensitive to consumption above and below the reference point as their difference increases. This property is shared by many other sensory and perceptual dimensions such as vision (Kahneman and Tversky, 1979). The ORD utility function captures these features (for small changes such that u is approximately linear) if an additional assumption is imposed on ϕ:

A.4 Reflection: ϕ is strictly concave for $y > 0$ and strictly convex for $y < 0$.

This assumption was first stated by Bowman et al. (1999). Since $\phi(y) > 1$ for $y < 0$, it must be the case that ϕ' becomes approximately linear for large losses.[12] The implication of assumption A.1-A.4 is highlighted in Figure 2 below.

[12] "The latter case [concavity in the loss region] may be more common since large losses often necessitate changes in life style." (Kahneman and Tversky, 1979, p.278)

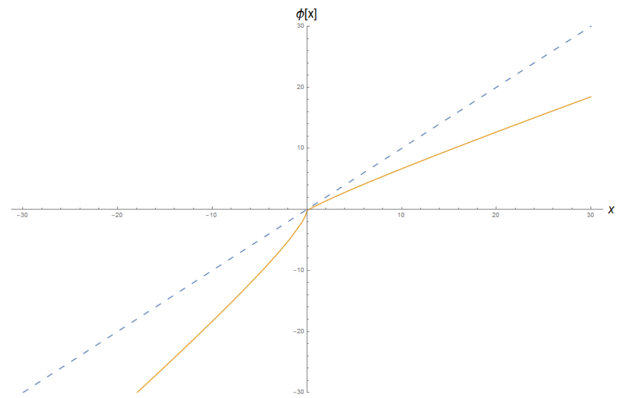

Figure 2: The value function $\phi(x)$ (solid) must lie below the 45-degree line (dashed) according to Assumption A.1-A.4.

The implication of this is that—if the DM already knows she will be consuming below (above) her reference level—it is beneficial to increase consumption in periods in which the reference level is close to the actual consumption level, since it is then the difference is the most noticeable. This implies that, if the DM is sufficiently patient, she prefers to lower her consumption after any wealth shock and gradually increasing it as the reference level is adjusting to the consumption level.

Corollary 7 *Given a wealth shock as in Proposition 6. Then consumption in any period $t \leq t' \leq \tau - 1$ is given by*

$$u'(x_{t'})\phi'(u(x_{t'}) - u(r_{t'})) = \beta u'(x_{t'+1})\phi'(u(x_{t'+1}) - u(r_{t'+1})).$$

Moreover, there exists $\beta^* \in (0, 1)$ such that if ϕ satisfies assumption A.4 then every consumption level $x_{t'}$ is increasing in t' for $t \leq t' \leq \tau$ whenever $\beta \in (\beta^*, 1)$.

Notice that Proposition 6, and Corollary 7 together indicate when such preferences over increasing consumption profiles should be observable, namely, when they are unexpected and the reference point is sufficiently sticky. Arguably, the former is the case in the hypothetical settings in the experiments where this behavior has been observed.

Corollary 7 can also be used to determine whether ϕ is convex for losses as proposed by prospect theory. If it is, one would observe a increasing consumption level just following a negative and unexpected wealth shock as opposed to, for example, decreasing consumption levels as would be the case if ϕ is concave for losses.

5. An Endowment Effect for Risk

An essential feature of the Kőszegi and Rabin (2006) model is that it can generate an endowment effect, not only for deterministic outcomes, but also for risk. To capture this behavior using the ORD model, the framework for the baseline model has to be extended to allow ex post choices to be explicitly modeled. In this framework, I consider the DM's ex post preference over lotteries as represented by a gain-loss function *given* some reference point. As I will show, an implication of the ORD model is that if the DM's reference point r^μ is optimal given some lottery μ, then she maximizes utility by choosing μ ex post over some other lottery ν, even though she would have preferred ν over μ if she expected to face it in ex ante (that is, given the optimal reference point r^ν).

The above situation can formally be stated as a surprise situation in which the DM ex ante expects to face a given lottery almost surely but where there is a vanishingly small probability that she will instead face choice between a set of lotteries. The situation described above is the one in which the choice situation happens to occur. Proposition 8 below states that the DM, in the ex post stage, is no more willing to accept a lottery μ on top of some wealth level x when her reference point is x, then she is to accept μ when she is already facing a lottery ν given any reference point y. Letting $y = r^\mu$ be the optimal reference point when facing μ with $\nu = \delta_x$, it is evident that the proposition implies an endowment effect for risk.

Proposition 8 *Suppose that ORD utility holds, ϕ is given by (6) and u is linear. For any lotteries μ and ν and any $x, y \in \mathbb{R}$, if $U_x(\mu) \geq U_x(x)$, then $U_y(\mu + \nu) \geq U_y(\nu)$.*

Proposition 8 establishes that stochasticity of the reference point is not necessary to generate an endowment effect for risk. The possibility of this behavior is described in Bleichrodt et al. (2001) who studies a similar utility function as the ORD function, albeit with the reference point given exogenously. Sprenger (2015) erroneously argues that this feature cannot be reconsolidated within the prospect theory framework with a deterministic reference point.

6. Relationships with Established Models

In this section, I will compare the ORD utility model with already established models in the literature.

6.1. Optimal Anticipation Models

There are two other models building on optimized risk preferences. The first model is Gollier and Muermann (2010) in which the DM trades off utility from ex ante anticipation with disutility from ex post disappointment. Let U denote some reference-dependent utility function $U : \mathbb{R}^2 \to \mathbb{R}$. As for the two-stage ORD model, the DM faces a lottery ex ante that resolves ex post. In contrast to my model, after choosing a lottery and before it has resolved, the DM forms subjective expectations regarding the outcome of the lottery.

Formally, if the DM faces a simple lottery f (i.e. a lottery with finite support) from the set $\Delta(S)$ of all the simple lotteries with support in the set $S \subset \mathbb{R}$, she can modify her expectations so that she believes that she is facing any simple lottery $g \in \Delta(S)$. These subjective probabilities, in turn, determine her reference point r by the implicit relationship $U(r, r) = \sum_x U(x, r)g(x)$. The DM distorts her beliefs about the lottery and her anticipation level to maximize the sum of anticipatory utility (weighted by some 'vividness factor' $k > 0$) and utility from actual consumption:

$$V_{GM}(f) = \max_{g \in \Delta(S), r \in S} k \sum_x U(x, r)g(x) + \sum_x U(x, r)f(x)$$
$$\text{subject to} \quad U(r, r) = \sum_x U(x, r)g(x) \quad (13)$$

In addition to the authors focusing on different aspects of reference point formation, there are two additional important differences between the two functions. First, the ex post utility function used in ORD model cannot be used for the representation defined by (13) as it is equal to zero when the outcome and the reference point coincide. Therefore, U has to include consumption utility that is independent of gain-loss utility. Second, the authors assume that U is twice differentiable everywhere, ruling out loss aversion.

A recent paper that is closely related to the present paper is Sarver (2017). The author axiomatize a strictly larger class of ex ante preferences for an infinite horizon model in the recursive setting of Epstein and Zin (1989). In an earlier working paper, the author discusses the interpretation explored in this paper, namely, as optimized reference-dependent preferences. However, his paper studies stationary preferences whereas this paper's focus is on nonstationary preferences. Many of the properties derived for the ORD utility class is based on results developed in Sarver's paper.

Another important difference between the ORD model with adaptive reference points and subclasses of Epstein and Zin (1989) recursive preferences studied by, e.g., Sarver (2017) and Artstein and Dillenberger (2015), is that the DM is sometimes information averse. For example, Artstein and Dillenberger show that splitting a lottery into several stages, such that information about the final outcome is released over time, reduces its value. The survey on information avoidance by Golman et al. (2017) criticizes these models on empirical grounds. By contrast, the ORD model does not feature preferences for information avoidance.

6.2. Prospect Theory

The ex post preferences considered in this paper generalizes the value function in prospect theory Kahneman and Tversky (1979) except that it requires that the derivatives at the kink exist and are finite.[13] Kahneman and Tversky assumes that the reference point is the status quo, so the value function v can be expressed as $U_0(x) = \phi(u(x) - u(0))$.[14]

The two-stage ORD function can be combined with nonlinear decision weights à la cumulative prospect theory (Tversky and Kahneman, 1992). Adding a probability weighting function that is independent of the reference point is straightforward, and the reference point is then determined with

[13]See the discussion in Köbberling and Wakker (2005, p.127-128) highlighting the inappropriateness of letting the derivate of the gain-loss function go to infinity at the kink.

[14]Kahneman and Tversky acknowledge that something along the lines of my generalization is appropriate: "The emphasis on changes as the carriers of value should not be taken to imply that the value of a particular change is independent of initial position. Strictly speaking, value should be treated as a function in two arguments: the asset position that serves as reference point, and the magnitude of the change (positive or negative) from that reference point." (Kahneman and Tversky, 1979, p.275)

respect to the transformed probabilities.[15] In this paper, I have chosen to completely ignore this important part of prospect theory to keep the focus on the impact of loss aversion relative to some endogenously formed reference point.

6.3. Kőszegi and Rabin Reference-Dependent Utility

I will also compare the ORD function with the model developed in Kőszegi and Rabin (2006, 2007, 2009), henceforth KR. The model features reference-dependent preference with endogenously given, and possibly stochastic, reference points. The KR model, in combination with the multi-period extension developed in Kőszegi and Rabin (2009), has become the workhorse model in the applied literature that models reference-dependent preferences with loss aversion.

I will now present a special case of the KR model called *Choice-Acclaiming Personal Equilibria* (Kőszegi and Rabin, 2007), or CPE, as formulated by Masatlioglu and Raymond (2016). This model is equivalent to the general KR model in the two-stage setting described above for the baseline ORD model. As will be highlighted below, the CPE function shares many similarities with the ORD function. This is especially clear when the gain-loss function is piecewise linear. A utility function V_{CPE} is a CPE function if there exists a differentiable and strictly increasing function u and ϕ given by equation (6) is piecewise linear with $\lambda \in \mathbb{R}$ and $\eta = 1$, such that

$$V_{CPE}(\mu) = \int u(x)d\mu(x) + \int\int \phi(u(x) - u(r))d\mu(x)d\mu(r). \tag{14}$$

As shown by Masatlioglu and Raymond (2016, Theorem 1, p.2766), a preference relation is represented by CPE with $\lambda \in [0, 2]$ if and only if it can be represented by both rank-dependent utility and quadratic utility (Chew et al., 1991). The probability weighting function w is then given by $w(x) = (2 - \lambda)x + (\lambda - 1)x^2$. Figure 3 below highlights similarities between the ORD and CPE functions when the gain-loss function is piecewise linear.

[15] It is less straightforward to add a weighting function changes depending on whether the outcome is above or below the reference point as in Tversky and Kahneman (1992). Moreover, it is also unclear how to do so in the dynamic model as the DM might have time-inconsistent preference in the presence of probability weighting.

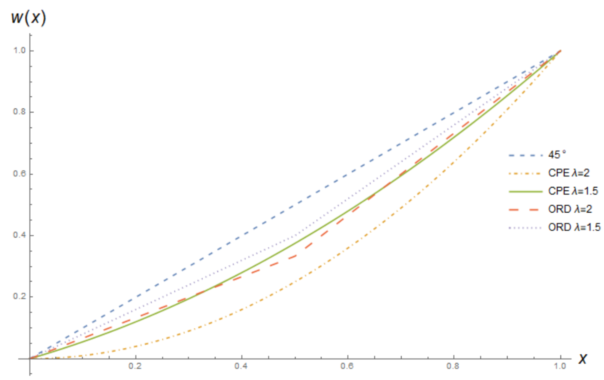

Figure 3: The ORD and CPE weighting functions with η set such that the kink is at the median.

Although the ORD model shares many similarities with the KR model, they differ in significant ways. I will outline four such differences. Firstly, as I have shown, ORD utility is monotone with respect to first order stochastic dominance. By contrast, the KR model is not for empirically identified values of λ. Secondly, in KR's model, for any sufficiently risky lottery μ, the DM is approximately risk neutral ex post if she was expecting to face μ. This is the case even when she, given the reference point induced by μ, ex post *knows* that she will not face it. The ORD model, by contrast, may assign the same optimal reference point to two lotteries that differ substantially in terms of riskiness. For example, when ϕ is piecewise linear the $\frac{1-\eta}{(\lambda-1)\eta}$-percentile of the lottery solely determines the reference point.

Thirdly, since the reference point in the KR model is determined by expectations of the induced probability distribution over wealth levels, the model requires a consistency condition to be fully specified. Kőszegi and Rabin (2006) introduced the concepts of personal equilibrium, or PE, and preferred personal equilibrium, or PPE. The former states that, given a stochastic reference point that is determined by her rational expectations of what she will do in the future, the expected behavior must be optimal for the DM (otherwise she would not make this choice). That is, the DM cannot 'surprise' herself. However, this formulation allows the existence of another choice that, if she had expected it, she would prefer making this choice over the former. Both of these choices would then be PEs, possible giving very different utilities. PPE is then defined as the PE that maximizes the utility of the decision maker. This consistency requirement leads to time-inconstant preferences as in the dynamic model by Kőszegi and Rabin (2009).

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Appendix A. Proofs of Propositions

Proof of Proposition 1. For sufficiency, assumption A.2 implies that it possible to write

$$W(\mu, r) = c(r) + k(r)\int \phi(u(x) - u(r))d\mu(x)$$

where c and k are two continuous and differentiable functions of r. Since assumption A.3 holds and $\lim_{y\to 0}\phi'(|y|) = \lim_{y\to 0}\phi'(-|y|)$, it must be the case that $\frac{\partial W(x,r)}{\partial r} = 0$ when $r = x$ if and only if $c'(r) = u'(r)k(r)\phi'(u(x) - u(r))$. Since deterministic choice ex ante is given by a utility function u by assumption A.1, it must be the case that $c'(r) = u'(r)$ since $\frac{\partial W(r,r)}{\partial r} = c'(r)$. This implies that $\frac{\partial W(x,r)}{\partial r} = u'(r) - u'(r)k(r)\phi'(u(x) - u(r))$ and $1 = k(r)\phi'(0) = k(r)$ meaning that $k(r) = 1$ for all $r \in \mathbb{R}$. Finally, notice that Assumption A.3 implies that $\frac{\partial\phi(y)}{\partial y} \leq 1$ for $y > 0$ and $\frac{\partial\phi(y)}{\partial y} \geq 1$ for $y < 0$ since $\frac{\partial W(x,r)}{\partial r} = u'(r)(1 - \frac{\partial U_r(x)}{\partial u(r)})$.

For necessity, it is immediate that the utility function given by equation (3) fulfills assumption A.1-A.3. □

Proof of Proposition 2. For statement 1., 2., and 4., Theorem S.1 (Sarver, 2017, p.2, Supplementary Appendix) implies that if $W : \Delta(X) \to \mathbb{R}$ for some compact metric space X (where $\Delta(X)$ is the set of Borel probability measures over X endowed with the topology of weak convergence), and suppose $\mathcal{C}$ is a set in the space of real-valued continuous functions $C(X)$. Then the following are equivalent: (i) V is is lower semicontinuous in the topology of weak convergence, convex, and monotone with respect to the same order of stochastic dominance in $\mathcal{C}$, and (ii) there exists a subset Φ of the closed convex hull of the set of all affine transformations of functions in $\mathcal{C}$ such that $V(\mu) = \sup_{\rho\in\Phi}\int \rho(x)d\mu(x)$. Since $\phi(u(x) - u(r))$ is increasing in x for a fixed r, this is indeed true. It is easy to see that V is concave if and only if both u and ϕ are concave.

For statement 3., notice that $\frac{\partial}{\partial r}U_r(\delta_x) = u'(r)\left(1 - \phi'(u(x) - u(r))\right)$. This is nonpositive if $r > x$ and nonnegative if $r < x$. If the lottery is nondegenerate, since U_r is linear in probabilities $\frac{\partial}{\partial r}U_r(\mu) < 0$ if $r > \bar{x}$ and $\frac{\partial}{\partial r}U_r(\mu) > 0$ if $r < \underline{x}$ where $\mu([y, z]) = 0$ if $y > \bar{x}$ or $z < \underline{x}$. □

Proof of Proposition 3. The risk premium is given implicitly by

$$u(\omega - \pi(\delta_\omega + \varepsilon\mu)) = u(r_\varepsilon) + \int \phi(u(\omega + \varepsilon x) - u(r_\varepsilon))d\mu(x). \qquad (15)$$

The optimal reference point is a function of ε as the (generically unique) solution to

$$\lim_{r\to r_\varepsilon^-}\int \phi'(u(\omega+\varepsilon x)-u(r))d\mu(x)\geq 1\geq \lim_{r\to r_\varepsilon^+}\int \phi'(u(\omega+\varepsilon x)-u(r))d\mu(x). \tag{16}$$

If both sides holds with equality any small impact of ε is of second order and, thus, has an negligible effect on r_ε. If not, it is optimal to set the reference point to $r_\varepsilon=\omega+\varepsilon x^*$ for some atom x^* of μ. By differentiating equation (15) then we get

$$\begin{aligned}\frac{\partial\pi(\delta_\omega+\varepsilon\mu)}{\partial\varepsilon} &= -(u'(\omega-\pi(\delta_\omega+\varepsilon\mu))^{-1}\Bigg[x^*u'(r_\varepsilon)\left[1-\int\phi'(u(\omega+\varepsilon x)-u(r))d\mu(x)\right]\\ &\quad+\int\phi'(u(\omega+\varepsilon x)-u(r))u'(\omega+\varepsilon x)xd\mu(x)\Bigg] \end{aligned}\tag{17}$$

Let ε tend to zero from above. Then, if equation (16) holds with equalities, the first two terms are zero and $\frac{\partial\pi(\delta_\omega+\varepsilon\mu)}{\partial\varepsilon}|_{\varepsilon=0^+}=[\lambda-1]\eta\int_{x>0}xd\mu(x)$. If not, we have

$$\begin{aligned}\frac{\partial\pi(\delta_\omega+\varepsilon\mu)}{\partial\varepsilon}|_{\varepsilon=0^+} &= -x^*+x^*\eta\int_{x>x^*}d\mu(x)+x^*\lambda\int_{x\leq x^*}d\mu(x)\\ &\quad-\eta\int_{x>x^*}xd\mu(x)-\lambda\eta\int_{x\leq x^*}xd\mu(x).\end{aligned}\tag{18}$$

There are two cases, either $x^*>0$ or $x^*<0$. I will start with the former case. The RHS of equation (18) can be reordered to give

$$\frac{\partial\pi(\delta_\omega+\varepsilon\mu)}{\partial\varepsilon}|_{\varepsilon=0^+}=x^*(\lambda\eta-1)-x^*(\lambda-1)\eta\int_{x>x^*}d\mu(x)+(\lambda-1)\eta\int_{x>x^*}xd\mu(x)>0$$

Finally, when $x^*<0$ the RHS of equation (18) can be reordered to give

$$\frac{\partial\pi(\delta_\omega+\varepsilon\mu)}{\partial\varepsilon}|_{\varepsilon=0^+}=-x^*(1-\eta)-x^*(\lambda-1)\eta\int_{x\leq x^*}d\mu(x)+(1-\lambda)\eta\int_{x\leq x^*}xd\mu(x)>0$$

□

Proof of Proposition 4. When ϕ is given by (6), the indirect utility given by equation (1) is obtained finding the maximum $r\in\mathbb{R}$ of

$$W(\mu,r)=u(r)+\int_{x\leq r}\lambda\eta[u(x)-u(r)]dF_\mu(x)+\int_{x>r}\eta[u(x)-u(r)]dF_\mu(x).$$

Differentiating with respect to r gives the first order condition

$$1 = \int_{x \leq r} \lambda \eta dF_\mu(x) + \int_{x > r} \eta dF_\mu(x)$$
$$\Leftrightarrow \frac{1-\eta}{(\lambda-1)\eta} = \int_{x \leq r} dF_\mu(x) = F_\mu(r).$$

$u(x)$ is strictly increasing, and so is its inverse, therefore the objective function is first increasing in r and then decreasing implying that the FOC equal to 0 is necessary and sufficient for the optimum.

By the above, the optimal reference point r^μ given a lottery μ is such that $\lim_{x \to r^\mu -} F_\mu(x) \leq \frac{1-\eta}{(\lambda-1)\eta} \leq F_\mu(r^\mu)$. To show that the ORD representation is a special case of RDU notice that, given an optimal reference point r, $V(\mu)$ is equal to

$$u(r^\mu) + \int_{x \leq r^\mu} \lambda \eta (u(x) - u(r^\mu)) dF_\mu(x) + \int_{x > r^\mu} \eta (u(x) - u(r^\mu)) dF_\mu(x)$$
$$= u(r^\mu) \left[1 - (\lambda - 1)\eta F_\mu(r^\mu) - \eta\right] + \int_{x \leq r^\mu} u(x) d\lambda \eta F_\mu(x) + \int_{x > r^\mu} u(x) d\eta F_\mu(x)$$
$$= u(r^\mu) \left[1 - (\lambda - 1)\eta F_\mu(r^\mu) - \eta\right] + u(r^\mu) \left[\lambda \eta F_\mu(r^\mu)) - \lambda \eta \lim_{x \to r^\mu -} F_\mu(x)\right]$$
$$+ \int_{x < r^\mu} u(x) d(w \circ F_\mu)(x) + \int_{x > r^\mu} u(x) d(w \circ F_\mu)(x)$$
$$= u(r^\mu) \left[1 - \eta + F_\mu(r^\mu) - \lambda \eta \lim_{x \to r^\mu -} F_\mu(x)\right] + \int_{x < r^\mu} u(x) d(w \circ F_\mu)(x) + \int_{x > r^\mu} u(x) d(w \circ F_\mu)(x)$$
$$= u(r^\mu) \left[w(F_\mu(r^\mu)) - w\left(\lim_{x \to r^\mu -} F_\mu(x)\right)\right] + \int_{x < r^\mu} u(x) d(w \circ F_\mu)(x) + \int_{x > r^\mu} u(x) d(w \circ F_\mu)(x)$$
$$= \int u(x) d(w \circ F_\mu)(x),$$

where w is defined as in the proposition. □

Proof of Proposition 5. Since the problem is recursive (except for the first period which is solved in a similar manner) and the optimal reference point can wlog be restricted to a compact set given any initial wealth level $\omega > 0$, the contraction mapping theorem implies that there is a unique value function in the optimal plan.

As the envelope theorem does not apply when ϕ has a kink and consumption is at this level, the derivation of the Euler equation is a bit involved. If the DM increases consumption in period t and takes it from period $t+1$, the optimal reference point in that period increases by δ^{-1}. The net effect is then:

$$\eta u'(c_t) - \delta^{-1}\delta\eta u'(r_t) + \delta^{-1}u'(a_t) = \delta^{-1}u'(a_t) \tag{19}$$

using that $r_t = c_t$. But then it is optimal to reduce the reference point in period $t+1$ by $\delta^{-1}(1-\delta) + \delta^{-1}$ implying a net effect:

$$\begin{aligned}
&\underbrace{-\beta\lambda\eta u'(c_{t+1}) + \delta^{-1}\delta\beta\lambda\eta u'(r_{t+1}) - \delta^{-1}\beta u'(a_{t+1})}_{\text{Optimal response due to the change in consumption in period } t+1} \\
&\underbrace{+\delta^{-1}(1-\delta)\delta\beta^2\eta u'(r_{t+2}) - \delta^{-1}(1-\delta)\delta\beta^2\eta u'(r_{t+2}) + \delta^{-1}(1-\delta)\beta^2 u'(a_{t+2})}_{\text{Optimal response to change in anticpation in period } t+1 \text{ to the change in consumption in period } t+1} \\
&\underbrace{-\delta^{-1}(1-\delta)\delta\beta\lambda\eta u'(r_{t+1}) + \delta^{-1}(1-\delta)\delta\beta\lambda\eta u'(r_{t+1}) - \delta^{-1}(1-\delta)\beta u'(a_{t+1})}_{\text{Optimal response due to the change in anticipation level in period } t} \\
&= -\delta^{-1}\beta u'(a_{t+1}) - \delta^{-1}(1-\delta)\beta u'(a_{t+1}) + \delta^{-1}(1-\delta)\beta^2 u'(a_{t+2})
\end{aligned} \tag{20}$$

since in period $t+2$ the reference point is optimally decreased by $\delta^{-1}(1-\delta)$. This gives:

$$u'(a_t) = \beta u'(a_{t+1}) + (1-\delta)\beta\left(u'(a_{t+1}) - \beta u'(a_{t+2})\right) \tag{21}$$

The solution to equation (21) is then given by $u'(a_t) = \beta u'(a_{t+1})$ if the consumption level is at the reference point in each period. This is readably verifiable to also be the case in period 1.

The DM's incentives to change the reference point is then (writing ϕ_t' for $\phi'(u(c_t) - u(r_t))$

$$u'(a_t) - \delta u'(r_t)\phi_t' - (1-\delta)\beta u'(a_{t+1}) \tag{22}$$

since it is optimal in period $t+1$ adjust the reference point to equate r_{t+1} with c_{t+1}. Therefore, she has incentives to equate the reference point with the consumption level in period t if $u'(a_t) \leq \lambda\eta u'(r_t)$ and $u'(a_t) \geq \eta u'(r_t)$. It is only the first inequality that is problematic as $a_t < c_t$ for all $t > 0$. Thus, we need to show that there exists a $\underline{\beta} \in (0,1)$ such that $u'(a_t) \leq \lambda\eta u'(r_t)$ in any plan with $u'(a_t) = \beta u'(a_{t+1})$ for $\beta \in (\underline{\beta}, 1)$.

To this end, define $f^1(x,\beta) = u'^{-1}(u'(x)\beta)$, $f^2(x,\beta) = f^1(f^1(x,\beta),\beta)$, and so on. Since intertemporal elasticity of substitution is bounded from above and from below by a number larger than 1, there exist scalars $\bar{f}(\beta)$ (although only when $c \to \infty$) and $\underline{f}(\beta)$ such that $\underline{f}(\beta)\cdot x \leq f^1(x,\beta) \leq \bar{f}(\beta)\cdot x$ for each $\beta \in (0,1)$. Moreover, as β tends to unity (from below) both $\underline{f}(\beta)$

and $\bar{f}(\beta)$ tend to unity from above.

Since $\lambda\eta u'(r_1) \geq u'(a_1)$ and we have

$$u'(r_{t+1}) = u'(\delta[a_{t+1} + (1-\delta)f^1(a_{t+1},\beta) + \ldots + (1-\delta)^t f^t(a_{t+1},\beta)])$$
$$\geq u'(\delta[a_{t+1}+(1-\delta)\bar{f}(\beta)\cdot a_{t+1}+\ldots+(1-\delta)^\infty \bar{f}(\beta)^\infty \cdot a_{t+1}]) = u'\left(\frac{\delta a_{t+1}}{1-(1-\delta)\bar{f}(\beta)}\right). \tag{23}$$

What is left to show is that $u'(a_{t+1}) \leq \lambda\eta u'\left(\frac{\delta a_{t+1}}{1-(1-\delta)\bar{f}(\beta)}\right)$ for all a_{t+1}. This holds if

$$f^1(a_{t+1},(\lambda\eta)^{-1}) \geq \underline{f}((\lambda\eta)^{-1}) \cdot a_{t+1} \geq \frac{\delta a_{t+1}}{1-(1-\delta)\bar{f}(\beta)} \tag{24}$$

which is true for any $a_{t+1} > 0$ if $\bar{f}(\beta)$ is close to 1. Thus, for any $\beta \in (\underline{\beta}, 1)$, where $\underline{\beta} \in (0,1)$ such that inequality (24) holds, it is optimal for the reference level to be at the consumption level in each period. □

Proof of Proposition 6*.* Assume as in Proposition 5 that the reference point is at the consumption level in each period. This is optimal in period t if $\phi'(x)u'(r_t) \geq u'(a_t) \geq \phi'(y)u'(r_t)$ for $x < 0 < y$. Since the consumption level in each period is monotonically increasing in the wealth level that the DM inherits from the last period, changing the wealth level is in principle the same as changing the consumption level in the present period. Thus, if r_t is large enough and ω_t is small enough we have $u'(a_t) = \lambda\eta u'(r_t)$ for $r_t > c_t$ (we need only to be concerned with $\phi' = \lambda\eta$ since $\phi(0^+) = \lambda\eta$). To see that this is possible, notice that $\bar{f}((\lambda\eta)^{-1})\cdot a_t \geq f(a_t,(\lambda\eta)^{-1}) = r_t$ ($\bar{f}((\lambda\eta)$ exists since we can make r_t arbitrary large) which implies that $r_t \geq \frac{\delta}{\bar{f}((\lambda\eta)^{-1})}r_t + (1-\delta)r_{t-1} \Leftrightarrow r_{t-1} \geq r_t > \left(1-\frac{\delta}{\bar{f}((\lambda\eta)^{-1})}\right)^{-1}(1-\delta)r_{t-1} > 0$ since $\bar{f}(x) > 1$ for all $x \in (0,1)$. Thus, if ω_t is close to zero, $c_t < \left(1-\frac{\delta}{\bar{f}((\lambda\eta)^{-1})}\right)^{-1}(1-\delta)r_{t-1}$ since r_{t-1} can be arbitrary large.

For situations when $r_t < c_t$ matters is slightly more complicated. It is possible to fix r_t and let ω_t be large enough so that $u'(a_t) = \eta u'(r_t)$ for $r_t > c_t$ but this will only work if η is close to 1. To see this, note that $r_t \cdot \bar{f}(\eta) \geq f(r_t,\eta) = a_t$ which gives $r_t \leq \delta\bar{f}(\eta)r_t + (1-\delta)r_{t-1} \Leftrightarrow 0 < r_{t-1} < r_t < \left(1-\delta\bar{f}(\eta)\right)^{-1}(1-\delta)r_{t-1}$ if $\delta\bar{f}(\eta) < 1$. Since $\bar{f}(\eta) \to 1$ from above as $\eta \to 1$ such an η exists for any $\delta \in (0,1)$.

First, assume that the wealth shock is positive. The candidate that I will verify has $a_{t'} > r_{t'}$ for each period $\tau \geq t' \geq t$ and thereafter $u'(a_{t'}) = \beta u'(a_{t'+1})$ with $a_{t'} < r_{t'}$. For each period $t < t'+1 \leq \tau$, consumption is given by $\phi'(u(c_{t'}) - u(r_{t'}))u'(c_{t'}) = \beta\phi'(u(c_{t'+1}) - u(r_{t'+1}))u'(c_{t'+1})$ and $\phi'(u(c_\tau) - u(r_\tau))u'(c_\tau) = \delta^{-1}\beta u'(a_{\tau+1})$ where the latter follows from equation (20). Note that the reference-point adjustment in period $t \leq \tau$ is of second order so, by the envelop theorem, the marginal utility from changing consumption is giving by the direct effect only.

Given the hypothesis of the optimal plan, it is optimal to set $a_t > r_t$ for all $t \leq \tau$ as

$$u'(a_t) > (\delta + (1-\delta)\beta)u'(a_t) \geq \delta u'(a_t) + \delta(1-\delta)\beta u'(a_t) + \ldots$$
$$+\delta[(1-\delta)\beta]^{\tau-t}u'(a_t) + \delta(1-\delta)^{\tau-t+1}\beta^{\tau-t}u'(a_t) + \ldots + \delta[(1-\delta)]^{\infty}\beta^{\tau-t}u'(a_t)$$

since $\phi'(y) \leq 1$ for $y > 0$ and $a_t = r_t < r_{t+t'}$ for any $\tau - t \geq t' \geq 1$ and $u'(a_{t'}) = \beta u'(a_{t'+1})$ for $t' \geq \tau$.

Since, wealth is bounded consumption has to decrease at some point. Therefore, if the reference point adjustment upwards is bounded from below, there must exist a finite period τ in which $c_\tau = r_\tau$. To get this lower bound, notice that $u'(a_t) \geq \delta\eta u'(r_t)$. Since intertemporal elasticity of substitution is bounded from below, it follows that $a_t \leq \underline{f}(\eta\delta)r_t$ implying that $r_t \leq \delta\underline{f}(\eta\delta)r_t + (1-\delta)r_{t-1}$ or $r_t \leq \frac{1-\delta}{1-\delta\underline{f}(\eta\delta)}r_{t-1}$. In period τ in which $c_\tau = r_\tau$ for the first time, it is optimal for the DM to keep the reference level at the consumption level since

$$\delta\eta u'(r_\tau) + (1-\delta)\beta u'(a_{\tau+1}) \geq u'(a_\tau) \geq \delta\lambda\eta u'(r_\tau) + (1-\delta)\beta u'(a_{\tau+1})$$
$$\Leftrightarrow \eta u'(r_\tau) \geq u'(a_\tau) \geq \lambda\eta u'(r_\tau)$$

since $u'(a_\tau) = \beta u'(a_{\tau+1})$, where the first inequality holds by assumption and the second because $a_\tau > r_\tau$.

For negative wealth shocks, the outline of the proof is similar. I will verify that the optimal plan has $a_{t'} < r_{t'}$ for each period and in any period $t' > \tau$ $c_t = r_t$ with $u'(a_{t'}) = \beta u'(a_{t'+1})$. For each period $t < t'+1 \leq \tau$, consumption is given by the same equation as above for positive wealth shocks. The reference point is decreasing in each period by the argument in Proposition 5.

First thing to do is find a lower bound on reference point adjustment when $r_t > c_t$. This FOC (that is now necessary and sufficient for optimality) since we are not at the kink of ϕ) is given by (with $\phi'(u(c_t) - u(r_t) = \phi_t)$

$$u'(a_t) \geq \delta\phi_t u'(r_t) + \ldots + \delta[(1-\delta)\beta]^{\infty}\phi_{\infty}u'(r_{\infty}) \geq \frac{\delta \min_{t' \geq t} \phi_{t'} u'(r_t)}{1-(1-\delta)\beta} = \gamma(\beta)^{-1}u'(r_t), \tag{25}$$

where the first weak inequality stems from that the DM could do better by adjusting the reference point in period τ and the second from that $r_{t'} > r_{t'+1}$ for $t' \geq t$. Since $\phi(y) > 1$ for $y < 0$, there exists a $\beta < 1$ such that $\gamma(\beta) < 1$. Thus, we have $\underline{f}(\gamma(\beta)) \cdot a_t = r_t \Leftrightarrow r_t = \frac{1-\delta}{1-\delta/\underline{f}(\gamma(\beta))} r_{t-1}$ and the reference point is decreasing with a rate that is bounded from below. Consumption is given by $u'(c_{t'}) = \beta\frac{\phi_{t+1}}{\phi_t}u'(c_{t'+1})$ when both r_t and c_t are small, $\phi_t \approx \phi_{t+1}$. Moreover, either there is some period t in which $c_t = 0$ or c_t and c_{t+1} are becoming arbitrary close to each other. Thus, since r_t is decreasing with a rate bounded from below, there exists a finite period τ in which $c_\tau = r_\tau$. □

Proof of Corollary 7. What needs to be done is to show that $\phi'(u(x_{t'}) - u(r_{t'})) < \beta\phi'(u(x_{t'+1}) - u(r_{t'+1}))$ for all $t \leq t' \leq \tau - 1$. Since the reference point is closing in on the consumption level in finite time and we can pick our β^* given a wealth shock, this is indeed possible to do as ϕ is either strictly concave or convex. □

Proof of Proposition 8. The DM prefers $y + \mu$ over μ when the reference point is y if and only if

$$\int \phi(x)dF_\mu(x) \geq 0.$$

She prefers $\mu + \nu$ over ν when the reference point is z if and only if

$$\int\int \phi(x + x' - z)dF_\mu(x)dF_\nu(x') \geq \int \phi(x' - z)dF_\nu(x').$$

or equivalently as

$$\int\int(\phi(x + x' - z) - \phi(x' - z))dF_\mu(x)dF_q(x') \geq 0 \tag{26}$$

As observed by Kőszegi and Rabin (2007, Proposition 1.) for $x \geq 0$, $\phi(x + x' - z) - \phi(x' - z) \geq \eta x$, and for $x \leq 0$, $\phi(x + x' - z) - \phi(x' - z) \geq \lambda\eta x$. Thus we can rewrite (26) as

$$\begin{aligned} &\int\int(\phi(x + x' - z) - \phi(x' - z))dF_\mu(x)dF_\nu(x') \\ \geq &\int\int\left(\eta\max\{x, 0\} + \lambda\eta\min\{x, 0\}\right)dF_\mu(x)dF_q(x') = \int\phi(x)dF_\mu(x) \geq 0. \end{aligned}$$

□