Viral Social Learning

David McAdams* Yangbo Song†

August 19, 2018

Abstract

We study social learning with a "viral" feature: on a continuous time line, a group of consumers need to each make a decision of whether to adopt a product, where awareness of the product is transmitted from adopting consumers to new ones. A consumer bases her action on the time she becomes aware at as well as her private signal about product quality. We find a unique equilibrium depicting the product's life cycle: consumers start with herding on adoption given high initial belief and being sensitive to signals given low initial belief, then use mixed strategies which keep beliefs constant, followed by a period of relying on signals, and finally rejecting the product once and for all when beliefs fall below a threshold. When strategic quality choice of the supply side is taken into account, we find an inverse relationship between quality improvement cost and average quality in the market. We also characterize conditions under which a competitive market is more or less socially preferred than a monopoly market.

Keywords: social learning, viral marketing, SIR model, information aggregation

JEL Classification: C72, D62, D83

*Fuqua School of Business and Economics Department, Duke University. Email: david.mcadams@duke.edu.

†The School of Management and Economics, CUHK Business School, Shenzhen Finance Institute, The Chinese University of Hong Kong, Shenzhen. Email: yangbosong@cuhk.edu.cn.

1 Introduction

A successful product launch nowadays is often associated with a "viral" aspect in marketing – early users of the product make others aware of it via word-of-mouth communication, emails, and social networking websites such as Facebook, Twitter, Instagram and Youtube. According to statistics from the Word of Mouth Marketing Association and TalkTrack[1], around 2.4 billion brand-related conversations take place every day in the US alone, and the average consumer mentions specific brand names 60 times per week in conversations. Given the speed and effectiveness of this peer-to-peer marketing approach, more and more producers begin to fund activities that generate more product exposure through consumers' social networks, for instance inviting star bloggers to write product reviews.

From the perspective of information aggregation among consumers, a product that "goes viral" presents a new problem. Different from the conventional setting that existence of the product is taken as given, a consumer now only perceives the product after she observe others using it, probably through random meeting within her social circle. In addition, and perhaps more realistically, non-adoption of the product does not spread awareness.

These features lead to at least two conceivable differences, in terms of epidemiological dynamics among consumers, from classical social learning models. First, the time that a consumer becomes aware of the product is endogenous, and the consumer's belief on quality hinges on it. Hearing about the product right after its launch and only after several months of the launch, for example, can generate very different even opposing beliefs. Second, due to such belief variation, contrasting behavior – herding on adoption, herding on non-adoption, and being sensitive to private information – may occur over time but may not persist forever. Therefore a product is likely to go through a "life cycle" after launch, whose characteristics are worth investigating. For instance, will a product with high initial belief manage to maintain the reputation? Conversely, will a product with low initial belief always remain unpopular? Furthermore, in a bigger picture where quality is endogenously determined by strategic producers, what can we say about average quality in the market and how is it affected by the market structure?

In this paper, we propose a first theoretical model that captures the essential characteristics of viral social learning, and answer these questions by providing an explicit characterization of equilibrium behavior. Our contribution to the literature is two-fold. On the demand side, we reveal how information aggregation evolves among consumers over time, and how it is determined by initial belief and precision of private

[1]Retrieved from: https://www.crewfire.com/50-peer-to-peer-marketing-statistics/.

information. Our results explain how a product gains initial attention, accumulates sales via peer-to-peer communication, gradually loses consumer trust, and finally dies out. On the supply side, we identify producers' choices of quality under different market structures, and point out the welfare-maximizing market structure. Similar to the traditional Cournot model, a producer's strategic behavior exhibits a regular and continuing pattern when its market power increases.

To model viral social learning, we adopt a framework that originally analyzes biological viruses: the susceptible-infected-recovered model, henceforward referred to as the SIR model. At time 0 on a continuous time line, a new product with unknown quality is launched among a small fraction of consumers, who decide whether to adopt it based on their private information. A consumer's private information consists of a noisy quality-dependent signal whose precision is higher than her initial belief or prior about quality, as in the standard setting of Bayesian social learning. The non-adopting consumers (the "recovered" or "immune") make no further move. The adopting consumers (the "infected") meet new consumers (the "susceptible") at the next time instant and make them aware of the product's existence; the new consumers then take into account this information – the time of their awareness – followed by their private signals, and decide whether to adopt the product. Afterwards, they enter either the infected or the recovered group, and the dynamic process continues. The game stops when every consumer has made their decision.

The consumers' epidemiological dynamics reflect the joint effect of three forces: precision of her private signal, time of awareness, and decisions by her predecessors. The signal structure is exogenously given and invariant, while the other two factors are endogenously determined. Predecessors' different behavior has different impacts on beliefs: neither type of herding provides new information on quality, while being sensitive to signals can be regarded *per se* as favorable to high quality because awareness essentially reveals another good signal. As a result, the effect of time on beliefs is not monotone. Although a good product spreads awareness faster than a bad one, which seems to suggest that a consumer who becomes aware earlier should hold a stronger belief that the product is good, it is also plausible that beliefs rise in time at least for a period when predecessors' actions rely on signals.

Our first main result, Theorem 1, depicts the counterbalance of these forces and characterizes the unique equilibrium among consumers. Right after the product launch, consumers will herd on adoption when their initial belief on quality is high, and be sensitive to signals when the belief is low. In this period of time, consumers' interim beliefs – that is, beliefs upon awareness but before private signals realize – fall in the former case while rise in the latter. When the value of interim beliefs reach $\frac{1}{2}$, consumers begin a period of using mixed strategies, during which beliefs remain unchanged. Afterwards, consumers become sensitive to private signals and their beliefs fall from $\frac{1}{2}$. Finally, as beliefs drop to a certain level, no new consumer will choose adoption and the product dies out. The product's life cycle thus formed can be readily computed numerically using simulation.

This result contributes to the understanding of social learning from two major aspects. Qualitatively, it implies that for products with intrinsically uncertain quality, fads are always transient even if consumers start with high hopes, while products that are initially not so popular may still enjoy a rising reputation and make considerable sales. Such phenomena are widely observed in real-life markets such as medicine and personal care, especially in the current information age where "viral" spread of brand names are prevalent. Quantitatively, the equilibrium dynamics of consumers' actions and beliefs can be readily computed using numerical simulation, in order to generate precise predictions.

Given the methodology for depicting consumers' equilibrium behavior, we then turn our focus to the supply side and analyze strategic producers. When producers have control over quality, the uncertainty faced by consumers is no longer intrinsic and exogenous, but results from the equilibrium among producers such that only a fraction of producers choose to launch a good product. We assume that a producer has the option of paying a cost to improve quality, and derive this equilibrium under three typical market structures: monopoly, oligopoly and perfect competition.

Theorem 2 summarizes our findings. When a producer has commitment power, it chooses high quality if and only if the cost is low. Qualitatively, this result is invariant regardless of the market structure. Without commitment, producers' behavior exhibits richer patterns. A monopoly producer never produces a good product in equilibrium however small the cost is. When there are multiple producers, we first observe a "Goldilocks effect", in the sense that high quality products can only appear in the market when the cost is neither too high nor to low. Second, within this range of cost levels, an inverse relation between cost and equilibrium proportion of high quality products emerges. This relation can also be interpreted as a demand for quality improvement, and is represented by a downward-sloping inverse-step function for oligopoly, which converges to a continuous function under perfect competition when the number of firms goes to infinity.

This result then sheds light on social welfare given different levels of market power. We compute and compare social welfare under two representative and opposite market structures: a monopoly with commitment power, versus competition without commitment power. When the quality improvement cost is low and a good product generates significantly higher utility than a bad one, monopoly is socially preferred because it ensures a good product for each consumer. Otherwise, competition yields higher social welfare because it leads to a mixture of good and bad products which turns out to be more cost-efficient than uniform quality.

The rest of the paper is organized as follows. Section 2 reviews related literature. Section 3 presents the model. Section 4 characterizes the unique equilibrium among consumers when quality uncertainty is exogenous. Section 5 characterizes producer equilibrium of endogenous quality, and also discusses producers' other possible choices such as price and spread speed. Section 6 concludes.

2 Literature Review

Early contributions to the literature of social learning by Bayesian agents include seminal papers by Bikhchandani et al. [1992], Banerjee [1992] and Smith and Sorensen [2000]. In these works, before an agent makes their own decision, he/she can observe both a private signal and the entire previous decision history. Herding behavior occurs when the "public" belief from the latter dominates the "private" belief from the former.

Subsequent research on social learning differs by their way of extending the basic model. One branch of literature, for instance Lee [1993], Banerjee [1993] and Celen and Kariv [2004], focuses on more complicated history observation such that agents may not observe the entire but only an independent subset of the history. A more recent paper, Acemoglu et al. [2011], can be regarded as a conclusive generalization of these works. It assumes that each agent observes (some of) their predecessors' actions according to a general stochastic process, and finds that when the private signal structure features unbounded belief, asymptotic learning occurs in each equilibrium if and only if agents always observe close predecessors. Lobel and Sadler [2015] adopt this model to analyze the pattern of learning when agents' observations are correlated. Other recent research in this area include Banerjee and Fudenberg [2004], Gale and Kariv [2003], Callander and Horner [2009] and Smith and Sorensen [2013], which differ from Acemoglu et al. [2011] mainly in relaxing the assumption of known decision order in observation, i.e. agents only observe the number of others taking an action but not their positions in the decision sequence. Guarino et al. [2011], Herrera and Horner [2013] and Monzon and Rapp [2014] consider the case where an agent does not even know their own position in the decision sequence.

Another branch of literature endogenizes the information acquisition process in different ways. On one hand, an agent may have access of direct information about the true state, by paying to acquire an informative signal or to sample an available option and know its value [Hendricks et al., 2012, Mueller-Frank and Pai, 2014, Ali, 2014]; on the the other hand, an agent may choose from different sources of indirect information by selecting which part of the previous history to observe [Kultti and Miettinen, 2006, 2007, Celen, 2008, Song, 2016].

Our paper contributes to the literature in two main aspects. First, an agent's awareness of decision, or the timing of him/her facing the choice between adoption and non-adoption, is endogenously determined via the "viral" nature of the model. This stands in contrast to the previous literature which assumes that the timing is exogenously given by a fixed sequence or a stochastic process, and allows us to capture the understudied feature of observational learning that, when more agents adopt a product, awareness of the product grows rapidly in time. We are then able to depict a product's typical life cycle in the unique equilibrium, the exact pattern of which differs by the initial belief and exposure.

We base our analysis on the classical susceptible-infectious-recovered (SIR) model, originated from Ross [1916] and Ross and Hudson [1917a,b] and developed by Kermack and McKendrick [1927, 1932, 1933]. The model was first proposed to simulate spread of contagious disease, and has been widely applied to study various epidemiological problems, such as disease spread on networks [Newman, 2002], antibiotic resistance [McAdams, 2017a,b], and vaccine scares [Bauch and Bhattacharyya, 2012]. Our model provides the first approach of adopting the SIR methodology in the context of observational learning by Bayesian agents.

Second, while the social learning literature mainly focuses on the learning behavior among consumers presented with a single product, our model provides a ready framework to study competition and market structure among producers who seek to maximize the proportion of consumers adopting their own product. Our approach relates to algorithmic diffusion models for viral marketing problems [Kempe et al., 2003, 2005, Mossel and Roch, 2010, Goyal et al., 2011, Borgs et al., 2014], in particular the ones that analyze competitive contagion (for instance Goyal et al. [2014]). These works focus on how the network effect – an agent's choice being directly determined by the choices of their neighbors – together with the network topology, determine product adoption rates in equilibrium. In contrast, the economic force underlying our results is the belief updating dynamics in a Bayesian framework; but similar to these models, our results offer explicit characterization of an easily computable equilibrium.

3 Model

3.1 Game Setup

Basic structure. There is a unit mass of consumers and a product that has uncertain quality: with probability $\alpha \in (0, 1)$ it is good (high quality) and with probability $1-\alpha$ it is bad (low quality). The quality is the unknown state in the model, denoted by $\omega \in \{g, b\}$, where g means good and b means bad. Each consumer has a unit demand for the product, and they must decide (once and for all) whether or not to pay price $p \geq 0$ to adopt the product when they are first exposed to it, getting a payoff $\bar{u} > 0$ from adopting a good product, $\underline{u} < \bar{u}$ from adopting a bad product, and 0 from not adopting. To establish a clear benchmark, we assume that consumers are indifferent between adoption and non-adoption if their belief on quality is $\frac{1}{2}$, i.e. $p = \frac{\bar{u}+\underline{u}}{2}$. Upon becoming aware of the product, consumers observe conditionally i.i.d. private signals $s_i \in \{G, B\}$ such that $\Pr(s_i = G|g) = \Pr(s_i = B|b) = \rho > 1/2$. Let "$G$ consumers" and "B consumers" denote consumers receiving signal G and B respectively.

Mass $\Delta > 0$ of consumers are exposed to the product at time $t = 0$ (i.e., at launch). Other consumers are exposed to the product over time by encountering those who have already adopted. Product awareness and adoption is captured by a standard SIR epidemiological model (details below). At time $t \geq 0$, each consumer either is unaware of the product ("susceptible state"), has previously chosen to adopt ("infected state"), or has previously chosen not to adopt ("recovered/immune state").

Epidemiological dynamics when the product is good. When the product has good quality, let $(S_g(t), I_g(t), R_g(t))$ denote the mass of consumers in each state at time t. Let $p(t; s_i)$ denote the probability of adoption at time t given private signal s_i. ($p(t; s_i)$ does not depend on true quality, because quality is unobserved.) Mass $I_g(t)$ adopters encounter other consumers at rate $\beta > 0$, fraction $S_g(t)$ of whom were not already aware of the product. Of these newly-aware consumers, fraction $p_g(t)$ adopt the product, where

$$p_g(t) = \rho p(t; G) + (1 - \rho)p(t; B).$$

Newly-exposed consumers who choose to adopt transition to the infected state, while those who choose not to adopt transition to the recovered state. Overall, then, epidemiological dynamics are characterized by the system

$$\begin{aligned} I_g'(t) &= \beta I_g(t) S_g(t) p_g(t) \\ R_g'(t) &= \beta I_g(t) S_g(t)(1 - p_g(t)) \\ S_g'(t) &= -\beta I_g(t) S_g(t) \end{aligned}$$

and $S_g(t) + I_g(t) + R_g(t) = 1$ for all t.

Epidemiological dynamics when the product is bad. Define similar notion for the case when the product is bad quality. The main difference is that, when the product is bad, consumers are more likely to receive bad signals. So, $p_b(t) = (1 - \rho)p(t; G) + \rho p(t; B)$.

Equilibrium Our solution concept of the game is *perfect Bayesian equilibrium*. As each consumer only takes a binary action once, her payoff is only affected by the product quality and her action, and interim beliefs are continuous in time, an equilibrium always exists.

3.2 Preliminary Analysis

Adoption decisions at time $t = 0$. It is easy to show that consumers who become aware of the product at time $t = 0$ will take actions according to the prior belief α:

$$\begin{cases} \text{Always adopt when } \alpha \in (\rho, 1); \\ \text{Adopt if } s_i = G \text{ but not if } s_i = B \text{ when } \alpha \in (1 - \rho, \rho); \\ \text{Never adopt when } \alpha \in (0, 1 - \rho). \end{cases}$$

We will skip the cases of $\alpha = \rho$ and $\alpha = 1 - \rho$ as they involve unnecessary discussion of consumers' multiple best responses but lead to no additional insight. In correspondence with the above strategies, the probability of adoption and the mass of susceptible, infected and immune consumers are summarized by the following lemma.

Lemma 1. *At $t = 0$:*

1. *$\alpha \in (\rho, 1)$: $p_g(0) = p_b(0) = 1$, $S_g(0) = S_b(0) = 1 - \Delta$, $I_g(0) = I_b(0) = \Delta$, $R_g(0) = R_b(0) = 0$.*

2. *$\alpha \in (1 - \rho, \rho)$: $p_g(0) = \rho$, $S_g(0) = 1 - \Delta$, $I_g(0) = \rho\Delta$, $R_g(0) = (1 - \rho)\Delta$; $p_b(0) = 1 - \rho$, $S_b(0) = 1 - \Delta$, $I_b(0) = (1 - \rho)\Delta$, $R_b(0) = \rho\Delta$.*

3. $\alpha \in (0, 1-\rho)$*:* $p_g(0) = p_b(0) = 0$*,* $S_g(0) = S_b(0) = 1 - \Delta$*,* $I_g(0) = I_b(0) = 0$*,* $R_g(0) = R_b(0) = \Delta$*.*

Note that in the second case, since $\rho > 1/2$, the good product is immediately adopted by a larger mass of consumers–spreading awareness of good products more quickly than bad products. In particular, a consumer who becomes aware of the product at time $t \approx 0$ will infer that she is more likely to have heard about the product if it is good.

Consumer beliefs and adoption decisions at time $t > 0$. The flow of agents who become aware of the product at time $t > 0$ is $\beta I_g(t)S_g(t)$ if the product is good or $\beta I_b(t)S_b(t)$ if the product is bad. Conditional on becoming aware at time t (before observing their private signal), the probability $q(t;\emptyset)$ that the product is good satisfies the condition

$$\frac{q(t;\emptyset)}{1-q(t;\emptyset)} = \frac{\alpha I_g(t)S_g(t)}{(1-\alpha)I_b(t)S_b(t)}.$$

We will call $q(t;\emptyset)$ the consumers' *interim belief* at time t. Updating by Bayes' Rule, an agent who observes private signal $s_i = G, B$ will then assess probability $q(t;s_i)$ that the product is good:

$$\begin{aligned} q(t;G) &= q(t;\emptyset)\frac{\rho}{\rho q(t;\emptyset) + (1-\rho)(1-q(t;\emptyset))} \\ q(t;B) &= q(t;\emptyset)\frac{1-\rho}{(1-\rho)q(t;\emptyset) + \rho(1-q(t;\emptyset))}. \end{aligned}$$

We will call $q(t;G)$ and $q(t;B)$ the consumers' *posterior belief* at time t, given signal G and B respectively.

Let $\overline{q}(t)$ be the level of $q(t;\emptyset)$ so that $q(t;B) = 50\%$ and let $\underline{q}(t)$ be the level of $q(t;\emptyset)$ so that $q(t;G) = 50\%$. Given different levels of interim belief, newly-aware consumers will choose one of the following strategies as their best response.

1. Herd on adoption: if $q(t;\emptyset) > \overline{q}(t)$ then all newly-aware consumers adopt, i.e., $I_g'(t) = \beta I_g(t)S_g(t)$, $R_g'(t) = 0$, $I_b'(t) = \beta I_b(t)S_b(t)$, and $R_b'(t) = 0$.

2. Be sensitive to signal: if $q(t;\emptyset) \in \big(\underline{q}(t), \overline{q}(t)\big)$, then all newly-aware G consumers adopt while all B consumers do not adopt, i.e., $I_g'(t) = \rho\beta I_g(t)S_g(t)$, $R_g'(t) = (1-\rho)\beta I_g(t)S_g(t)$, $I_b'(t) = (1-\rho)\beta I_b(t)S_b(t)$, and $R_b'(t) = \rho\beta I_b(t)S_b(t)$.

3. Use mixed strategies: if $q(t;\emptyset) = \overline{q}(t)$ then all newly aware G consumers adopt while all B consumers mix between adoption and not. In this case, $I_g'(t) = (\rho + (1-\rho)x)\beta I_g(t)S_g(t)$, $R_g'(t) = (1-\rho)(1-x)\beta I_g(t)S_g(t)$, $I_b'(t) = (1-\rho+\rho x)\beta I_b(t)S_b(t)$, and $R_b'(t) = \rho(1-x)\beta I_b(t)S_b(t)$, for some $x \in [0,1]$. If $q(t;\emptyset) = \underline{q}(t)$ then all newly aware G consumers mix between adoption and not while no B consumer adopts. In this case, $I_g'(t) = \rho x\beta I_g(t)S_g(t)$, $R_g'(t) = (1-\rho+\rho(1-x))\beta I_g(t)S_g(t)$, $I_b'(t) = (1-\rho)x\beta I_b(t)S_b(t)$, and $R_b'(t) = (\rho+(1-\rho)(1-x))\beta I_b(t)S_b(t)$, for some $x \in [0,1]$.

4. Herd on non-adoption: if $q(t;\emptyset) < \underline{q}(t)$ then no newly-aware consumer adopt, i.e., $I_g'(t) = 0$, $R_g'(t) = \beta I_g(t)S_g(t)$, $I_b'(t) = 0$, and $R_b'(t) = \beta I_b(t)S_b(t)$.

When $\alpha > \rho$ or $\alpha < 1-\rho$, consumers' behavior at $t > 0$ is straight forward. We state it in the result below.

Proposition 1. *When $\alpha > \rho$, consumers always herd on adoption; when $\alpha < 1-\rho$, consumers never adopt the product.*

Proof. Suppose that $\alpha > \rho$. By Lemma 1 we have

$$\frac{q(t;\emptyset)}{1-q(t;\emptyset)} \approx \frac{\alpha I_g(0)S_g(0)}{(1-\alpha)I_b(0)S_b(0)} = \frac{\alpha}{1-\alpha}$$

for $t \approx 0$. Hence, $q(t;\emptyset) > \overline{q}(t) = \frac{1}{2}$ which means that consumers will herd on adoption for $t \approx 0$. It then follows that $p_g(t) = p_b(t) = 1$, $S_g(t) = S_b(t)$ and $I_g(t) = I_b(t)$. Therefore, consumers' interim beliefs stay unchanged and the argument applies to all subsequent $t > 0$ until all consumers have adopted the product.

Suppose that $\alpha < 1-\rho$. As $I_g(0) = I_b(0) = 0$ by Lemma 1, awareness can never be spread so no consumer will adopt the product. □

In the next section, we will characterize the dynamics of consumers' equilibrium behavior and beliefs in the more interesting case of $\alpha \in (1-\rho, \rho)$.

4 Equilibrium Characterization

Theorem 1 presents our first main result. We show that the product has a clear life cycle among consumers, starting with herding on adoption under high initial belief and being sensitive to signals under low initial belief, followed by a period of using mixed strategies and another period of being sensitive to signals, and finally ending with herding on non-adoption.

Theorem 1. *The equilibrium is generically unique*[2]*: at $t = 0$, consumers are sensitive to signals. Afterwards, there exist time cutoffs t_1, t_2, t_3 such that:*

1. *For all $t \in (0, t_1)$: when $\alpha \in (\frac{1}{2}, \rho)$, consumers herd on adoption; when $\alpha \in (1-\rho, \frac{1}{2})$, consumers are sensitive to signals; t_1 is characterized by $q(t_1; \emptyset) = \rho$. At $t = t_1 > 0$, G consumers always adopt while B consumers adopt with an arbitrary probability. When $\alpha = \frac{1}{2}$, $t_1 = 0$.*

2. *For all $t \in (t_1, t_2)$: G consumers always adopt while B consumers adopt with probability $x(t) = \frac{\rho S_g(t)-(1-\rho)S_b(t)-(I_g(t)-I_b(t))}{\rho S_b(t)-(1-\rho)S_g(t)}$; t_2 is characterized by $\rho S_g(t_2) - I_g(t_2) = (1-\rho)S_b(t_2) - I_b(t_2)$. At $t = t_2$, G consumers always adopt while B consumers adopt with an arbitrary probability.*

3. *For all $t \in (t_2, t_3)$: consumers are sensitive to signals; t_3 is characterized by $q(t_3; \emptyset) = 1 - \rho$. At $t = t_3$, G consumers adopt with an arbitrary probability while B consumers never adopt.*

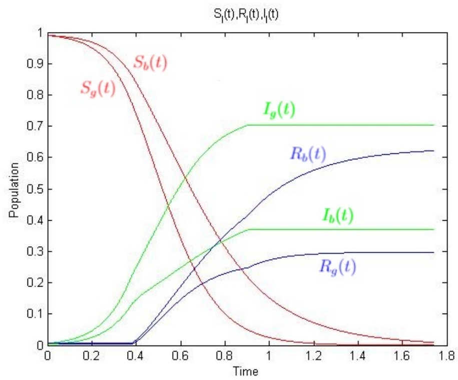
4. *For all $t > t_3$, consumers herd on non-adoption.*

Proof. By Lemmas 2-8 below. □

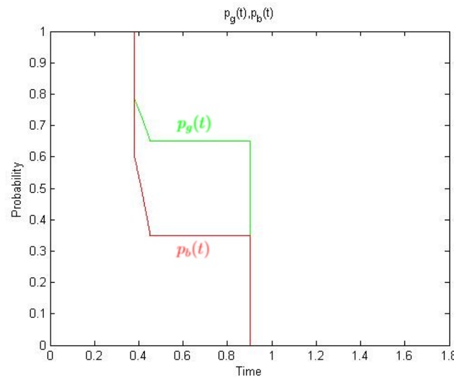
Given the parameter set $(\alpha, \rho, \beta, \Delta)$, the unique equilibrium can be easily simulated by numerical methods. The following figures illustrate how consumer behavior and interim beliefs evolve over time for different initial beliefs[3].

[2] As our equilibrium is perfect Bayesian equilibrium on continuous time, we regard uniqueness as up to variations on a zero measure of time instants.

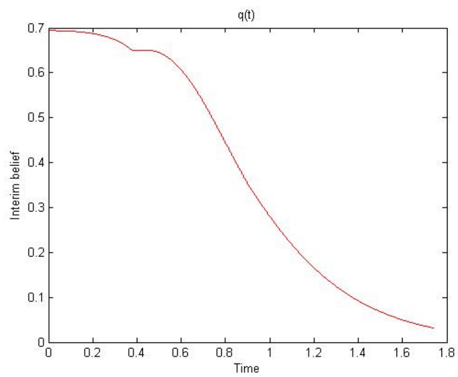
[3] In the simulation, we set $(\rho, \beta, \Delta) = (0.65, 10, 0.001)$. $\alpha = 0.55$ for Figure 1, 0.5 for Figure 2 and 0.45 for Figure 3.



(a) Population of susceptible, infected and recovered consumers

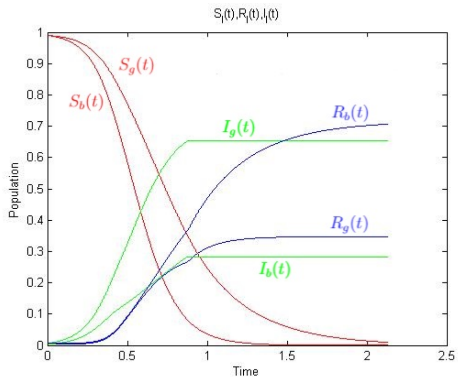


(b) Probability of product adoption

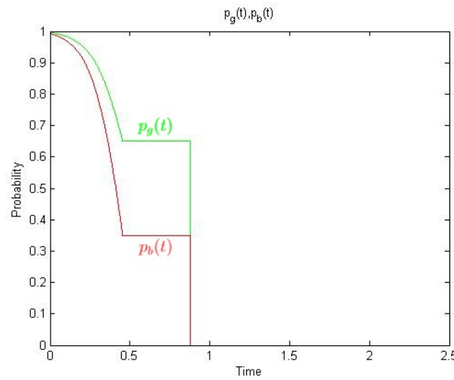


(c) Interim beliefs

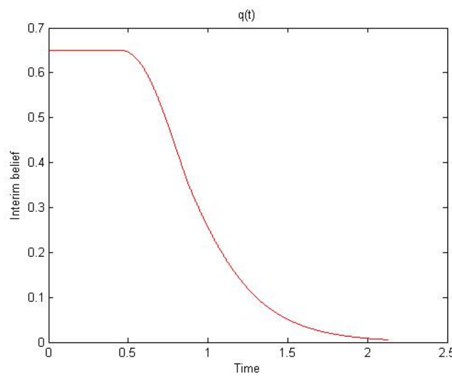
Figure 1: Simulation results when $\alpha \in (\frac{1}{2}, \rho)$



(a) Population of susceptible, infected and recovered consumers

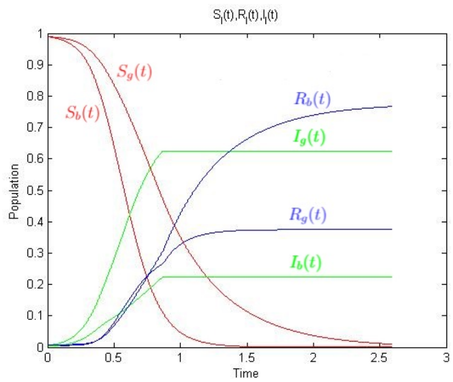


(b) Probability of product adoption

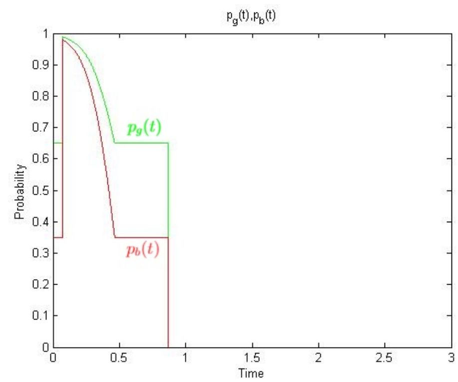


(c) Interim beliefs

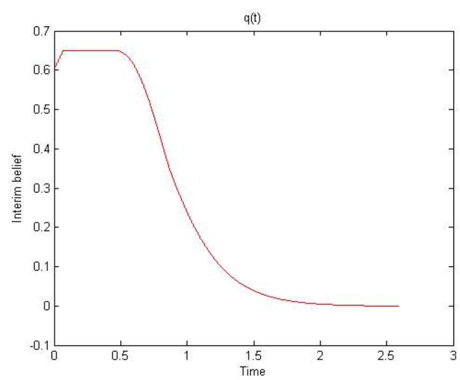
Figure 2: Simulation results when $\alpha = \frac{1}{2}$



(a) Population of susceptible, infected and recovered consumers



(b) Probability of product adoption



(c) Interim beliefs

Figure 3: Simulation results when $\alpha \in (1-\rho, \frac{1}{2})$

In subsections 4.1-4.3, we explain step by step the mathematical argument and economic intuition behind the result.

4.1 Early Behavior and Evolution of Beliefs

We begin by discussing consumers' early equilibrium behavior given different values of α and how it affects successors' interim beliefs when t is close to 0. The key to determining consumers' optimal action over time is to trace the value of $\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)}$, which we will call the ratio between transmission rates. It starts at the value of $\frac{\rho}{1-\rho}$ at $t = 0$ regardless of α, and is proportional to $\frac{q(t;\emptyset)}{1-q(t;\emptyset)} = \frac{\alpha I_g(t)S_g(t)}{(1-\alpha)I_b(t)S_b(t)}$. This term, which is also equal to $\frac{-S_g'(t)}{-S_b'(t)}$, captures how fast a good product spreads relative to a bad one at time t; the higher it is, the higher incentive a newly aware consumer has to adopt the product. In the proof of the following lemmas, we will demonstrate how different values of the ratio causes consumers to behave differently as time passes.

Our first observation is that a high initial belief causes herding initially, which in turn lowers interim beliefs over time.

Lemma 2. *Suppose that $\alpha \in (\frac{1}{2}, \rho)$. In every equilibrium, there exists $t_1 > 0$ such that consumers herd on adoption and interim belief strictly decreases in t for all $t \in (0, t_1)$.*

Proof. In every equilibrium, at $t \approx 0$ we have

$$\begin{aligned}\frac{q(t;\emptyset)}{1-q(t;\emptyset)} &\approx \frac{\alpha I_g(0)S_g(0)}{(1-\alpha)I_b(0)S_b(0)} \\ &= \frac{\alpha\rho\Delta(1-\Delta)}{(1-\alpha)(1-\rho)\Delta(1-\Delta)} = \frac{\alpha\rho}{(1-\alpha)(1-\rho)} > \frac{\rho}{1-\rho},\end{aligned}$$

so herding must occur when t is sufficiently close to 0. When consumers herd on adoption at t, for $l = g, b$

$$\begin{aligned}I_l'(t) &= \beta I_l(t)S_l(t) \\ S_l'(t) &= -I_l'(t).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{(\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)})'}{\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)}} &= \frac{I_g'(t)}{I_g(t)} + \frac{S_g'(t)}{S_g(t)} - \frac{I_b'(t)}{I_b(t)} - \frac{S_b'(t)}{S_b(t)} \\ &= \beta(S_g(t) - S_b(t) + I_b(t) - I_g(t)) \\ &= \beta((S_g(t) - I_g(t)) - (S_b(t) - I_b(t)))\end{aligned}$$

Since $S_g(0) - I_g(0) = 1 - \Delta - \rho\Delta < 1 - \Delta - (1-\rho)\Delta = S_b(0) - I_b(0)$, $(S_g(t) - I_g(t)) - (S_b(t) - I_b(t)) < 0$ when $t \approx 0$. This implies that $(\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)})' < 0$, i.e. the interim belief strictly decreases in t. □

Early after product launch, consumers' interim beliefs are still close to their initial belief which is higher than ρ. It means that even if a consumer receives a bad signal, her posterior belief will still be higher than $\frac{1}{2}$, which induces her to herd on adoption. However, herding results in lower interim beliefs for her successors, to whom herding reveals no new information but late awareness is unfavorable for quality.

Our next result shows a contrasting pattern with a low initial belief[4].

Lemma 3. *Suppose that* $\alpha \in (1-\rho, \frac{1}{2})$. *In every equilibrium, there exists* $t_1 > 0$ *such that consumers are sensitive to signals and interim belief strictly increases in* t *for all* $t \in (0, t_1)$.

Proof. In every equilibrium, at $t \approx 0$ we have

$$\begin{aligned}\frac{q(t;\emptyset)}{1-q(t;\emptyset)} &\approx \frac{\alpha I_g(0)S_g(0)}{(1-\alpha)I_b(0)S_b(0)}\\ &= \frac{\alpha\rho\Delta(1-\Delta)}{(1-\alpha)(1-\rho)\Delta(1-\Delta)} = \frac{\alpha\rho}{(1-\alpha)(1-\rho)} \in (\frac{1-\rho}{\rho}, \frac{\rho}{1-\rho}),\end{aligned}$$

so consumers must be sensitive to signals when t is sufficiently close to 0. We have

$$\begin{aligned}I_g'(t) &= \beta\rho I_g(t)S_g(t)\\ S_g'(t) &= -\beta I_g(t)S_g(t)\\ I_b'(t) &= \beta(1-\rho)I_b(t)S_b(t)\\ S_g'(t) &= -\beta I_b(t)S_b(t).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{(\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)})'}{\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)}} &= \frac{I_g'(t)}{I_g(t)} + \frac{S_g'(t)}{S_g(t)} - \frac{I_b'(t)}{I_b(t)} - \frac{S_b'(t)}{S_b(t)}\\ &= \beta(\rho S_g(t) - (1-\rho)S_b(t) + I_b(t) - I_g(t))\\ &= \beta((\rho S_g(t) - I_g(t)) - ((1-\rho)S_b(t) - I_b(t))).\end{aligned}$$

Note that

$$(\rho S_g(0) - I_g(0)) - ((1-\rho)S_b(0) - I_b(0)) = (2\rho-1)(1-2\Delta) > 0.$$

Hence, $(\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)})' > 0$, i.e. the interim belief strictly increases in t, when t is close to 0. □

In this case, a longer period of time before awareness alone is still considered bad news for quality. Nevertheless, as consumers are sensitive to signals, awareness itself

[4]Unless otherwise specified, it is assumed henceforth that Δ is sufficiently small. We will discuss equilibrium characterization for a large Δ in subsection 4.4.

implies a series of good signals, which outweighs the negative effect of time. Hence in aggregation, interim beliefs rise after product launch.

Will interim beliefs keep increasing/decreasing? The following lemma shows that they will indeed, during which consumers' behavior stays unchanged, until interim beliefs reach the value of ρ.

Lemma 4. *$|(\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)})'|$ has a positive lower bound before $q(t;\emptyset)=\rho$.*

Proof. When $\alpha=\frac{1}{2}$, $q(0;\emptyset)=\rho$ and the above statement is trivially true.

When $\alpha\in(\frac{1}{2},\rho)$, $q(0;\emptyset)>\rho$ and we know from the proof of Lemma 2 that

$$(\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)})'=\beta((S_g(t)-I_g(t))-(S_b(t)-I_b(t)))\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)}$$

as long as consumers herd on adoption, and note that in this case

$$((S_g(t)-I_g(t))-(S_b(t)-I_b(t)))'=-2(I_g(t)S_g(t)-I_b(t)S_b(t))$$

Before $q(t;\emptyset)=\rho$, or equivalently $\frac{q(t;\emptyset)}{1-q(t;\emptyset)}=\frac{\alpha I_g(t)S_g(t)}{(1-\alpha)I_b(t)S_b(t)}=\frac{\rho}{1-\rho}$, $\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)}\geq\frac{(1-\alpha)\rho}{\alpha(1-\rho)}>1$, and thus $(S_g(t)-I_g(t))-(S_b(t)-I_b(t))<(S_g(0)-I_g(0))-(S_b(0)-I_b(0))=-(2\rho-1)\Delta$. A lower bound of $|(\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)})'|$ is $\beta(2\rho-1)\Delta$.

When $\alpha\in(1-\rho,\frac{1}{2})$, $q(0;\emptyset)<\rho$ and from the proof of Lemma 3 we have

$$(\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)})'=\beta((\rho S_g(t)-I_g(t))-((1-\rho)S_b(t)-I_b(t)))\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)}$$

and

$$((\rho S_g(t)-I_g(t))-((1-\rho)S_b(t)-I_b(t)))'=-2(\rho S_g(t)I_g(t)-(1-\rho)S_b(t)I_b(t)).$$

Let $\hat{t}=\frac{\frac{\rho}{1-\rho}-\frac{\alpha\rho}{(1-\alpha)(1-\rho)}}{\beta(\frac{1}{2}\rho-\frac{1}{2}(1-\rho))\frac{\rho}{1-\rho}}>0$. When Δ is sufficiently small, $-(\rho S_g(t)I_g(t)-(1-\rho)S_b(t)I_b(t))$ is also sufficiently small for $t\in(0,\hat{t})$ and thus $(\rho S_g(t)-I_g(t))-((1-\rho)S_b(t)-I_b(t))>\frac{1}{2}\rho-\frac{1}{2}(1-\rho)>0$ for $t\in(0,\hat{t})$. Since $\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)}$ is increasing, we also have $\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)}>\frac{I_g(0)S_g(0)}{I_b(0)S_b(0)}=\frac{\rho}{1-\rho}$ for $t\in(0,\hat{t})$. Therefore, a lower bound of $|(\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)})'|$ is $\beta(\frac{1}{2}\rho-\frac{1}{2}(1-\rho))\frac{\rho}{1-\rho}$ and $\frac{q(t;\emptyset)}{1-q(t;\emptyset)}=\frac{\alpha I_g(t)S_g(t)}{(1-\alpha)I_b(t)S_b(t)}$ will reach $\frac{\rho}{1-\rho}$ within time $\hat{t}$. □

The interim belief ρ – whether it is reached initially when $\alpha=\frac{1}{2}$, from above when $\alpha\in(\frac{1}{2},\rho)$, or from below when $\alpha\in(1-\rho,\frac{1}{2})$ – marks the first switching point in both beliefs and actions of consumers. With this belief, a G consumer will adopt the product while a B consumer becomes indifferent, which seems to produce a plethora of candidates for equilibrium behavior. In the next subsection, we prove that consumers will only use one particular strategy in equilibrium, with explicit characterization of the strategy.

4.2 Mixed Strategies when $q(t;\emptyset) = \rho$

By Lemmas 2-4, after the product launch, interim beliefs will decrease (increase) until they reach ρ when the initial belief is above (below) $\frac{1}{2}$. Let t_1 be the time at which $q(t;\emptyset) = \rho$ for the first time; by Lemma 4, as long as the dynamic process in the special case of $\alpha = \frac{1}{2}$, $t_1 = 0$. To make the problem simpler, we may as well view the dynamic process as starting from t_1 with an initial belief of $\frac{\rho}{1-\rho}$ and corresponding $S_l(t_1)$, $I_l(t_1)$ and $R_l(t_1)$ $(l = g, b)$.

We first identify a condition under which consumers will use mixed strategies in every equilibrium right after t_1, which is $\rho S_g(t_1) - I_g(t_1) > (1-\rho)S_b(t_1) - I_b(t_1)$. This condition signifies that time is still early when interim beliefs first reach ρ (as it must hold at $t = 0$), and is always satisfied when $\alpha = \frac{1}{2}$ or when $\alpha \in (1-\rho, \frac{1}{2})$ and Δ is sufficiently small; we will assume for subsections 4.2-4.3 that it is satisfied, and discuss equilibrium characterization when it is violated as a special case in subsection 4.4.

The following lemma can be regarded as a corollary of Lemmas 2, 3 and 4.

Lemma 5. *In every equilibrium, there exists $t_2 > t_1$ such that for all $t \in (t_1, t_2)$, G consumers adopt the product while B consumers mix between adoption and non-adoption.*

Proof. Suppose that in some equilibrium the consumers do not act as proposed above, then there are two possibilities: (1) consumers herd on adoption for all $t \in (t_1, t_1 + \epsilon)$ for some $\epsilon > 0$; (2) consumers are sensitive to signals for all $t \in (t_1, t_1 + \epsilon)$ for some $\epsilon > 0$.

Consider the first possibility. By the proof of Lemmas 2 and 4, we know that if consumers herd on adoption for all $t \in (t_1, t_1+\epsilon)$, $(\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)})' < 0$ when ϵ is sufficiently small. It means that $q(t;\emptyset) < \rho$ and consumers should be sensitive to signals instead, a contradiction.

Consider the second possibility. By the proof of Lemmas 2 and 3, as long as $\rho S_g(t_1) - I_g(t_1) > (1-\rho)S_b(t_1) - I_b(t_1)$ we have $(\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)})' > 0$ when ϵ is small. Hence $q(t;\emptyset) > \rho$ and consumers should herd on adoption, a contradiction. $\square$

The condition $\rho S_g(t_1) - I_g(t_1) > (1-\rho)S_b(t_1) - I_b(t_1)$ is only used in the proof for the hypothetical case of consumers being sensitive to signals. This implies that being sensitive to signals has different effects on interim beliefs at different times. Intuitively, early in the game when the condition must be satisfied, the difference in spread speed between good and bad products is still mild. Therefore being sensitive to signals is good news in general because a good signal contains favorable information which is stronger than the adverse information from an instant of elapsed time. However, as the difference in spread speed widens over time and finally breaks the condition, the unfavorable information from elapsed time dominates and interim belief will drop as a result.

Consumers using mixed strategies on (t_1, t_2) immediately implies that interim beliefs stay at ρ; otherwise, B consumers will deviate from mixing. Nevertheless, just as the initial pattern of consumer behavior does not persist forever, neither do such mixed strategies. The following lemma characterizes the mixing probability with which B consumers adopt, and shows that it drops to 0 over time.

Lemma 6. *In every equilibrium, after t_1, G consumers always adopt while B consumers adopt with probability $x(t) = \frac{\rho S_g(t)-(1-\rho)S_b(t)-(I_g(t)-I_b(t))}{\rho S_b(t)-(1-\rho)S_g(t)}$, until t_2 such that $\rho S_g(t_2) - I_g(t_2) = (1-\rho)S_b(t_2) - I_b(t_2)$.*

Proof. Lemma 5 has shown that B consumers will mix between adoption and non-adoption right after t_1, which means that $q(t;\emptyset)$ must stay constant at ρ. Combine this result with the condition characterizing $q(t;\emptyset)$:

$$\frac{\rho}{1-\rho} = \frac{\alpha I_g(t)S_g(t)}{(1-\alpha)I_b(t)S_b(t)}.$$

The above condition needs to hold for every $t \in (t_1, t_2)$ for some $t_2 > t_1$, and we know that it holds for $t = t_1$, and hence the ratio of derivatives with respect to t has to be constant at $\frac{\rho}{1-\rho}$ as well for every $t \in (0, t_1)$, i.e.

$$\begin{aligned}\frac{\rho}{1-\rho} &= \frac{\alpha}{1-\alpha}\frac{I_g'(t)S_g(t) + I_g(t)S_g'(t)}{I_b'(t)S_b(t) + I_b(t)S_b'(t)} \\ &= \frac{\alpha}{1-\alpha}\frac{\beta I_g(t)S_g^2(t)p_g(t) - \beta I_g^2(t)S_g(t)}{\beta I_b(t)S_b^2(t)p_b(t) - \beta I_b^2(t)S_b(t)} \\ &= \frac{\alpha}{1-\alpha}\frac{I_g(t)S_g(t)(S_g(t)p_g(t) - I_g(t))}{I_b(t)S_b(t)(S_b(t)p_b(t) - I_b(t))}.\end{aligned}$$

Equivalently, we must have

$$S_g(t)p_g(t) - I_g(t) = S_b(t)p_b(t) - I_b(t).$$

Let $x(t)$ be the probability with which B consumers adopt at t, then we have

$$S_g(t)(\rho + (1-\rho)x(t)) - I_g(t) = S_b(t)((1-\rho) + \rho x(t)) - I_b(t)$$
$$x(t) = \frac{\rho S_g(t) - (1-\rho)S_b(t) - (I_g(t) - I_b(t))}{\rho S_b(t) - (1-\rho)S_g(t)} \in [0,1).$$

Applying Lemma 5 once more, as long as $\rho S_g(t) - I_g(t) > (1-\rho)S_b(t) - I_b(t)$, B consumers will adopt with probability $x(t')$ for all $t' \in (t, t+\epsilon)$ for some $\epsilon > 0$. Therefore in every (piecewise continuous) equilibrium, B consumers will adopt with probability $x(t)$ if $\rho S_g(t) - I_g(t) > (1-\rho)S_b(t) - I_b(t)$.

It now remains to prove that such mixing will not persist, i.e. $x(t)$ will drop to 0 given sufficient time. Note that

$$x'(t) = \frac{\begin{matrix}(\rho S'_g(t) - (1-\rho)S'_b(t) - (I'_g(t) - I'_b(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ -(\rho S_g(t) - (1-\rho)S_b(t) - (I_g(t) - I_b(t)))(\rho S'_b(t) - (1-\rho)S'_g(t))\end{matrix}}{(\rho S_b(t) - (1-\rho)S_g(t))^2}.$$

Rearranging and simplifying the numerator, we have

$$\begin{aligned}\text{numerator} =& (\rho^2 - (1-\rho)^2)(S'_g(t)S_b(t) - S'_b(t)S_g(t)) \\ &- (I'_g(t) - I'_b(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ &+ (I_g(t) - I_b(t))(\rho S'_b(t) - (1-\rho)S'_g(t)).\end{aligned}$$

The second term above is

$$\begin{aligned}&- (I'_g(t) - I'_b(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ =& - \beta(I_g(t)S_g(t)(\rho + (1-\rho)x(t)) - I_b(t)S_b(t)(1-\rho+\rho x(t)))(\rho S_b(t) - (1-\rho)S_g(t)) \\ =& - \beta I_b(t)(I_g(t) - I_b(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ &- \beta(I_g(t) - I_b(t))S_g(t)(\rho + (1-\rho)x(t))(\rho S_b(t) - (1-\rho)S_g(t))\end{aligned}$$

The third term above is

$$\begin{aligned}&(I_g(t) - I_b(t))(\rho S'_b(t) - (1-\rho)S'_g(t)) \\ =& - \beta(I_g(t) - I_b(t))(\rho I_b(t)S_b(t) - (1-\rho)I_g(t)S_g(t)) \\ =& - \beta I_b(t)(I_g(t) - I_b(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ &+ \beta(I_g(t) - I_b(t))S_g(t)(1-\rho)(I_g(t) - I_b(t)).\end{aligned}$$

We know that from time 0 to t, $\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)} > 1$, which, together with the initial conditions $I_g(0) > I_b(0)$ and $S_g(0) = S_b(0)$, means that $I_g(t) > I_b(t)$ and $S_g(t) < S_b(t)$. In addition, since $\rho+(1-\rho)x(t) > 1-\rho$ and $\rho S_b(t)-(1-\rho)S_g(t) > \rho S_g(t)-(1-\rho)S_b(t) > I_g(t) - I_b(t)$, we can conclude that the sum of the second term and the third term is negative. For the first term, $S_g'(t)S_b(t)-S_b'(t)S_g(t) = -\beta S_g(t)S_b(t)(I_g(t)-I_b(t)) < 0$, and hence the numerator is negative, which means that $x'(t) < 0$.

Finally, as both $S_g(t)$ and $S_b(t)$ will become 0 given sufficient time, and $I_g(t) > I_b(t)$ as long as $x(t) > 0$ as shown above, we know that $x(t)$ must drop to 0 given sufficient time as well. This completes the proof.

□

After t_1, as the negative effect of elapsed time accumulates, B consumers must mix with a smaller and smaller probability on adoption, creating more positive information from signals to keep interim belief at ρ. This finally causes consumers to become sensitive to signals, which indicates the product's final "living" period. We describe the subsequent equilibrium behavior in the next subsection.

4.3 Signal Dependence and Non-Adoption

We have shown previously that consumers stop using mixed strategies when $\rho S_g(t) - I_g(t) = (1-\rho)S_b(t) - I_b(t)$. The following result proves that consumers become sensitive to signals for a period after t_2.

Lemma 7. *Let t_2 be the time at which $\rho S_g(t) - I_g(t) = (1-\rho)S_b(t) - I_b(t)$ for the first time. In every equilibrium, there exists $t_3 > t_2$ such that for all $t \in (t_2, t_3)$, consumers are sensitive to signals.*

Proof. From the proof of Lemma 4, we know that it is not possible in any equilibrium that consumers herd on adoption on $t \in (t_1, t_1+\epsilon)$ for any $\epsilon > 0$. Also, note that

$$\begin{aligned}&((\rho S_g(t) - I_g(t)) - ((1-\rho)S_b(t) - I_b(t)))' \\ =&-((\rho + p_g(t))S_g(t)I_g(t) - ((1-\rho) + p_b(t))S_b(t)I_b(t))\end{aligned}$$

and that $\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)} > 1$ from time 0 to t_2, which means that $(\rho S_g(t) - I_g(t)) - ((1-\rho)S_b(t) - I_b(t))$ is strictly decreasing at t_2, and thus it is also not possible in any equilibrium that B consumers mix with positive probability on adoption on $t \in (t_1, t_1+\epsilon)$ for any $\epsilon > 0$.

When consumers are sensitive to signals on (t_2, t_3) for some $t_3 > t_2$, we know that $(\rho S_g(t) - I_g(t)) < ((1-\rho)S_b(t) - I_b(t))$ when $t_3 \approx t_2$. Therefore $q(t;\emptyset) < \rho$ and being sensitive to signals is indeed optimal for consumers given their interim beliefs.

□

Compared with Lemma 3, the effect of being sensitive to signals on interim beliefs is exactly the opposite. While awareness still implies a series of good signals, the negative impact on beliefs from a longer time from launch to awareness can no longer be balanced out. Clearly, as interim beliefs keep falling if consumers are sensitive to signals, such behavior will not persist. We now characterize the time that it stops, as well as the equilibrium behavior afterwards.

Lemma 8. *Let t_2 be the time at which $\rho S_g(t) - I_g(t) = (1-\rho)S_b(t) - I_b(t)$ for the first time. In every equilibrium, after t_2, consumers are sensitive to signals until t_3 such that $q(t_3, \emptyset) = 1-\rho$, and herd on non-adoption afterwards.*

Proof. When $\alpha \in (1-\rho, \frac{1}{2}]$: for $t \in (0, t_3)$, $\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)} > \frac{1-\rho}{\rho}$ and hence $((\rho S_g(t) - I_g(t)) - ((1-\rho)S_b(t) - I_b(t)))' < 0$. Therefore, consumers will be sensitive to signals at least until t_3 when $q(t_3, \emptyset) = 1-\rho$.

Next, we prove that consumers must herd on non-adoption right after t_3. Suppose that there exists $\epsilon > 0$ such that in some equilibrium consumers are sensitive to signals on $(t_3, t_3+\epsilon)$. As $\rho S_g(t) - I_g(t) < (1-\rho)S_b(t) - I_b(t)$ at $t \approx t_3$, $q(t; \emptyset) < 1-\rho$ and consumers should herd on non-adoption instead, a contradiction.

Alternatively, suppose that there exists $\epsilon > 0$ such that in some equilibrium the G consumers use a mixed strategy with probability $y(t) \in (0,1)$ on $(t_3, t_2+\epsilon)$. We have

$$\frac{[I_g(t)S_g(t)]'}{[I_b(t)S_b(t)]'} = \frac{q(t;\emptyset)}{1-q(t;\emptyset)} \frac{\rho y(t) S_g(t) - I_g(t)}{(1-\rho)y(t)S_b(t) - I_b(t)}$$
$$y(t) = \frac{I_g(t) - I_b(t)}{\rho S_g(t) - (1-\rho)S_b(t)}.$$

However, when ϵ is small, $(\rho S_g(t) - I_g(t)) < ((1-\rho)S_b(t) - I_b(t))$; also, $\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)} > \frac{1-\rho}{\rho}$ for $t \in (0, t_3)$ implies that $I_g(t) > I_b(t)$ when $t \approx t_3$. From these two conditions, we know that either $y(t) < 0$ or $y(t) > 1$, a contradiction.

Hence, the only possible equilibrium behavior is that the consumers herd on non-adoption on $(t_3, t_2+\epsilon)$. In this case we have

$$\begin{aligned}
\frac{\left(\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)}\right)'}{\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)}} &= \frac{I_g'(t)}{I_g(t)} + \frac{S_g'(t)}{S_g(t)} - \frac{I_b'(t)}{I_b(t)} - \frac{S_b'(t)}{S_b(t)} \\
&= \beta(I_b(t) - I_g(t)) \\
&< 0,
\end{aligned}$$

which is consistent with the above equilibrium behavior. Also, as $I_b(t)$ and $I_g(t)$ stay constant when consumers herd on non-adoption, this means that consumers will not change their behavior after t_3.

When $\alpha \in (\frac{1}{2}, \rho)$: we prove that after t_2, interim beliefs will keep decreasing as long as consumers are sensitive to signals. Suppose not, then there exists $t' > t_2$ such that $\rho S_g(t') - I_g(t') = (1-\rho)S_b(t') - I_b(t')$ for the first time after t_2. It means further that there exists $t'' \in (t_2, t')$ such that $((\rho S_g(t) - I_g(t)) - ((1-\rho)S_b(t) - I_b(t)))' = 0$, i.e. $\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)} = \frac{1-\rho}{\rho}$. Therefore, $\frac{I_g(t')S_g(t')}{I_b(t')S_b(t')} < \frac{1-\rho}{\rho}$.

Let $a = S_g(t_2)$ and $b = S_b(t_2)$, and let $c = \rho S_g(t_2) - I_g(t_2) = (1-\rho)S_b(t_2) - I_b(t_2)$, $d = -(\rho S_g(t') - I_g(t')) = -((1-\rho)S_b(t') - I_b(t'))$. Since $S_l(t)$ is decreasing in t and $I_l(t)$ is increasing in t for $l = g, b$, we know that $c \geq 0$ and $d > 0$. We know that

$$\begin{aligned} c + d &= (\rho S_g(t_2) - I_g(t_2)) - (\rho S_g(t') - I_g(t')) \\ &= \int_{t_2}^{t'} 2\beta\rho I_g(t)S_g(t)dt = 2(I_g(t') - I_g(t_2)) = -2\rho(S_g(t') - S_g(t_2)) \\ &= \int_{t_2}^{t'} 2\beta(1-\rho)I_b(t)S_b(t)dt = 2(I_b(t') - I_b(t_2)) = -2(1-\rho)(S_b(t') - S_b(t_2)), \end{aligned}$$

which implies that

$$\begin{aligned} I_g(t') - I_g(t_2) = I_b(t') - I_b(t_2) &= \frac{c+d}{2} \\ S_g(t') - S_g(t_2) &= -\frac{c+d}{2\rho} \\ S_b(t') - S_b(t_2) &= -\frac{c+d}{2(1-\rho)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{I_g(t')S_g(t')}{I_b(t')S_b(t')} &= \frac{(I_g(t_2) + I_g(t') - I_g(t_2))(S_g(t_2) + S_g(t') - S_g(t_2))}{(I_b(t_2) + I_b(t') - I_b(t_2))(S_b(t_2) + S_b(t') - S_b(t_2))} \\ &= \frac{(a - \frac{c+d}{2\rho})((a\rho - c) + \frac{c+d}{2})}{(b - \frac{c+d}{2(1-\rho)}))((b(1-\rho) - c) + \frac{c+d}{2})} \\ &= \frac{a(a\rho - c) + \frac{c^2-d^2}{4\rho}}{b(b(1-\rho) - c) + \frac{c^2-d^2}{4(1-\rho)}} \end{aligned}$$

We already know that $\frac{a(a\rho-c)}{b(b(1-\rho)-c)} = \frac{(1-\alpha)\rho}{\alpha(1-\rho)} > 1$. Hence, no matter whether $c^2 - d^2 \geq 0$ or $c^2 - d^2 < 0$, $\frac{I_g(t')S_g(t')}{I_b(t')S_b(t')} > \frac{1-\rho}{\rho}$, a contradiction. This proves that interim beliefs will keep decreasing until t_3 such that $q(t_3; \emptyset) = 1 - \rho$.

As $I_g(t_2) > I_b(t_2)$ and $(\rho S_g(t_3) - I_g(t_3)) - ((1-\rho)S_b(t_3) - I_b(t_3)) < (\rho S_g(t_2) - I_g(t_2)) - ((1-\rho)S_b(t_2) - I_b(t_2)) = 0$, which implies that $I_g(t_3) - I_g(t_2) > I_b(t_3) - I_b(t_2)$, we know that $I_g(t_3) > I_b(t_3)$, and by our previous argument consumers must herd on non-adoption after t_3.

□

After t_2, interim beliefs go down with time even though consumers use the most informative strategy of being sensitive to signals. Hence when $q(t;\emptyset)$ reaches $1-\rho$, which means that G consumers become indifferent, there will not be a second period of mixed strategies, but consumers lose interest in the product once and for all.

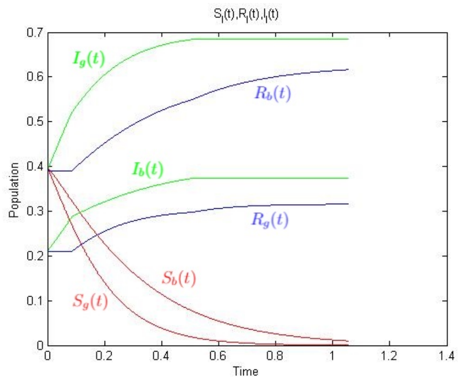
4.4 Special Cases in Equilibrium Characterization

Our analysis so far, which establishes different time cutoffs $t_1 < t_2 < t_3$, is based on two assumptions: (1) a sufficiently small Δ, and (2) $\rho S_g(t_1) - I_g(t_1) > (1-\rho)S_b(t_1) - I_b(t_1)$. In this subsection, we illustrate the consumer equilibrium assuming violation of either assumption.

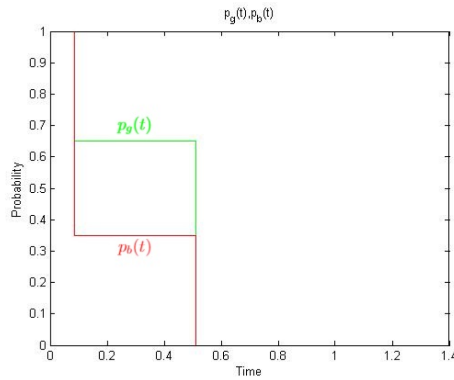
As it turns out, the two conditions are inter-connected: when Δ increases from infinitesimal, it must cause $\rho S_g(t) - I_g(t) < (1-\rho)S_b(t) - I_b(t)$ to occur when $q(t;\emptyset) = \rho$ (if $\alpha \in [\frac{1}{2}, \rho)$) or before $q(t;\emptyset) = \rho$ (if $\alpha \in (1-\rho, \frac{1}{2})$), before Δ reaches its maximum possible value 1. Alternatively, violation of (2) above may also occur if α is very close to ρ, which makes both $S_g(t)$ and $S_b(t)$ relatively small when $q(t;\emptyset) = \rho$.

On one hand, the equilibrium now is characterized by only two cutoffs as mixed strategies on $[t_1, t_2]$ in Theorem 1 have vanished. On the other hand, consumers' behavior around the first cutoff differs by α. When $\alpha \in (\frac{1}{2}, \rho)$, it means that consumers start being sensitive to signals immediately after their interim beliefs reach ρ; when $\alpha = \frac{1}{2}$, it means that consumers start being sensitive to signals immediately after $t = 0$; when $\alpha \in (1-\rho, \frac{1}{2})$, it means that consumers start being sensitive to signals as soon as $\rho S_g(t) - I_g(t) < (1-\rho)S_b(t) - I_b(t)$, which is before interim beliefs reach ρ. The following figures illustrate these three cases[5].

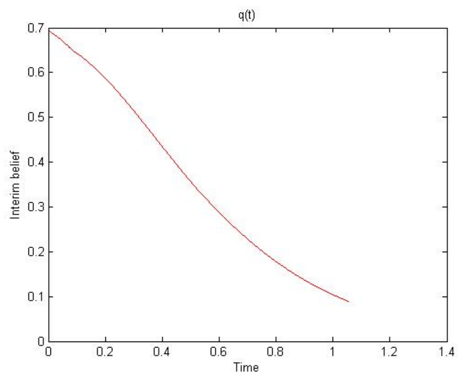
[5] In the simulation, $\alpha = 0.55$ for Figure 4, 0.5 for Figure 5 and 0.45 for Figure 6. We set $(\rho, \beta, \Delta) = (0.65, 10, 0.6)$ for Figures 4 and 5, and $(\rho, \beta, \Delta) = (0.65, 10, 0.3)$ for Figure 6. The difference in Δ is because a larger Δ is needed for consumers to start with being sensitive to signals when $\alpha = 0.5$, but the same Δ would have made Figures 5 and 6 identical. Hence we picked a smaller Δ when $\alpha \in (1-\rho, \frac{1}{2})$ to show that $q(t;\emptyset)$ may still increase initially even though $\rho S_g(t_1) - I_g(t_1) < (1-\rho)S_b(t_1) - I_b(t_1)$.



(a) Population of susceptible, infected and recovered consumers

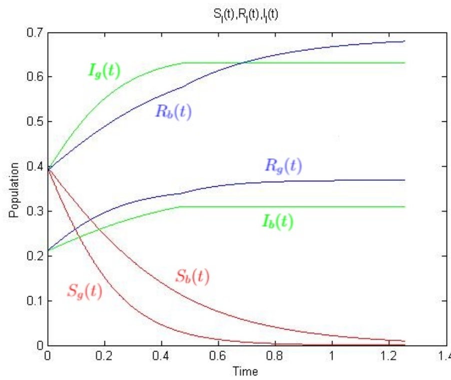


(b) Probability of product adoption

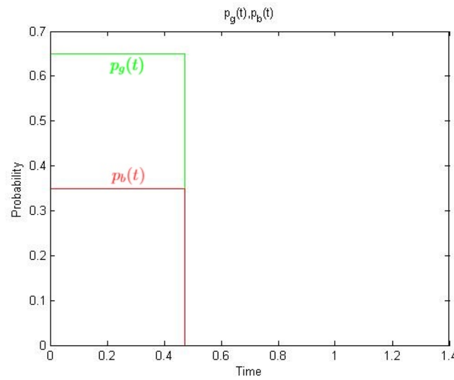


(c) Interim beliefs

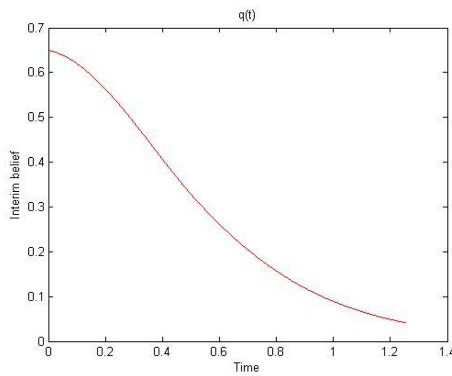
Figure 4: Simulation results when $\alpha \in (\frac{1}{2}, \rho)$



(a) Population of susceptible, infected and recovered consumers

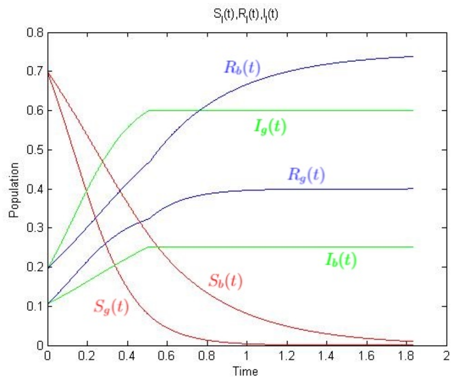


(b) Probability of product adoption

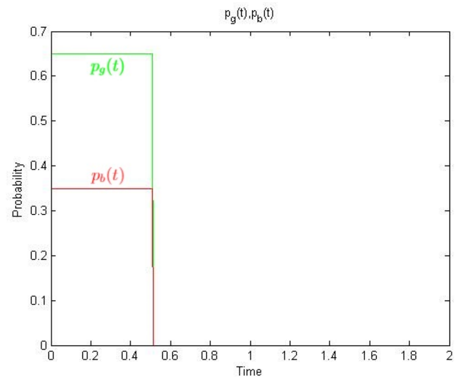


(c) Interim beliefs

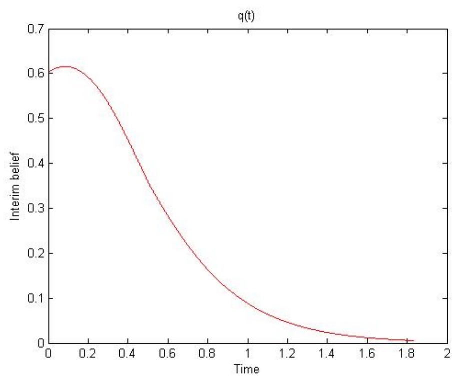
Figure 5: Simulation results when $\alpha = \frac{1}{2}$



(a) Population of susceptible, infected and recovered consumers



(b) Probability of product adoption



(c) Interim beliefs

Figure 6: Simulation results when $\alpha \in (1-\rho, \frac{1}{2})$

Our results so far depict a complete picture of a product's life cycle, given that uncertainty in quality is intrinsic and exogenously determined. Here consumers' beliefs quantify the perception that the true state is one of the possible two, i.e. the product has good quality.

The natural contrast to this case is that the market consists of many products of the same kind, the average quality of which is not exogenously given but stems from strategic decisions of their producers. Now there is only one true state – the number or fraction of producers that produce good products – but each consumer still faces uncertainty when exposed to a product: she does not know whether it originally came from a good or bad producer. The next section focuses on the epidemiological dynamics in this scenario, together with a number of other model extensions with strategic producers.

5 Learning Dynamics with Strategic Producers

In a world with competing producers, the most natural model appears to be one where a consumer may encounter the same kind of product, possibly produced by different producers, many times before she can make a final decision of which one to adopt. Such a model involves consumers' additional incentives of strategic delay or switching a currently adopting product for a better one in expectation. These issues are significant and demand future study, but will complicate the analysis to a great extent. Therefore, in this section we maintain the assumption that each consumer decides once and for all whether to adopt the product when she becomes aware of it. This assumption allows us to focus on how producer decisions affect the subsequent epidemiological dynamics among consumers, and study a canonical model that can lay a foundation for analyzing richer strategic behavior in future research.

Suppose that there can be multiple producers in the market, among which some produce good products and others produce bad products. The number of producers can be one, discretely many or a continuum, depending on the market structure that will be specified below. Each launches its product among a small measure of consumers, the sum of which is equal to $\Delta > 0$. Products are identical in appearance across producers, so consumers have no information on the producer's identity when they become aware. Therefore, a consumer's initial belief, if exposed to a product at time 0, becomes the fraction of good producers in the producer population. Without loss of generality, we also assume that each producer's variable cost, i.e. the cost incurred for producing more of the same product, is negligible. We will independently analyze three types of choice made by a strategic producer: quality, spread speed, and price. The numbers of good and bad producers are endogenously determined when quality can be chosen, while exogenously given in the other two cases.

5.1 Quality

Suppose that a producer can only produce a bad product initially, but can pay a cost $c > 0$ to produce a good product instead before launching it into the market[6]. The quality decision is unobservable by the other producers and the consumers. For notational simplicity we assume that the product's price is 1.

Consider $n \in \mathbb{N}^+$ producers, n_g being good ones and $n_b = n - n_g$ being bad ones. Given the quality choices which determine n_g and n_b, The epidemiological dynamics for each producer are governed by the following equations:

[6] To make a producer's revenue and cost comparable, we fix c and normalize the revenue for each market structure accordingly.

$$I_l'(t) = \beta I_l(t)S(t)p_l(t)$$
$$R_l'(t) = \beta I_l(t)S(t)(1 - p_l(t))$$
$$S'(t) = -\beta I(t)S(t),$$

where $l = g, b$ denotes its product's quality, and $S(t) = n_g S_g(t) + n_b S_b(t)$ and $I(t) = n_g I_g(t) + n_b I_b(t)$ are the corresponding aggregate measures of susceptible and infected consumers.

An equilibrium among producers essentially means that for each producer, its quality choice generates a weakly higher profit than the alternative given the other producers' choices. We focus on pure strategy equilibria. Conceivably, the equilibrium characterization hinges on the market structure, i.e. the total number of producers n. We analyze the three typical structures – monopoly, oligopoly and perfect competition – below.

Monopoly: $n = 1$. A producer's optimal choice depends on whether it has commitment power, or equivalently, whether a newly exposed consumer knows the number of good and bad producers in the market and hence forms a correct belief. It is straight forward to derive the equilibrium as follows.

Proposition 2. *The producer equilibrium in a monopoly market is unique. When the producer has no commitment power: the producer produces a bad product which is never launched. When the producer has commitment power, a good product is always launched if $c < 1$ and is never launched if $c > 1$.*

Proof. Suppose that the producer has no commitment power, and suppose that there exists an equilibrium where it produces a good product. Since the initial belief is 1, consumers will herd on adoption forever, which in turn means that the producer should produce a bad product and save the cost instead, a contradiction.

Suppose that the producer has commitment power. Producing a good product generates revenue 1 while a bad product 0; hence, the producer should launch a good product if $c > 1$ and a bad one if $c < 1$. □

Oligopoly: $n \geq 2$. Given n, an equilibrium is characterized by the number n_g. We start with the case that producers have no commitment power.

Our first observation is that, different from the case with exogenous quality uncertainty, interim beliefs stay constant when consumers herd on adoption while always increase in time when consumers are sensitive to signals.

Lemma 9. *Consider arbitrary $t > 0$ and $\epsilon > 0$. For any time period $(t, t+\epsilon)$, if consumers herd on adoption, $q(t';\emptyset) = q(t;\emptyset)$; if consumers are sensitive to signals or B consumers mix with probability $x(t) \in (0,1)$ on adoption, $q(t';\emptyset)$ is increasing in t.*

Proof. Note that

$$(\frac{I_g(t)S(t)}{I_b(t)S(t)})' = (\frac{I_g(t)}{I_b(t)})' = \frac{I_g(t)}{I_b(t)}S(t)(p_g(t) - p_b(t)).$$

If consumers herd on adoption, $p_g(t) = p_b(t) = 1$. Then $(\frac{I_g(t)S(t)}{I_b(t)S(t)})' = 0$, which means that interim beliefs stay unchanged. If consumers are sensitive to signals, $p_g(t) = \rho + (1-\rho)x(t)$ while $p_b(t) = 1 - \rho + \rho x(t)$, and thus $p_g(t) > p_b(t)$ and $(\frac{I_g(t)S(t)}{I_b(t)S(t)})' > 0$. Therefore, interim beliefs increase in t. □

As mentioned before, there is only one true state, characterized by n_g, when quality of each product is endogenously determined. This essentially eliminates the effect of elapsed time on interim beliefs, so the only update a consumer conducts when aware of the product is from her predecessors' strategies.

Lemma 9 provides a clear characterization of consumer equilibrium after quality decisions are made, as well as a producer's payoff from each decision. If $\frac{n_g}{n} \geq \frac{1}{2}$, herding on adoption starts right after $t = 0$. When the initial exposure Δ is sufficiently small, each good producer earns approximately $\frac{\rho}{\frac{n_g}{n}\rho + (1-\frac{n_g}{n})(1-\rho)}$ and each bad producer earns approximately $\frac{1-\rho}{\frac{n_g}{n}\rho + (1-\frac{n_g}{n})(1-\rho)}$. If a good producer chooses to be a bad one instead, it saves cost c but earns $\frac{1-\rho}{\frac{n_g-1}{n}\rho + (1-\frac{n_g-1}{n})(1-\rho)}$; if a bad producer chooses to be a good one instead, it pays cost c but earns $\frac{\rho}{\frac{n_g+1}{n}\rho + (1-\frac{n_g+1}{n})(1-\rho)}$.

On the other hand, suppose that $\frac{n_g}{n} \in (1-\rho, \frac{1}{2})$. Initially, consumers are sensitive to signals. When Δ is sufficiently small, interim beliefs will rise above ρ when $S(t)$ is still sufficiently close to 1 and then become constant. Thus each good producer earns approximately $\frac{(1-\frac{n_g}{n})\rho}{\frac{n_g}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g}{n})\frac{n_g}{n}(1-\rho)} = \frac{\rho}{\frac{n_g}{n}}$ and each bad producer earns approximately $\frac{\frac{n_g}{n}(1-\rho)}{\frac{n_g}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g}{n})\frac{n_g}{n}(1-\rho)} = \frac{1-\rho}{1-\frac{n_g}{n}}$. If a good producer chooses to be a bad one instead, it saves cost c but earns $\frac{\frac{n_g}{n}(1-\rho)}{\frac{n_g-1}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g-1}{n})\frac{n_g}{n}(1-\rho)}$; if a bad producer chooses to be a good one instead, it pays cost c but earns $\frac{(1-\frac{n_g}{n})\rho}{\frac{n_g+1}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g+1}{n})\frac{n_g}{n}(1-\rho)}$.

Now we are ready to characterize producer equilibria. Let $\lfloor \cdot \rfloor$ denote the floor function, and $\lceil \cdot \rceil$ the ceiling function; let $\bar{n}_g := \lfloor n\rho \rfloor$, $\underline{n}_g := \lceil n(1-\rho) \rceil$ and $n_g^* := \lceil \frac{n}{2} \rceil$. The following result summarizes our findings.

Proposition 3. *In an oligopoly market, when producers have no commitment power:*

1. *There always exists a null equilibrium where* $n_g = 0$*.*

2. *When*

$$c \in \Big(\frac{\rho}{\frac{\bar{n}_g+1}{n}\rho + (1-\frac{\bar{n}_g+1}{n})(1-\rho)} - \frac{1-\rho}{\frac{\bar{n}_g}{n}\rho + (1-\frac{\bar{n}_g}{n})(1-\rho)}, \frac{\rho}{\frac{n_g^*}{n}\rho + (1-\frac{n_g^*}{n})(1-\rho)} - \frac{1-\rho}{\frac{n_g^*-1}{n}\rho + (1-\frac{n_g^*-1}{n})(1-\rho)}\Big],$$

the equilibria besides $n_g = 0$ *are characterized by the following condition:*

$$c \in \Big[\frac{\rho}{\frac{n_g+1}{n}\rho + (1-\frac{n_g+1}{n})(1-\rho)} - \frac{1-\rho}{\frac{n_g}{n}\rho + (1-\frac{n_g}{n})(1-\rho)}, \frac{\rho}{\frac{n_g}{n}\rho + (1-\frac{n_g}{n})(1-\rho)} - \frac{1-\rho}{\frac{n_g-1}{n}\rho + (1-\frac{n_g-1}{n})(1-\rho)}\Big].$$

Generically, there is a unique such equilibrium.

3. *When*

$$c \in \Big(\frac{\rho}{\frac{n_g^*}{n}\rho + (1-\frac{n_g^*}{n})(1-\rho)} - \frac{1-\rho}{\frac{n_g^*-1}{n}\rho + (1-\frac{n_g^*-1}{n})(1-\rho)}, \frac{(1-\frac{n_g^*-1}{n})\rho}{\frac{n_g^*}{n}(1-\frac{n_g^*-1}{n})\rho + (1-\frac{n_g^*}{n})\frac{n_g^*-1}{n}(1-\rho)} - \frac{\frac{n_g^*-1}{n}(1-\rho)}{\frac{n_g^*-1}{n}(1-\frac{n_g^*-1}{n})\rho + (1-\frac{n_g^*-1}{n})\frac{n_g^*-1}{n}(1-\rho)}\Big),$$

$n_g = 0$ *is the only equilibrium.*

4. *When*

$$c \in \Big[\frac{(1-\frac{n_g^*-1}{n})\rho}{\frac{n_g^*}{n}(1-\frac{n_g^*-1}{n})\rho + (1-\frac{n_g^*}{n})\frac{n_g^*-1}{n}(1-\rho)} - \frac{\frac{n_g^*-1}{n}(1-\rho)}{\frac{n_g^*-1}{n}(1-\frac{n_g^*-1}{n})\rho + (1-\frac{n_g^*-1}{n})\frac{n_g^*-1}{n}(1-\rho)}, \frac{(1-\frac{n_g}{n})\rho}{\frac{n_g}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g}{n})\frac{n_g}{n}(1-\rho)} - \frac{\frac{n_g}{n}(1-\rho)}{\frac{n_g-1}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g-1}{n})\frac{n_g}{n}(1-\rho)}\Big),$$

the equilibria besides $n_g = 0$ *are characterized by the following condition:*

$$c \in \Big[\frac{(1-\frac{n_g}{n})\rho}{\frac{n_g+1}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g+1}{n})\frac{n_g}{n}(1-\rho)} - \frac{\frac{n_g}{n}(1-\rho)}{\frac{n_g}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g}{n})\frac{n_g}{n}(1-\rho)}, \frac{(1-\frac{n_g}{n})\rho}{\frac{n_g}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g}{n})\frac{n_g}{n}(1-\rho)} - \frac{\frac{n_g}{n}(1-\rho)}{\frac{n_g-1}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g-1}{n})\frac{n_g}{n}(1-\rho)}\Big].$$

Generically, there is a unique such equilibrium.

5. *When*

$$c > \frac{(1-\frac{n_g}{n})\rho}{\frac{n_g}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g}{n})\frac{n_g}{n}(1-\rho)} - \frac{\frac{n_g}{n}(1-\rho)}{\frac{n_g-1}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g-1}{n})\frac{n_g}{n}(1-\rho)}$$

or

$$c < \frac{\rho}{\frac{\bar{n}_g+1}{n}\rho + (1-\frac{\bar{n}_g+1}{n})(1-\rho)} - \frac{1-\rho}{\frac{\bar{n}_g}{n}\rho + (1-\frac{\bar{n}_g}{n})(1-\rho)},$$

$n_g = 0$ *is the only equilibrium.*

Proof. Clearly, since producers have no commitment power, there is always a null equilibrium where $n_g = 0$. This proves 1.

To prove 2, suppose that $\frac{n_g}{n} \in [\frac{1}{2}, \rho)$. Note that

$$\begin{aligned}\frac{\rho}{1-\rho} &\geq \frac{\frac{n_g+1}{n}\rho + (1-\frac{n_g+1}{n})(1-\rho)}{\frac{n_g-1}{n}\rho + (1-\frac{n_g-1}{n})(1-\rho)} \\ &= \frac{(\frac{n_g+1}{n}\rho + (1-\frac{n_g+1}{n})(1-\rho))(\frac{n_g}{n}\rho + (1-\frac{n_g}{n})(1-\rho))}{(\frac{n_g-1}{n}\rho + (1-\frac{n_g-1}{n})(1-\rho))(\frac{n_g}{n}\rho + (1-\frac{n_g}{n})(1-\rho))},\end{aligned}$$

which implies that

$$\begin{aligned}&\frac{\rho}{\frac{n_g}{n}\rho + (1-\frac{n_g}{n})(1-\rho)} - \frac{\rho}{\frac{n_g+1}{n}\rho + (1-\frac{n_g+1}{n})(1-\rho)} \\ \geq &\frac{1-\rho}{\frac{n_g-1}{n}\rho + (1-\frac{n_g-1}{n})(1-\rho)} - \frac{1-\rho}{\frac{n_g}{n}\rho + (1-\frac{n_g}{n})(1-\rho)} \\ &\frac{\rho}{\frac{n_g}{n}\rho + (1-\frac{n_g}{n})(1-\rho)} - \frac{1-\rho}{\frac{n_g-1}{n}\rho + (1-\frac{n_g-1}{n})(1-\rho)} \\ \geq &\frac{\rho}{\frac{n_g+1}{n}\rho + (1-\frac{n_g+1}{n})(1-\rho)} - \frac{1-\rho}{\frac{n_g}{n}\rho + (1-\frac{n_g}{n})(1-\rho)}.\end{aligned}$$

Hence, a sufficient and necessary condition for n_g to be an equilibrium is

$$\begin{aligned}c \in &[\frac{\rho}{\frac{n_g+1}{n}\rho + (1-\frac{n_g+1}{n})(1-\rho)} - \frac{1-\rho}{\frac{n_g}{n}\rho + (1-\frac{n_g}{n})(1-\rho)}, \\ &\frac{\rho}{\frac{n_g}{n}\rho + (1-\frac{n_g}{n})(1-\rho)} - \frac{1-\rho}{\frac{n_g-1}{n}\rho + (1-\frac{n_g-1}{n})(1-\rho)}].\end{aligned}$$

To prove 3, suppose that $\frac{n_g}{n} \in (1-\rho, \frac{1}{2})$. Note that

$$\frac{(1-\frac{n_g}{n})\rho}{\frac{n_g}{n}(1-\rho)} \geq \frac{\frac{n_g+1}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g+1}{n})\frac{n_g}{n}(1-\rho)}{\frac{n_g-1}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g-1}{n})\frac{n_g}{n}(1-\rho)}$$
$$= \frac{(\frac{n_g+1}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g+1}{n})\frac{n_g}{n}(1-\rho))(\frac{n_g}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g}{n})\frac{n_g}{n}(1-\rho))}{(\frac{n_g-1}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g-1}{n})\frac{n_g}{n}(1-\rho))(\frac{n_g}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g}{n})\frac{n_g}{n}(1-\rho))},$$

which implies that

$$\frac{(1-\frac{n_g}{n})\rho}{\frac{n_g}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g}{n})\frac{n_g}{n}(1-\rho)} - \frac{(1-\frac{n_g}{n})\rho}{\frac{n_g+1}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g+1}{n})\frac{n_g}{n}(1-\rho)}$$
$$\geq \frac{\frac{n_g}{n}(1-\rho)}{\frac{n_g-1}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g-1}{n})\frac{n_g}{n}(1-\rho)} - \frac{\frac{n_g}{n}(1-\rho)}{\frac{n_g}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g}{n})\frac{n_g}{n}(1-\rho)}$$
$$\frac{(1-\frac{n_g}{n})\rho}{\frac{n_g}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g}{n})\frac{n_g}{n}(1-\rho)} - \frac{\frac{n_g}{n}(1-\rho)}{\frac{n_g-1}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g-1}{n})\frac{n_g}{n}(1-\rho)}$$
$$\geq \frac{(1-\frac{n_g}{n})\rho}{\frac{n_g+1}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g+1}{n})\frac{n_g}{n}(1-\rho)} - \frac{\frac{n_g}{n}(1-\rho)}{\frac{n_g}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g}{n})\frac{n_g}{n}(1-\rho)}.$$

Hence, a sufficient and necessary condition for n_g to be an equilibrium is

$$c \in [\frac{(1-\frac{n_g}{n})\rho}{\frac{n_g+1}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g+1}{n})\frac{n_g}{n}(1-\rho)} - \frac{\frac{n_g}{n}(1-\rho)}{\frac{n_g}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g}{n})\frac{n_g}{n}(1-\rho)},$$
$$\frac{(1-\frac{n_g}{n})\rho}{\frac{n_g}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g}{n})\frac{n_g}{n}(1-\rho)} - \frac{\frac{n_g}{n}(1-\rho)}{\frac{n_g-1}{n}(1-\frac{n_g}{n})\rho + (1-\frac{n_g-1}{n})\frac{n_g}{n}(1-\rho)}].$$

Finally, it remains to determine whether the largest possible value of $\frac{\rho}{\frac{n_g}{n}\rho+(1-\frac{n_g}{n})(1-\rho)} - \frac{1-\rho}{\frac{n_g-1}{n}\rho+(1-\frac{n_g-1}{n})(1-\rho)}$ (when $\frac{n_g}{n} \geq \frac{1}{2}$) is greater or smaller than the smallest possible value of $\frac{(1-\frac{n_g}{n})\rho}{\frac{n_g+1}{n}(1-\frac{n_g}{n})\rho+(1-\frac{n_g+1}{n})\frac{n_g}{n}(1-\rho)} - \frac{\frac{n_g}{n}(1-\rho)}{\frac{n_g}{n}(1-\frac{n_g}{n})\rho+(1-\frac{n_g}{n})\frac{n_g}{n}(1-\rho)}$ (when $\frac{n_g}{n} < \frac{1}{2}$). There are two possible cases.

Case 1: N is even. The former value is equal to $2\rho - \frac{1-\rho}{(\frac{1}{2}-\frac{1}{n})\rho+(\frac{1}{2}+\frac{1}{n})(1-\rho)}$, and the latter is equal to $\frac{(\frac{1}{2}+\frac{1}{n})2\rho}{(\frac{1}{2}+\frac{1}{n})\rho+(\frac{1}{2}-\frac{1}{n})(1-\rho)} - \frac{1-\rho}{\frac{1}{2}+\frac{1}{n}}$. Since $\frac{(\frac{1}{2}+\frac{1}{n})}{(\frac{1}{2}+\frac{1}{n})\rho+(\frac{1}{2}-\frac{1}{n})(1-\rho)} > 1$ while $\frac{1}{2}+\frac{1}{n} > (\frac{1}{2}-\frac{1}{n})\rho + (\frac{1}{2}+\frac{1}{n})(1-\rho)$, the latter value is greater.

Case 2: N is odd. The former value is equal to $\frac{\rho}{(\frac{1}{2}+\frac{1}{2n})\rho+(\frac{1}{2}-\frac{1}{2n})(1-\rho)} - \frac{1-\rho}{(\frac{1}{2}-\frac{1}{2n})\rho+(\frac{1}{2}+\frac{1}{2n})(1-\rho)}$, and the latter is equal to $\frac{(\frac{1}{2}+\frac{1}{2n})\rho}{(\frac{1}{2}+\frac{1}{2n})^2\rho+(\frac{1}{2}-\frac{1}{2n})^2(1-\rho)} - \frac{1-\rho}{\frac{1}{2}+\frac{1}{2n}}$. Since $\frac{\frac{1}{2}+\frac{1}{2n}}{(\frac{1}{2}+\frac{1}{2n})^2\rho+(\frac{1}{2}-\frac{1}{2n})^2(1-\rho)} >$

$\frac{1}{(\frac{1}{2}+\frac{1}{2n})\rho+(\frac{1}{2}-\frac{1}{2n})(1-\rho)}$ while $\frac{1}{2}+\frac{1}{2n} > (\frac{1}{2}-\frac{1}{2n})\rho+(\frac{1}{2}+\frac{1}{2n})(1-\rho)$, the latter value is greater. Hence 4 is proved.

Finally, when the cost c falls into the regions specified in 5, an equilibrium n_g can only be such that either $\frac{n_g}{n} > \rho$ or $\frac{n_g}{n} < 1-\rho$. The former case is impossible as every good producer would deviate, earning the same revenue while saving the cost; the latter case is only possible when $n_g = 0$, as no producer is willing to pay the cost but make no sales. This completes the proof. □

One way to interpret this equilibrium is a from a sequential entry context: consider many potential producers, and suppose that there is a period of time before the product launch when each producer can choose whether to enter the market at any instant. Entry is observable by all other producers. Therefore, an equilibrium n_g is such that after the first n_g producers enter, neither will any entrant regret its decision nor will any additional producer has an incentive to enter.

When the competing producers have commitment power, it only makes a difference when the decision of one producer can switch the value of $\frac{n_g}{n}$ across the four regions $(0, 1-\rho)$, $(1-\rho, \frac{1}{2})$, $[\frac{1}{2}, \rho)$ and $(\rho, 1)$. This property is not generic and will become impossible when n gets large, so we choose not to discuss it in this paper.

Perfect competition: continuum of producers. Suppose that there is a unit mass of initially identical producers. With some abuse of notation, let α be the fraction of good ones. In the epidemiological dynamics, $S(t) = \alpha S_g(t) + (1-\alpha)S_b(t)$ and $I(t) = \alpha I_g(t) + (1-\alpha)I_b(t)$ after normalization. Now it does not matter whether a producer has commitment power since it can affect neither consumers' beliefs nor the aggregate SIR dynamics. An equilibrium is characterized by the fraction of good producers α.

We find that, similar to the classical Cournot model, the equilibrium among producers (besides $\alpha = 0$) under perfect competition is essentially the limit of the equilibrium in an oligopoly market as $n_g \to \infty$.

Proposition 4. *In a perfectly competitive market:*

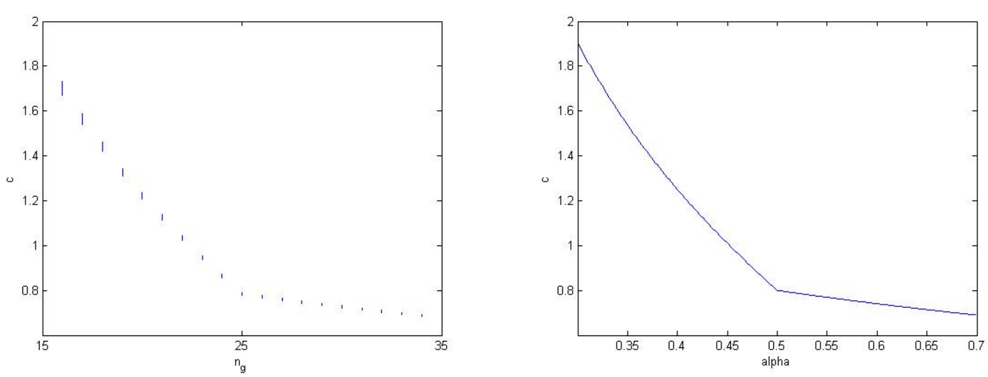
> *1. When* $c > \frac{2\rho-1}{2\rho(1-\rho)}$ *or* $c < \frac{2\rho-1}{\rho^2+(1-\rho)^2}$, $\alpha = 0$ *is the only equilibrium.*
>
> *2. When* $c \in (\frac{2\rho-1}{\rho^2+(1-\rho)^2}, \frac{2\rho-1}{\rho(1-\rho)})$, *there exists one and only one equilibrium besides* $\alpha = 0$. *In this equilibrium,* α *is (approximately) characterized by* $f(\alpha) = c$, *where* $f(\alpha)$ *is*
>
> $$\begin{cases} \frac{\rho-\alpha}{\alpha(1-\alpha)} \text{ if } \alpha \in (1-\rho, \frac{1}{2}) \\ \frac{2\rho-1}{\alpha\rho+(1-\alpha)(1-\rho)} \text{ if } \alpha \in [\frac{1}{2}, \rho). \end{cases}$$

Proof. The first observation is that α can never exceed ρ in equilibrium; otherwise, every good producer will deviate and produce a bad product. Similarly, it is not possible that $\alpha \in (0, 1-\rho)$ in equilibrium. Second, there is always a null equilibrium where $\alpha = 0$.

Therefore, if $\alpha \in [\frac{1}{2}, \rho)$, herding on adoption occurs immediately after time 0 and persists afterwards. In this case, when the initial exposure Δ is sufficiently small, each good producer earns approximately $\frac{\rho}{\alpha\rho+(1-\alpha)(1-\rho)}$ and each bad producer earns approximately $\frac{1-\rho}{\alpha\rho+(1-\alpha)(1-\rho)}$. Hence, a sufficient and necessary condition for α to be an equilibrium is $\frac{2\rho-1}{\alpha\rho+(1-\alpha)(1-\rho)} = c$.

If $\alpha \in (1-\rho, \frac{1}{2})$, when the initial exposure Δ is sufficiently small, the consumers' belief $\frac{q(t;\emptyset)}{1-q(t;\emptyset)} = \frac{\alpha I_g(t)S(t)}{(1-\alpha)I_b(t)S(t)}$ will rise above $\frac{\rho}{1-\rho}$ when $S(t)$ is still sufficiently close to 1, and thus each good producer earns approximately $\frac{(1-\alpha)\rho}{\alpha(1-\alpha)\rho+(1-\alpha)\alpha(1-\rho)} = \frac{\rho}{\alpha}$ and each bad producer earns approximately $\frac{\alpha(1-\rho)}{\alpha(1-\alpha)\rho+(1-\alpha)\alpha(1-\rho)} = \frac{1-\rho}{1-\alpha}$. Hence, a sufficient and necessary condition for α to be an equilibrium is $\frac{\rho-\alpha}{\alpha(1-\alpha)} = c$. Note that $\frac{2\rho-1}{\alpha\rho+(1-\alpha)(1-\rho)}$ and $\frac{\rho-\alpha}{\alpha(1-\alpha)}$ are decreasing in α, and that $\frac{2\rho-1}{\frac{1}{2}\rho+(1-\frac{1}{2})(1-\rho)} = \frac{\rho-\frac{1}{2}}{\frac{1}{2}(1-\frac{1}{2})}$. □

The following figures illustrate the equilibrium derived in Propositions 5 and 6[7]. The relation between c and the equilibrium fraction of good producers constitutes a "demand curve of quality improvement", and it is also clear from the figure that the oligopoly equilibrium converges to the perfectly competitive one as n goes to infinity.



(a) Oligopoly equilibrium (b) Perfectly competitive equilibrium

Figure 7: Producer equilibrium with competition

[7]We set $\rho = 0.7$, and $n = 50$ for Figure 7(a).

Welfare. With viral social learning, when does competition among producers improves welfare, and when does it not? We compare social welfare in two representative cases: a monopoly with commitment power and perfect competition. It is without much loss of generality to skip the case of oligopoly as it simply lies between the two extremes.

We assume that the quality difference is sufficient for a monopoly producer choosing high quality to be socially desirable, i.e. $\bar{u} - \underline{u} \geq 1$. Social welfare under monopoly, denoted V_m, is

$$\begin{cases} \bar{u} - c \text{ if } c \leq 1 \\ \underline{u} \text{ if } c > 1. \end{cases}$$

Social welfare under perfect competition, denoted V_c, is

$$\begin{cases} \frac{\alpha\rho\bar{u} + (1-\alpha)(1-\rho)\underline{u}}{\alpha\rho + (1-\alpha)(1-\rho)} - \alpha c \text{ if } c \in (f^{-1}(\rho), f^{-1}(\frac{1}{2})] \\ \rho\bar{u} + (1-\rho)\underline{u} - \alpha c \text{ if } c \in (f^{-1}(\frac{1}{2}), f^{-1}(1-\rho)) \\ \underline{u} \text{ if } c > f^{-1}(1-\rho) \text{ or } c < f^{-1}(\rho), \end{cases}$$

where $\alpha = f^{-1}(c)$ with $f(\cdot)$ as defined in Proposition 4. Alternatively, we can express V_c in terms of equilibrium α, and after simplification we get

$$\begin{cases} \underline{u} + \frac{\alpha}{\alpha\rho + (1-\alpha)(1-\rho)}(\rho(\bar{u} - \underline{u}) - (2\rho - 1)) \text{ if } \alpha \in [\frac{1}{2}, \rho) \\ \underline{u} + \rho(\bar{u} - \underline{u}) - (2\rho - 1) - \frac{(1-2\alpha)(1-\rho)}{1-\alpha} \text{ if } \alpha \in (1-\rho, \frac{1}{2}) \\ \underline{u} \text{ if } c > f^{-1}(1-\rho) \text{ or } c < f^{-1}(\rho). \end{cases}$$

In the following result, we show how the socially preferred market structure depends on the value of c.

Proposition 5. *The comparison of social welfare between monopoly and perfect competition is as follows.*

1. *When $c < f^{-1}(\rho)$, $V_m > V_c$.*
2. *When $c > f^{-1}(1-\rho)$, $V_m = V_c$.*
3. *When $c \in (f^{-1}(\rho), f^{-1}(1-\rho))$:*
   - *If $(1-\rho)(\bar{u} - \underline{u}) < (2\rho - 1)$, $V_m < V_c$.*
   - *If $(1-\rho)(\bar{u} - \underline{u}) > (2\rho - 1)$, $V_m > V_c$ when $c < \min\{1, f^{-1}(\frac{\rho}{(1-\rho)(\bar{u}-\underline{u})+1})\}$ and $V_m < V_c$ when $c > \min\{1, f^{-1}(\frac{\rho}{(1-\rho)(\bar{u}-\underline{u})+1})\}$.*

Proof. Note that $\frac{\alpha}{\alpha\rho+(1-\alpha)(1-\rho)}$ is increasing in α and is greater than 1 when $\alpha \geq \frac{1}{2}$, and that $\frac{(1-2\alpha)(1-\rho)}{1-\alpha}$ is decreasing in α and greater than 0 when $\alpha < \frac{1}{2}$.

When $c > f^{-1}(1-\rho)$ or $c < f^{-1}(\rho)$: $V_c = \underline{u}$.

When $c \in (f^{-1}(\rho), f^{-1}(1-\rho))$: V_c is increasing in α. We first compare V_c with $\underline{u}$, the value of V_m when $c > 1$. Since $\bar{u} - \underline{u} \geq 1$ by assumption, $\rho(\bar{u} - \underline{u}) - (2\rho - 1) \geq 1 - \rho > 0$. Also, as $\rho(\bar{u} - \underline{u}) - (2\rho - 1) > \frac{(1-2\alpha)(1-\rho)}{1-\alpha}$ is equivalent to $\rho(\bar{u} - \underline{u}) - \rho > (\rho(\bar{u} - \underline{u}) - 1)\alpha$, which is always satisfied for $\alpha \in (1-\rho, \frac{1}{2})$, we know that $V_c > \underline{u}$.

We then compare social welfare under perfect competition with $\bar{u} - c$, the value of V_m when $c < 1$. When c is such that $\alpha \in [\frac{1}{2}, \rho)$, the difference $V_m - V_c$ is

$$\begin{aligned}
&\bar{u} - c - (\frac{\alpha\rho\bar{u} + (1-\alpha)(1-\rho)\underline{u}}{\alpha\rho + (1-\alpha)(1-\rho)} - \alpha c) \\
=&\frac{(1-\alpha)(1-\rho)(\bar{u} - \underline{u})}{\alpha\rho + (1-\alpha)(1-\rho)} - (1-\alpha)\frac{2\rho - 1}{\alpha\rho + (1-\alpha)(1-\rho)} \\
=&\frac{(1-\alpha)((1-\rho)(\bar{u} - \underline{u}) - (2\rho - 1))}{\alpha\rho + (1-\alpha)(1-\rho)}.
\end{aligned}$$

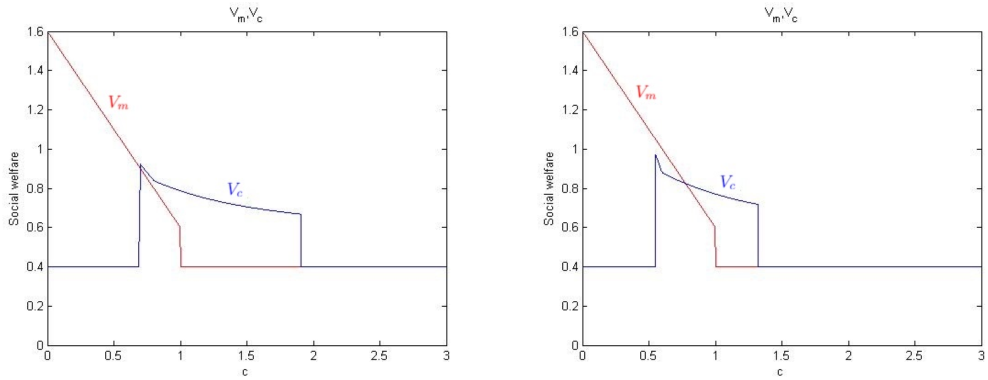
This difference is positive when $(1-\rho)(\bar{u} - \underline{u}) > (2\rho - 1)$ and negative when $(1-\rho)(\bar{u} - \underline{u}) < (2\rho - 1)$. When c is such that $\alpha \in (1-\rho, \frac{1}{2})$, $V_m - V_c$ is

$$\begin{aligned}
&\bar{u} - c - (\rho\bar{u} + (1-\rho)\underline{u} - \alpha c) \\
=&(1-\rho)(\bar{u} - \underline{u}) - (\frac{\rho}{\alpha} - 1).
\end{aligned}$$

If $(1-\rho)(\bar{u} - \underline{u}) < (2\rho - 1)$, the difference is negative for all $\alpha \in (1-\rho, \frac{1}{2})$; if $(1-\rho)(\bar{u} - \underline{u}) > (2\rho - 1)$, the difference is positive when $\alpha > \frac{\rho}{(1-\rho)(\bar{u}-\underline{u})+1}$ and negative when $\alpha < \frac{\rho}{(1-\rho)(\bar{u}-\underline{u})+1}$. □

The following figures depict how social welfare changes with c under the two market structures[8].

[8] We set $\bar{u} = 1.6$ and $\underline{u} = 0.4$. The value of ρ is 0.7 for Figure 8(a) and 0.65 for Figure 8(b).



(a) $(1-\rho)(\bar{u}-\underline{u}) < (2\rho-1)$ (b) $(1-\rho)(\bar{u}-\underline{u}) > (2\rho-1)$

Figure 8: Social welfare under two market structures

The first and second observations in Proposition 5 are quite clear. When quality can be improved at a low cost, a monopoly producer avoids the free-riding incentive problem faced by competing producers and can ensure uniform good quality. When the cost is too high, under neither market structure is a producer willing to launch a good product.

For the third one, note that with competition, the higher the proportion of bad products, the more socially desirable a good product is: a higher fraction of bad products means that a good product earns a higher market share, which then results in provision of $\bar{u}$ to more consumers at the same cost. Hence, perfect competition is better than monopoly in two circumstances: (1) the difference in quality is not very large so that it is cost-efficient to have a mixture of good and bad products in the market, but under monopoly it is either all good products or all bad ones; (2) the quality improvement cost is high, which induces a monopoly to stop producing high quality products, but it is still socially beneficial to have some good products which can be achieved under perfect competition.

5.2 Spread Speed

As shown in the previous subsection, commitment power has no significant impact on equilibrium for most cases with competing producers, and equilibrium behavior under oligopoly and perfect competition is similar. Hence, in this subsection and the next, we will focus on the case of perfect competition for a convenient exposition of results.

Suppose now that the choice variable of a producer at $t = 0$ is its spread speed

β. In particular, a producer can spread its product at speed β by paying cost $C(\beta)$. We assume regular conditions on $C(\beta)$: $C(0) = C'(0) = 0$, $C'(\beta)$ and $C''(\beta)$ are both positive.

We focus on symmetric equilibria, i.e. each producer chooses β_g and each bad producer choose β_b. Profits of the two types of producers can be written as

$$\pi_g(\beta_g) = I_g(0) + \beta_g \int_0^\infty (I_g(t)S(t)p_g(t))dt - C(\beta_g)$$
$$\pi_b(\beta_b) = I_b(0) + \beta_b \int_0^\infty (I_b(t)S(t)p_b(t))dt - C(\beta_b).$$

Hence their profit-maximizing conditions are

$$\int_0^\infty (I_g(t)S(t)p_g(t))dt = C'(\beta_g)$$
$$\int_0^\infty (I_b(t)S(t)p_b(t))dt = C'(\beta_b),$$

which can be further approximated as

$$I_g(\infty) = \beta_g C'(\beta_g)$$
$$I_b(\infty) = \beta_b C'(\beta_b)$$

when Δ is sufficiently small (so that $I_g(0)$ and $I_b(0)$ are sufficiently small). Note that the right-hand side of both equations are increasing in β given our assumptions on the cost function.

We find that good and bad producers will not choose similar spread speed in equilibrium unless the fraction of good producers is particularly high.

Proposition 6. *Assume that $\alpha < \rho$. In a producer equilibrium with positive β_g or β_b, either $\beta_g > \beta_b$ or $\rho\beta_g < (1-\rho)\beta_b$.*

Proof. Suppose otherwise that $\frac{\beta_g}{\beta_b} \in [\frac{1-\rho}{\rho}, 1]$. We discuss two possible cases.

First, suppose that consumers herd on non-adoption at $t = 0$. Clearly the profit of each producer is zero and no producer will choose a positive spread speed.

Second, suppose that G consumers adopt (with positive probability) at $t = 0$, i.e. $\alpha \in [1-\rho, \rho)$. Then $\frac{I_g(0)}{I_b(0)} = \frac{\rho}{1-\rho}$. Consider the following three scenarios.

(1) $\frac{\alpha\beta_g}{(1-\alpha)\beta_b} \geq \frac{\rho}{1-\rho}$: this is not possible as $\frac{\alpha(1-\rho)}{(1-\alpha)\rho} < 1$.

(2) $\frac{\alpha\beta_g}{(1-\alpha)\beta_b} < \frac{1-\rho}{\rho}$: observe that

$$(\frac{I_g(t)}{I_b(t)})' = \frac{I_g(t)}{I_b(t)} S(t)(\beta_g p_g(t) - \beta_b p_b(t)).$$

Hence $p_g(t) = p_b(t) = 0 \ \forall \ t > 0$, and $\frac{I_g(\infty)}{I_b(\infty)} = \frac{I_g(0)}{I_b(0)} > 1$. It implies that $\beta_g C'(\beta_g) > \beta_b C'(\beta_b)$, a contradiction to $\beta_g \le \beta_b$.

(3) $\frac{\alpha\beta_g}{(1-\alpha)\beta_b} \in [\frac{1-\rho}{\rho}, \frac{\rho}{1-\rho})$: as $\frac{\beta_g}{\beta_b} \in [\frac{1-\rho}{\rho}, 1]$, we know that $\frac{I_g(t)}{I_b(t)}$ is weakly increasing in time when consumers are sensitive to signals, and weakly decreasing in time when consumers herd on adoption. Also, we know that herding on adoption will never occur at t if $\frac{\alpha\beta_g I_g(t)}{(1-\alpha)\beta_b I_b(t)} < \frac{\rho}{1-\rho}$.

Note that $\frac{\alpha\beta_g}{(1-\alpha)\beta_b} < \frac{\rho}{1-\rho}$ means that $\frac{\rho(1-\alpha)\beta_b}{(1-\rho)\alpha\beta_g} > 1$, namely when $\frac{\alpha\beta_g I_g(t)}{(1-\alpha)\beta_b I_b(t)} = \frac{\rho}{1-\rho}$, $\frac{I_g(t)}{I_b(t)} = \frac{\rho(1-\alpha)\beta_b}{(1-\rho)\alpha\beta_g} > 1$. It then suffices to prove that $\frac{I_g(t)}{I_b(t)}$ will not decrease further in time when $\frac{\alpha\beta_g I_g(t)}{(1-\alpha)\beta_b I_b(t)} = \frac{\rho}{1-\rho}$. Suppose that $\frac{I_g(t)}{I_b(t)}$ decreases at t, then it must be that consumers herd on adoption on $(t, t+\epsilon)$ for $\epsilon > 0$ and $\beta_g < \beta_b$. However, this implies that $\frac{\alpha\beta_g I_g(t')}{(1-\alpha)\beta_b I_b(t')} < \frac{\rho}{1-\rho}$ for $t' \in (t, t+\epsilon)$, which means that consumers should be sensitive to signals instead, a contradiction. This completes the proof. □

$\beta_g > \beta_b$ means that interim beliefs are always increasing whether consumers herd on adoption or are sensitive to signals. In this case good consumers enjoy higher marginal returns from increasing spread speed, and thus are willing to invest more. On the other hand, $\rho\beta_g < (1-\rho)\beta_b$ may occur in equilibrium because the high spread speed of bad products compensates for the disadvantage from higher frequency of bad signals. When a fast-spreading bad product makes higher marginal profits than a slow-spreading good product, it becomes worthwhile to spend more on increasing product awareness.

Relatively similar speed, reflected by $\frac{\beta_g}{\beta_b} \in [\frac{1-\rho}{\rho}, 1]$, can exist when α is so high that consumers herd on adoption from the very beginning. In this case, a mild but positive difference between spread speed of bad and good producers maintains a corresponding mild advantage in marginal profits for the former. This is not possible if consumers start with a lower initial belief and hence a period of being sensitive to signals, which would only broaden the gap between profits if spread speed does not differ by much.

5.3 Price

Suppose that each producer can choose its price at $t = 0$. Again we focus on symmetric equilibria, i.e. each good producer chooses the same price P_g and each bad producer P_b. We assume that consumers hold a prudent off-equilibrium belief, which means that they believe that the product must be bad if its price is neither P_g or P_b.

It is clear that a separating equilibrium $P_g \neq P_b$ never exists, so we only consider pooling equilibria where $P_g = P_b = P$. There is always a pooling equilibrium where initial consumers herd on adoption (for instance, when $P = \underline{u}$) as well as one where initial consumers herd on non-adoption (for instance, when $P = \bar{u}$). Our main goal is then identifying conditions for the existence of pooling equilibria where consumers start with being sensitive to signals. We find that such equilibria exist when and only when the utility difference between a good product and a bad one is sufficiently large.

Proposition 7. *There exists a pooling equilibrium among producers, where initial consumers are sensitive to signals, if and only if* $\alpha \geq \frac{\rho \underline{u}}{(1-\rho)(\bar{u}-\underline{u})}$.

Proof. Given α, the expected payoff from adoption for a consumer without private signal, a G consumer, and a B consumer respectively, is $\alpha\bar{u}+(1-\alpha)\underline{u}$, $\frac{\alpha\rho}{\alpha\rho+(1-\alpha)(1-\rho)}\bar{u}+\frac{(1-\alpha)(1-\rho)}{\alpha\rho+(1-\alpha)(1-\rho)}\underline{u}$ and $\frac{\alpha(1-\rho)}{\alpha(1-\rho)+(1-\alpha)\rho}\bar{u}+\frac{(1-\alpha)\rho}{\alpha(1-\rho)+(1-\alpha)\rho}\underline{u}$. Therefore, for initial consumers to be sensitive to signals, it must be the case that $\alpha \in [\frac{P-\underline{u}}{\bar{u}-\underline{u}}, \frac{P-\underline{u}}{\frac{1-\rho}{\rho}(\bar{u}-P)+(P-\underline{u})})$ or $\alpha \in (\frac{P-\underline{u}}{\frac{\rho}{1-\rho}(\bar{u}-p)+(P-\underline{u})}, \frac{P-\underline{u}}{\bar{u}-\underline{u}})$.

In the former case, consumers herd on adoption for all $t > 0$, with each good producer earning approximately $\frac{\rho}{\alpha\rho+(1-\alpha)(1-\rho)}P$ and each bad producer earning approximately $\frac{1-\rho}{\alpha\rho+(1-\alpha)(1-\rho)}P$. If a producer deviates, it can at best charge price $\underline{u}$ and consumers will herd on adoption from $t = 0$ onwards. Hence, it attracts $\frac{1}{\rho}$ times as many consumers as a good producer does, and $\frac{1}{1-\rho}$ times as many consumers as a bad one does. A necessary and sufficient condition for p to be an equilibrium price is that a bad producer has no incentive to deviate, i.e.

$$\frac{1-\rho}{\alpha\rho+(1-\alpha)(1-\rho)}P \geq \frac{1}{\alpha\rho+(1-\alpha)(1-\rho)}\underline{u},$$

or equivalently $P \geq \frac{\underline{u}}{1-\rho}$. Note that the lower bound for α, $\frac{P-\underline{u}}{\bar{u}-\underline{u}}$, is increasing in P, and that $\frac{\frac{\underline{u}}{1-\rho}-\underline{u}}{\bar{u}-\underline{u}} > 1$ if $\frac{\underline{u}}{1-\rho} > \bar{u}$. Thus it is necessary and sufficient for the existence of an equilibrium price P, that $\alpha \geq \frac{\frac{\underline{u}}{1-\rho}-\underline{u}}{\bar{u}-\underline{u}} = \frac{\rho\underline{u}}{(1-\rho)(\bar{u}-\underline{u})}$.

In the latter case, consumers are sensitive to signals until $q(t;\emptyset) = \frac{P-\underline{u}}{\bar{u}-\underline{u}}$, after which they herd on adoption. Each good producer earns approximately $\frac{\rho(P-\underline{u})}{\alpha(\rho(P-\underline{u})+(1-\rho)(\bar{u}-P))}$ and each bad producer earns approximately $\frac{(1-\rho)(\bar{u}-P)}{(1-\alpha)(\rho(P-\underline{u})+(1-\rho)(\bar{u}-P))}$. Similarly, if a producer deviates, it can at best charge price $\underline{u}$ and consumers will herd on adoption from $t = 0$ onwards. Let $I_d(t)$ denote the number of consumers that have bought from a deviating producer at t, and for $l = g, b$ we have

$$(\frac{I_d(t)}{I_l(t)})' = \frac{I_d(t)}{I_l(t)} S(t)(1 - p_l(t)).$$

Hence, the deviating producer attracts $k_1 > \frac{1}{\rho}$ times as many consumers as a good producer does, and $k_2 > \frac{1}{1-\rho}$ times as many consumers as a bad one does. A necessary and sufficient condition for P to be an equilibrium price is that a bad producer has no incentive to deviate, i.e.

$$\frac{(1-\rho)(\bar{u}-P)}{(1-\alpha)(\rho(P-\underline{u})+(1-\rho)(\bar{u}-P))} P \geq \frac{(1-\rho)(\bar{u}-P)}{(1-\alpha)(\rho(P-\underline{u})+(1-\rho)(\bar{u}-P))} k_2 \underline{u},$$

or equivalently $P \geq k_2 \underline{u}$. Note that the lower bound for α, $\frac{P-\underline{u}}{\frac{\rho}{1-\rho}(\bar{u}-P)+(P-\underline{u})}$, is increasing in P. A necessary condition then for the existence of such p is that $P \geq \frac{k_2\underline{u}-\underline{u}}{\frac{\rho}{1-\rho}(\bar{u}-k_2\underline{u})+(k_2\underline{u}-\underline{u})}$. Since $k_2 > \frac{1}{1-\rho}$, this condition is already stricter than the previous one $\alpha \geq \frac{\rho \underline{u}}{(1-\rho)(\bar{u}-\underline{u})}$.

□

From the proof, we can also see that for a wider range of parameters, the equilibrium price P is low so that consumers at $t > 0$ start with herding on adoption. This is because with a high price and consumers being sensitive to signals, bad producers are at a greater disadvantage due to more frequent bad signals, and thus they have more incentives to just stop pretending and charge a low price for more sales.

6 Conclusion

In this paper, we have built a continuous-time social learning model to study the epidemiological dynamics among consumers when information about a product is spread via viral marketing. Our results depict a unique equilibrium life cycle of the product, in which various types of consumer behavior – herding on adoption, being sensitive to signals, and mix strategies – occur and switch from one to another when beliefs evolve over time. We also take strategic producers into account and illustrate how their decisions on quality, spread speed and price affect equilibrium dynamics.

We hope that our work can lay a foundation for the study of richer strategic behavior in the social learning process. Possible future directions that extend the current model include active acquisition of information by consumers, different topologies of consumers' social network, time-varying quality improvement by producers, etc.

References

Daron Acemoglu, Munther A. Dahleh, Ilan Lobel, and Asuman Ozdaglar. Bayesian learning in social networks. *Review of Economic Studies*, 78(4):1201–1236, 2011.

S. Nageeb Ali. Social learning with endogenous information. *Working Paper*, 2014.

Abhijit Banerjee and Drew Fudenberg. Word-of-mouth learning. *Games and Economic Behavior*, 46(1):1–22, 2004.

Abhijit V. Banerjee. A simple model of herd behavior. *The Quarterly Journal of Economics*, 107(3):797–817, 1992.

Abhijit V. Banerjee. The economics of rumours. *The Review of Economic Studies*, 60(2):309–327, 1993.

Chris T. Bauch and Samit Bhattacharyya. Evolutionary game theory and social learning can determine how vaccine scares unfold. *PLOS Computational Biology*, 8(4), 2012. URL `https://doi.org/10.1371/journal.pcbi.1002452`.

Sushil Bikhchandani, David Hirshleifer, and Ivo Welch. A theory of fads, fashion, custom, and cultural change as informational cascades. *Journal of Political Economy*, 100(5):992–1026, 1992.

Christian Borgs, Michael Brautbar, Jennifer Chayes, and Brendan Lucier. Maximizing social influence in nearly optimal time. In *Proceedings of the ACM-SIAM Symposium on Discrete Algorithms*, 2014.

Steven Callander and Johannes Horner. The wisdom of the minority. *Journal of Economic Theory*, 144(4):1421–1439, 2009.

Bogachan Celen. Does costly information acquisition improve welfare in a social learning environment? *Working Paper*, 2008.

Bogachan Celen and Shachar Kariv. Observational learning under imperfect information. *Games and Economic Behavior*, 47(1):72–86, 2004.

Douglas Gale and Shachar Kariv. Bayesian learning in social networks. *Games and Economic Behavior*, 45(2):329–346, 2003.

Amit Goyal, Wei Lu, and Laks V. S. Lakshmanan. Simpath: An efcient algorithm for inuence maximization under the linear threshold model. In *Proceedings of the IEEE International Conference on Data Mining*, 2011.

Sanjeev Goyal, Hoda Heidari, and Michael Kearns. Competitive contagion in networks. *Games and Economic Behavior*, 2014. URL https://doi.org/10.1016/j.geb.2014.09.002.

Antonio Guarino, Heike Harmgart, and Steffen Huck. Aggregate information cascades. *Games and Economic Behavior*, 73(1):167–185, 2011.

Kenneth Hendricks, Alan Sorensen, and Thomas Wiseman. Observational learning and demand for search goods. *American Economic Journal: Microeconomics*, 4 (1):1–31, 2012.

Helios Herrera and Johannes Horner. Biased social learning. *Games and Economic Behavior*, 80:131–146, 2013.

David Kempe, Jon Kleinberg, and Eva Tardos. Maximizing the spread of influence through a social network. In *Proceedings of the ACM SIGKDD Conference on Knowledge Discovery and Data Mining*, 2003.

David Kempe, Jon Kleinberg, and Eva Tardos. Influential nodes in a diffusion model for social networks. In *Proceedings of the International Colloquium on Automata, Languages, and Programming*, 2005.

William Ogilvy Kermack and Anderson Gray McKendrick. A contribution to the mathematical theory of epidemics. *Proceedings of the Royal Society London A*, 115(772):700–721, 1927.

William Ogilvy Kermack and Anderson Gray McKendrick. A contribution to the mathematical theory of epidemics: II. *Proceedings of the Royal Society London A*, 138(834):55–83, 1932.

William Ogilvy Kermack and Anderson Gray McKendrick. A contribution to the mathematical theory of epidemics: III. *Proceedings of the Royal Society London A*, 141(843):94–122, 1933.

Klaus K. Kultti and Paavo A. Miettinen. Herding with costly information. *International Game Theory Review*, 8(2):21–33, 2006.

Klaus K. Kultti and Paavo A. Miettinen. Herding with costly observation. *The B.E. Journal of Theoretical Economics*, 7(1 (Topics)):Article 28, 2007.

In Ho Lee. On the convergence of informational cascades. *Journal of Economic Theory*, 61(2):395–411, 1993.

Ilan Lobel and Evan Sadler. Information diffusion in networks through social learning. *Theoretical Economics*, 10:807–851, 2015.

David McAdams. Resistance diagnosis and the changing epidemiology of antibiotic resistance. *Antimicrobial Therapeutics Reviews*, 1388(1):5–17, 2017a.

David McAdams. Resistance diagnosis and the changing economics of antibiotic discovery. *Antimicrobial Therapeutics Reviews*, 1388(1):18–25, 2017b.

Ignacio Monzon and Michael Rapp. Observational learning with position uncertainty. *Journal of Economic Theory*, 154:375–402, 2014.

Elchanan Mossel and Sebastien Roch. Submodularity of influence in social networks: From local to global. *SIAM Journal of Computing*, 39(6):2176–2188, 2010.

Manuel Mueller-Frank and Mallesh M. Pai. Social learning with costly search. *Working Paper*, 2014.

Mark E. J. Newman. Spread of epidemic disease on networks. *Physical Review E*, 66(1):016128, 2002.

Ronald Ross. An application of the theory of probabilities to the study of a priori pathometry: I. *Proceedings of the Royal Society London A*, 92(638):204–230, 1916.

Ronald Ross and Hilda Hudson. An application of the theory of probabilities to the study of a priori pathometry: II. *Proceedings of the Royal Society London A*, 93 (650):212–225, 1917a.

Ronald Ross and Hilda Hudson. An application of the theory of probabilities to the study of a priori pathometry: III. *Proceedings of the Royal Society London A*, 93 (650):225–240, 1917b.

Lones Smith and Peter Sorensen. Pathological outcomes of observational learning. *Econometrica*, 68(2):371–398, 2000.

Lones Smith and Peter Sorensen. Rational social learning with random sampling. *Working Paper*, 2013.

Yangbo Song. Social learning with endogenous observation. *Journal of Economic Theory*, 166:324–333, 2016.