Stock Price Cycles
and
Business Cycles

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Abstract

We present a unified and quantitatively credible explanation for the joint behavior of stock prices and business cycles. We consider a frictionless production economy with time-separable consumption preferences in which investors extrapolate past stock price gains, in line with the available empirical evidence (Adam, Marcet, and Beutel, 2017). Expectations about all other payoff-relevant variables are rational and households maximize utility given their beliefs (internal rationality). The model replicates a standard set of asset pricing and business cycle moments, as well as moments capturing the interaction between business cycles and stock prices. The model generates belief-driven stock price booms that are associated with a corresponding boom in hours worked and investment. Once the boom turns into a bust, output, hours and investment remain persistently depressed. The model predicts that 75% of the observed fluctuations in the price-dividend ratio and 13% of observed output fluctuations are attributable to subjective price extrapolation by investors.

1 Introduction

We present a simple economic model that quantitatively replicates the joint behavior of business cycles and stock price cycles. The model matches standard data moments characterizing business cycle behavior, standard moments capturing stock price behavior, as well as data moments that link stock prices with business cycle variables. The model also gives rise to cycles in stock price valuation of the kind observed in the data.

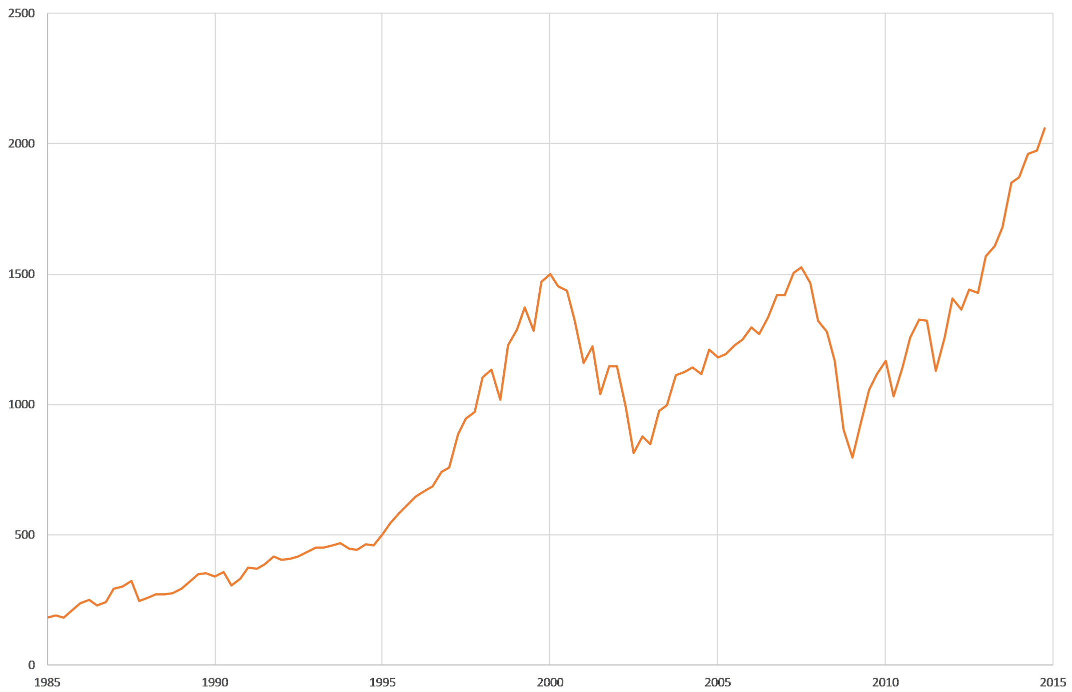

Figure 1. Cycles in the S&P 500: Q1:1985-Q4:2014, nominal index values

Stock price cycles have – in comparison to business cycles – received relatively little attention in the literature. Somewhat surprisingly, this is the case even though stock price cycles are easily discernible in the data. Figure 1 depicts the S&P500 stock index from 1985 to 2014.[1] Over the considered thirty-year period, stock prices displayed three significant run-ups and two large reversals, with the price reversal amounting to a close to 50% price drop when compared to the previous peak. Similar run-ups and reversals can be observed in the stock markets of other advanced economies. We refer to this repeated boom-bust pattern in stock prices as stock price cycles.

Figure 2 presents an alternative approach for capturing stock price cycles: it depicts the empirical distribution of the quarterly price-dividend (PD) ratio of the S&P500.[2] While the mode of the PD ratio is slightly below 125, the PD distribution displays a long right tail with values of almost three times the mode being in the support of the distribution. The observed positive skewness and the large support of the empirical PD distribution are an alternative way to capture the presence of occasional stock price run-ups and reversals.

[1] Figure 1 depicts the nominal value of the S&P500, but similar conclusions emerge if one deflates the nominal value by the consumer price index.

[2] The quarterly PD ratio is defined as the end-of-quarter price over a deseasonalized measure of quarterly dividend payouts, see Appendix A.1 for details.

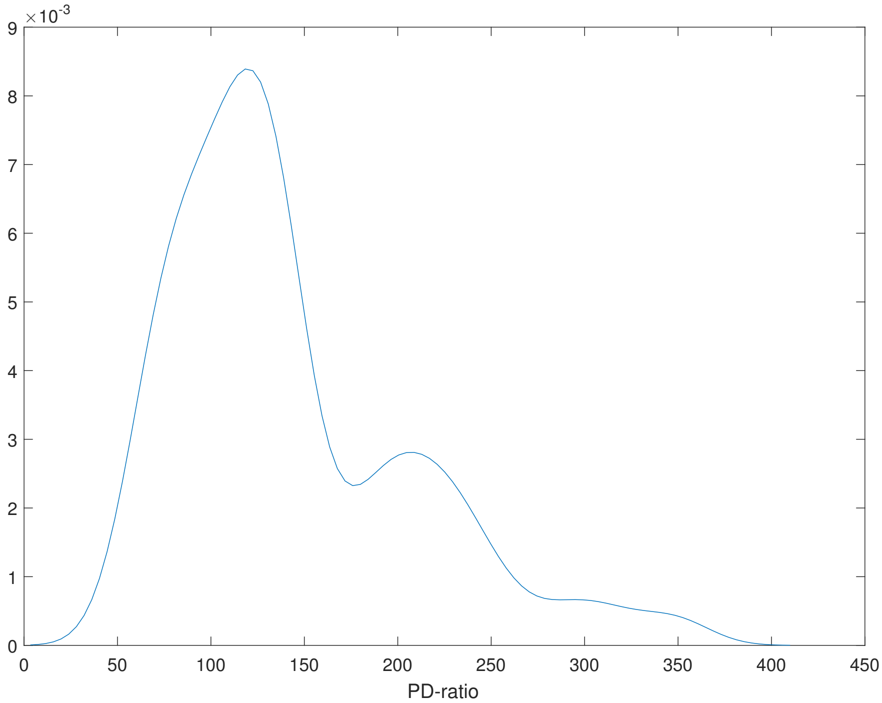

Figure 2. Postwar distribution of the quarterly PD ratio of the S&P 500 (kernel density estimate, Q1:1955-Q4:2014)

Making economic sense of these large stock price movements remains, however, challenging. This is especially true when seeking a joint explanation for stock price behavior and the behavior of the business cycle. While the latter is relatively smooth, stock prices are rather volatile. Without addressing stock price cycles, the existing literature typically relies on two approaches for reconciling the smoothness of the business cycles with the volatility of stock prices.

The first explanation combines preferences featuring a low elasticity of intertemporal substitution (EIS) with adjustment frictions.[3] A low EIS creates a strong desire for intertemporal consumption smoothing, while adjustment frictions prevent such smoothing from fully taking place. For agents to be willing to accept the observed moderate consumption fluctuations, asset prices then need to adjust strongly in equilibrium. Since one possible adjustment margin is agents' labor supply, too strong adjustments in the number of hours worked need to be prevented in such settings.[4] Therefore, EIS-based

[3]Typical examples are habit preferences (e.g. Jermann, 1998; Boldrin, Christiano, and Fisher, 2001; Uhlig, 2007; Jaccard, 2014), though sometimes also recursive preferences with a low EIS are used (e.g. Guvenen, 2009).

[4]Otherwise the high desire to smooth intertemporal fluctuations would lead to a strong adjustment in

explanations include labor market frictions of some form (adjustment frictions, inflexible labor supply through preferences, real wage frictions), causing labor market frictions to become key for explaining asset price behavior. The centrality of labor market frictions is puzzling, in particular in light of the fact that labor market institutions differ considerably across advanced economies, while stock prices are very volatile in all these economies.

A second explanation reconciling smooth business cycles with volatile stock prices relies on specifying recursive preferences in conjunction with an additional source of exogenous uncertainty, e.g. long-run growth risk or disaster risk.[5] Such shocks have large pricing implications under recursive preferences, provided the coefficient of risk aversion is larger than the inverse EIS. These shocks then generate a large equity premium in the presence of realistic consumption dynamics. Time variation in the equity premium leads to substantial volatility in stock prices and returns, though typically less than what can be observed in the data.[6] As recently pointed out by Epstein, Farhi, and Strzalecki (2014), standard calibrations of recursive preferences imply very large consumption premia for the resolution of uncertainty and the verdict on the quantitative plausibility of these premia is still outstanding.

The present paper proposes an alternative explanation for stock price and business cycle behavior. In contrast to the existing approaches, it relies on a standard separable preference specification and also features a frictionless labor market and perfectly elastic labor supply from households.[7] Despite these features, the model can jointly replicate the quantitative behavior of business cycles and stock prices using (reasonably sized) shocks to total factor productivity as its only source of exogenous random variation. The model also generates a stable risk-free interest rate and gives rise to large and persistent stock price boom-bust patterns of the kind observed in the data. The most notable dimension along which the model falls short of fully matching the data is the equity premium: it generates only about one third of the empirically observed premium. Since we consider a setting with time-separable utility and a logarithmic utility function for consumption, this still represents a respectable achievement.

Key to the empirical success of the model is a departure from the rational expectations hypothesis (REH). Following Adam, Marcet, and Nicolini (2016) and Adam, Marcet, and Beutel (2017), we consider subjectively Bayesian investors who seek to filter the long-term trend component of capital gains from observed capital gains and as a result extrapolate

hours worked and a very smooth consumption profile, which in turn would largely eliminate asset price fluctuations.

[5]See Gourio (2012), Croce (2014)

[6]Labor market frictions or the elasticity of labor supply are usually not discussed in this strand of literature. While they are certainly not as crucial as they are for the low-EIS-based explanations, flexible labor supply in a frictionless labor market is still a powerful tool for households to insure against consumption fluctuations. The fact that the asset pricing models in this second strand of literature tend to generate too little volatility of hours worked is an indication that the chosen preference specification make labor supply not flexible enough.

[7]We consider households with a linear disutility of work.

(to some extent) past capital gains into the future. Adam, Marcet, and Beutel (2017) show that such extrapolative behavior is in fact consistent with the available survey evidence on investors' return expectations, whereas the REH is strongly rejected by the data.[8] Adam, Marcet, and Nicolini (2016) show how subjective beliefs of such kind substantially improve the asset pricing predictions of a standard Lucas (1978) endowment economy. The present setting significantly extends their framework to one with endogenous production, featuring endogenous labor and investment choices. In doing so, we also present a conceptual approach for dealing with dynamic decision settings in which agents' expectations deviate from rational expectations along some dimension (future asset prices), but are consistent with the REH along other dimensions (all other variables beyond agents' control).

We estimate the subjective belief model using the simulated method of moments and also estimate a version with fully rational expectations. We show that the empirical improvements associated with a departure from the assumption of rational stock price expectations is significant, both along the business cycle dimension and – even more importantly – along the stock price dimension. Under fully rational expectations, the model fails to fully replicate business cycle moments, because we do not consider investment-specific productivity shocks. We show that in a setting with subjective price beliefs such shocks are not required.

We also consider the welfare effects associated with stock price fluctuations that are driven by fluctuations in investors' subjective beliefs. Stock price movements induce volatility of investment and hours worked, but these variations are not exclusively driven by productivity developments. As a result, average consumption is higher and average hours worked lower when imposing rational stock price expectations, as investment choices then fully reflect underlying productivity. The volatility of hours and investment significantly falls under fully rational expectations and there is a dramatic drop in the volatility of stock prices. The welfare gains – measured in terms of ex-post realized utility – amount to a permanent increase in consumption of 0.29%, thus are in the order of magnitude associated with eliminating the business cycle.

The paper is structured as follows. Section 2 discusses the related literature. Section 3 presents key facts about business cycles, stock prices and their interaction in the United States. Section 4 describes our model with the exception of beliefs, which are discussed in detail in Section 5. All equilibrium conditions of the model are summarized in Section 6. Section 7 derives analytical insights and discusses the dynamics of stock prices, price beliefs and macro aggregates. Section 8 outlines our estimation procedure and assesses the empirical performance of our model. There, we also discuss how subjective price beliefs contribute to the empirical success of the model by contrasting our results with those of a model version where agents hold fully rational expectations. Section 9 compares our

[8] Adam, Matveev, and Nagel (2018) test to what extent survey expectations are consistent with various expectations hypotheses entertained in the asset pricing literature.

quantitative results to two closely related papers, Boldrin, Christiano, and Fisher (2001) and Adam, Marcet, and Beutel (2017). Section 10 discusses the welfare implications of belief-driven stock price cycles. Section 11 concludes.

2 Related Literature

Early attempts of jointly modeling business cycles and stock prices have relied on a combination of habit preferences and adjustment frictions to generate high stock price volatility and equity premia (Jermann, 1998; Boldrin, Christiano, and Fisher, 2001; Uhlig, 2007). Habit preferences create a low elasticity of intertemporal substitution (EIS) and thereby a strong desire to smooth consumption, which leads to volatile stock prices in a setting with empirically plausible consumption fluctuations. The intertemporal substitution channel, however, causes all asset prices to be volatile and thus generates a counterfactually high volatility of the risk-free interest rate.[9] An exception is Uhlig (2007) who considers external habits, which create strong fluctuations in risk aversion[10] and thereby allow for relatively volatile stock prices and a stable risk-free interest rate.[11] To prevent consumption smoothing, low EIS models also crucially rely on labor market frictions or inelastic labor supply: labor supply in Jermann (1998) is fully inelastic, while a timing friction forces households to choose labor supply one period in advance in Boldrin, Christiano, and Fisher (2001); Uhlig (2007) introduces a real wage rigidity that leads to labor supply rationing following negative productivity shocks. Jaccard (2014) augments the model of Jermann (1998) and allows for adjustments in labor supply, but introduces a habit specification over a composite of consumption and leisure which makes labor supply adjustments unattractive for the purpose of smoothing consumption.[12]

Models with a low EIS also typically generate the equity premium via a counterfactually high term premium. For example, the premium on long-term bonds in Jermann's (1998) model is 92% of that on stocks, compared to only 28% in the data.

Another line of literature jointly considers business cycle dynamics and stock price behavior using models with limited asset market participation.[13] In these models, a limited set of agents has access to the stock market and in addition insures the consumption of

[9]The risk-free rate volatility in Jermann (1998) is twice as high as in the data; in Boldrin, Christiano, and Fisher (2001) it is almost five times as high.

[10]See Boldrin, Christiano, and Fisher (1997) for a discussion of the differing risk aversion implications of internal and external habit specifications.

[11]It is unclear, however, whether Uhlig (2007) matches the volatility of stock returns as he only reports the sharpe ratio.

[12]As a result, the standard deviation of the growth rate of hours worked in Jaccard (2014) is only about one half of the value observed in the data.

[13]As mentioned in Guvenen (2009), stock market participation increased substantially during the 1990's. From 1989 to 2002 the number of households who owned stocks increased by 74% so that half of U.S. households had become stock owners by the year 2002.

non-participating agents via other contracts. An early example is Danthine and Donaldson (2002), who consider shareholders and workers, where the latter do not participate in financial markets at all. Shareholders then offer workers a labor contract that gives rise to "operating leverage" in the sense that it causes the cash flows of shareholders to become even more volatile and procyclical. As a result, the model gives rise to an equity premium and volatile stock returns, albeit at the cost of creating too much volatility for shareholder's consumption.[14]

Guvenen (2009) considers a model with limited stock market participation in which all agents participate in the bond market. When stock market investors have a higher EIS than non-participating agents, they optimally insure the latter against income fluctuations via bond market transactions, thereby channeling most labor income risk to a small set of stock market participants. As a result, their consumption is strongly procyclical and gives rise to both a high equity premium and high volatility of returns. The model assumes the EIS to be low – even for shareholders – thus generates additional stock price volatility through the same channels as the habit models. The model performs quantitatively very well on the financial dimension, except for a slightly too volatile risk-free interest rate. Performance along the business cycle dimension is more mixed, as consumption is too volatile and investment and hours worked too smooth.

Tallarini (2000), Gourio (2012), Croce (2014) and Hirshleifer, Li, and Yu (2015) discuss the asset pricing predictions of the real business cycle models under Epstein-Zin preferences (Epstein and Zin, 1989), assuming that the coefficient of risk aversion is larger than the inverse EIS. Tallarini (2000) shows that increasing risk aversion while keeping the EIS fixed at one barely affects business cycle dynamics, but has substantial effects on the price of risk. While the model can give rise to a high Sharpe ratio, in line with the value observed in the data, it considerably undershoots the equity premium and the volatility of returns. The model thus falls short of reconciling the volatility of stock prices with the smoothness of business cycle dynamics. Gourio (2012) considers preferences with a larger EIS and moderate risk aversion and enriches the model by time-varying disaster risk.[15] Whereas constant disaster risk has little effect on the model dynamics, time-variation in disaster risk combined with preferences for early resolution of uncertainty generate a high equity premium and return volatility, while at the same time keeping the dynamics of macro aggregates relatively smooth, provided the disaster does not realize. While framed as a rational expectations model applied to a disaster-free data sample, one may alternatively interpret the model as a subjective belief model and the exogenous shock

[14] In Table 6 in Danthine and Donaldson (2002), which is the specification with the best overall empirical fit, shareholder consumption volatility is about 10 times as large as aggregate consumption volatility. Guvenen (2009) reviews the empirical evidence on stockholders' relative consumption volatility and concludes that stockholders' consumption is about 1.5-2 times as volatile as non-stockholders' and thus even less high in relative terms when compared to aggregate consumption volatility

[15] A disaster is a potentially persistent event in which the economy experiences for the duration of the disaster in each period a negative productivity shock and a capital depreciation shock.

to the disaster probability as a shock to agents' expectations. Under this interpretation, subjective beliefs drive asset price dynamics, as is the case in the present paper. However, subjectively expected returns in his model are negatively related to the PD ratio, unlike in the data and unlike in the present model.

Another channel through which preferences for early resolution of uncertainty can generate realistic asset pricing predictions is long-run consumption growth risk as in Bansal and Yaron (2004). Croce (2014) considers a production economy with long-run productivity growth risk and shows that this translates into long-run consumption risk, thereby transferring the asset pricing predictions of the endowment economy of Bansal and Yaron (2004) to a real business cycle setting. His model can match well the equity premium and low and stable risk-free rate, but is only partially able to generate price and return volatility.

Hirshleifer, Li, and Yu (2015) also consider an economy with preferences for early resolution of uncertainty but assume inelastic labor supply. Agents know the productivity process only imperfectly and over-extrapolate recent productivity observations in their filtering problem. This mechanism endogenously generates long-run variations in perceived technology growth and thus perceived consumption growth, despite there being only short-run technology shocks present in the data-generating process. If the extrapolation bias is small but sufficiently persistent, then the model produces a sizeable equity premium and about 50% of the observed volatility of stock returns. The model also matches a range of business cycle quantities, except for the volatility of hours worked.

In independent and complementary work, Winkler (2018) considers the asset pricing implications of a rich DSGE model featuring subjective price beliefs of the kind we also consider. His model features financial frictions, price and wage rigidities, and limited stock market participation, from which the present paper abstracts. In addition, Winkler (2018) considers a setup with "conditionally model-consistent expectations", which requires agents' beliefs to be consistent with both the subjective law of motion for prices, other agents' decision functions and a maximum number of market clearing conditions.[16] This setup allows for deviations from rational expectations also for other variables than just stock prices and differs from our setting with "partially rational expectations", which endows agents with the correct statistical model about decision-relevant variables other than prices.

Learning about stock prices in our model does not only improve stock market predictions, but helps quantitatively also along the business cycle dimension. The idea that learning can improve the business cycle predictions of the standard real business cycle model is also present in Eusepi and Preston (2011), who consider a setting where agents are learning about wages and rental rates. They show that this generates significant amplification and persistence in effects of neutral technology shocks, so that expectations

[16]Due to Walras' Law one market clearing condition can be dropped. Winkler (2018) must drop a second one, as not all markets can clear under the subjective plans.

ultimately drive the business cycle. In our setting, beliefs about wages and rental rates are assumed to be rational, but subjective stock price expectations do affect business cycle outcomes. Specifically, fluctuations in expectations affect capital valuations and the demand for investment goods, thereby generating fluctuations in investment and labor demand.

The paper is also related to the literature on rational stock market bubbles, as for instance derived in classic work by Froot and Obstfeld (1991). While rational bubbles provide an alternative approach for generating stock market volatility, they seem inconsistent with empirical evidence along two important dimensions. First, the assumption of rational return expectations is strongly at odds with survey measures of return expectations, which clearly favors the subjective belief specifications we consider in the present paper (compare Adam, Marcet, and Beutel, 2017). Second, Giglio, Maggiori, and Stroebel (2016) show that there is very little evidence supporting the notion that violations of the transversality condition drive asset price fluctuations, unlike suggested by the rational bubble hypothesis.

3 Stock Prices and Business Cycles: Key Facts

This section presents key data moments that characterize U.S. business cycles and stock price behavior and that we focus on in our empirical analysis. We consider quarterly U.S. data for the period Q1:1955-Q4:2014. The start date of the sample is determined by the availability of the aggregate hours worked series. Details of the data sources are reported in Appendix A.1.

Table 1 presents a standard set of business cycle moments for output (Y), consumption (C), investment (I) and hours worked (H).[17] These quantities have been divided by the working age population so as to take into account demographic changes in the U.S. population over the sample period. The second to last column in Table 1 reports the data moment and the last column the standard deviation of the estimated moment. We will use the latter in our simulated methods of moments estimation and for computing t-statistics.[18]

The picture that emerges from Table 1 is a familiar one: output fluctuations are relatively small, consumption is considerably less volatile than output, while investment is considerably more volatile; hours worked are roughly as volatile as output. Consumption, investment and hours all correlate strongly with output. A major quantitative challenge

[17] As is standard in the business cycle literature, we compute business cycle moments using logged and subsequently HP-filtered data with a smoothing parameter of 1600. All other data moments will rely on unfiltered (level) data. We HP filter model variables when comparing to filtered moments in the data and use unfiltered model moments otherwise.

[18] All standard deviations of moments reported in this section are computed by a procedure combining Newey-West estimators with the delta method as in Adam, Marcet, and Nicolini (2016). We refer to Appendix F of that paper for details.

Table 1
U.S. business cycle moments (quarterly real values, Q1:1955-Q4:2014)

	Symbol	Data moment	Std. dev. data moment
Std. dev. of output	$\sigma(Y)$	1.72	0.25
Relative std. dev. of consumption	$\sigma(C)/\sigma(Y)$	0.61	0.03
Relative std. dev. of investment	$\sigma(I)/\sigma(Y)$	2.90	0.35
Relative std. dev. of hours worked	$\sigma(H)/\sigma(Y)$	1.08	0.13
Correlation output and consumption	$\rho(Y,C)$	0.88	0.02
Correlation output and investment	$\rho(Y,I)$	0.86	0.03
Correlation output and hours worked	$\rho(Y,H)$	0.75	0.03

will be to simultaneously replicate the relative smoothness of the business cycle with the much larger fluctuations in stock prices to which we turn next.

Table 2 presents a standard set of moments characterizing U.S. stock price behavior. The first three moments summarize the behavior of the PD ratio:[19] the PD ratio is rather large and implies a dividend yield of just 66 basis points per quarter. The PD ratio is also very volatile: the standard deviation of the PD ratio is more than 40% of its mean value and fluctuations in the PD ratio are very persistent, as documented by the high quarterly auto-correlation of the PD ratio. Table 2 also reports the average real stock return, which is high and close to 2% per quarter. Stock returns are also very volatile: the standard deviation of stock returns is about four times its mean value. This contrasts with the behavior of the short-term risk-free interest rate documented in Table 2. The risk-free interest rate is very low and very stable. The standard deviation of the risk-free interest rate in Table 2 is likely even overstated, as ex-post realized inflation rates have been used to transform nominal safe rates into a real rate. Table 2 also reports the standard deviation of dividend growth. Dividend growth is relatively smooth, especially when compared to the much larger fluctuation in equity returns. This fact is hard to reconcile with the observed large fluctuations in stock prices (Shiller, 1981).

Table 3 presents data moments that link the PD ratio to business cycle variables. It shows that stock prices are pro-cyclical: (1) the PD ratio correlates positively with hours worked; (2) stock prices also correlate positively with the investment to output ratio, but the correlation is surprisingly weak and also estimated very imprecisely. Table 4 below shows why this is the case: the investment to output ratio correlates positively with the PD ratio over the second half of the sample period (1985-2014), i.e., in the period with large stock price cycles, but negatively in the first half of the sample period (1955-1984).

[19]The PD ratio is defined as the end-of-quarter stock price divided by dividend payments over the quarter. Following standard practice, dividends are deseasonalized by averaging dividends over the last four quarters.

Table 2
Key moments of stock prices, risk-free rates and dividends (U.S., quarterly real values, Q1:1955-Q4:2014)

	Symbol	Data moment	Std. dev. data moment
Average PD ratio	$E[P/D]$	152.3	25.3
Std. dev. PD ratio	$\sigma(P/D)$	63.39	12.39
Auto-correlation PD ratio	$\rho(P/D)$	0.98	0.003
Average equity return (%)	$E[r^e]$	1.87	0.45
Std. dev. equity return (%)	$\sigma(r^e)$	7.98	0.35
Average risk-free rate (%)	$E[r^f]$	0.25	0.13
Std. dev. risk-free rate (%)	$\sigma(r^f)$	0.82	0.12
Std. dev. dividend growth (%)	$\sigma(D_{t+1}/D_t)$	1.75	0.38

Table 3
Comovement of stock prices with real variables and survey return expectations (U.S., quarterly real values, Q1:1955-Q4:2014)

Correlations	Symbol	Data moment	Std. dev. data moment
Hours & PD ratio	$\rho(H, P/D)$	0.51	0.17
Investment-output & PD ratio	$\rho(I/Y, P/D)$	0.19	0.31
Survey expect. & PD ratio	$\rho(E^{\mathcal{P}}[r^e], P/D)$	0.79	0.07

Table 4 also shows that the overall investment to output ratio correlates much more strongly with the PD ratio if one excludes non-residential investment and investment in non-residential structure. The negative correlation in the first half of the sample period is, however, a robust feature of the data.[20]

Table 3 also reports the correlation of the PD ratio with the one-year-ahead expected real stock market return of private U.S. investors. It shows that investors are optimistic about future holding period returns when the PD ratio is high already. As shown in Adam, Marcet, and Beutel (2017), this feature of the data is inconsistent with the notion that investors hold rational return expectations. Since we will consider an asset pricing model with subjective return expectations, we include this data moment into our analysis.

[20]The correlation of the hours worked series with the PD ratio is positive in the first and second half of the sample period.

Table 4
Stock prices and investment: alternative measures and sample periods (U.S., quarterly real values)

$corr(I/Y, P/D)$	1955-2014	1955-1984	1985-2014
Fixed investment	0,19	-0,64	0,40
Fixed investment, less residential inv. and nonresidential structures:	0,58	-0,66	0,77

4 Asset Pricing in a Production Economy

We build our analysis on a stripped-down version of the representative agent model of Boldrin, Christiano, and Fisher (2001). This model features a consumption goods producing sector and an investment goods producing sector. Both sectors produce output using a neoclassical production function with capital and labor as input factors. Output from the investment goods sector can be invested to increase the capital stock.

We deviate from Boldrin, Christiano, and Fisher (2001) by using a standard time-separable specification for consumption preferences instead of postulating consumption habits. In addition, we remove all labor market frictions on the firm side by making hours worked perfectly flexible.

Given a linear specification for the disutility of labor, the labor market is then perfectly flexible and competitive. While frictions are arguably present in U.S. labor markets, we prefer the fully flexible specification to illustrate that our asset pricing implications do not depend on assuming labor market frictions. This feature distinguishes the present analysis from much of the earlier work.

We furthermore simplify our setup relative to the one studied in Boldrin, Christiano, and Fisher (2001) by specifying an exogenous capital accumulation process in the investment goods sector, in line with a balanced growth path solution. This helps with analytical tractability of the model, but also insures that the supply of new capital goods is sufficiently inelastic, so that the model has a chance of replicating the large persistent swings in stock prices that can be observed in the data.[21]

4.1 Production Technology

There are two sectors, one producing a perishable consumption good (consumption sector), the other producing an investment good that can be used to increase the capital stock in the consumption sector (investment sector). The representative firm in each sector hires labor and rents capital, so as to produce its respective output good according

[21] In Boldrin, Christiano, and Fisher (2001), capital prices do not display large and persistent fluctuations.

to standard Cobb-Douglas production functions,

$$Y_{c,t} = K_{c,t}^{\alpha_c} \left(Z_t H_{c,t}\right)^{1-\alpha_c}, \qquad Y_{i,t} = K_{i,t}^{\alpha_i} \left(Z_t H_{i,t}\right)^{1-\alpha_i}, \tag{1}$$

where $K_{c,t}$, $K_{i,t}$ denote capital inputs and $H_{c,t}$, $H_{i,t}$ labor inputs in the consumption and the investment sector, respectively, and $\alpha_c \in (0,1)$ and $\alpha_i \in (0,1)$ the respective capital shares in production. Z_t is an exogenous labor-augmenting level of productivity and the only source of exogenous variation in the model. Productivity follows

$$Z_t = \gamma Z_{t-1} \varepsilon_t, \qquad \log \varepsilon_t \sim ii\mathcal{N}\left(-\frac{\sigma^2}{2}, \sigma^2\right), \tag{2}$$

with $\gamma \geq 1$ denoting the mean growth rate of technology and $\sigma > 0$ the standard deviation of log technology growth.

Labor is perfectly flexible across sectors, but capital is sector-specific. The output of investment goods firms increases next period's capital in the consumption goods sector, so that

$$K_{c,t+1} = (1-\delta_c) K_{c,t} + Y_{i,t}, \tag{3}$$

where $\delta_c \in (0,1)$ denotes the depreciation rate. Capital in the investment goods sector evolves according to

$$K_{i,t+1} = (1-\delta_i) K_{i,t} + X_t \tag{4}$$

where X_t is an exogenous endowment of new capital in the investment goods sector and $\delta_i \in (0,1)$ the capital depreciation rate. We set X_t such that $K_{i,t+1} \propto Z_t$, which insures that the model remains consistent with balanced growth.[22] The assumed capital stock dynamics in the investment goods sector insures that capital good production that deviates from the balanced growth path is subject to decreasing returns to scale. This allows for persistent price fluctuations in the price of consumption capital around the balanced growth path.

4.2 Households

The representative household each period chooses consumption $C_t \geq 0$, hours worked $H_t \geq 0$, the end-of-period capital stocks $K_{c,t+1} \geq 0$ and $K_{i,t+1} \geq 0$ to maximize

$$E_0^{\mathcal{P}}\left[\sum_{t=0}^{\infty} \beta^t \left(\log C_t - H_t\right)\right], \tag{5}$$

[22]The model allocation is then identical to a setting in which investment is produced with labor only, i.e., $Y_{i,t} \propto Z_t \varepsilon_t^{-\alpha_i} (H_{i,t})^{1-\alpha_i}$. We prefer a specification that includes capital, capital depreciation and an exogenous investment input, as this allows us to define capital values in both sectors in a symmetric fashion.

where the operator $E_0^{\mathcal{P}}$ denotes the agent's expectations in some probability space $(\Omega, \mathcal{S}, \mathcal{P})$. Here, Ω is the space of realizations, $\mathcal{S}$ the corresponding σ-Algebra, and $\mathcal{P}$ a subjective probability measure over $(\Omega, \mathcal{S})$. As usual, the probability measure $\mathcal{P}$ is a model primitive and given to agents. The special case with rational expectations is nested in this specification, as explained below.

Household choices are subject to the flow budget constraint

$$C_t + K_{c,t+1}Q_{c,t} + K_{i,t+1}Q_{i,t} = W_tH_t + X_tQ_{i,t} + K_{c,t}\left((1-\delta_c)Q_{c,t} + R_{c,t}\right) + K_{i,t}\left((1-\delta_i)Q_{i,t} + R_{i,t}\right), \tag{6}$$

for all $t \geq 0$, where $Q_{c,t}$ and $Q_{i,t}$ denote the prices of consumption-sector and investment-sector capital, respectively, and $R_{c,t}$ and $R_{i,t}$ the rental rates earned by renting out capital to firms in the consumption and investment sector, respectively; W_t denotes the wage rate and X_t the endowment of new investment-sector capital.

To allow for subjective price beliefs, we shall consider an extended probability space relative to the case with rational expectations. In its most general form, households' probability space is spanned by all external processes, i.e. by all variables that are beyond their control. These are given by the process $\{Z_t, X_t, W_t, R_{c,t}, R_{i,t}, Q_{c,t}, Q_{i,t}\}_{t=0}^{\infty}$, so that the space of realizations is

$$\Omega := \Omega_Z \times \Omega_X \times \Omega_W \times \Omega_{R,c} \times \Omega_{R,i} \times \Omega_{Q,c} \times \Omega_{Q,i},$$

where $\Omega_{\mathcal{X}} = \prod_{t=0}^{\infty} \mathbb{R}$ with $\mathcal{X} \in \{Z, X, W, R_c, R_i, Q_c, Q_i\}$. Letting $\mathcal{S}$ denote the sigma-algebra of all Borel subsets of Ω, beliefs will be specified by a well-defined probability measure $\mathcal{P}$ over $(\Omega, \mathcal{S})$. Letting Ω^t denote the set of all partial histories up to period t, households' decision functions can then be written as

$$(C_t, H_t, K_{c,t+1}, K_{i,t+1}) : \Omega^t \longrightarrow \mathbb{R}^4. \tag{7}$$

We assume that households choose these functions, so as to maximize (5) subject to the constraints (6).

In the special case with rational expectations, $(X, W, R_c, R_i, Q_c, Q_i)$ are typically redundant elements of the probability space Ω, because households are assumed to know that these variables can at time $t \geq 0$ be expressed as *known deterministic equilibrium functions* of the history of fundamentals Z^t only.[23] Without loss of generality, one can then exclude these elements from the probability space and write:[24]

$$(C_t, H_t, K_{c,t+1}, K_{i,t+1}) : \Omega_Z^t \longrightarrow \mathbb{R}^4,$$

[23] This assumes that there are no sunspot fluctuations in the rational expectations equilibrium.

[24] Sunspot fluctuation require either keeping some of the endogenous variables or including the sunspot variables into the probability space.

where $\Omega_Z^t = \prod_{s=0}^t \mathbb{R}$ is the space of all realizations of $Z^t = (Z_0, Z_1, ..., Z_t)$. This routinely performed simplification implies that households perfectly know how the markets determine the excluded variables as a function of the history of shocks. By introducing subjective beliefs, we will step away from this assumption.

To insure that the household's maximization problem remains well-defined in the presence of the kind of subjective price beliefs introduced below, we impose additional capital holding constraints of the form $K_{c,t+1} \leq \overline{K}_{c,t+1}$ and $K_{i,t+1} \leq \overline{K}_{i,t+1}$, for all $t \geq 0$, where the bounds $\left(\overline{K}_{c,t+1}, \overline{K}_{i,t+1}\right)$ are assumed to increase in line with the balanced growth path and are assumed sufficiently large, such that they never bind in equilibrium.[25]

4.3 Competitive Equilibrium

The competitive equilibrium of an economy in which households hold subjective beliefs is defined as follows:

Definition 1. For given initial conditions $(K_{c,-1}, K_{i,-1})$, a competitive equilibrium with subjective household beliefs $\mathcal{P}$ consists of allocations $\{C_t, H_t, H_{c,t}, H_{i,t}, K_{c,t+1}, K_{i,t+1}\}_{t=0}^{\infty}$ and prices $\{Q_{c,t}, Q_{i,t}, R_{c,t}, R_{i,t}, W_t\}_{t=0}^{\infty}$, all of which are measurable functions of the process $\{Z_t\}_{t=0}^{\infty}$, such that for all partial histories $Z^t = (Z_0, Z_1, ...Z_t)$ and all $t \geq 0$, prices and allocations are consistent with

1. profit maximizing choices by firms,
2. the subjective utility maximizing choices for households decision functions (7), and
3. market clearing for consumption goods ($C_t = Y_{c,t}$), hours worked ($H_t = H_{c,t} + H_{i,t}$), and the two capital goods (equations (3) and (4)).

The equilibrium requirements are weaker than what is required in a competitive rational expectations equilibrium, because household beliefs are not restricted to be rational. For the special case where $\mathcal{P}$ incorporates rational expectations, the previous definition defines a standard competitive rational expectations equilibrium.

4.4 Connecting Model Variables to Data Moments

In order to compare our model to the data, we need to define the real variables (investment, output) and stock market variables (stock prices, dividends) in our production economy. For the real variables, this is relatively straightforward. We follow Boldrin,

[25]These capital holding constraints are required for subjective price beliefs to be consistent with internal rationality in a setting with an effectively exogenous stochastic discount factor, see Adam and Marcet (2011) for details. While they must be chosen sufficiently large so as to never bind in equilibrium, their precise values do not matter for equilibrium determination.

Christiano, and Fisher (2001) and define investment as being proportional to the quantity of capital produced and use the steady state capital price Q_c^{ss} as a base price, so that fluctuations in the price of capital do not contribute to fluctuations in real investment. Investment is thus given by

$$I_t = Q_c^{ss} Y_{i,t}$$

and output correspondingly by

$$Y_t = C_t + I_t.$$

To define stock prices and dividends, we consider a setup where (investment- and consumption-sector) capital can be securitized via shares and where shares and capital can be jointly created or jointly destroyed at no cost. The absence of arbitrage opportunities then implies that the price of shares is determined by the price of the capital it securitizes. We consider a representative consumption-sector share and a representative investment-sector share. The only free parameter in this extended setup is then the dividend policy of stocks, which is indeterminate (Miller and Modigliani, 1961). To obtain a parsimonious setting, we assume a time-invariant profit payout share $p \in (0,1)$: a share p of rental income/profits per share is paid out as dividends each period and in both sectors, with the remaining share $1-p$ being reinvested in the capital stock that the share securitizes.

We now describe the setting for the consumption sector in greater detail. The setup for the investment sector is identical up to an exchange of subscripts. Let $k_{c,t}$ denote the units of (beginning-of-period t) capital held per unit of shares issued in the consumption sector. The capital is used for production and earns a rental income/profit of $k_{c,t}R_{c,t}$. Given the payout ratio $p \in (0,1)$, dividends per share are given by

$$D_{c,t} = p k_{c,t} R_{c,t}.$$

Retained profits are reinvested to purchase $(1-p)\, k_{c,t}R_{ct}/Q_{c,t}$ units of new capital per share.[26] The end-of-period capital per share then consists of the depreciated beginning-of-period capital stock and purchases of new capital from retained profits. The end-of-period share price $P_{c,t}$ is thus equal to[27]

$$P_{c,t} = (1-\delta) k_{c,t} Q_{c,t} + (1-p) k_{c,t} R_{c,t},$$

and the end-of-period PD ratio is given by

$$\frac{P_{c,t}}{D_{c,t}} = \frac{1-\delta_c}{p}\frac{Q_{c,t}}{R_{c,t}} + \frac{1-p}{p}.$$

[26] In case the aggregate capital supply differs from capital demand implied by the existing number of shares, new shares are created or existing shares repurchased to equilibrate capital demand and supply.

[27] We compute end-of-period share prices, because this is the way prices have been computed in the data.

Since the last term is small for reasonable payout ratios p, the end-of-period PD ratio is approximately proportional to the capital price over rental price ratio $(Q_{c,t}/R_{c,t})$. Moreover, as is easily verified, the equity return per unit of stock $R^e_{c,t} = (P_{c,t} + D_{c,t})/P_{c,t-1}$ is equal to the return per unit of capital $R^k_{c,t} = ((1-\delta_c)Q_{c,t} + R_{c,t})/Q_{c,t-1}$.

Given sectoral stock prices and PD ratios, we can define the aggregate PD ratio using a value-weighted portfolio of the sectoral investments. Let

$$w_{c,t-1} = \frac{Q_{c,t-1}K_{c,t}}{Q_{c,t-1}K_{c,t} + Q_{i,t-1}K_{i,t}}$$

denote the end-of-period $t-1$ value share of the consumption sector. The value share of the investment sector is then $1 - w_{c,t-1}$. A portfolio with total value P_{t-1} at the end of period $t-1$ and value shares $w_{c,t-1}$ and $1 - w_{c,t-1}$ in the consumption and investment-sector, respectively, must contain $w_{c,t-1}P_{t-1}/P_{c,t-1}$ consumption shares and $(1-w_{c,t-1})\,P_{t-1}/P_{i,t-1}$ investment shares. The end-of-period t value of this portfolio is then given by

$$P_t = \frac{w_{c,t-1}P_{t-1}}{P_{c,t-1}}P_{c,t} + \frac{(1-w_{c,t-1})\,P_{t-1}}{P_{i,t-1}}P_{i,t} \tag{8}$$

and period t dividend payments for this portfolio are

$$D_t = \frac{w_{c,t-1}P_{t-1}}{P_{c,t-1}}D_{c,t} + \frac{(1-w_{c,t-1})\,P_{t-1}}{P_{i,t-1}}D_{i,t}. \tag{9}$$

Using the previous two equations, the aggregate PD ratio can be expressed as a weighted mean of the sectoral PD ratios where the weights are given by the share of portfolio dividends coming from each sector:

$$\frac{P_t}{D_t} = \frac{w_{c,t-1}\frac{D_{c,t}}{P_{c,t-1}}}{w_{c,t-1}\frac{D_{c,t}}{P_{c,t-1}} + (1-w_{c,t-1})\frac{D_{i,t}}{P_{i,t-1}}}\frac{P_{c,t}}{D_{ct}} + \frac{(1-w_{c,t-1})\frac{D_{i,t}}{P_{i,t-1}}}{w_{c,t-1}\frac{D_{c,t}}{P_{c,t-1}} + (1-w_{c,t-1})\frac{D_{i,t}}{P_{i,t-1}}}\frac{P_{i,t}}{D_{i,t}}.$$

Note that the PD ratio is independent of the initial portfolio value P_{t-1}. Aggregate dividend growth can similarly be expressed using equations (9) and (8) as a weighted average of the sectoral dividend growth rates. This completes our definition of model variables.

5 Price Beliefs, Probability Space and State Space

In specifying household beliefs $\mathcal{P}$, we shall consider two alternative belief specifications: (1) a standard setting in which *all* expectations are rational and (2) a setting in which households hold subjective beliefs about future capital prices $(Q_{c,t+j}, Q_{i,t+j})$, in line with the belief setup considered in Adam, Marcet, and Beutel (2017), but rational expectations about all remaining variables $(Z_{t+j}, X_{t+j}, W_{t+j}, R_{c,t+j}, R_{i,t+j})$.

We consider the setting with fully rational expectations as a point of reference. It is well known that under rational expectations the asset pricing implications of the model are strongly at odds with the data. Considering this setup, however, allows highlighting the empirical improvements achieved by introducing subjective price beliefs.

Two considerations motivate us to keep rational expectations about variables other than prices: first, we do not want to deviate from the rational expectations assumption by more than in Adam, Marcet, and Beutel (2017), so as to illustrate that the same deviation that can be used to explain stock price behavior in an endowment setting can explain stock price behavior and business cycle dynamics; second, investor expectations about future stock prices can be observed relatively easily from investor survey data, which allows disciplining the subjective belief choice. Observing beliefs about future values of $(X_{t+j}, W_{t+j}, R_{c,t+j}, R_{i,t+j})$ is a much harder task, which prompts us to keep the standard assumption of rational expectations.

Under rational expectations, decision functions in period t depend only on the history of fundamental shocks Z^t, as discussed in Section 4.2. Given the Markov-structure for shocks Z_t, there is furthermore a recursive time-invariant form of the decision functions, where decisions depend only on current shock Z_t and the beginning-of-period capital stocks $(Z_t, K_{c,t}, K_{i,t})$. Using this fact, we can standardly solve for the nonlinear rational expectations equilibrium using global approximation methods.[28]

We consider subjective capital price expectations of the form previously introduced in Adam, Marcet, and Beutel (2017). Specifically, we assume that agents perceive the end-of-period capital prices $Q_{s,t}$ $(s = c, i)$ to evolve as

$$\log Q_{s,t} = \log Q_{s,t-1} + \log \beta_{s,t} + \log \varepsilon_{s,t},$$

with $\varepsilon_{s,t}$ denoting a transitory shock to price growth and $\beta_{s,t}$ a persistent component, which is given by

$$\log \beta_{s,t} = \log \beta_{s,t-1} + \log \nu_{s,t}.$$

The innovations $(\varepsilon_{s,t}, \nu_{s,t})$ are independent of each other, with $\log \varepsilon_{s,t} \sim ii\mathcal{N}(-\sigma_\varepsilon^2/2, \sigma_\varepsilon^2)$ and $\log \nu_{s,t} \sim ii\mathcal{N}(-\sigma_\nu^2/2, \sigma_\nu^2)$, and also independent of all other variables.[29]

Each period t, agents only observe capital prices $\log Q_s$ up to period t and estimate the persistent component driving capital gains. Let $\log \beta_{s,t-1} \sim \mathcal{N}(\log m_{s,t-1}, \sigma^2)$ denote the period $t-1$ prior belief about the persistent component, where σ denotes the steady state Kalman filter uncertainty.[30] In period t, agents observe the new capital price $Q_{s,t}$ and update their beliefs using Bayes' law. Their period t beliefs are then given by $\log \beta_{s,t} \sim$

[28]This requires a standard transformation of variables, so as to render them stationary.

[29]The random variables $\varepsilon_{s,t+1}$, $\nu_{s,t+1}$ are not defined on the probabiltiy space Ω, but on a larger auxiliary space $\overline{\Omega}$, as they contain random variation that is independent of variation in both prices and other observables in equilibrium. Decision function can still be specified to be functions of Ω only.

[30]We have $2\sigma^2 \equiv -\sigma_\nu^2 + \sqrt{(\sigma_\nu^2)^2 + 4\sigma_\nu^2\sigma_\varepsilon^2}$.

$\mathcal{N}(\log m_{s,t}, \sigma^2)$ with

$$\log m_{s,t} = \log m_{s,t-1} - \frac{\sigma_v^2}{2} + g\left(\log Q_{s,t} - \log Q_{s,t-1} + \frac{\sigma_\varepsilon^2 + \sigma_v^2}{2} - \log m_{s,t-1}\right), \tag{10}$$

where the Kalman gain is given by

$$g = \frac{\sigma^2}{\sigma^2 + \sigma_\varepsilon^2}. \tag{11}$$

Agents' beliefs can thus be parsimoniously summarized by the variables $(m_{c,t}, m_{i,t})$, which capture *agents' degree of optimism* about future capital gains. Equation (10) shows how agents' capital gain expectations are driven by observed past capital gains, whenever $g > 0$. Adam, Marcet, and Beutel (2017) show that for values of g around 2% this delivers an empirically plausible specification describing the dynamics of investors' subjective capital gain beliefs over time.

To avoid simultaneous determination of price beliefs and prices, we follow Adam, Marcet, and Beutel (2017) and use a slightly modified information setup in which agents receive at time t information about the transitory component in $t-1$, $\log \varepsilon_{s,t-1}$. The modification causes the updating equation (10) to contain only lagged price growth and no variance correction terms. Capital gain beliefs then evolve according to

$$\log m_{s,t} = \log m_{s,t-1} + g\left(\log Q_{s,t-1} - \log Q_{s,t-2} - \log m_{s,t-1}\right) + g \log \varepsilon_{s,t}^1 \tag{12}$$

where $\log \varepsilon_{s,t}^1 \sim ii\mathcal{N}(\frac{-\sigma_\varepsilon^2}{2}, \sigma_\varepsilon^2)$ is a time t innovation to agents' information set (unpredictable using information available to agents up to period $t-1$), which captures information that agents receive in period t about the transitory price growth component $\log \varepsilon_{s,t-1}$.[31] Agents' expectations are then given by

$$E_t^{\mathcal{P}}\left[\frac{Q_{s,t+1}}{Q_{s,t}}\right] = m_{s,t}. \tag{13}$$

Given the specific subjective price beliefs and the assumption of rational expectations about $(Z_{t+j}, X_{t+j}, W_{t+j}, R_{c,t+j}, R_{i,t+j})$, we can work with the probability space

$$\widetilde{\Omega} \equiv \Omega_Z \times \Omega_{Q,c} \times \Omega_{Q,i}$$

[31]However, we follow the baseline specification of Adam, Marcet, and Beutel (2017) and assume that along the equilibrium path no actual information is observed, instead set $\ln \varepsilon_{s,t}^1$ to 0 each period. It is easy to see in our specific model, that – given the solution to the filtering problem – beliefs about future m values do not matter for policy functions. For this reason we work with a reduced updating equation

$$\log m_{s,t} = \log m_{s,t-1} + g\left(\log Q_{s,t-1} - \log Q_{s,t-2} - \log m_{s,t-1}\right)$$

and the reduced outcome space $\tilde{\Omega}$ below, which does not include observable innovations $\varepsilon_{c,t}^1$ and $\varepsilon_{i,t}^1$.

and consider decision functions of the form

$$(C_t, H_t, K_{c,t+1}, K_{i,t+1}) : \widetilde{\Omega}^t \longrightarrow R^4. \tag{14}$$

Moreover, the subjective belief setup allows us to summarize the history of capital prices in the two sectors using the two state variables $(m_{c,t}, m_{it}) \in \mathbb{R}^2$, so that household decision functions are time-invariant function of an extended set of state variables $(Z_t, K_{c,t}, K_{i,t}, m_{c,t}, m_{it})$. The state S_t describing the aggregate economy at time t can be summarized by the vector $S_t = (Z_t, K_{c,t}, K_{i,t}, m_{c,t}, m_{it}, Q_{c,t-1}, Q_{i,t-1})$, where the latter two variables are required to be able to describe the equilibrium dynamics of beliefs over time, as implied by equation (12). The economy's equilibrium dynamics can then be described by a nonlinear state transition function $G(\cdot)$ mapping current states and future technology into future states

$$S_{t+1} = G(S_t, Z_{t+1}),$$

and by an outcome function $F(\cdot)$ mapping states into economic outcomes for the remaining variables

$$(W_t, R_{c,t}, R_{i,t}, Y_t, C_t, I_t, H_t) = F(S_t).$$

We endow households with beliefs about future values $(W_{t+j}, R_{c,t+j}, R_{i,t+j})$ consistent with these equilibrium mappings.[32] Since the mappings depend themselves on the solution to the household problem and the solution to the household problem on the assumed mappings, one needs to solve for a fixed point. Because the economy features seven state variables, it is generally not feasible to solve for the fixed point of this problem at a speed that would allow estimating the subjective belief model using the simulated methods of moments. Yet, as we explain below, it becomes computationally feasible for our log-linear specification for household preferences.

6 Equilibrium Conditions

This section derives the equations characterizing the competitive equilibrium. These equations hold independently of the assumed belief structure. From the household's

[32] That is, households beliefs $\mathcal{P}$ over the full set of exogenous variables Ω are jointly described by the equilibrium mappings F and G and a measure $\tilde{\mathcal{P}}$ on $\tilde{\Omega}$ that is consistent with the exogenous law of motion for Z and the perceived laws of motion for Q_i and Q_c implied by the learning specification above. Appendix A.3.1 provides details on how $\mathcal{P}$ can be recovered from $\tilde{\mathcal{P}}$ and the mappings F and G. Appendix A.3.2 proves that there is a unique measure $\tilde{\mathcal{P}}$ consistent with the subjective belief description given in this section.

first-order conditions we have[33]

$$C_t = W_t, \tag{15}$$

$$Q_{c,t} = \beta E_t^{\mathcal{P}}\left[\frac{W_t}{W_{t+1}}\left(\left(1-\delta_c\right)Q_{c,t+1} + R_{c,t+1}\right)\right], \tag{16}$$

$$Q_{i,t} = \beta E_t^{\mathcal{P}}\left[\frac{W_t}{W_{t+1}}\left(\left(1-\delta_i\right)Q_{i,t+1} + R_{i,t+1}\right)\right], \tag{17}$$

for all $t \geq 0$. The first-order conditions of consumption-sector firms are

$$W_t = \frac{\left(1-\alpha_c\right)Y_{c,t}}{H_{c,t}}, \tag{18}$$

$$R_{c,t} = \frac{\alpha_c Y_{c,t}}{K_{c,t}}, \tag{19}$$

and the optimality conditions of investment-sector firms

$$W_t = \left(1-\alpha_i\right)Q_{c,t}K_{i,t}^{\alpha_i}Z_t^{1-\alpha_i}H_{i,t}^{-\alpha_i}, \tag{20}$$

$$R_{i,t} = \alpha_i Q_{c,t}K_{i,t}^{\alpha_i-1}Z_t^{1-\alpha_i}H_{i,t}^{1-\alpha_i}. \tag{21}$$

The market clearing conditions are

$$C_t = Y_{c,t}, \tag{22}$$

$$H_t = H_{c,t} + H_{i,t}, \tag{23}$$

$$K_{c,t+1} = \left(1-\delta_c\right)K_{c,t} + Y_{i,t}, \tag{24}$$

$$K_{i,t+1} = \left(1-\delta_i\right)K_{i,t} + X_t. \tag{25}$$

For the case with subjective beliefs, we have in addition (for $s = c, i$)

$$E_t^{\mathcal{P}}\left[Q_{s,t+1}\right] = m_{s,t}Q_{s,t}, \tag{26}$$

where $m_{s,t}$ evolves according to (12) and rational expectations about all other variables. For the case with rational expectations, we have $E_t^{\mathcal{P}}\left[\cdot\right] = E_t\left[\cdot\right]$.

Equations (15), (18) and (22) show that labor in the consumption sector is constant and given by

$$H_{c,t} = 1 - \alpha_c. \tag{27}$$

Consumption variations are thus driven entirely by productivity variations and by the dynamics of capital accumulation in the consumption sector. This feature will allow

[33] The household budget constraint will hold automatically, as we will keep all market clearing conditions in the set of equations characterizing equilibrium.

the model to replicate the smoothness of the aggregate consumption series. For the investment sector, we obtain from equation (20) that

$$H_{i,t} = K_{i,t} Z_t^{\frac{1-\alpha_i}{\alpha_i}} \left((1-\alpha_i) \frac{Q_{c,t}}{W_t} \right)^{\frac{1}{\alpha_i}}, \tag{28}$$

which shows that high prices for consumption capital ($Q_{c,t}$) induce – ceteris paribus – high labor demand by firms in the investment sector and thus drive up investment in the consumption sector. This feature will allow the model to generate a positive comovement between capital prices, hours worked and investment. In addition, it follows from equation (20) that high prices for consumption capital ($Q_{c,t}$) and high labor input ($H_{i,t}$) induce high rental rates in the investment sector. From equation (17) it then follows that high prices for consumption capital ($Q_{c,t}$) transmit via this channel to higher prices for investment capital ($Q_{c,t}$), so that the asset prices in the two sectors comove with each other.

7 Equilibrium Dynamics under Subjective Beliefs

This section provides analytic insights into the equilibrium dynamics of the model under subjective beliefs. These analytic results show how the model can give rise to persistent swings in capital prices and thus generate volatile and persistent stock price dynamics, while simultaneously producing smooth dynamics for the remaining variables.

We start by studying the prices dynamics of consumption-sector capital. Given the assumption of rational dividend and wage expectations and given the assumed subjective price beliefs (26), we obtain from equation (16)

$$Q_{c,t} = \beta E_t \left[\frac{W_t}{W_{t+1}} \right] (1-\delta_c)\, m_{c,t} Q_{c,t} + \beta E_t \left[\frac{W_t}{W_{t+1}} R_{c,t+1} \right]. \tag{29}$$

Using the equilibrium relationships (19), (22) and (15) allows expressing the discounted expected rental rate (the last term in equation (29)) using period t variables:

$$E_t \left[\frac{W_t}{W_{t+1}} R_{c,t+1} \right] = \alpha_c \frac{W_t}{K_{c,t+1}} \tag{30}$$

Substituting into equation (29) and solving for $Q_{c,t}$ delivers

$$Q_{c,t} = \frac{\beta \alpha_c}{\frac{1}{W_t} - \beta (1-\delta_c) E_t \left[\frac{1}{W_{t+1}} \right] m_{c,t}} \frac{1}{K_{c,t+1}}. \tag{31}$$

To illustrate the asset price dynamics implied by this equation, suppose for a moment that the capital stock ($K_{c,t+1}$) and expected future wages ($E_t[1/W_{t+1}]$) are constant over time.

We shall consider the additional effects associated with movements in these variables below. The asset pricing equation (31) then takes the particularly simple form

$$Q_{c,t} = \frac{A}{\frac{1}{W_t} - Bm_{c,t}}, \tag{32}$$

where A and B are positive constants. From

$$W_t = C_t = Y_{c,t} = Z_t K_{c,t}^{\alpha_c} (1-\alpha_c)^{1-\alpha_c}, \tag{33}$$

we get that wages are determined by productivity (Z_t) and the consumption-sector capital stock ($K_{c,t}$). This shows that that capital prices in equation (32) increase with productivity (Z_t), the capital stock ($K_{c,t}$) and with subjective optimism ($m_{c,t}$). Optimism is itself driven by past observed capital gains according to equation (12). Independent of the source of an initial increase in the capital price $Q_{c,t,}$, it follows from equation (32) that it will propagate over time through two self-reinforcing effects: (1) any observed capital gain today increases – via the belief updating equation (12) – optimism tomorrow ($m_{c,t+1}$) and thus tomorrow's capital prices; (2) an increase in the current capital price $Q_{c,t}$, increases hours worked in the investment sector (equation (28)), thereby tomorrow's consumption-sector capital and thus tomorrow's wages (W_{t+1}) (equation (33)).[34] An increased wage tomorrow raises tomorrow's capital price (equation (32)).

The previous arguments shows how – for given end-of-period capital stock levels ($K_{c,t+1}$) and given expectations about future wages ($E_t[1/W_{t+1}]$) – asset price increases are followed by further increases.[35] Yet, along a path where $Q_{c,t}$ rises, investment in consumption-sector capital also rises and thus the capital stock $K_{c,t+1}$ and future wages W_{t+1}. This sets in motion dampening forces that work against the capital price increase. The last term on the r.h.s of equation (31) shows that an increased capital stock starts to directly dampen the capital price increase. This effect emerges because an expanded capital stock leads – ceteris paribus – to falling rental rates (equation (30)). Similarly, since wages will increase as the capital stock expands (equation (33)) inverse wage expectations $E_t\left[1/W_{t+1}\right]$ in equation (31) start to fall. This exerts an additional dampening effect on asset prices (equation (31)).

It turns out that the dampening forces associated with rising capital stocks and rising expected wages are not always strong enough to terminate capital price booms. To insure that capital prices remain finite, we follow Adam, Marcet, and Nicolini (2016)

[34]This investment increase is obvious, if it is triggered by an increase in $m_{c,t}$, as the wage W_t does then not react. If instead the initial price increase is due to an increase in productivity, we can substitute (32) into (28) to obtain $H_{i,t} = K_{i,t} Z_t^{\frac{1-\alpha_i}{\alpha_i}} \left((1-\alpha_i)\frac{A}{1-W_t Bm_{c,t}}\right)^{\frac{1}{\alpha_i}}$. This shows that the joint wage and capital price increase still implies an increase in hours worked in the investment sector.

[35]Similarly, asset price decreases are followed by further decreases.

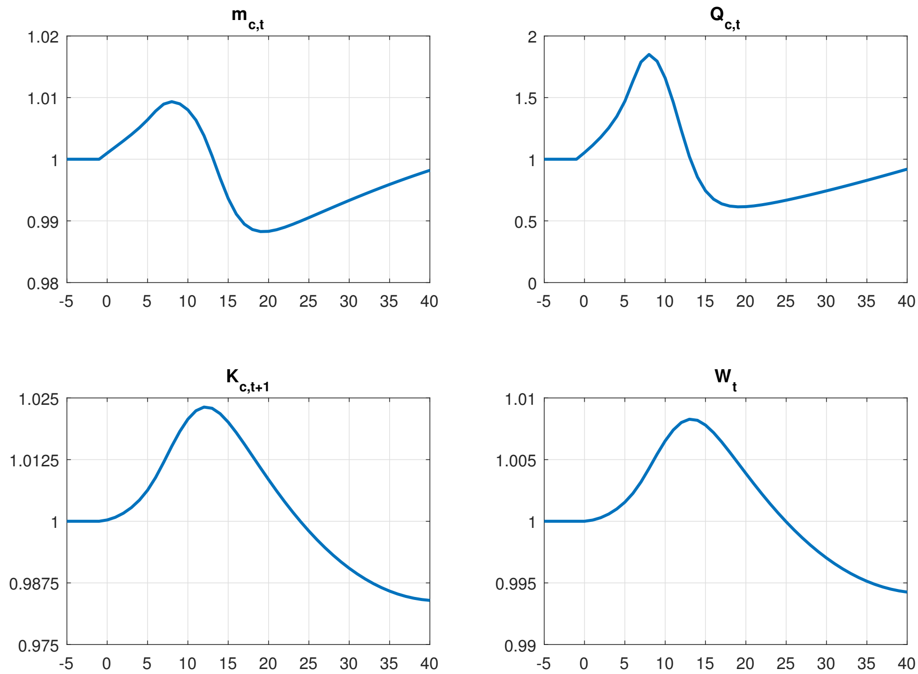

Figure 3. Impulse response to a +10bps shock to expected capital gains in consumption-sector capital ($m_{c,t}$)

and impose a projection facility that eventually bounds upward belief revisions.[36] The projection facility can be interpreted as an approximate implementation of a Bayesian updating scheme where agents have a truncated prior about the support of $\beta_{s,t}$. The details of the projection facility are spelled out in Appendix A.2.

Figure 3 illustrates the quantitative model dynamics by displaying the impulse responses to an exogenous one-time increase in the quarterly expected growth rate of the price of consumption sector capital ($m_{c,t}$) by 10 basis points in period zero.[37] The figure uses the estimated parameters from Section 8 and normalizes all variables relative to the deterministic balanced growth path values that would emerge in the absence of

[36]The issue of bounding beliefs so as to ensure that expected utility remains finite is present in many applications of both Bayesian and adaptive learning to asset prices. The literature has typically dealt with this issue by using a projection facility, assuming that agents simply *ignore* observations that would imply updating beliefs beyond the required bound. See Timmermann (1993, 1996), Marcet and Sargent (1989), or Evans and Honkapohja (2001). This approach has two problems. First, it does not arise from Bayesian updating. Second, it introduces a discontinuity in the simulated moments and creates difficulties for our MSM estimation. To avoid this, we follow the differentiable approach to bounding beliefs used in Adam, Marcet, and Nicolini (2016).

[37]We report the impulse response to a technology shock in Section 8.

the shock. Figure 3 shows that the period zero increase in optimism and the associated small increase in the capital price is followed by eight quarters of further increases in optimism and capital prices. The increased capital price leads to additional investment in consumption-sector capital and also to a wage increase, as discussed before. These reactions eventually dampen the price increase and capital prices start to revert direction. Prices significantly undershoot their balanced growth path value before slowly recovering. As capital prices become depressed, investment falls and consumption-sector capital persistently undershoots its balanced growth path value. Associated with the reduction in capital is a reduction in wages.

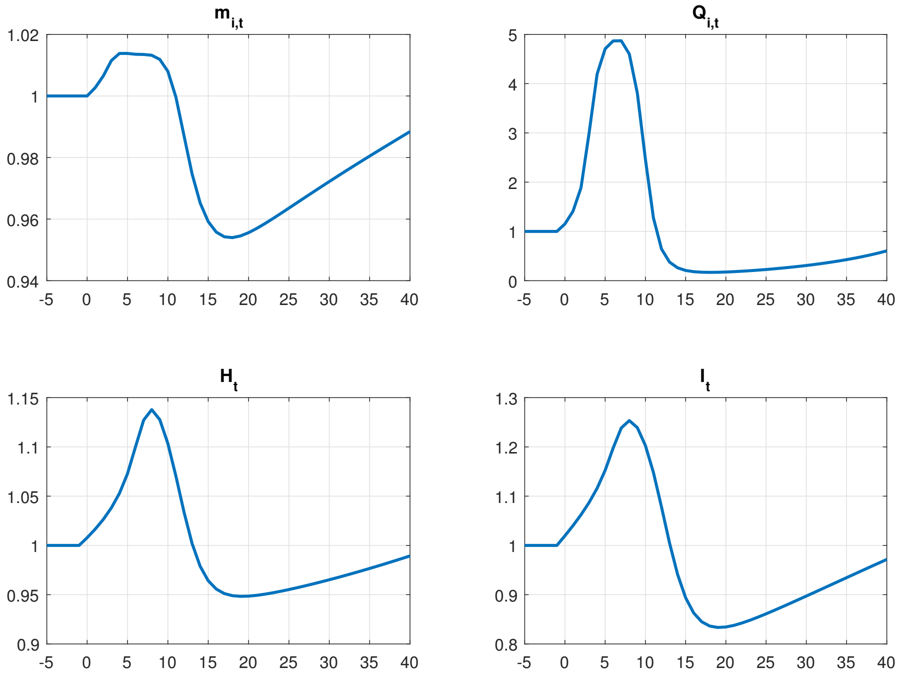

Figure 4. Impulse response to a +10bps shock to expected capital gains in consumption-sector capital ($m_{c,t}$)

Figure 4 illustrates how the dynamics in the consumption sector spill over into the investment sector. To understand the channels through which this spillover happens, note that we obtain from equation (17) and the assumed subjective price beliefs (26) the following relationship:

$$Q_{i,t} = \frac{\beta E_t \left[\frac{W_t}{W_{t+1}} R_{i,t+1} \right]}{1 - \beta \left(1 - \delta_i\right) E_t \left[\frac{W_t}{W_{t+1}} \right] m_{i,t}}. \tag{34}$$

Since wages increase during the price boom in consumption sector capital, see Figure 3, the drop of $E\left[W_t/W_{t+1}\right]$ exerts downward pressure on the price of investment-sector capital in both the numerator and the denominator of equation (34). Yet, equation (20) shows that expected rental rates $R_{i,t+1}$ increase strongly following the increase in the price of consumption-sector capital (and also as a result of the increase in hours worked in the investment sector). Due to the strong response of $R_{i,t+1}$, the overall price effect at time zero is positive and the price boom in the consumption-sector capital spills over into an increase in the price of investment-sector capital. The initial price increase then propagates via the belief updating mechanism (12) and the positive dependence of capital prices on $m_{i,t}$ (equation (34)) into a persistent boom for the price of investment-sector capital. This shows that capital prices in the two sectors have a tendency to comove over time.

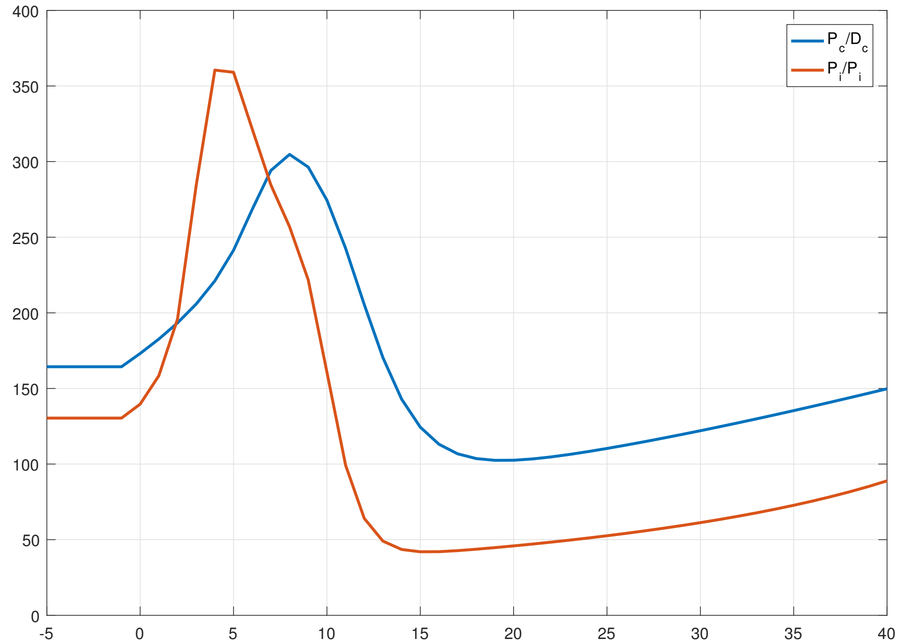

Figure 5. Impulse response to a +10bps shock to expected capital gains in consumption-sector capital ($m_{c,t}$)

Figures 3 and 4 show that capital prices in the investment sector move considerably more than in the consumption sector. Figure 5 shows that this difference is far less pronounced for a standard stock market valuation measure such as the PD ratio. Since dividends in the investment sector increase strongly during the price boom (equation (21)) the difference in the response of the PD ratio across sectors is much more muted. Overall, the model is consistent with empirical observation that stock prices in the investment sector are more volatile than stock prices in the consumption sector.

While the PD ratios approximately double in response to the considered optimism shock, the response of real quantities is one or two orders of magnitude smaller. Consumption, for instance, increases in line with the increase in the stock of consumption sector capital and moves by less than +/- 1% over the boom-bust cycle, see Figure 3. Hours worked are more variable and increase by around 13% while investment increases by up to 25%. This relatively muted response of real variables allows the model to reconcile the relative smoothness of the business cycle with the observed high volatility of stock prices.

8 Quantitative Performance: Subjective versus Rational Price Beliefs

This section estimates the model using the simulated methods of moments (SMM). In the setting with subjective price beliefs, the model has nine parameters that need to be estimated, $(\beta, \alpha_c, \alpha_i, \delta_c, \delta_i, \gamma, \sigma, p, g)$. There is one parameter less under fully rational expectations, as the Kalman gain g then drops out.

In our baseline approach, we estimate the model using 13 target moments and a diagonal weighting matrix with the data-implied variance of the estimated data moment as diagonal entries. For the subjective belief model, these target moments include the seven business cycle moments listed in Table 1 and all asset pricing moments listed in Table 2, except for the mean and standard deviation of the risk-free interest rate. We exclude the risk-free rate moments from the set of targeted moments because the model has a hard time fully matching the equity premium. We shall nevertheless report the risk-free rate moments implied by the estimated model. For the rational expectations model, we furthermore exclude the equity return moments ($E[r^e]$, $\sigma(r^e)$) and the autocorrelation of the PD ratio ($\rho(P/D)$), as including these variables significantly deteriorates the model fit along the business cycle dimension without improving it substantially for the excluded moments.[38,39]

With fully rational expectations, equilibrium prices and equilibrium quantities are

[38]Excluding these moments does not conceal the true potential of the rational expectations model to match the equity return moments. Even if we give the model the best chance to match $E[r^e]$ and $\sigma(r^e)$ by making them the only estimation targets, the resulting model moments are just 0.74% and 0.50%, respectively, as opposed to the values 0.77% and 0.16% reported in Table 6. Relative to their empirical counterparts (1.87% and 7.98%), these numbers are small and similar in magnitude.

[39]We keep the level and standard deviation of the PD ratio as estimation targets for the RE model, because otherwise the payout ratio p would be unidentified. Excluding them has almost no impact on the remaining estimated parameters.

unique.[40] For the case with subjective price beliefs, this is far from obvious, but the following result shows that we have indeed uniqueness of equilibrium allocations and prices in our setting with subjective price beliefs:

Proposition 1. *There are unique functions G and F and a unique measure $\tilde{\mathcal{P}}$ on $\tilde{\Omega}$, such that*

1. *$\tilde{\mathcal{P}}$ describes the joint beliefs of households about technology and capital prices as defined in Section 5*
2. *G is a state transition function, $S_{t+1} = G(S_t, Z_{t+1})$, and F an outcome function, $(W_t, R_{c,t}, R_{i,t}, Y_t, C_t, I_t, H_t) = F(S_t)$, such that S_t, S_{t+1} and $(W_t, R_{c,t}, R_{i,t}, Y_t, C_t, I_t, H_t)$ are consistent with*
   (a) All equilibrium conditions (see Section 6)
   (b) Households' belief updating equations (equation (12) for $s \in \{c, i\}$)[41]

The proof of Proposition 1 is given in Appendix A.3.3. There, we also provide an explicit characterization of the mappings G and F based on the equations collected in Section 6. Essentially, this characterization determines those functions up to a static equation system (stated in Lemma 4 in the appendix). In our numerical solution procedure, we solve this equation system and construct the mapping G mirroring the construction in the proof of Proposition 1. Having recovered G and F, this essentially determines equilibrium.[42]

When estimating the learning model we impose the additional restriction that the impulse responses of capital prices to fundamental shocks display an exponential decay rate of 1.16% per quarter.[43] We do so to avoid that the estimation selects parameter values that would imply deterministic equilibrium cycles. Clearly, imposing this additional restriction can only deteriorate the fit of the targeted moments. The model moments that we present thus represent a lower bound on the fit that could be achieved in the absence of this restriction.

Table 5 reports the estimated parameter values and Table 6 target moments. We first discuss business cycle moments, which are reported in the upper half of Table 6. Along this dimension, our estimated subjective belief model matches the targets very well. Instead, the rational expectations model displays somewhat too much consumption

[40] As is well-known, the market allocation is equivalent to the solution of a social planning problem, with the latter featuring a concave objective function and a convex set of constraints.

[41] With projection adjustments as described in Appendix A.2, where necessary.

[42] Moments are computed based on a Monte Carlo simulation of model dynamics using the state transition function G. Simulated paths for all parameter combinations start in the balanced growth path and are based on the same random draw of 10,000 productivity shocks. The first 500 observations of the simulation are dropped before computing moments.

[43] Details of this restriction are provided in Appendix A.4.

Table 5
Estimated parameters

Parameter	Subjective Belief Model	RE Model w/o inv. shocks	RE Model w inv. shocks
β	0.996	0.997	0.997
α_c	0.36	0.36	0.36
α_i	0.73	0.35	0.35
δ_c	0.010	0.020	0.010
δ_i	0.014	0.030	0.030
γ	1.004	1.005	1.003
σ	0.015	0.018	0.015
p	0.350	0.195	0.337
g	0.019	–	–
σ_i	–	–	0.015

Table 6
Model fit for targeted moments (13 / 10 targets, 9/8 model parameters for the subjective belief/RE model)

	Data (std.dev.)	Subj. Belief Model	RE Model w/o inv. shocks	RE Model w inv. shocks
	Business Cycle Moments			
$\sigma(Y)$	1.72 (0.25)	1.83	1.90	1.85
$\sigma(C)/\sigma(Y)$	0.61 (0.03)	0.67	0.75	0.66
$\sigma(I)/\sigma(Y)$	2.90 (0.35)	2.90	1.88	2.79
$\sigma(H)/\sigma(Y)$	1.08 (0.13)	1.06	0.31	0.56
$\rho(Y,C)$	0.88 (0.02)	0.84	0.98	0.86
$\rho(Y,I)$	0.86 (0.03)	0.89	0.97	0.90
$\rho(Y,H)$	0.75 (0.03)	0.70	0.89	0.80
	Financial Moments			
$E[P/D]$	152.3 (25.3)	150.0	174.6	166.0
$\sigma(P/D)$	63.39 (12.39)	44.96	7.00	8.28
$\rho(P/D)$	0.98 (0.003)	0.97	0.96	0.95
$E[r^e]$	1.87 (0.45)	1.25	0.77	0.57
$\sigma(r^e)$	7.98 (0.35)	7.07	0.16	0.16
$\sigma(D_{t+1}/D_t)$	1.75 (0.38)	2.46	1.19	1.69

volatility and considerably too little investment and hours volatility. In addition, the degree of comovement of the macro aggregates is higher in the rational expectations model than both in the data and in the subjective belief model.

The poorer performance of the rational expectations model can be fixed by introducing additional investment-specific shocks.[44] While the literature has mostly focused on persistent investment-specific technology processes, we add here just a parsimonious iid component to the investment-sector production function, so as to keep the changes to the model described above minimal and to have only one additional parameter. Specifically, we replace $Y_{i,t}$ from equation (1) by

$$Y_{i,t} = \varepsilon_t^I K_{i,t}^{\alpha_i} \left(Z_t H_{i,t}\right)^{1-\alpha_i}$$

with $\log \varepsilon_t^I \sim \mathcal{N}(-\frac{\sigma_i^2}{2}, \sigma_i^2)$. We solve and re-estimate this augmented model under rational expectations using the same procedure and targets as for the model without investment-specific shocks. The resulting parameter estimates and model moments are reported in the third columns of Tables 5 and 6, respectively. With investment shocks, the business cycle predictions of the model are much closer to the data and overall comparable with our subjective belief model. While this approach is in principle capable of generating good business cycle predictions, the required volatility σ_i of the investment-specific shock is as high as the volatility σ of the neutral shock (see Table 5). Interestingly, this additional source of exogenous randomness is not required in our subjective belief model to generate the same amount of investment volatility (and even more hours volatility), despite σ being the same in both specifications.

We next turn to the financial target moments reported in the lower part of Table 6. The model with subjective beliefs is able to match the level and persistence of the PD ratio and the PD ratio in the model also displays substantial variability, although its volatility is somewhat lower than in the data. Equity returns in the model are also very volatile, generating almost 90% of the volatility observed in the data, despite a moderate – though somewhat overstated – dividend growth volatility. The model can match the level of equity returns only partially with the mean equity return in the model being only two thirds of the 1.87% per quarter earned on average on stock market investments in our sample period.

The two versions of the rational expectations model, reported in the last two columns, perform significantly worse along the financial dimension. While the level of the PD ratio (by choice of p), its persistence and the volatility of dividend growth are roughly in line with the data, the rational expectations model generates almost no price and return volatility and equity returns are considerably lower than in the subjective belief model, regardless of whether investment shocks are present or not.[45]

[44] Investment-specific technology shocks are found to be important drivers of business cycles in estimated models (compare Justiniano, Primiceri, and Tambalotti, 2010, 2011) and thus are a natural

Table 7
Model fit for untargeted moments

Moment	Data	Subjective Belief Model	RE Model w/o inv. shocks	RE Model w inv. shocks
$E[r^f]$	0.25 (0.13)	0.78	0.77	0.57
$\sigma(r^f)$	0.82 (0.12)	0.06	0.09	0.16
$\rho(H, P/D)$	0.51 (0.17)	0.79	-0.97	-0.95
$\rho(I/Y, P/D)$	0.19 (0.31)	0.69	-0.97	-0.94
$\rho(E^{\mathcal{P}}[r^e], P/D)$	0.79 (0.07)	0.52	-0.99	-0.98

Table 7 reports all moments from Section 3 which where not targeted by the estimation. The first two rows show the mean and volatility of the risk-free rate. The model is consistent with a stable risk-free rate, both under subjective beliefs and rational expectations, but tends to overstate its level.[46] Yet, despite the too low mean stock return and the too high mean risk-free rate, the subjective belief model generates an equity premium of 0.47%. While only around 30% of the historical equity premium, this result is still remarkable, given the low risk aversion implicit in our preference specification.[47] In comparison, there is literally no equity premium in both rational expectations versions of the model.

The subjective belief model also generates positive comovement between the business cycle and stock prices as measured by the correlations of H and I/Y with the PD ratio. In contrast, these correlations are strongly negative under rational expectations. Furthermore, the model matches the positive correlation between expected returns and the PD ratio. While quantitatively this correlation is almost four standard errors lower than in the data, it is still in the range of values that Adam, Marcet, and Beutel (2017) report for other surveys than the UBS Gallup survey, which our data moment is based on, see their Appendix A.2. We interpret this as evidence that our model generates overall realistic comovement of stock prices and expected returns. Again, rational expectations are strongly at odds with the data: because the PD ratio tends to mean-revert and expected returns must on average forecast realized returns, the correlation is strongly negative in both rational expectation specifications.

candidate to improve business cycle predictions of the real business cycle model.

[45]The poor ability to match $E[r^e]$ and $\sigma(r^e)$ is not because these are untargeted moments in the rational expectations estimation, compare footnote 38.

[46]The somewhat lower value for the model with investment shocks is due to the lower growth rate (γ) in that model.

[47]Due to linear disutility of labor the effective risk aversion is even lower than what is implied by logarithmic utility in consumption: agents are *risk-neutral* with respect to all risk that is uncorrelated with wages. Thus only assets that yield low payoffs in states where wages are low carry a positive risk-premium.

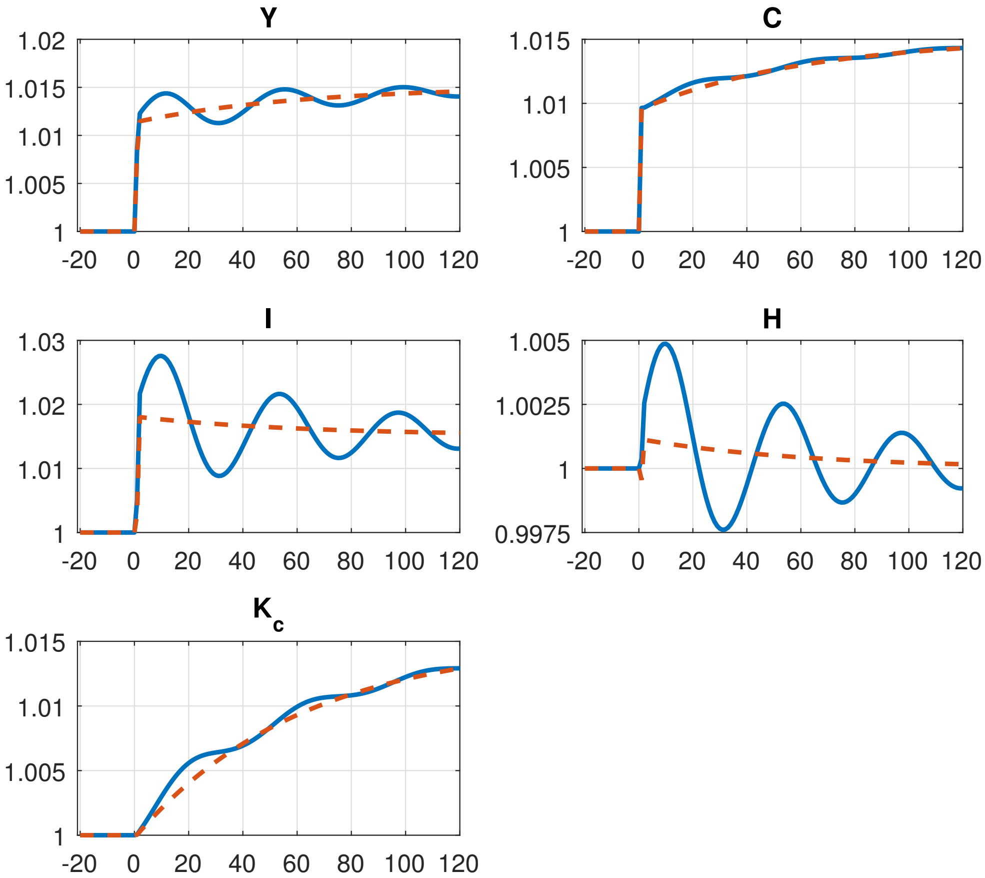

Figure 6. Impulse response functions of macro aggregates to a technology shock for subjective belief model (solid lines) and RE model (dashed lines)

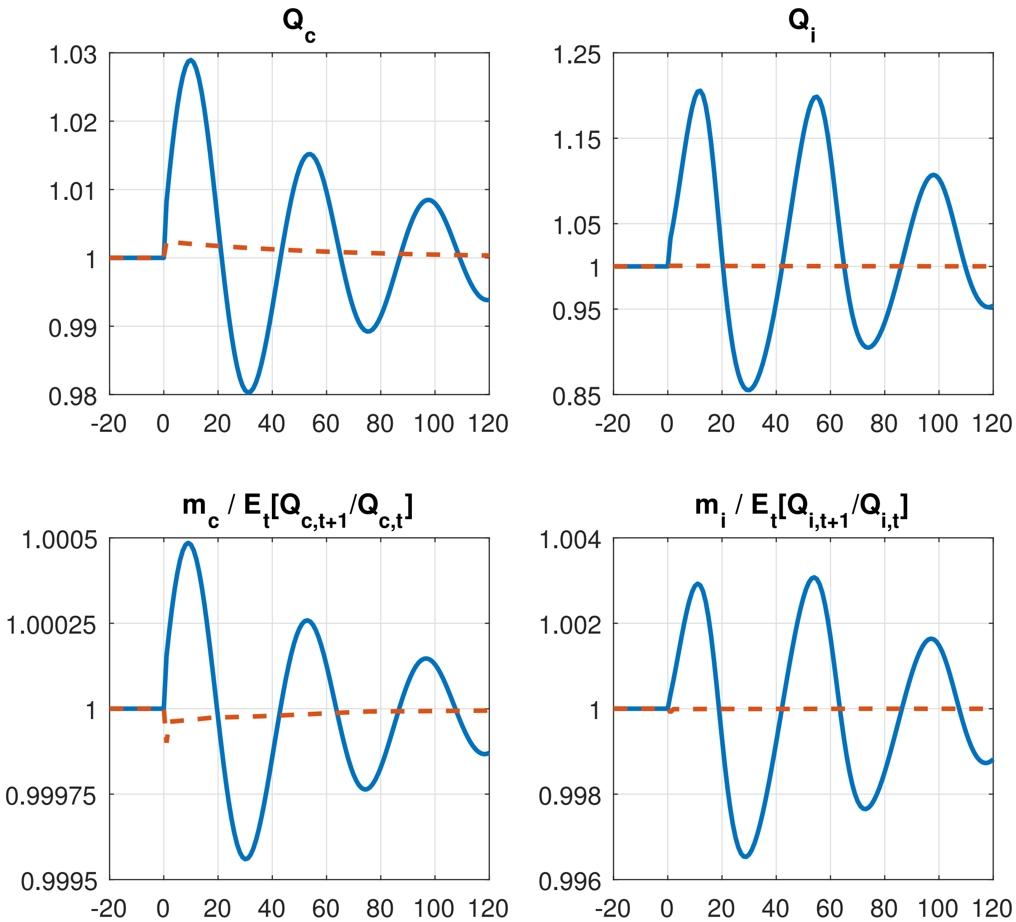

Figure 7. Impulse response functions of prices and beliefs to a technology shock for subjective belief model (solid lines) and RE model (dashed lines)

The central mechanism underlying the dynamics in the subjective belief model has already been discussed in Section 7. To illustrate this mechanism further, Figures 6 and 7 depict impulse response functions of the model with subjective beliefs to a technology shock (solid lines) for macro aggregates and capital prices/expectations, respectively.[48] For comparison purposes, the figures also plot impulse responses of the rational expectations model with an identical parameterization as the subjective belief model (dashed lines). We first discuss the latter as the dynamics under rational expectations are also informative for the response under subjective beliefs and the difference between the two explains well, why our subjective belief model can account better for the joint dynamics of business cycles and stock prices than the rational expectations model.

The dashed lines in Figure 6 show familiar adjustment dynamics present in a real business cycle model with capital adjustment frictions. On impact, the higher technology level leads to an increase in output, consumption and investment and, with one period lag,[49] also in hours worked. The increased technology increases the marginal product of (consumption-sector) capital. This in turn increases capital prices in both sectors above their steady state level (see Figure 7),[50] which sets in motion a capital accumulation process that lasts beyond the impact period (lower panel in Figure 6). As capital increases over time, investment declines and consumption and output increase to their new balanced growth path, on which all three variables are permanently higher (relative to their deterministic trend) than on the pre-shock balanced growth path. As the capital stock expands, marginal products of capital and capital prices return back to steady state. The two lower plots in Figure 7, which show the expected capital price growth, $E_t\left[Q_{s,t+1}/Q_{s,t}\right]$ for $s \in \{c, i\}$, mirror the response of decreasing capital prices along the adjustment path.

Next, we turn to the impulse responses of the subjective belief model. Here, two channels determine the adjustment dynamics after a technology shock. First, the economy reacts on impact almost identical to the rational expectations economy[51] and, again,

[48]We start the model in a deterministic balanced growth path and plot deterministic impulse responses to a one-time one-standard-deviation positive innovation to technology, shutting down all randomness in the dynamics thereafter. For trending quantities (Y, C, I, K), the plotted dynamics are normalized by dividing by γ^t to eliminate the deterministic time trend.

[49]This lag is because of our particular timing assumption about the exogenous capital stock in the investment sector according to which the end-of-period capital stock $K_{i,t+1}$ is held proportional to Z_t. As it is not possible to reallocate capital from the consumption to the investment sector, the marginal product of labor in the investment sector (measured in capital units) grows on impact only due to the technology increase itself. With $\alpha_c < \alpha_i$, this increases is smaller than the increase in the wage, which even leads to a small decline of hours on impact, unless it is offset by a sufficiently strong increase in Q_c (recall, that hours in the consumption sector are constant, so all the dynamics in hours are due to adjustments in labor supply of the investment sector).

[50]While capital in the investment sector exogenously appears as a response to the technology shock, $Q_{i,t}$ still increases due to the increased output price $Q_{c,t}$ of the investment sector. The response is, however, quantitatively small and thus hardly visible in the figure.

[51]The only difference is that under rational expectations agents expect a subsequent decline in capital prices, which dampens the initial increase. Under subjective beliefs, no such decline is expected, which

the shock creates a situation in which the capital stock is below its desired level, so low frequency movements of the economy display an adjustment process very close to the one observed under rational expectations. But second, as capital supply is not fully elastic, the higher capital demand on impact increases capital prices and this has substantially different implications under subjective beliefs than under rational expectations. Under rational expectations, the capital price just indicates the relative scarcity of capital and reverts back to steady state as capital is accumulated. Instead, under subjective beliefs, the initial surprise increase in prices $Q_{c,t}$, $Q_{i,t}$ triggers an upward revision in beliefs $m_{c,t+1}$, $m_{i,t+1}$ through the belief updating equation (12). The exogenous technology shock thus endogenously triggers a "belief shock" of the kind we have studied in Section 7 to illustrate the equilibrium dynamics.[52] As a result, all impulse responses cyclically fluctuate around the long-run adjustment path under rational expectations. Quantitatively, these cyclical deviations from their long-run paths are small for consumption and capital, somewhat larger for hours and output, even larger for investment and substantially larger for capital prices and price-growth expectations. This explains why the model with subjective beliefs can generate considerably more volatility in investment, hours and, particularly, stock prices, without overstating the overall volatility of consumption and output. The additional belief-driven cyclical fluctuations in the adjustment dynamics after the shock also break the almost perfect comovement of output, consumption, investment and hours under rational expectations, instead bring the correlations $\rho(Y,C)$, $\rho(Y,I)$ and $\rho(Y,H)$ closer to the data.

While these impulse responses display cyclical price variations, the fluctuations in the PD ratio (not shown) implied by a single one-standard-deviation technology shock are quantitatively small. To generate sizeable fluctuations in the PD ratio as in Figure 5, larger amounts of capital gains optimism are required than generated by a single shock.[53] However, a sequence of positive technology shock can occasionally create a sufficient amount of optimism for dynamics to display large stock price cycles. Figure 8, which plots simulated sample paths for output and the PD ratio over 200 quarters[54], shows two such price cycles in the first half of the sample (lower panel). The first, larger, price cycle is accompanied by an output boom that ends in a deep recession as prices drop sharply, mainly due to a strong decline in investment activity (not shown). The subsequent recovery is followed by a second, smaller cycle in output and prices. In the second half of the sample, prices fluctuate only little and output fluctuations are mainly

leads to a larger impact response in Q_c, translating into an additional positive effect on hours worked and investment.

[52] While we have shown there a "large" increase in optimism to illustrate the emergence of a boom-bust cycle, the effect is quantitatively much smaller here. This explains the differences between the Figures 6 and 7 here and the Figures 3 and 4 there.

[53] Note that the peak level in m_c in Figure 7 is just 5 basis points, whereas we have studied a one-time increase in optimism of 10 basis points in Section 7

[54] The figure plots output relative to its stochastic trend.

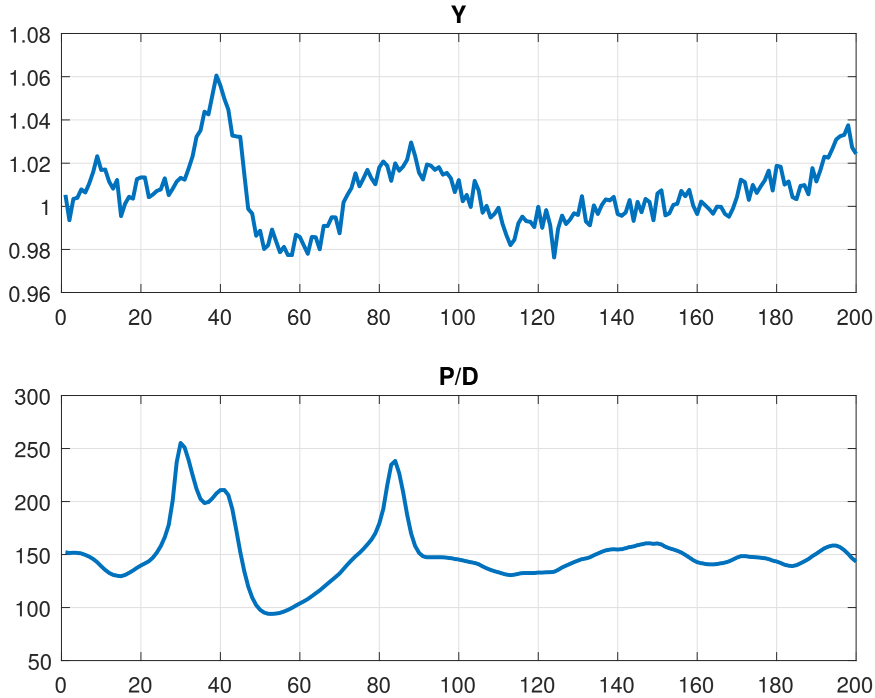

Figure 8. Simulated equilibrium paths of (detrended) output and the price-dividend ratio

driven by technology shocks and capital adjustment dynamics. From this discussion, we conclude that our model gives rise to quantitatively realistic stock price cycles, not unlike the ones depicted in Figure 1, and that these cycles are linked to economic activity.

The existence of stock price cycles in our model also implies that the model economy spends a considerably amount of time in states of capital gains optimism, in which PD ratios are substantially above their long-run mean. There is thus a large amount of mass in the right tail of the ergodic P/D distribution, which is illustrated in Figure 9. The figure plots once again the empirical kernel density of the P/D distribution in the data as in Figure 2, but now in addition the distribution in the model. As is visible there, the model generates a right tail that is remarkably close to the one observed in the data. However, in the model more mass is concentrated around the mode, while very small PD ratios are rare compared to the data. The reason for this asymmetry is that belief revisions have much larger effects on prices in states of optimism than in states of pessimism as is easily observable in the equations (31) and (34), such that additional negative shocks in a bust do not drive down prices as much as additional positive shocks in a boom drive them further up. The lack of mass in the left tail of the distribution also explains why we cannot match the full scale of the volatility of the PD ratio in Table 5 without overstating the mean.

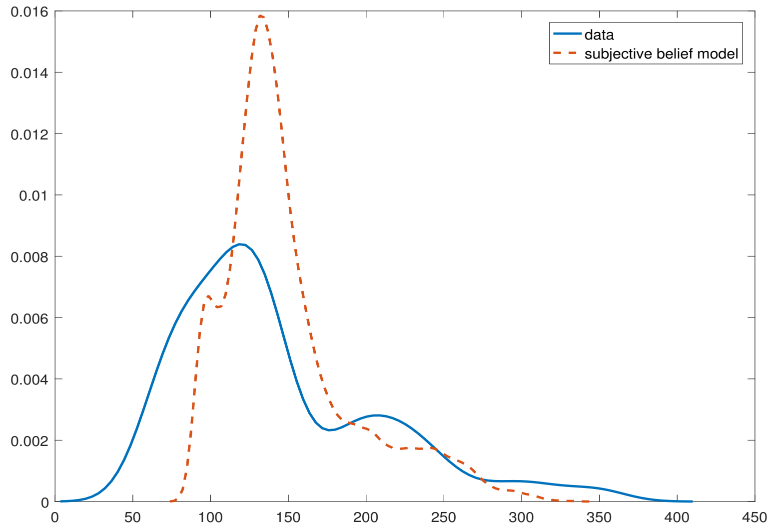

Figure 9. Unconditional density of PD ratio: subjective belief model versus data (not targeted in estimation, kernel estimates)

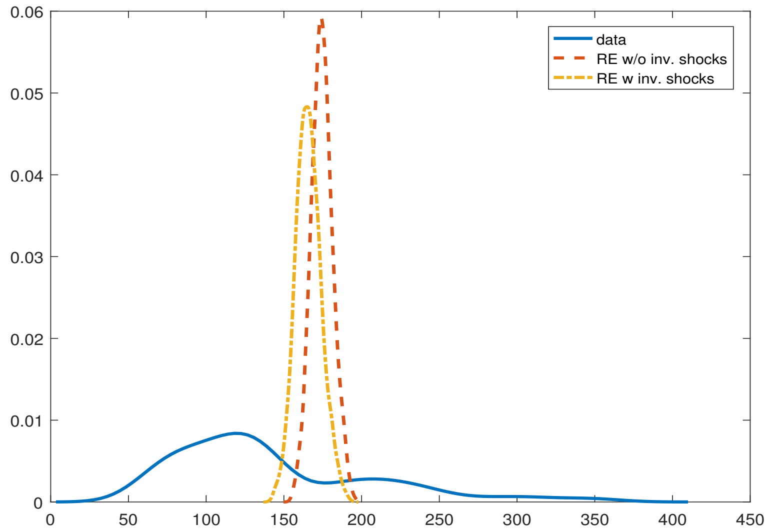

Figure 10. Unconditional density of PD ratio: RE models vs data (not targeted in estimation, kernel estimates)

To further illustrate by how much the shape of the P/D distribution is affected by subjective beliefs, Figure 10 plots the distribution of the PD ratio in the data together with the distributions implied by the two versions of the rational expectations model. In both cases, the distribution is unimodal and symmetric, with a substantial amount of mass closely centered around the mean.

9 Comparison with Boldrin, Christiano, and Fisher (2001) and Adam, Marcet, and Beutel (2017)

In this section we compare the quantitative implications of our model with Boldrin, Christiano, and Fisher (2001) and Adam, Marcet, and Beutel (2017). We choose to compare our model to Boldrin, Christiano, and Fisher (2001), because it is one of the leading joint explanations of stock prices and business cycles under rational expectations in the literature and the one closest to ours. We compare our model to Adam, Marcet, and Beutel (2017), because they study the same belief specification as we do, but in an endowment economy.

Boldrin, Christiano, and Fisher (2001) calibrate their model to a different data sample than we consider. In order to make the comparison as fair as possible, we use their reported data moments as a benchmark,[55] not the ones reported in Section 3, but re-estimate our model to fit those moments using the procedure outlined in Section 8. As estimation targets we use all moments reported in Table 8 with the exception of the standard deviation of the risk-free rate.[56] Boldrin, Christiano, and Fisher (2001) report annual instead of quarterly return moments despite their model being quarterly, which necessitates another change to make the two models comparable. As return autocorrelations are not fully in line with the data in either model,[57] we transform the annual return moments reported in Boldrin, Christiano, and Fisher (2001) both for the data and the model to quarterly frequency under the assumption of no return autocorrelation at that frequency.[58]

[55]For financial moments, they consider U.S. data covering the period 1892–1987, for business cycle moments U.S. data covering 1964:Q1–1988:Q2.

[56]The standard deviation of the risk-free rate is clearly lower in our model than in the data, particularly relative to the sample starting in 1892 used by Boldrin, Christiano, and Fisher (2001). We do not consider this as a serious shortcoming, since the data moment is likely overstated as argued in Section 3. For this reason, we do not attempt to match this number perfectly.

[57]In our model, returns at the quarterly frequency are positively autocorrelated. This is a known weakness of subjective price belief models of the kind studied here in the absence of transitory shocks. See Adam, Marcet, and Beutel (2017) Section VIII.A for a discussion and solution of this issue. Instead, in Boldrin, Christiano, and Fisher (2001) quarterly returns are strongly negatively correlated.

[58]This assumption is (approximately) correct for the data and it transforms – counterfactually – the good fit of the Boldrin, Christiano, and Fisher (2001) model to quarterly returns, so as to not bias the model comparison towards our model. It implies that means are divided by 4 and standard deviations

Table 8
Parameters for subjective belief model and model of Boldrin, Christiano, and Fisher (2001)

Parameter	Subjective Belief Model	BCF
β	0.997	0.999999
α_c	0.36	0.36
α_i	0.6	0.36
δ_c	0.15	0.021
δ_i	0.01	0.021
γ	1.004	1.004
σ	0.015	0.018
g	0.025	–
p	0.4	(0.248)

Note: BCF refers to Boldrin, Christiano, and Fisher (2001).

Table 8 shows the estimated parameters in comparison with the parameters used by Boldrin, Christiano, and Fisher (2001). Due to the similar structure of the two models, the parameters in their model have an identical interpretation as in ours. They are also quantitatively not too different. Table 9 reports a standard set of business cycle and financial return moments in the data and in both models. The second column reports moments for our model, the third column moments for Boldrin, Christiano, and Fisher (2001). Overall, both models match the set of business cycle moments well. Where there are differences between the two, our model tends to do slightly better. Along the financial dimension, the model by Boldrin, Christiano, and Fisher (2001) is able to match the average levels of the risk-free rate and stock returns perfectly and generates stock return volatility close to the one observed in the data. Our model is less successful with the former two moments, but its prediction lie within a one-standard-error interval around the point estimates in the data. As Boldrin, Christiano, and Fisher (2001), we are able to generate high stock return volatility, but unlike in their paper, this is not achieved by generating counterfactually high volatility in the risk-free rate.

Table 10 reports additional evidence for the two models. The first four moments are statistics reported in Tables 1 and 2 of Boldrin, Christiano, and Fisher (2001) which have not been discussed in Section 3 of the present paper. $\rho(\Delta Y_t)$, $\rho(\Delta C_t)$ denote the autocorrelation of log output growth and log consumption growth, respectively, $\sigma(P_{hp})$ is the standard deviation of logged and HP-filtered quarterly stock prices and $\rho(Y, P_{hp})$ is the correlation of output and stock prices, both logged and HP-filtered. The two models perform similarly for the first three moments, both matching the persistence of output

are divided by 2. Similarly, we divide standard errors of means by 2 and standard errors of standard deviations by $\sqrt{2}$.

Table 9
Model comparison with Boldrin, Christiano, and Fisher (2001): targeted moments

	Data (std.dev.)	Subj. Belief Model	BCF
Business Cycle Moments			
$\sigma(Y)$	1.89 (0.21)	1.83	1.97
$\sigma(C)/\sigma(Y)$	0.40 (0.04)	0.67	0.69
$\sigma(I)/\sigma(Y)$	2.39 (0.06)	2.46	1.67
$\sigma(H)/\sigma(Y)$	0.80 (0.05)	0.80	0.51
$\rho(Y,C)$	0.76 (0.05)	0.91	0.95
$\rho(Y,I)$	0.96 (0.01)	0.93	0.69
$\rho(Y,H)$	0.78 (0.05)	0.74	0.86
Financial Moments			
$E[r^f]$	0.30 (0.41)	0.68	0.30
$E[r^e - r^f]$	1.66 (0.89)	0.79	1.66
$\sigma(r^f)$	2.64 (0.52)	0.07	12.30
$\sigma(r^e)$	9.70 (1.10)	9.21	9.20

Notes: BCF refers to Boldrin, Christiano, and Fisher (2001); data and standard errors are taken from BCF and refer to their sample; moments and standard errors for financial moments are transformed to quarterly frequency by the procedure described in the main text.

Table 10
Model comparison with Boldrin, Christiano, and Fisher (2001): additional moments

	Data (std. dev.)	Subj. Belief Model	BCF
Additional Moments reported by BCF			
$\rho(\Delta \log Y)$	0.34 (0.07)	0.39	0.36
$\rho(\Delta \log C)$	0.24 (0.09)	-0.02	-0.05
$\sigma(P_{hp})$	8.56 (0.85)	19.7	12.1
$\rho(Y, P_{hp})$	0.30 (0.08)	0.32	0.16
Dividend and P/D Moments			
$\sigma(D_{t+1}/D_t)$	1.75 (0.38)	1.49	6.87
$E[P/D]$	152.3 (25.3)	110.5	162.4
$\sigma(P/D)$	63.39 (12.39)	32.95	13.20
$\rho(P/D)$	0.98 (0.003)	0.91	0.18
$\rho(H, P/D)$	0.51 (0.17)	0.29	-0.60
$\rho(I/Y, P/D)$	0.19 (0.31)	0.20	-0.60

Notes: BCF refers to Boldrin, Christiano, and Fisher (2001); p estimated to minimize standard-error-weighted squared distance of mean and std of P/D from data, estimate is $p = 0.24782$; data and standard errors for first four moments are taken from BCF and refer to their sample; data and standard errors for remaining moments are not reported in BCF, we therefore take the ones from Section 3.

growth, but underpredicting the persistence of consumption growth and overstating the volatility of HP-filtered stock prices. Our model generates somewhat more comovement of stock prices with output than Boldrin, Christiano, and Fisher (2001), in line with the data.

Table 10 also shows statistics that relate to the behavior of dividends and the PD ratio. These statistics are not reported by Boldrin, Christiano, and Fisher (2001). We therefore report our own estimates discussed in Section 3 in the data column and compute the respective moments in the Boldrin, Christiano, and Fisher (2001) model ourselves.[59] For the definition of dividends and prices we use the same convention as for our own model. Namely, firms pay out a fixed fraction p of capital rental income each period as dividends and reinvest the remaining fraction $1-p$ into new capital. Then, dividend growth equals the growth rate of the (sector-weighted) capital rental rates and the PD ratio is an affine linear function of the capital-price-to-rental-rate ratio, as in our model. For this reason, only the moments $E[P/D]$ and $\sigma(P/D)$ depend on the value of the parameter p, which is not present in Boldrin, Christiano, and Fisher (2001). We choose this parameter so as to minimize the sum of squared t statistics for two moments.[60] The resulting value is $p = 0.248$. In our model, dividends and the PD ratio behave qualitatively as discussed in Section 8, although the overall quantitative fit is somewhat worse, because the PD ratio was not targeted by the estimation. Yet, for any of the reported moments, our model clearly outperforms Boldrin, Christiano, and Fisher (2001). Dividend growth is far too volatile in their model and the PD ratio is neither as volatile nor as persistent as in the data and in our model. In addition, the PD ratio in Boldrin, Christiano, and Fisher (2001) displays negative comovement with hours worked and the investment-to-output ratio. We have encountered such negative correlations also in the rational expectations version of our model above. The reason for this negative correlation is, that capital prices are much less persistent than H and I/Y in Boldrin, Christiano, and Fisher (2001) and thus only mildly procyclical, but rental rates are similarly persistent as macro aggregates. This leads to a negative correlation of H (I/Y) and the PD ratio, despite the fact that they both move in the same direction on impact in response to a technology shock.

Next, we compare the financial moments of our model to the baseline specification of the endowment-economy model of Adam, Marcet, and Beutel (2017) reported in their Table 3 (last column, labeled "diagonal Σ matrix"). As their data sample is almost identical to ours, we consider again the estimated model from Section 8. Table 11 reports the results. Not unexpected, our production economy matches the moments less well than the endowment economy studied in Adam, Marcet, and Beutel (2017). Yet, over-

[59]We solve the Boldrin, Christiano, and Fisher (2001) model by solving the associated planner problem using standard Bellman-equation-based iteration methods. We evaluate the accuracy of our solution by computing Euler errors and replicating the model moments reported in Boldrin, Christiano, and Fisher (2001). Additional moments reported by us are computed using the same method as described in footnote a in all of their tables (based on 500 simulations of sample size 200).

[60]This mirrors our estimation procedure for the subjective belief model.

all the performance of our model is relatively close to that of the Adam, Marcet, and Beutel (2017) model. Given that our model has at the same time realistic business cycle implications, we consider this a substantial achievement.

Table 11
Model comparison with Adam, Marcet, and Beutel (2017)

Moment	Data	Model	AMB
$E[P/D]$	152.3 (25.3)	149.95	115.2
$\sigma(P/D)$	63.39 (12.39)	44.96	88.20
$\rho(P/D)$	0.98 (0.003)	0.97	0.98
$E[r^e]$	1.87 (0.45)	1.25	1.82
$\sigma(r^e)$	7.98 (0.35)	7.07	7.74
$E[r^f]$	0.25 (0.13)	0.78	0.99
$\sigma(r^f)$	0.82 (0.12)	0.06	0.83
$\sigma(D_{t+1}/D_t)$	1.75 (0.38)	2.46	1.92
$\rho(E^{\mathcal{P}}[r^e], P/D)$	0.79 (0.07)	0.52	0.79

Note: Column "Model" refers to our model, column "AMB" refers to Adam, Marcet, and Beutel (2017).

10 Welfare Implications of Belief-Driven Stock Price Booms

This section studies to what extent the presence of belief-driven boom and bust cycles affects allocations and household welfare. To this end, it considers the estimated subjective belief model from Section 8 and the allocative and welfare implications associated with imposing fully rational expectations on households. In the absence of subjective belief distortions, the competitive allocation is efficient and thus represents a natural welfare benchmark against which one can judge the costs of belief-driven boom and bust cycles. When evaluating welfare in the subjective belief model, we consider ex-post realized welfare rather than ex-ante expected welfare, as the latter would depend also on the degree of subjective optimism.

Table 12 reports business cycle and financial moments for a setting with and without subjective beliefs.[61] It shows that the elimination of subjective price beliefs decreases output volatility by about 13%. This is mainly driven by a substantial fall in the volatility of investment and a dramatic fall in the volatility of hours worked. These effects are partly

[61]The results for both models are based on the estimated parameters for the subjective belief model in Table 5.

Table 12
The effects of shutting down subjective price beliefs

	Data	Subjective Belief Model	REE Implied by Subj. Belief Model
	Business Cycle Moments		
$\sigma(Y)$	1.72 (0.25)	1.83	1.60
$\sigma(C)/\sigma(Y)$	0.61 (0.03)	0.67	0.89
$\sigma(I)/\sigma(Y)$	2.90 (0.35)	2.90	1.59
$\sigma(H)/\sigma(Y)$	1.08 (0.13)	1.06	0.12
$\rho(Y, C)$	0.88 (0.02)	0.84	0.96
$\rho(Y, I)$	0.86 (0.03)	0.89	0.91
$\rho(Y, H)$	0.75 (0.03)	0.70	0.70
	Financial Moments		
$E[P/D]$	152.3 (25.3)	150.0	199.7
$\sigma(P/D)$	63.39 (12.39)	44.96	8.99
$\rho(P/D)$	0.98 (0.003)	0.97	0.99
$E[r^e]$	1.87 (0.45)	1.25	0.68
$\sigma(r^e)$	7.98 (0.35)	7.07	0.19
$E[r^f]$	0.25 (0.13)	0.78	0.68
$\sigma(r^f)$	0.82 (0.12)	0.06	0.06
$\sigma(D_{t+1}/D_t)$	1.75 (0.38)	2.46	0.92

compensated by an increase in consumption volatility, in both absolute and relative terms. The elimination of subjective beliefs has, however, its strongest effect on the volatility of financial variables. The standard deviation of the PD ratio decreases by more than three quarters and the standard deviation of stock returns falls even more dramatically. Dividend growth volatility also falls considerably.

The increase in consumption volatility and the fact that the volatility of hours is welfare neutral – given our linear specification for the disutility of work – suggests that the welfare implications of eliminating subjective beliefs are likely to be small. Indeed, we find that the utility gain associated with the elimination of subjective beliefs amounts to 0.29% of consumption. This is of the same order of magnitude as the welfare gains associated with the elimination of the business cycle.

The welfare gains associated with the elimination of subjective beliefs arise mainly through changes in the *mean* levels of consumption and work. Table 13 depicts the mean value of (detrended) consumption and labor under the two belief specifications.[62] It shows that average consumption is higher and average labor is lower in a setting with

[62] The utility from the common consumption trend, which is a function of the productivity process $\{Z_t\}$ only, enters utility separately.

Table 13
Mean values of (detrended) consumption and labor

	Subjective Belief Model	REE Implied by Subj. Belief Model
$E[C^{detr}]$	2.7719	2.7768
$E[L]$	0.7152	0.7141

fully rational expectations. This is the case because with rational expectations asset prices correctly signal the value of investment opportunities. Instead, under subjective beliefs, investment can be triggered by a belief-driven price boom, so that high levels of hours worked and high levels of investment might occur during times in which new capital is not particularly productive. As a result, agents can work on average more but achieve lower average consumption levels.

We conclude this section with two remarks. First, one reason for the low welfare effects of fluctuations induced by price cycles is the low curvature in assumed preferences. As curvature increases, agents would be willing to give up more consumption to eliminate fluctuations. However, the welfare gains associated with the elimination of fluctuations induced by stock price cycles are bounded above by the welfare gains associated with the elimination of all fluctuations and thus the cost of business cycles. Unless higher curvature in preferences also increases the mean effects reported in Table 13, we expect our result to generalize that costs of belief-driven price cycles are small and of the same order of magnitude as the costs of business cycles.

Second, the representative agent assumption is critical for the result that welfare effects of stock price cycles are small. Adam, Beutel, Marcet, and Merkel (2016) show that a belief-driven boom-bust cycle can have substantial redistributive effects in the presence of belief heterogeneity. In their model agents trade with each other at prices that are – relative to a setting without belief distortions – too high or low. In the present model, prices only affect allocations through aggregate investment. Likely, our welfare assessment would change in the presence of heterogeneity that leads to trading among agents over the course of stock price cycles, if the welfare objective was to include distributional concerns.

11 Conclusions

We have presented a simple theory of the joint behavior of stock prices and business cycles. This theory is quantitatively consistent with key facts about the business cycle, stock price behavior and the interaction between the two. In particular, it can reconcile smooth business cycles and volatile stock prices. Relative to the previous literature, no labor market frictions or inseparable preferences are required to achieve this. Instead, in our model stock prices are volatile because agents hold subjective stock price expectations and extrapolate observed past capital gains into the future as in Adam, Marcet, and Beutel (2017). Extrapolation creates price-belief feedback loops that can occasionally disconnect prices from fundamentals and lead to large and persistence stock price cycles. In the model, such price cycles have allocative implications, because they lead to cyclical variation in investment demand. Our theory suggests that this variation is an important source of the business cycle fluctuations of investment and hours worked, but has only minor effects on consumption variability.

We also assess the welfare implications of belief distortions in stock markets. In the presence of such distortions, prices are only imperfect signals of the relative scarcity of capital goods. Consequently, in times of optimism there can be over-investment, in times of pessimism under-investment. Inefficient timing of investment leads to a lower average level of consumption despite a higher average amount of hours worked. However, we find the welfare costs of belief-driven stock price cycles to be relatively moderate. Several considerations absent in our model might overturn this result. First, effects of price cycles could be much larger in the presence of heterogeneity and trading in a setting where the wealth distribution matters. Second, in a richer model with a large cross-section of sectors in which sectoral price cycles only partially comove fluctuations in relative firm valuations may lead to additional misallocation, absent in our model. Third, if asset prices play an important role for the balance sheet dynamics of a financially constrained sector,[63] fluctuations in asset prices may have larger effects on allocations than in the model presented in this paper. Exploring these channels appears to be an interesting avenue for future research.

Conceptually, we show how to specify expectations that combine subjective beliefs for some model variables with rational expectations along other dimensions. The proposed expectation concept that is "partially rational" with respect to a subset of model-endogenous variables can be generalized beyond our specific setting. In particular, it may prove fruitful in other macroeconomic applications where beliefs can be disciplined empirically along some dimensions, but one wishes to maintain the rational expectations hypothesis along others.

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[63] Arguably, this is less relevant for stocks than for other asset classes such as real estate. House prices also display large and persistent cycles in many advanced economies. To the extent that such cycles are also driven by the same belief dynamics analyzed here, one could easily extend the analysis to include housing.

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A Appendix

A.1 Data Sources

Data on Macro Aggregates Data series on macro aggregates and related variables have all been downloaded from the Federal Reserve Economic Data (FRED) database maintained by the federal reserve bank of St. Louis (https://fred.stlouisfed.org). All time series refer to the United States, are at quarterly frequency and cover the sample period Q1:1955 to Q4:2014. Data on output, consumption and investment is from the National Income and Product Accounts of the BEA. We measure nominal output by gross domestic product (FRED Code "GDP"), nominal consumption by personal consumption expenditures in nondurable goods ("PCND") and services ("PCESV") and nominal investment by fixed private investment ("FPI"). Nominal values of investment subcomponents for Table 4 are private residential fixed investment ("PRFI") and private nonresidential fixed investment in structures ("B009RC1Q027SBEA"), equipment ("Y033RC1Q027SBEA") and intellectual property products ("Y001RC1Q027SBEA"). These nominal series have been deflated by the consumer price index for all urban consumers ("CPIAUCSL") from the BLS, which is consistent with the deflating procedure used for stock prices and interest rates by Adam, Marcet, and Beutel (2017) and us. Hours worked are based on an index of nonfarm business sector hours ("HOANBS") published by the BLS. Working age population is based on data published by the OECD ("LFWA64TTUSQ647N").

Stock Prices, Interest Rates and Investor Expectations We use identical data sources as in Adam, Marcet, and Beutel (2017) and refer to their data appendix for details. They use 'The Global Financial Database' to obtain data on stock prices and interest rates until Q1:2012. We extend their stock price and interest rate data to Q4:2014 using identical data sources as they do.

We compute dividends based on price and total return index data of the SP 500 index by the procedure outlines there, resulting in a dividend series $\{D_t\}$, where t runs through all quarters from Q1:1955 to Q4:2014. Given the price series from the price index $\{P_t\}$, where P_t is the closing price of the last trading day in quarter t, we define the price-dividend ratio as the ratio P_t/D_t of the end-of-quarter closing price P_t and the within quarter dividend D_t .

A.2 Details of the Projection Facility

Following Adam, Marcet, and Nicolini (2016), we modify the belief updating equation (12) to

$$\ln m_{s,t} = w_{s,t}\left(\ln m_{s,t-1} + g\left(\ln Q_{s,t-1} - \ln Q_{s,t} - \ln m_{s,t-1}\right) + g \ln \varepsilon^1_{s,t}\right)$$

where $w_{s,t}(\cdot)$ is a differentiable function satisfying $w_{s,t}(x) = x$ for $x \leq \underline{m}_{s,t}$ and $w_{s,t}(x) \leq \overline{m}_{s,t}$ for all x, with $\overline{m}_{s,t} > \underline{m}_{s,t}$. Beliefs are thus bounded below $\overline{m}_{s,t}$, but evolve as described in the main text as long as they remain below $\underline{m}_{s,t}$. Following Adam, Marcet, and Nicolini (2016) we consider the function

$$w_{s,t}(x) = \begin{cases} x & \text{if } x \leq \underline{m}_{s,t} \\ \underline{m}_{s,t} + \frac{x - \underline{m}_{s,t}}{x + \overline{m}_{s,t} - 2\underline{m}_{s,t}}\left(\overline{m}_{s,t} - \underline{m}_{s,t}\right) & \text{if } \underline{m}_{s,t} < x. \end{cases} \tag{35}$$

and calibrate the critical values $\left(\underline{m}_{s,t}, \overline{m}_{s,t}\right)$ in both sectors $s = c, i$ such that $\underline{m}_{s,t}$ is the degree of optimism that implies a quarterly PD ratio of 250 and $\overline{m}_{s,t}$ is the degree of optimism implying a PD ratio of 500. The critical PD values of 250 and 500 are taken from Adam, Marcet, and Nicolini (2016).

We now explain how these critical values can be computed. The Euler equation for capital in sector s is

$$\begin{aligned} Q_{s,t} &= \beta\left(1-\delta_s\right) E_t^{\mathcal{P}}\left[\frac{W_t}{W_{t+1}} Q_{s,t+1}\right] + \beta E_t^{\mathcal{P}}\left[\frac{W_t}{W_{t+1}} R_{s,t+1}\right] \\ &= \beta\left(1-\delta_s\right) E_t^{\mathcal{P}}\left[\frac{W_t}{W_{t+1}}\right] m_{s,t} Q_{s,t} + \beta E_t^{\mathcal{P}}\left[\frac{W_t}{W_{t+1}} R_{s,t+1}\right] \end{aligned}$$

implying

$$Q_{s,t} = \frac{\beta E_t^{\mathcal{P}}\left[\frac{W_t}{W_{t+1}} R_{s,t+1}\right]}{1 - \beta\left(1-\delta_s\right) E_t^{\mathcal{P}}\left[\frac{W_t}{W_{t+1}}\right] m_{s,t}}$$

and thus the price-dividend ratio is

$$\frac{P_{s,t}}{D_{s,t}} = \frac{1-\delta}{p}\frac{Q_{s,t}}{R_{s,t}} + \frac{1-p}{p} = \frac{1}{p}\frac{\left(1-\delta\right)\beta E_t^{\mathcal{P}}\left[\frac{W_t}{W_{t+1}}\frac{R_{s,t+1}}{R_{s,t}}\right]}{1 - \beta\left(1-\delta\right) E_t^{\mathcal{P}}\left[\frac{W_t}{W_{t+1}}\right] m_{s,t}} + \frac{1-p}{p}. \tag{36}$$

The value $\underline{m}_{s,t}$ is the value for $m_{s,t}$ in the preceding equation that causes the PD to be equal to 250; likewise, $\overline{m}_{s,t}$ is the value that causes the PD ratio to be equal to 500.[64] Since

[64] In addition, we do not allow $\overline{m}_{c,t}$ to exceed the theoretical upper bound on beliefs for which uniqueness has been proven in Appendix A.3.3 (see Lemma 4), irrespective of the implied PD ratio. This is done to insure equilibrium uniqueness, but is a purely theoretical concern. In practice, we encountered not a single case in which this upper bound was binding in our numerical simulations.

the expectations of W_{t+1} and $R_{s,t+1}$ both depend on $K_{c,t+1}$ (this follows from equations (18) and (19) and the fact that $Y_{c,t+1} = K_{c,t+1}^{\alpha_c} (Z_{t+1}(1-\alpha))^{1-\alpha_c}$), solving for the values $(\underline{m}_{c,t}, \overline{m}_{c,t})$ thus requires simultaneously solving for the optimal investment $Y_{i,t}$ in period t. Since the capital stock dynamics in the investment sector are exogenous, $\underline{m}_{i,t}, \overline{m}_{i,t}$ can be computed once the new belief $m_{c,t}$ that incorporates the projection facility has been determined.

A.3 Details on the Representation of Beliefs and Equilibrium Existence

A.3.1 Belief Representations on Ω and $\tilde{\Omega}$

We provide here details on the remarks made in footnote 32. To ease the discussion, we first introduce some notation that will also be used in Sections A.3.2 and A.3.3.

Notation For each random (upper case) symbol A, denote elements of Ω_A by a (these are real sequences) and their t-th component by a_t (these are real numbers).[65] For each random sequence $\{A_t\}_{t=0}^{\infty}$, write shorter just A. On the domain $\tilde{\Omega} = \Omega_Z \times \Omega_{Q,c} \times \Omega_{Q,i}$ we define the random variables (sequences) $\tilde{Z}$, $\tilde{Q}_c$ and $\tilde{Q}_i$ as projections on the first, second and third component, respectively,

$$\tilde{Z}(z, q_c, q_i) = z, \qquad \tilde{Q}_c(z, q_c, q_i) = q_c, \qquad \tilde{Q}_i(z, q_c, q_i) = q_i.$$

Similarly, we define the random variables Z, X, W, R_c, R_i, Q_c, Q_i as projections from the domain $\Omega = \Omega_Z \times \Omega_X \times \Omega_W \times \Omega_{R,c} \times \Omega_{R,i} \times \Omega_{Q,c} \times \Omega_{Q,i}$ to the respective factor.[66] We make the difference between $\tilde{Z}$ (defined on $\tilde{\Omega}$ consisting of typical elements $\tilde{\omega} = (z, q_c, q_i)$) and Z (defined on Ω consisting of typical elements $\omega = (z, x, w, r_c, r_i, q_c, q_i)$) explicit to avoid any ambiguity and confusion arising in the arguments below. In the main text, we regularly do not make these distinctions and use the same symbols, whenever a variable has the same interpretation, no matter on which space it is defined and whether it is a random variable or a realization.

We regularly have to work with the big vector $(Z, X, W, R_c, R_i, Q_c, Q_i)$ of random sequences on Ω.[67] For space reasons, we denote this vector just by O ("observables vector").

Furthermore, we use the following two conventions for the outcome mapping $F : s_t \mapsto (w_t, r_{c,t}, r_{i,t}, y_t, c_t, i_t, h_t)$ (where $s_t = (z_t, k_{c,t}, k_{i,t}, m_{c,t}, m_{i,t}, q_{c,t-1}, q_{i,t-1})$) introduced at the end of Section 5

[65]Because the variables m_c and m_i are lower case in the model, those symbols can denote both random variables and realizations. This ambiguity should not lead to any confusion below.

[66]E.g. $Z(z, x, w, r_c, r_i, q_c, q_i) = z$.

[67]As a mapping, this vector just agrees with the identity id_Ω, but this notation conceals the economic interpretation of the random variables.

- We denote by F_W, $F_{R,c}$, $F_{R,i}$ etc. the components of that function
- By a slight abuse of notation, we denote by F also the mapping from the full state sequence $s = \{s_t\}_{t=0}^{\infty}$ into the full outcome sequence $(w, r_c, r_i, y, c, i, h) = \{(w_t, r_{c,t}, r_{i,t}, y_t, c_t, i_t, h_t)\}_{t=0}^{\infty}$, similarly we use the component functions F_W, $F_{R,c}$, $R_{R,i}$ etc.

Mapping a Recursive Evolution into a Sequence Representation Consider a fixed initial state $s_0 = (z_0, k_{c,0}, k_{i,0}, m_{c,0}, m_{i,0}, q_{c,-1}, q_{i,-1})$ and a measurable function G describing a recursive state evolution as in the main text.[68] For any sequence $z \in \Omega_Z$ such that z_0 is consistent with s_0, the recursion

$$s_{t+1} = G(s_t, z_{t+1}) \qquad t = 0, 1, 2, \ldots$$

defines then a unique sequence $s = \{s_t\}_{t=0}^{\infty}$ in $\Sigma := \prod_{t=0}^{\infty} \mathbb{R}^7$ (Σ is the space of all state sequences). The procedure just outlined defines therefore a function $H : \Omega_Z \to \Sigma$ which maps technology sequences z into state sequences s.[69] Obviously, if G is measurable and Σ is endowed with the usual product σ-field, then H is also a measurable function. Call H the sequence representation associated with G (and the initial state s_0[70]).

Consistency of Beliefs with an Evolution The following definition clarifies the notion in the main text that beliefs about wages and rental rates be consistent with the equilibrium mappings G and F. The definition is formulated for arbitrary mappings G and F that do not necessarily need to correspond to the equilibrium mappings of the model.

Definition 2. For a given measurable state evolution G and a given measurable outcome function F, we say that a measure $\mathcal{P}$ on Ω implies beliefs about (W, R_c, R_i) consistent with mappings G and F, if

$$W = F_W \circ H \circ Z, \qquad R_c = F_{R,i} \circ H \circ Z, \qquad R_i = F_{R,i} \circ H \circ Z$$

$\mathcal{P}$-a.s. Here, H is the sequence representation associated with G.

[68] The function G is allowed to be arbitrary here, but should have the same domain and codomain as in Section 5. There is no need for it to conform with any notion of equilibrium or model consistency.

[69] Strictly speaking, the mapping is only partially defined due to the requirement that z_0 has to equal the first component of s_0. This is not important for anything to follow and we will ignore it from now on (alternatively, one could just keep the last 6 components of s_0 fixed and specify that H always assigns the value z_0 to the first component of s_0).

[70] From now on we consider a fixed initial state throughout without explicitly mentioning this anymore.

Constructing Consistent Beliefs from G, F and $\tilde{\mathcal{P}}$ The following lemma is the main result of this section and provides the justification why it is sufficient to work with the smaller probability space $\tilde{\Omega}$ instead of Ω.

Lemma 1. *For any given measure $\tilde{\mathcal{P}}$ on $\tilde{\Omega}$ and measurable G and F, there is a unique measure $\mathcal{P}$ on Ω with the following properties:*

1. *The distribution of (Z, Q_c, Q_i) under $\mathcal{P}$ equals $\tilde{\mathcal{P}}$;*
2. *The joint distribution of (Z, X) under $\mathcal{P}$ is consistent with the exogenous relationship between Z and X;*
3. *$\mathcal{P}$ implies beliefs about (W, R_c, R_i) that are consistent with the mappings G and F.*

Proof. We give an explicit construction of the measure $\mathcal{P}$ as the distribution of a set of suitable random variables on $\tilde{\Omega}$ under $\tilde{\mathcal{P}}$. First, the capital accumulation equation (25) for investment-sector capital and the assumption $K_{i,t+1} \propto Z_t$ imply

$$\bar{k}_i Z_t = (1-\delta)\,\bar{k}_i Z_{t-1} + X_t \Rightarrow X_t = \bar{k}_i\left(Z_t - (1-\delta_i)\,Z_{t-1}\right), \tag{37}$$

where $\bar{k}_i$ is the (fixed) proportionality constant.[71] The second requirement that the joint distribution of (Z, X) under $\mathcal{P}$ be consistent with the exogenous relationship between those variables means that equation (37) has to hold $\mathcal{P}$-a.s. for all t. We thus define a random variable $\tilde{X} : \tilde{\Omega} \to \Omega_X$ in a way that is consistent with the analog equation (37) on the $\tilde{\Omega}$ domain:[72]

$$\tilde{X}_t := \bar{k}_i\left(\tilde{Z}_t - (1-\delta_i)\,\tilde{Z}_{t-1}\right).$$

Next, let the function $H : \Omega_Z \to \Sigma$ be the sequence representation associated with G. As for X, we simply define random variables for wages, $\tilde{W} : \tilde{\Omega} \to \Omega_W$, and rental rates, $\tilde{R}_c : \tilde{\Omega} \to \Omega_{R,c}$, $\tilde{R}_i : \tilde{\Omega} \to \Omega_{R,i}$, on the domain $\tilde{\Omega}$ in a way that they satisfy the analog of the consistency condition for the probability space $(\tilde{\Omega}, \tilde{\mathcal{S}}, \tilde{\mathcal{P}})$, namely

$$\tilde{W} = F_W \circ H \circ \tilde{Z}, \qquad \tilde{R}_c = F_{R,i} \circ H \circ \tilde{Z}, \qquad \tilde{R}_i = F_{R,i} \circ H \circ \tilde{Z}.$$

With these definitions the random (observables) vector

$$\tilde{O} := (\tilde{Z}, \tilde{X}, \tilde{W}, \tilde{R}_c, \tilde{R}_i, \tilde{Q}_c, \tilde{Q}_i)$$

is a measurable mapping from $\tilde{\Omega}$ to Ω and thus its distribution defines a measure $\mathcal{P}$ on Ω. We claim that this measure satisfies the three conditions in the assertion and is the only measure to do so:

[71]Strictly speaking, $\bar{k}_i$ is a model parameter. However, the choice of this parameter is not discussed in Section 8, because its value does not matter for any results reported in the main text. Intuitively, changing $\bar{k}_i$ just scales up or down the production of consumption-sector capital and thereby changes the units in which this capital is measured, which has no economic relevance.

[72]For $t = 0$ one must back out $\tilde{Z}_{t-1} = z_{-1}$ from the entry $k_{i,0}$ of the initial state: $z_{-1} = \frac{k_{i,0}}{\bar{k}_i}$.

1. Z is the projection defined by $Z(z, x, w, r_c, r_i, q_c, q_i) = z$, so $Z(\tilde{O}) = \tilde{Z}$ and a similar argument shows $Q_c(\tilde{O}) = \tilde{Q}_c$ and $Q_i(\tilde{O}) = \tilde{Q}_i$. So we get

$$(Z, Q_c, Q_i)(\tilde{O}) = \left(\tilde{Z}, \tilde{Q}_c, \tilde{Q}_i\right) = \mathrm{id}_{\tilde{\Omega}}.$$

Because $\mathcal{P}$ is the distribution of $\tilde{O}$ under $\tilde{\mathcal{P}}$, this equation implies that the distribution of (Z, Q_c, Q_i) under $\mathcal{P}$ and the distribution of $\mathrm{id}_{\tilde{\Omega}}$ under $\tilde{\mathcal{P}}$ must be identical. As the latter distribution is $\tilde{\mathcal{P}}$ itself, this proves the first property.

2. The following equation holds by definition of the random variables X, Z and $\tilde{X}$ (for all $\tilde{\omega} \in \tilde{\Omega}$)

$$\begin{aligned} X_t(\tilde{O}) = \tilde{X}_t &= \bar{k}_i \left(\tilde{Z}_t - (1 - \delta_i)\, \tilde{Z}_{t-1}\right) \\ &= \bar{k}_i \left(Z_t(\tilde{O}) - (1 - \delta_i)\, Z_{t-1}(\tilde{O})\right). \\ &= \left[\bar{k}_i \left(Z_t - (1 - \delta_i)\, Z_{t-1}\right)\right](\tilde{O}) \end{aligned}$$

As this equation holds on $\tilde{\Omega}$, it must in particular hold $\tilde{\mathcal{P}}$-a.s., so X_t and $\bar{k}_i \left(Z_t - (1 - \delta_i)\, Z_{t-1}\right)$ must coincide a.s. with respect to the distribution of $\tilde{O}$ under $\tilde{\mathcal{P}}$, which is exactly $\mathcal{P}$. Hence, equation (37) holds $\mathcal{P}$-a.s.

3. The consistency proof works along the same lines as the proof that (37) has to hold $\mathcal{P}$-a.s. by reducing it to the analogous consistency condition in the tilde space for the tilde variables. The argument is omitted for this reason.

For uniqueness, suppose that $\mathcal{P}'$ is another (arbitrary) measure on Ω such that properties 1-3 are satisfied. Then in particular equation (37) holds $\mathcal{P}'$-a.s. for all t and by the definition of $\tilde{X}$ and $\tilde{Z}$ we obtain for all t

$$\begin{aligned} \tilde{X}_t\,(Z, Q_c, Q_i) &= \bar{k}_i \left(\tilde{Z}_t - (1 - \delta_i)\, \tilde{Z}_{t-1}\right)(Z, Q_c, Q_i) \\ &= \bar{k}_i \left(Z_t - (1 - \delta_i)\, Z_{t-1}\right) \\ &= X_t \qquad\qquad \mathcal{P}'\text{-a.s.} \end{aligned}$$

(here, all equalities except for the last hold even ω-by-ω and the last is just equation (37)). Similarly, from $W = F_W \circ H \circ Z$ $\mathcal{P}'$-a.s. (the consistency condition for wage beliefs under $\mathcal{P}'$) and the definition of $\tilde{W}$ and $\tilde{Z}$ we can conclude

$$\tilde{W}\,(Z, Q_c, Q_i) = F_W \circ H \circ \tilde{Z}\,(Z, Q_c, Q_i) = F_W \circ H \circ Z = W \qquad \mathcal{P}'\text{-a.s.}$$

Identical arguments also yield

$$\begin{aligned} \tilde{R}_c\,(Z, Q_c, Q_i) &= R_c \qquad \mathcal{P}'\text{-a.s.} \\ \tilde{R}_i\,(Z, Q_c, Q_i) &= R_i \qquad \mathcal{P}'\text{-a.s.} \end{aligned}$$

Combining those results with the obvious equations $\tilde{Z}(Z, Q_c, Q_i) = Z$, $\tilde{Q}_c(Z, Q_c, Q_i) = Q_c$, $\tilde{Q}_i(Z, Q_c, Q_i) = Q_i$ implies

$$\mathrm{id}_\Omega = O = \tilde{O}\,(Z, Q_c, Q_i) \qquad \mathcal{P}'\text{-a.s.}$$

But by property 1, the distribution of (Z, Q_c, Q_i) under $\mathcal{P}'$ must be $\tilde{\mathcal{P}}$ and thus the equation shows that the distribution of id_Ω under $\mathcal{P}'$ must be equal to the distribution of $\tilde{O}$ under $\tilde{\mathcal{P}}$, which is by definition $\mathcal{P}$. Hence, $\mathcal{P}' = \mathcal{P}$. □

A.3.2 Construction and Uniqueness of $\tilde{\mathcal{P}}$

We first construct a measure and show that it has all the properties that any candidate for $\tilde{\mathcal{P}}$ has to have. We then argue why it is the only such measure. First, the following two auxiliary constructions are required. As always, we assume implicitly, that an initial state $s_0 = (z_0, k_{c,0}, k_{i,0}, m_{c,0}, m_{i,0}, q_{c,-1}, q_{i,-1})$ is fixed.

1. Let $\mathcal{P}_Z$ be a measure on Ω_Z that describes the exogenous evolution of Z, i.e. under $\mathcal{P}_Z$ for all $t \geq 1$
$$\log \varepsilon_t := \log\left(\frac{Z_t}{\gamma Z_{t-1}}\right)$$
is iid normal with mean $-\frac{\sigma^2}{2}$ and variance σ^2 and $Z_0 = z_0$ $\mathcal{P}_Z$-a.s. Clearly, a unique measure with this property exists.

2. For $s \in \{c, i\}$ let $\mathcal{P}_{Q,s}$ be a measure on $\Omega_{Q,s}$ that describes the subjective evolution of Q_s under learning, i.e. under $\mathcal{P}_{Q,s}$ for all $t \geq 1$
$$\log \varepsilon_t^{Q,s} := \log\left(\frac{Q_{s,t}}{m_{s,t-1}Q_{s,t-1}}\right)$$
is iid normal with mean $-\frac{\sigma_Q^2}{2}$ and variance σ_Q^2 and $\sigma_Q^2 = \sigma_\varepsilon^2$. Here $m_{s,t}$ is recursively defined by equation (12), for $t \geq 1$[73]
$$m_{s,t} = m_{s,t-1}\left(\frac{Q_{s,t-1}}{m_{s,t-1}Q_{s,t-2}}\right)^g.$$
In addition, $Q_{s,-1} = q_{s,-1}$ $\mathcal{P}_{Q,s}$-a.s. Also, the measure $\mathcal{P}_{Q,c}$ is uniquely defined by these properties.

It is obvious, that $\mathcal{P}_Z$ indeed describes the exogenous evolution of Z as defined in equation (2) and that $\mathcal{P}_{Q,c}$ and $\mathcal{P}_{Q,i}$ represent the marginal distribution of Q_c and Q_i, respectively,

[73]Note that we here ignore the additional observable innovation present in that equation in line with footnote 31.

under agents' beliefs, if they are to be consistent with the Bayesian learning formulation in Section 5. So any measure $\tilde{\mathcal{P}}$ on $\tilde{\Omega}$ that correctly represents subjective beliefs as defined in Section 5 must imply the marginal measures $\mathcal{P}_Z$, $\mathcal{P}_{Q,c}$ and $\mathcal{P}_{Q,i}$ on Ω_Z, $\Omega_{Q,c}$ and $\Omega_{Q,i}$, respectively. The only issue left to discuss is thus which assumptions about the dependence structure of the processes Z, Q_c and Q_i lead to a valid belief measure $\tilde{\mathcal{P}}$ in line with the assumptions made in Section 5. We claim that only independence does and thus $\tilde{\mathcal{P}}$ must be given by $\tilde{\mathcal{P}} = \mathcal{P}_Z \otimes \mathcal{P}_{Q,c} \otimes \mathcal{P}_{Q,i}$.

To see this, note that on the extended probability space that includes latent variables in households' filtering problem, agents must think that the four equations

$$\begin{aligned}
\log Q_{c,t} &= \log Q_{c,t-1} + \log \beta_{c,t} + \log \varepsilon_{c,t} \\
\log \beta_{c,t} &= \log \beta_{c,t-1} + \log \nu_{c,t} \\
\log Q_{i,t} &= \log Q_{i,t-1} + \log \beta_{i,t} + \log \varepsilon_{i,t} \\
\log \beta_{i,t} &= \log \beta_{i,t-1} + \log \nu_{i,t}
\end{aligned}$$

hold with probability 1 for all $t \geq 1$. In addition, as stated in Section 5, $\{\varepsilon_{c,t}\}_{t=1}^{\infty}$, $\{\nu_{c,t}\}_{t=1}^{\infty}$, $\{\varepsilon_{i,t}\}_{t=1}^{\infty}$, $\{\nu_{i,t}\}_{t=1}^{\infty}$ are independent stochastic processes and independent of all other model variables, including the process Z. But those two facts can only be simultaneously true, if the three processes Z, Q_c and Q_i are independent.[74]

A.3.3 Existence and Uniqueness of G and F

The goal of this section is to prove Proposition 1. We start with a result that collects important equations and gives an explicit characterization of the function F – up to the presence of the argument $Q_{c,t}$ which is not part of the state S_t.

[74] Formally, this would require a simple induction proof over time, which is not explicitly spelled out here.

Lemma 2. *In any equilibrium, irrespective of beliefs, the following equations have to hold*

$$
\begin{aligned}
W_t &= K_{c,t}^{\alpha_c} Z_t^{1-\alpha_c} (1-\alpha_c)^{1-\alpha_c} \\
R_{c,t} &= \alpha_c K_{c,t}^{\alpha_c - 1} Z_t^{1-\alpha_c} (1-\alpha_c)^{1-\alpha_c} \\
R_{i,t} &= \alpha_i \left(\frac{1-\alpha_i}{W_t} \right)^{\frac{1-\alpha_i}{\alpha_i}} Z_t^{\frac{1-\alpha_i}{\alpha_i}} Q_{c,t}^{\frac{1}{\alpha_i}} \\
Y_t &= K_{c,t}^{\alpha_c} Z_t^{1-\alpha_c} (1-\alpha_c)^{1-\alpha_c} + Q_c^{ss} K_{i,t} Z_t^{\frac{1}{\alpha_i}} \left(\frac{(1-\alpha_i) Q_{c,t}}{W_t} \right)^{\frac{1-\alpha_i}{\alpha_i}} \\
C_t &= K_{c,t}^{\alpha_c} Z_t^{1-\alpha_c} (1-\alpha_c)^{1-\alpha_c} \\
I_t &= Q_c^{ss} K_{i,t} Z_t^{\frac{1}{\alpha_i}} \left(\frac{(1-\alpha_i) Q_{c,t}}{W_t} \right)^{\frac{1-\alpha_i}{\alpha_i}} \\
H_t &= 1 - \alpha_c + K_{i,t} Z_t^{\frac{1-\alpha_i}{\alpha_i}} \left((1-\alpha_i) \frac{Q_{c,t}}{W_t} \right)^{\frac{1}{\alpha_i}}
\end{aligned}
$$

for all $t \geq 0$*. In particular,* $(W_t, R_{c,t}, R_{i,t}, Y_t, C_t, I_t, H_t)$ *is a deterministic function of* $(Z_t, K_{c,t}, K_{i,t}, Q_{c,t})$*.*

Conversely, if these equations hold, and $H_{c,t}$*,* $H_{i,t}$ *are given by equations (27) and (28), then allocations* $(C, H, H_c, H_i, K_c, K_i)$ *and prices* (Q_c, Q_i, R_c, R_i, W) *are consistent with all equilibrium conditions (equations (15)-(25)), except for the two Euler equations (equations (16) and (17)) and the two capital accumulation equations (equations (24) and (25)).*

Proof. In any competitive equilibrium with subjective beliefs, the equilibrium equations (15)-(25) stated in Section 6 have to hold. Based on these equations and definitions in the model description, the expressions in the assertion can be computed. The wage is given by

$$W_t \overset{(15)}{=} C_t \overset{(22)}{=} Y_{c,t} \overset{(1)}{=} K_{c,t}^{\alpha_c} Z_t^{1-\alpha_c} H_{c,t}^{1-\alpha_c} \overset{(27)}{=} (1-\alpha_c)^{1-\alpha_c} K_{c,t}^{\alpha_c} Z_t^{1-\alpha_c}.$$

Similarly, the rental rate in the consumption sector is

$$R_{c,t} \overset{(19)}{=} \frac{\alpha_c Y_{c,t}}{K_{c,t}} \overset{(1),(27)}{=} \alpha_c (1-\alpha_c)^{1-\alpha_c} K_{c,t}^{\alpha_c - 1} Z_t^{1-\alpha_c}.$$

Substituting $H_{i,t}$ as given by equation (28) into the capital first-order condition of investment firms (21) yields for the rental rate in the investment sector

$$
\begin{aligned}
R_{i,t} &= \alpha_i Q_{c,t} K_{i,t}^{\alpha_i - 1} Z_t^{1-\alpha_i} \left(K_{i,t} Z_t^{\frac{1-\alpha_i}{\alpha_i}} \left((1-\alpha_i) \frac{Q_{c,t}}{W_t} \right)^{\frac{1}{\alpha_i}} \right)^{1-\alpha_i} \\
&= \alpha_i \left(\frac{1-\alpha_i}{W_t} \right)^{\frac{1-\alpha_i}{\alpha_i}} Z_t^{\frac{1-\alpha_i}{\alpha_i}} Q_{c,t}^{\frac{1}{\alpha_i}}.
\end{aligned}
$$

Output is defined as $Y_t = C_t + I_t$, so the asserted output equation follows from the equations for C_t and I_t. The equation for C_t follows from $C_t = W_t$ (15) and the equation for W_t already proven. The equation for $I_t = Q_c^{ss} Y_{i,t}$ is obtained by substituting $H_{i,t}$ stated in (28) into the investment-sector production function (1). Finally, the expression for H_t just combines hours in the two sectors (given by (27) and (28)) with labor market clearing (23).

For the second part of the lemma, we just remark that all the equations (15), (18), (19), (20), (21), (22) and (23) have been used in deriving the equations in the first part of the lemma and inverting the arguments used there shows that these equations also necessarily need to hold, if the equations in the lemma and (27) and (28) hold.

□

The functional relationships in Lemma 2 contain $Q_{c,t}$ as the sole argument that is not contained in the state of time t. $Q_{c,t}$ and $K_{c,t+1}$ are simultaneously determined by the consumption-sector Euler equation (16) and the accumulation equation of consumption-sector capital (24). It has been shown in the main text how the former equation can be solved for $Q_{c,t}$ under the assumption of subjective price beliefs, compare equation (31). In this equation, still a conditional expectation $E_t\left[\frac{1}{W_{t+1}}\right]$ appears. However, given the representation of W in Lemma 2, this conditional expectations can be easily solved explicitly

$$\begin{aligned} E_t\left[\frac{1}{W_{t+1}}\right] &= E_t\left[\frac{1}{K_{c,t+1}^{\alpha_c} Z_{t+1}^{1-\alpha_c}(1-\alpha_c)^{1-\alpha_c}}\right] \\ &= \frac{1}{(1-\alpha_c)^{1-\alpha_c} K_{c,t+1}^{\alpha_c}} E_t\left[(\gamma Z_t \varepsilon_{t+1})^{\alpha_c - 1}\right] \\ &= \frac{1}{(1-\alpha_c)^{1-\alpha_c} K_{c,t+1}^{\alpha_c}} \frac{E\left[\varepsilon_{t+1}^{\alpha_c-1}\right]}{\gamma^{1-\alpha_c} Z_t^{1-\alpha_c}}, \end{aligned}$$

where it has been used that $K_{c,t+1}$ is known at the end of period t and $Z_{t+1} = \gamma Z_t \varepsilon_{t+1}$. The expectation $E\left[\varepsilon_{t+1}^{\alpha_c-1}\right]$ of the log-normal variable $\varepsilon_{t+1}^{\alpha_c-1}$ is simply given by $e^{(\alpha_c-1)(\alpha_c-2)\frac{\sigma^2}{2}}$. Combining this result with equation (31), the price of consumption capital must be given by equation (38) in the next lemma, which is key for the argument to follow. We first formulate a version under the assumption of predetermined beliefs $m_{c,t}$ (no projection facility applied).[75] As in Appendix A.3.1, we use in the following lower case letters to denote realizations of random variables.

[75] Note also, that the second equation (39) is just a combination of the capital accumulation equation (24) with $Y_{c,t} = \frac{I_t}{Q_c}$ and the expression for I_t from Lemma 2, so this equation has to hold along any equilibrium path as well.

Lemma 3. *For any given* $(z_t, k_{c,t}, k_{i,t}, m_{c,t})$ *with* $z_t, k_{c,t}, k_{i,t}, m_{c,t} > 0$*, the equations*

$$q_{c,t} = \frac{\alpha_c \beta}{1 - \beta (1-\delta_c) \frac{w_t}{k_{c,t+1}^{\alpha_c} (1-\alpha_c)^{1-\alpha_c}} \frac{e^{(\alpha_c-1)(\alpha_c-2)\frac{\sigma^2}{2}}}{\gamma^{1-\alpha_c} z_t^{1-\alpha_c}} m_{c,t}} \frac{w_t}{k_{c,t+1}} \tag{38}$$

$$k_{c,t+1} = (1-\delta_c) k_{c,t} + z_t^{\frac{1}{\alpha_i}} k_{i,t} \left(\frac{(1-\alpha_i) q_{c,t}}{w_t} \right)^{\frac{1-\alpha_i}{\alpha_i}} \tag{39}$$

have a unique solution $(q_{c,t}, k_{c,t+1})$ *with* $q_{c,t}, k_{c,t+1} > 0$*. Here,* $w_t = k_{c,t}^{\alpha} (1-\alpha)^{1-\alpha} z_t^{1-\alpha}$ *is a function of* $k_{c,t}$ *and* z_t*.*

Proof. Equation (38) expresses $q_{c,t}$ as a function of $k_{c,t+1}$, $q_{c,t} = f(k_{c,t+1})$, equation (39) expresses $k_{c,t+1}$ as a function of $q_{c,t}$, $k_{c,t+1} = g(q_{c,t})$. Clearly, due to $w_t, z_t > 0$ and $\alpha_i < 1$ the function g is strictly increasing on the domain $(0, \infty)$. Furthermore, as long as the denominator on the left of equation (38) is positive, i.e. for $k_{c,t+1} \in (\underline{K}, \infty)$ with

$$\underline{K} := \left(\beta (1-\delta_c) \frac{w_t}{(1-\alpha_c)^{1-\alpha_c}} \frac{e^{(\alpha_c-1)(\alpha_c-2)\frac{\sigma^2}{2}}}{\gamma^{1-\alpha_c} z_t^{1-\alpha_c}} m_{c,t} \right)^{\frac{1}{\alpha_c}},$$

$f(k_{c,t+1})$ is strictly decreasing in $k_{c,t+1}$.[76] Hence, also the function

$$h : (\underline{K}, \infty) \to ((1-\delta_c) k_{c,t}, \infty), k_{c,t+1} \mapsto g(f(k_{c,t+1}))$$

must be strictly decreasing and thus there is at most one fixed point $k^*_{c,t+1}$ (satisfying $k^*_{c,t+1} = h(k^*_{c,t+1})$). As any solution $(q_{c,t}, k_{c,t+1})$ to (38) and (39) must satisfy $q_{c,t} = f(k_{c,t+1})$ and $k_{c,t+1} = g(q_{c,t})$, any such $k_{c,t+1}$ must necessarily be a fixed point of h. Thus, there can be at most one (positive) solution to (38) and (39).

Conversely, for any fixed point $k^*_{c,t+1}$ of h, the pair $(q^*_{c,t}, k^*_{c,t+1}) = (f(k^*_{c,t+1}), k^*_{c,t+1})$ is obviously a (positive) solution to (38) and (39). It is thus left to show that a fixed point always exists. The function h is continuous and the following limit considerations show the existence of a fixed point by the intermediate value theorem:

- $f(k) \to 0$ as $k \to \infty$, so $h(k) \to g(0) = (1-\delta_c) k_{c,t}$ as $k \to \infty$, hence for large k $h(k) - k$ is negative
- $f(k) \to \infty$ as $k \searrow \underline{K}$ and $g(q) \to \infty$ as $q \to \infty$, so $h(k) \to \infty$ as $k \searrow \underline{K}$, hence for small k close to $\underline{K}$, $h(k) - k$ is positive

□

[76] When looking for a positive solution $(q_{c,t}, k_{c,t+1})$ to (38) and (39), we can restrict attention to $k_{c,t+1} > \underline{K}$, because otherwise $q_{c,t} = f(k_{c,t+1})$ becomes negative.

Unfortunately, the above result treats $m_{c,t}$ as fixed, but due to the projection facility described in Appendix A.2, $m_{c,t}$ is not fully predetermined in period t, instead might be projected downward, if it implies a too high PD ratio. Hence, the simple result above cannot always be applied. The following technical lemma deals with the more general case.[77]

Lemma 4. *For any given positive values of z_t, $k_{c,t}$, $k_{i,t}$, $m_{c,t}$ consider the three equations (again, $w_t = (1-\alpha_c)^{1-\alpha_c} k_{c,t}^{\alpha_c} z_t^{1-\alpha_c}$)*

$$q_{c,t} = \frac{\alpha_c \beta}{1 - \beta (1-\delta_c) \frac{w_t}{k_{c,t+1}^{\alpha_c} (1-\alpha_c)^{1-\alpha_c}} \frac{e^{(\alpha_c - 1)(\alpha_c - 2)\frac{\sigma^2}{2}}}{\gamma^{1-\alpha_c} z_t^{1-\alpha_c}} m_{c,t}^p} \frac{w_t}{k_{c,t+1}} \tag{40}$$

$$k_{c,t+1} = (1-\delta_c) k_{c,t} + z_t^{\frac{1}{\alpha_i}} k_{i,t} \left(\frac{(1-\alpha_i) q_{c,t}}{w_t} \right)^{\frac{1-\alpha_i}{\alpha_i}} \tag{41}$$

$$m_{c,t}^p = \begin{cases} m_{c,t}, & m_{c,t} \leq \underline{m}(k_{c,t+1}) \\ \underline{m}(k_{c,t+1}) + \frac{(\overline{m} - \underline{m})(k_{c,t+1})}{m_{c,t} + (\overline{m} - 2\underline{m})(k_{c,t+1})} \left(m_{c,t} - \underline{m}(k_{c,t+1}) \right), & m_{c,t} \geq \underline{m}(k_{c,t+1}) \end{cases} \tag{42}$$

where the functions $\underline{m}$, $\overline{m}$ are the projection thresholds for m_c as defined in Appendix A.2.[78] *If*

$$m_{c,t} < \frac{1}{\beta (1-\alpha_c)} \left(\frac{\gamma}{(1-\delta_c) e^{\frac{(2-\alpha_c)\sigma^2}{2}}} \right)^{1-\alpha_c}, \tag{43}$$

then the equation system has a unique solution $(q_{c,t}, k_{c,t+1}, m_{c,t}^p)$.

Proof. Ignore the capital accumulation equation (41) and first consider equations (40) and (42). To transfer the proof from Lemma 3, we need to show that after substituting $m_{c,t}^p$ into the first equation, this still defines a decreasing relationship between $k_{c,t+1}$ and $q_{c,t}$. As $\underline{m}(k_{c,t+1})$ is strictly increasing in $k_{c,t+1}$ (and approaching $-\infty$ as $k_{c,t+1} \to 0$), there is some threshold $\hat{k}$, such that $\underline{m}(k_{c,t+1}) \leq m_{c,t}$ for $k_{c,t+1} \leq \hat{k}$ and $\underline{m}(k_{c,t+1}) \geq m_{c,t}$ for $k_{c,t+1} \geq \hat{k}$. We consider the two cases separately:

1. If $k_{c,t+1} \leq \hat{k}$, then the projection is actually used, so we have

$$m_{c,t}^p = \underline{m} + \frac{\overline{m} - \underline{m}}{m_{c,t} + \overline{m} - 2\underline{m}} (m_{c,t} - \underline{m})$$

[77] The additional upper bound on beliefs stated in equation (43) is of little practical relevance. In all calibrations we consider, the upper bound is approximately equal to $\frac{1}{1-\alpha_c} \approx 1.5$, which implies an expected appreciation in the capital price of 50% within the next quarter and is much larger than any value m_c ever attained in our numerical simulations. We nevertheless tighten the projection upper bound $\overline{m}_{c,t}$ to be always consistent with equation (43), see footnote 64. This additional modification does not invalidate the proof of the lemma.

[78] $\underline{m}$, $\overline{m}$ also depend on z_t, w_t and $k_{c,t}$, which is suppressed in the notation.

from the third equation. For notational convenience define the constants

$$
\begin{aligned}
A &= \frac{1}{\beta(1-\delta_c)\frac{w_t \exp\left((1-\alpha_c)(2-\alpha_c)\frac{\sigma^2}{2}\right)}{\gamma^{1-\alpha_c}(1-\alpha_c)^{1-\alpha_c} z_t^{1-\alpha_c}}} \\
\underline{B} &= \frac{\beta\left(1-\delta_c\right)k_{c,t}}{\left(p\left(1+\underline{PD}\right)-1\right)\beta\left(1-\delta_c\right)\frac{w_t \exp\left((1-\alpha_c)(2-\alpha_c)\frac{\sigma^2}{2}\right)}{\gamma^{1-\alpha_c}(1-\alpha_c)^{1-\alpha_c} z_t^{1-\alpha_c}}} = \frac{\beta\left(1-\delta_c\right)k_{c,t}}{p\left(1+\underline{PD}\right)-1}A \\
\overline{B} &= \frac{\beta\left(1-\delta_c\right)k_{c,t}}{\left(p\left(1+\overline{PD}\right)-1\right)\beta\left(1-\delta_c\right)\frac{w_t \exp\left((1-\alpha_c)(2-\alpha_c)\frac{\sigma^2}{2}\right)}{\gamma^{1-\alpha_c}(1-\alpha_c)^{1-\alpha_c} z_t^{1-\alpha_c}}} = \frac{\beta\left(1-\delta_c\right)k_{c,t}}{p\left(1+\overline{PD}\right)-1}A \\
C &= \beta(1-\delta_c)\frac{w_t \exp\left((1-\alpha_c)(2-\alpha_c)\frac{\sigma^2}{2}\right)}{\gamma^{1-\alpha_c}(1-\alpha_c)^{1-\alpha_c} z_t^{1-\alpha_c}} = A^{-1}
\end{aligned}
$$

and drop all subscripts for the following argument (k refers to $k_{c,t+1}$, q to $q_{c,t}$ and m to $m_{c,t}$).

Then we have (this follows from equations (35) and (36) for $s = c$)

$$\underline{m} = Ak^{\alpha_c} - \underline{B}k^{\alpha_c - 1}, \qquad \overline{m} = Ak^{\alpha_c} - \overline{B}k^{\alpha_c - 1}$$

and (from equation (40))

$$q = \frac{\text{const}}{1 - Ck^{\alpha_c} m^p}\frac{1}{k}.$$

Using $m^p = \underline{m} + \frac{\overline{m}-\underline{m}}{m+\overline{m}-2\underline{m}}\left(m - \underline{m}\right)$, we obtain

$$
\begin{aligned}
\frac{\text{const}}{q} &= k - Ck^{1-\alpha_c} m^p \\
&= k - Ck^{1-\alpha_c}\left(\underline{m} + \frac{\overline{m}-\underline{m}}{m+\overline{m}-2\underline{m}}\left(m-\underline{m}\right)\right) \\
&= k - Ck^{1-\alpha_c}\Bigg(Ak^{\alpha_c} - \underline{B}k^{\alpha_c-1} \\
&\qquad + \frac{Ak^{\alpha_c} - \overline{B}k^{\alpha_c-1} - Ak^{\alpha_c} + \underline{B}k^{\alpha_c-1}}{m + Ak^{\alpha_c} - \overline{B}k^{\alpha_c-1} - 2Ak^{\alpha_c} + 2\underline{B}k^{\alpha_c-1}}\left(m - Ak^{\alpha_c} + \underline{B}k^{\alpha_c-1}\right)\Bigg) \\
&= C\underline{B} - C\left(\underline{B} - \overline{B}\right)\frac{m - Ak^{\alpha_c} + \underline{B}k^{\alpha_c-1}}{m - Ak^{\alpha_c} + \left(2\underline{B} - \overline{B}\right)k^{\alpha_c-1}}
\end{aligned}
$$

$C\underline{B}$ and $C\left(\underline{B} - \overline{B}\right)$ are positive constants, so q is decreasing in k, if and only if the expression $\frac{m-Ak^{\alpha_c}+\underline{B}k^{\alpha_c-1}}{m-Ak^{\alpha_c}+\left(2\underline{B}-\overline{B}\right)k^{\alpha_c-1}}$ is. Using that the derivative of $x \mapsto \frac{u(x)}{u(x)+v(x)}$ is given by $\frac{u'(x)v(x)-v'(x)u(x)}{(u(x)+v(x))^2}$, we find that $\frac{m-Ak^{\alpha_c}+\underline{B}k^{\alpha_c-1}}{m-Ak^{\alpha_c}+\left(2\underline{B}-\overline{B}\right)k^{\alpha_c-1}}$ is (strictly) decreasing in k, if and only if[79]

$$\begin{aligned}&\left(-A\alpha_c k^{\alpha_c-1}+(\alpha_c-1)\,\underline{B}k^{\alpha_c-2}\right)\left(\underline{B}-\overline{B}\right)k^{\alpha_c-1}\\ &<(\alpha_c-1)\left(\underline{B}-\overline{B}\right)k^{\alpha_c-2}\left(m-Ak^{\alpha_c}+\underline{B}k^{\alpha_c-1}\right)\end{aligned}$$

After expanding the products on both sides and canceling common terms, this inequality simplifies to

$$\begin{aligned}0\quad &<(\alpha_c-1)\left(\underline{B}-\overline{B}\right)mk^{\alpha_c-2}+A\left(\underline{B}-\overline{B}\right)k^{2\alpha_c-2}\\ \Leftrightarrow\quad m\quad &<\frac{Ak^{\alpha_c}}{1-\alpha_c}.\end{aligned}$$

Finally, using the definition of A, we obtain the condition (from now on subscripts are added back again for clarity about the timing of variables)

$$\begin{aligned}m_{c,t}&<\frac{\gamma^{1-\alpha_c}(1-\alpha_c)^{1-\alpha_c}z_t^{1-\alpha_c}}{\beta(1-\delta_c)w_t\exp\left((1-\alpha_c)(2-\alpha_c)\frac{\sigma^2}{2}\right)}\frac{k_{c,t+1}^{\alpha_c}}{1-\alpha_c}\\ &\overset{(33)}{=}\frac{\gamma^{1-\alpha_c}(1-\alpha_c)^{1-\alpha_c}z_t^{1-\alpha_c}}{\beta(1-\delta_c)(1-\alpha_c)^{1-\alpha_c}k_{c,t}^{\alpha_c}z_t^{1-\alpha_c}\exp\left((1-\alpha_c)(2-\alpha_c)\frac{\sigma^2}{2}\right)}\frac{k_{c,t+1}^{\alpha_c}}{1-\alpha_c}\\ &=\frac{\gamma^{1-\alpha_c}}{\beta\left(1-\delta_c\right)\left(1-\alpha_c\right)\exp\left((1-\alpha_c)(2-\alpha_c)\frac{\sigma^2}{2}\right)}\left(\frac{k_{c,t+1}}{k_{c,t}}\right)^{\alpha_c}\end{aligned}$$

This condition is tighter, the smaller is $k_{c,t+1}$. The smallest possible value of $k_{c,t+1}$ given $k_{c,t}$ is $k_{c,t+1}=(1-\delta_c)k_{c,t}$ (otherwise equation (41) is inconsistent with a positive $q_{c,t}$), so the above condition on $m_{c,t}$ is certainly satisfied, if

$$m_{c,t}<\frac{1}{\beta\left(1-\alpha_c\right)}\left(\frac{\gamma}{\left(1-\delta_c\right)e^{\frac{(2-\alpha_c)\sigma^2}{2}}}\right)^{1-\alpha_c},$$

which is exactly the condition required in the assertion. So as long as $k_{c,t+1}\leq\hat{k}$, the third and first equation define a strictly decreasing relationship between $k_{c,t+1}$ and $q_{c,t}$.

2. If $k_{c,t+1}\geq\hat{k}$, then $m_{c,t}^p=m_{c,t}$ does not depend on the level on $k_{c,t+1}$ anymore and thus the third and first equation define a strictly decreasing relationship between $k_{c,t+1}$ and $q_{c,t}$ by arguments made in the proof of Lemma 3.

[79] Here, $u(x)=m-Ax^{\alpha_c}+\underline{B}x^{\alpha_c-1}\Rightarrow u'(x)=-A\alpha_c x^{\alpha_c-1}+(\alpha_c-1)\,\underline{B}x^{\alpha_c-2}$ and $v(x)=\left(\underline{B}-\overline{B}\right)x^{\alpha_c-1}\Rightarrow v'(x)=(\alpha_c-1)\left(\underline{B}-\overline{B}\right)x^{\alpha_c-2}$

After substituting the third equation into the first, we have as before two functional relationships $q_{c,t} = f(k_{c,t+1})$ and $k_{c,t+1} = g(q_{c,t})$ with f strictly decreasing and g strictly increasing. The same arguments made in the proof of Lemma 3 guarantee a unique solution. The associated level of $m^p_{c,t}$ can then be computed from the third equation.

□

We have now all the tools available to prove Proposition 1.

Proof of Proposition 1. Existence and uniqueness of the measure $\tilde{\mathcal{P}}$ has already been discussed in Section A.3.2 of this appendix. Here, it is left to construct the mappings G and F. Throughout the construction only necessary equilibrium conditions are used and no construction step admits several choices. For this reason, the construction will also yield uniqueness of the mappings G and F.

Suppose $S_t = s_t$ for an arbitrary (fixed) state $s_t = (z_t, k_{c,t}, k_{i,t}, m_{c,t}, m_{i,t}, q_{c,t-1}, q_{i,t-1})$ such that $m_{c,t}$ respects the upper bound of Lemma 4. For the following argument it is also useful to define the "reduced state" $\hat{s}_t = (z_t, k_{c,t}, k_{i,t}, m_{c,t}, q_{c,t-1})$ that does not include $m_{i,t}$ and $q_{i,t-1}$. We first show the existence of a unique state transition function $\hat{G}$ for this reduced state. In any equilibrium, the wage is a function of $K_{c,t}$ and Z_t only, compare Lemma 2. Hence, conditional on $\hat{S}_t = \hat{s}_t$, $W_t = \tilde{F}_W(\hat{s}_t)$ with some deterministic function $\tilde{F}_W$ (whose explicit form is given in Lemma 2). Furthermore, in any equilibrium (with beliefs as specified in Section 5) the equations (40), (41), (42) have to hold along any equilibrium path. Lemma 4 thus implies the existence of a unique vector $(q_{c,t}, k_{c,t+1}, m^p_{c,t})$, given z_t, $k_{c,t}$, $k_{i,t}$, $m_{c,t}$ and $w_t = \tilde{F}_W(\hat{s}_t)$, which are all uniquely determined by the reduced state $\hat{s}_t$, such that all three equations hold. The equations thus implicitly define three functions

$$\tilde{G}_{Q,c}(\hat{s}_t) = q_{c,t}, \qquad \tilde{G}_{K,c}(\hat{s}_t) = k_{c,t+1}, \qquad \tilde{G}_{m^p,c}(\hat{s}_t) = m^p_{c,t}$$

and the argument given so far implies that along any equilibrium path $q_{c,t}$, $k_{c,t+1}$ and $m^p_{c,t}$ (the value of $m_{c,t}$ after projection) must necessarily be related to $\hat{s}_t$ as described by these three equations.

Next, $m_{c,t+1}$ must satisfy the belief updating equation (compare (12) for the consumption sector[80])

$$\ln m_{c,t+1} = \ln m^p_{c,t} + g\left(\ln q_{c,t} - \ln q_{c,t-1} - \ln m^p_{c,t}\right) =: \ln \tilde{G}_{m,c}(\hat{s}_t),$$

where the right-hand side is a function of $\hat{s}_t$, because $q_{c,t} = \tilde{G}_{Q,c}(\hat{s}_t)$ and $m^p_{c,t} = \tilde{G}_{m^p,c}(\hat{s}_t)$ are and $q_{c,t-1}$ is a component of $\hat{s}_t$.

In addition, $K_{i,t+1} \propto Z_t$ by definition. Let $\bar{k}_i$ be the proportionality constant, i.e. $K_{i,t+1} = \bar{k}_i Z_t$. This implies for realizations conditional on $\hat{S}_t = \hat{s}_t$ that $k_{i,t+1} = \bar{k}_i z_t =: \tilde{G}_{K,i}(\hat{s}_t)$.

[80] As remarked in footnote 31, we set the additional shock to agents' information set to 0 in all periods.

In total, we obtain from the discussion so far that $\hat{s}_{t+1} = (z_{t+1}, k_{c,t+1}, k_{i,t+1}, m_{c,t+1}, q_{c,t})$ must necessarily satisfy

$$\hat{s}_{t+1} = (z_{t+1}, \tilde{G}_{K,c}(\hat{s}_t), \tilde{G}_{K,i}(\hat{s}_t), \tilde{G}_{m,c}(\hat{s}_t), \tilde{G}_{Q,c}(\hat{s}_t)) =: \hat{G}\left(\hat{s}_t, z_{t+1}\right).$$

Thus, in any equilibrium, the evolution of the reduced state $\hat{s}_{t+1}$ must be governed by the transition function $\hat{G}$. From our derivation it is also clear that the evolution of $\hat{G}$ is consistent with the equations (40), (41), (42), the belief updating equation (12) in the consumption sector and the exogenous evolution of K_i. In particular, $\hat{G}$ is then consistent with the consumption-sector Euler equation (16) and with the capital accumulation equations (24), (25) in both sectors.

Next, we let $\hat{F}$ be the mapping from reduced states $\hat{s}_t$ to outcomes $(w_t, r_{c,t}, r_{i,t}, y_t, c_t, i_t, h_t)$ defined by the formulas given in Lemma 2, if $q_{c,t}$ is everywhere replaced by $\tilde{G}_{Q,c}(\hat{s}_t)$.[81] Combining this with the obvious state reduction mapping $s_t \mapsto \hat{s}_t$, we can define $F(s_t) := \hat{F}(\hat{s}_t)$.[82] Lemma 2 tells us then that this choice of F is the only possible choice consistent with equilibrium and in turn this F is consistent with all equilibrium equations other than the two Euler equations and the two capital accumulation equations. Consequently, F and $\hat{G}$ together are consistent with all the equilibrium conditions (15)-(25) except for the investment-sector Euler equation (17). In addition, $\hat{G}$ is also consistent with the belief updating equation for consumption-sector capital prices.

To complete the existence proof, it is left to show that $\hat{G}$ can be extended to a full state transition mapping G that is in addition consistent with the investment-sector Euler equation and the belief updating equation for investment-sector capital prices. First, consider the conditional expectations in equation (34), which is a partially solved version of the investment-sector Euler equation (17) from the main text. These conditional

[81] This means,

$$\begin{aligned}
\hat{F}_W(\hat{s}_t) &= k_{c,t}^{\alpha_c} z_t^{1-\alpha_c}(1-\alpha_c)^{1-\alpha_c} \\
\hat{F}_{R,c}(\hat{s}_t) &= \alpha_c k_{c,t}^{\alpha_c - 1} z_t^{1-\alpha_c}(1-\alpha_c)^{1-\alpha_c} \\
\hat{F}_{R,i}(\hat{s}_t) &= \alpha_i \left(\frac{1-\alpha_i}{\hat{F}_W(\hat{s}_t)}\right)^{\frac{1-\alpha_i}{\alpha_i}} z_t^{\frac{1-\alpha_i}{\alpha_i}} \left(\tilde{G}_{Q,c}(\hat{s}_t)\right)^{\frac{1}{\alpha_i}} \\
&\vdots
\end{aligned}$$

[82] We keep the separate mapping $\hat{F}$ as an auxiliary device for the extension of $\hat{G}$ to G below.

expectations can be written as

$$
\begin{aligned}
E_t\left[\frac{W_t}{W_{t+1}}R_{i,t+1}\right] &= E\left[\frac{\hat{F}_W(\hat{S}_t)}{\hat{F}_W(\hat{S}_{t+1})}\hat{F}_{R,i}(\hat{S}_{t+1})\Big| S_t, S_{t-1},\dots\right] \\
&= E\left[\frac{\hat{F}_W(\hat{S}_t)}{\hat{F}_W\left(\hat{G}\left(\hat{S}_t, Z_{t+1}\right)\right)}\hat{F}_{R,i}\left(\hat{G}\left(\hat{S}_t, Z_{t+1}\right)\right)\Big|\hat{S}_t\right],
\end{aligned}
$$

$$
\begin{aligned}
E_t\left[\frac{W_t}{W_{t+1}}\right] &= E\left[\frac{\tilde{F}_W(\hat{S}_t)}{\tilde{F}_W(\hat{S}_{t+1})} \mid S_t, S_{t-1},\dots\right] \\
&= E\left[\frac{\tilde{F}_W(\hat{S}_t)}{\tilde{F}_W\left(\hat{G}\left(\hat{S}_t, Z_{t+1}\right)\right)} \mid \hat{S}_t\right],
\end{aligned}
$$

where in each case the second equality follows from the fact that all information in the history S^t not already contained in $\hat{S}_t$ is redundant for predicting Z_{t+1} and $\hat{S}_t$. Hence, both conditional expectations are deterministic functions of the current reduced state $\hat{S}_t$. As these two conditional expectations and $R_{i,t} = \hat{F}_{R,i}(\hat{S}_t)$ are the only relevant variables to compute the projection bounds in the investment sector, compare Appendix A.2, conditional on $\hat{S}_t = \hat{s}_t$, the projected belief in the investment sector is given by $m^p_{i,t} = \tilde{G}_{m^p,i}(\hat{s}_t, m_{i,t})$ with some deterministic function $\tilde{G}_{m^p,i}$. By equation (34), $Q_{i,t}$ must then assume in equilibrium the value (conditional on $\hat{S}_t = \hat{s}_t$ and $m_{i,t}$)

$$
q_{i,t} = \frac{\beta E\left[\frac{W_t}{W_{t+1}}R_{i,t+1} \mid \hat{S}_t = \hat{s}_t\right]}{1 - \beta E_t\left[\frac{W_t}{W_{t+1}} \mid \hat{S}_t = \hat{s}_t\right](1-\delta)\,\tilde{G}_{m^p,i}(\hat{s}_t, m_{i,t})}.
$$

The right-hand side is a function of $\hat{s}_t$ and $m_{i,t}$ and therefore of the full state s_t. Denote it by $\tilde{G}_{Q,i}(s_t)$. Finally, the belief updating equation for the investment sector defines $m_{i,t+1}$ as a function of the current state,

$$
m_{i,t+1} = \tilde{G}_{m,i}(s_t) := \tilde{G}_{m^p,i}(\hat{s}_t, m_{i,t})\left(\frac{\tilde{G}_{Q,i}(s_t)}{\tilde{G}_{m^p,i}(\hat{s}_t, m_{i,t})q_{i,t-1}}\right)^g.
$$

Define thus

$$
\begin{aligned}
G(s_t, z_{t+1}) : = \Big(&\hat{G}_Z(\hat{s}_t, z_{t+1}), \hat{G}_{K,c}(\hat{s}_t, z_{t+1}), \hat{G}_{K,i}(\hat{s}_t, z_{t+1}), \hat{G}_{m,c}(\hat{s}_t, z_{t+1}), \\
&\tilde{G}_{m,i}(s_t), \hat{G}_{Q,c}(\hat{s}_t, z_{t+1}), \tilde{G}_{Q,i}(s_t)\Big).
\end{aligned}
$$

By construction, G is then (together with F) consistent with all equations that $\hat{G}$ is and in addition with the investment-sector Euler equation (17) and with the belief updating equation for m_i. This finishes the proof of the proposition. □

A.4 Details on the IRF Decay Restrictions in Estimation

Capital prices in the subjective belief model show a cyclical pattern in response to a technology shock: as prices fall from a boom to steady state, agents may be already slightly pessimistic and prices undershoot, as prices recover from a bust back to steady state, agents may already be slightly optimistic, triggering another boom. If these dynamics are too strong, deterministic cycles can exist or the cyclical dynamics can even be self-amplifying (small shocks lead to a sequence of cycles of increasing magnitude). We impose a decay restriction in the estimation to rule out such dynamics. The IRF decay restriction does not impose a certain speed of decay within one cycle, but rather requires the decay of subsequent cycle peaks over time to be sufficiently fast.

Specifically, for each sector $s \in \{c, i\}$ we consider a long deterministic impulse response path (400 quarters) for the capital price Q_s to a one-standard-deviation technology shock starting in the steady state. We identify all peaks (local maxima) of the resulting path $\{Q_{s,t}\}_{t=0}^{400}$, where at t there is a "peak", if $Q_{s,t} > Q_{s,t-1}, Q_{s,t+1}$. Let T_p be the set of all peak times of the impulse response path. If $|T_p| \leq 1$, we set the peak decay rate to ∞, thereby always admitting such a parameter combination. If $|T_p| \geq 2$, we fit an exponential function through the points $\{Q_{s,t}\}_{t \in T_p}$, specifically we estimate the least-squares regression

$$\log\left(\frac{Q_{s,t}}{Q_s^{ss}} - 1\right) = a + bt + \varepsilon_t, \qquad t \in T_p,$$

where Q_s^{ss} is the steady-state value of Q_s. We call $-b$ the peak decay rate of the impulse response. Figure 11 illustrates the procedure graphically: the blue solid line is the impulse response path, the red circles mark the peaks, i.e. the points $(t, Q_{s,t}/Q_s^{ss})$ for $t \in T_p$, and the yellow dashed line represents the fitted exponential.

On the b parameter we impose the restriction $-b \geq 1.16\%$. This implies a half-life of at most 60 quarters or – given a typical distance of approximately 40 quarters between two peaks – a size reduction of at least one third from one peak to the next.

Parameter combinations are admitted in the estimation, if they satisfy this condition for both Q_c and Q_i.

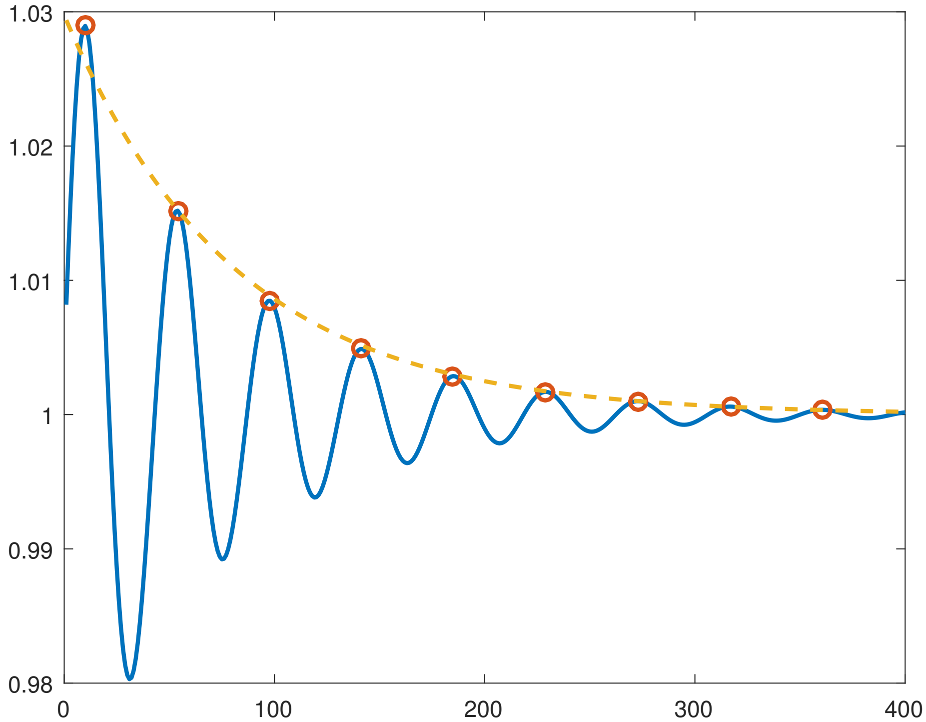

Figure 11. Illustration of the decay rate definition