Wage Dynamics with Developing Asymmetric Information

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Abstract

It has been recognized that the key features of the labor market include long duration of waiting for a match (search friction) and gradual learning of match quality over time (learning friction). This paper analyses the effect of the two friction in determining the market's expectation about a worker's quality and how these factors map to a worker's labor market outcome. The driving mechanism of our results is the asymmetry of information: current employer has better information about the worker's quality than outside firms that only have access to public information, such as the worker's past job history. We first show how wage negotiation schemes, together with worker's on-the-job search shapes equilibrium inference problem regarding the worker's type. Using this result, we see how the evolution of belief affects worker's life-time wage profile. We show that the friction hinders wage growth for high quality workers, while allowing low quality workers to receive excess wage. Our model also generates inertia that hinders wage growth for high quality workers with initial unlucky draws. We also suggest a new channel through which changes in technology and policy can affect an individual's labor market outcome.

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1 Introduction

Consider a recruitment manager (he) interviewing an applicant (she) who has been working in another company for some time. The manager can review her CV and credentials to determine whether she is a good fit for the posted job. But there are always some things that cannot be read off from CVs; most importantly, why is she applying to this job in the first place? Is she applying for this job because she knows she is likely to be fired from her previous job? It might also be that she is applying for this job merely to increase bargaining power in her previous job. In fact, if it is the former, then the manager has to worry about winner's curse: the company might be overpaying for this new worker who her previous employer does not want to keep. At the same time, as a recruitment manager, he has to worry about his own objective: recruiting (or poaching) good workers at lowest cost, hence he wants to avoid being too pessimistic when bidding for this worker.

Indeed, in many industries, bringing in a competitor and negotiating wage is widely accepted as a norm for driving up a worker's income. One anecdote from consulting industry records that the average frequency a consultant triggers wage renegotiation following search on-the-job is once in three years.[1] Empirically, bulk of new hires come from other jobs, and on average, job-to-job transitions is roughly twice as large as flows from non-employment to employment [11]; in US, among 122 million stock of employed, there was roughly 4 million employees transferring to another job in a given month. From these examples, we emphasize that on-the-job search is an important force driving wage dynamics (as noted early in works such as [2], [13]), and tries to identify the channel through which on-the-job search affects individual wage profile and equilibrium inference on worker's quality.

In the model, information about worker's quality is revealed gradually over time, and the learning is confined within the matched party, the worker and his/her current employer. The information is modelled with a Poisson news process that generates arrival over time, with rate α. Outside firms, other than the matched party, do not observe this news process. However, we assume that worker's separation from a job, or shift to another job are public information, as these events are recorded in a credible and verfiable CV that the worker carries. An employee working for this firm (incumbent) generates meeting with new firm (poacher) at Poisson rate λ, which denotes the worker's on-the-job search intensity (following the notations in [2], [13].) Once brought into contact, the new firm gets to see the worker's CV. The new firm, using the information on CV and the knowledge about the information structure, makes his Bayes-rational inference about the worker's quality. When worker's quality is correlated between firms, the new firm cares about what the previous employers of this worker know about the worker's quality. In essence, the new firm forms belief over the beliefs about the worker's quality in other matches.

[1] I thank Jay Park for this comment.

We first focus on the informational content of worker's employment history. When the current employer knows more about the quality of the worker, the new firm faces the problem of adverse selection – the low quality worker that the current employer does not have high willingness to pay are the ones who are most likely to be attracted by the poaching firm. Therefore, low quality workers shift jobs more often, while high quality workers selectively stay longer in the match. In turn, average quality of workers increases with tenure. This result is in line with other models of dynamic adverse selection, such as [9]

Next, we analyse how the worker's history feeds to determination of wages. We show that, under the search and learning frictions, wage gradually increases with tenure for high quality workers. The worker's wage does not jump up directly to his/her marginal productivity because the worker's outside offer grows gradually with tenure, as poachers get more optimistic about the quality of the worker. This is an informational explanation of positive wage-tenure relationship which can offer different insight into positive wage-tenure correlation. Past theories suggested that firms optimally back-load wages in order to increase the retention rate of workers ([1], [14]) or that the return to tenure is due to unobserved human capital accumulation during the tenure.

Due to the gradual learning, our model predicts initially large number of job shifts, while the transitions become smaller as the worker quality is revealed to the firm, and the firms start to retain this worker. This also predicts that wage growth from job shifts would be smaller in the early phase of career, where job shift is initiated by a worker trying to escape from a bad match, while the wage growth after the learning has occured is more likely due to the competition for the qualified worker. If the initial wage growth is indicative of worker's quality, we can project early career outcomes to the lifetime wage growth of an individual.[2]

It is also documented that there is considerable 'stigma effect' attached to a worker's timing of entry into the job market. There is evidence that controlling for observable traits, workers who shifted the job many times before exhibits low wage growth than the comparable individual. In fact, even in US, there is evidence that companies viewed past history of job hopping undesirable which matches the prediction of our model. Given a worker's life-time path of wages, we will be able to test whether such inertia exists, and our model supplies one possible explanation.

Lastly, in this setting, we analyse how the market outcomes for workers are differentially affected by changes in economic environment. When learning the worker's quality is a gradual process, increasing the worker and firm's meeting rate serves as a more frequent assessment of the worker's quality, and the information content of the worker's history increases. In turn, inequality in market outcome between high quality workers and low quality workers exacerbates. Intuitively, when the market's inference about the worker is accurate, firms hiring a high quality worker can no more sustain informational advantage over the competitors

[2] I thank Ioana Marinescu for this comment.

who want to poach the worker; and bad quality workers can no more hide bad news from the previous match. We believe that this can serve as an explanation for rising income inequality, and return to skills among workers after on-line job search platforms became popular.

1.1 Literature Review

One of the most successful applications of the dynamic economic model was in the field of labor economics, culminating in the award of Nobel Prize in 2010 to three economists: Diamond, Mortensen and Pissarides, for their 'analysis of markets with search frictions'. Search friction is still a relevant consideration for labor market, where there is huge heterogeneity in workers and jobs, asymmetry of information and strategic incentives. We believe that our work contributes to the understanding of wage-tenure profile as in [1], [14], and the Bayesian inference problem in labor market, such as [3].

In symmetric setting, [11] models the poaching auction as an ascending auction. In their paper, the worker shifts to another firm if and only if the surplus of the poacher is greater than the surplus of the current match. In case the poacher's surplus is smaller, the auction does not affect the continuation outcome.

At the same time, the paper uses ascending auction game, where the equilibrium exhibits (1) the outcome of the auction is efficient, and (2) the loser does not bid, and the winner just bids the future value of the worker in the current match. The incumbent's deviation to bid positive value is deterred by the poacher's threat of countering the incumbent's offer. Since the winner of the auction is known in the beginning, a small cost in participating in the bidding precludes the incumbent from bidding. This is rather surprising result because job-to-job transition typically leads to jump or fall in wages.

[13] uses another equilibrium such that the worker's rent from a meeting is maximized. In their paper, any opponent's bids that are below the current surplus of the match is out-bid by the incumbent. The unique Nash equilibrium of the Bertrand competition exhibits the same efficient allocation, but the winner pays the valuation of the match. As a follow-up, [4] justifies this equilibrium by formulating an alternating-offers bargaining game and taking an appropriate limit of the parameters such that the limit converges to the equilibrium.

In that regard, the wage determination mechanism in our paper is similar to that of [2] where the firms are assumed to post fixed wage contract and cannot respond. The paper was the first to show that the possibility of worker's job-to-job transition lead to (1) non-degenerate wage distribution in the economy, and (2) allow workers to earn postive share of profit.

The paper is also closely related to the literature on dynamic adverse selection, notably, papers such as [9] and [6]. Their papers focus on the seller's incentive to wait in order to sell the bad good at attractive price, and solves for equilibrium where separating/pooling behavior is endogenous as functions of time. Our paper's approach is a little different because we also keep track of worker's belief and wages after a single sale (auction). However, the main mechanism follows through to our model in terms of drift of beliefs.

On the technical side, we use the first-price auction model with asymmetrically informed bidders as in [5], and [10]. Rather than working on the static setting, our underlying information structure of an auction is evolving dynamically. Papers like [15] have asked questions about values of auction in case there are random number of bidders attracted over time. Our paper also looks at the value of object sold in dynamic market, following the evolution of beliefs after one consummated trade.

2 Model

2.1 Basic Setting

Time is continuous, $t \in [0, \infty)$. The model is set in continuous time, in order to keep track of the evolution of beliefs, firms' profit and workers' value over time.

Since we are focusing on adverse selection problem, we keep the firm types identical, while assuming that workers are either type H or L, standing for $High$ (peach) and Low (lemon) quality.

A worker's type is learned through a *good news* process that generates news with rate α for H workers but generates no news for L workers. This can be thought of as the arrival of surplus stream $y > 0$, that only H worker generates. For references later, we call the arrival of this process a '*breakthrough*', which we can think of a worker fully adapting to a job and starting to generate extra profit to the employer.

We consider two cases of the arrival of surplus.

- **(Case 1) Flow payoff following a breakthrough:** When a worker generates a breakthrough, he starts producing flow output $y > 0$ onward, which is also immediately observed by the new employer when hired.
- **(Case 2) Poisson arrival of lump-sum payoffs:** The good news payoff arrives intermittently, and every new employment relation restarts the learning process.

We will switch around the argument for beliefs as either calendar time t or tenure in the firm τ, but we will consistently use the notation p for the firm's belief about the worker's type (H), $p : [0, \infty) \to [0, 1]$. In the good news case, starting from initial belief p_0, the drift of the belief when there was no arrival until time t elapsed is given by the well-known differential equation:[3]

$$-p'(t) = \alpha p(t)(1 - p(t)), \quad p(0) = p_0$$

[3]Derivation is contained in Appendix 5.1 for the readers who are not familiar with continuous time belief process.

which has the unique solution

$$p(t) = \frac{p_0 e^{-\alpha t}}{p_0 e^{-\alpha t} + (1 - p_0)}.$$

Intuitively, for a Poisson process with arrival rate α, the probability that there is no arrival for $t > 0$ interval of time is $e^{-\alpha t}$. The solution is a simple application of Bayes' rule that calculates posterior belief.

It was shown that the belief is continuously drifting down without the arrival of news. When the news does arrive, it reveals perfectly that the worker is H type; belief immediately jumps up to 1, and stays there forever. At any point t in tenure, the firm's belief can take only two values, $p(t)$ or 1.

Meeting and Information Outside firms do not observe the news process and hence, forms belief about the state of the match. The beliefs condition on the history of the worker that the firms can observe. The information that is available for both firms is the history of the worker up until worker's time in the market t. The universe of space of histories consist of partitions of age t, into a vector of past tenures in n firms that the worker was employed at. Formally, we denote the full set of public histories of a worker of age t by

$$\mathcal{H}(t) = \{(\tau_1, \tau_2, \ldots, \tau_n; n) \mid \sum_{i=1}^{n} \tau_i = t\},$$

and a particular history by $h(t) \in \mathcal{H}(t)$. Potentially, there may be other elements of worker's history that are also observable, such as wages.

The firms $1, 2, \ldots, n$ are chronically ordered so that the n-th firm is the last firm to employ the worker: the current employer, incumbent firm. The incumbent's belief over the worker's type P_n is a mapping

$$P_n : \mathcal{H}(t) \times N(\tau_n) \to [0, 1]$$

where $N(\tau_n)$ is the random variable whose image is the number of arrivals for the Poisson news process within the interval $[0, \tau_n]$. Conditional on the worker being H type, its distribution is:

$$Pr(\{N(\tau_n) = k\}) = \frac{(\alpha \tau_n)^k}{k!} e^{-\alpha \tau_n}, \quad k = 0, 1, 2, \ldots$$

Since the arrival of good news, $N(\tau_n) > 0$, makes the belief jump to 1, starting from the initial belief $p_n(0|h(t))$, the incumbent's belief at τ_n takes value on two points. That is,

$P_n(\tau_n|h(t))$ is a binary random variable such that

$$P_n(\tau_n|h(t)) = \begin{cases} 1 & \text{if } N(\tau_n) > 0 \\ p_n(\tau_n|h(t)) := \frac{p_n(0|h(t))e^{-\alpha\tau_n}}{p_n(0|h(t))e^{-\alpha\tau_n}+(1-p_n(0|h(t)))} & \text{if } N(\tau_n) = 0 \end{cases}$$

In order to simplify notation, we will dispense with the random variable notation P_n of the beliefs, and focus on the lower bound $p_n(\tau_n|h(t))$, as long as there is no source of confusion.

With the good news process, incumbent's belief is degenerate, and takes at most two points at τ_n, over the support $\{1, p_n(\tau_n|h(t))\}$. We denote the outside firm's belief over the firm's belief by

$$x(\tau_n|h(t)) = Pr(\{P_n(\tau_n|h(t)) = 1\}).$$

It is convenient to minimize the dimension of this object, outside firm's belief is over the incumbent's belief, which might have continuous distribution. We will focus on the cases where disagreements are restricted in some ways. First of all, we will have all firms agree on the quality of a no-news worker p, while the only disagreement is on whether the incumbent's belief is on p or 1. Secondly, we want the incumbent and the poacher to agree on where x started from with the beginning of tenure, $x(0|h(t))$. In our model, it is sufficient to have the following informational assumption in order to achieve this goal.

Assumption 1. When a poacher wins, the firm's winning bid becomes public.

In the flow-payoff case, p is only the function of calendar time and the above assumption applies to the initial belief $x(0|h(t))$. In the lump-sum payoff case, $x(0|h(t))$ is fixed at 0, and the assumption allows firms to agree on $p(0|h(t))$.

We can dispense with the assumption with the cost of notational complexity. If the winning bid is not revealed, then since the poacher have better estimate about the worker's type (through the exact bid that attracted this worker), this introduces additional inference problem that requires outside firms to guess what this estimate is. We state here the following result that is true according to our model mechanism.

Lemma 1. Assume that $p_n(0|h(t))$ and $x_n(0|h(t))$ are common knowledge. Then, in equilibrium, the sufficient statistic for the outside firm's belief is τ_n, and the belief can be written as a deterministic function

$$x(\cdot|h(t)) : [0, \infty] \to [0, 1].$$

This will be our objective of interest. Here we state a brief summary of the notations regarding information structure, and proceed to the introduction of wage negotiation game.

Definition 2. The following notations are used in congruence with the information structure of the game:

- The history of a worker at age t is an element $h(t) \in \mathcal{H}(t) := \{(\tau_1, \dots, \tau_n; n) : \sum_{i=1}^{n} \tau_i = t\}$, and is a public information.
- Belief of an incumbent firm with private news, is either 1 or $p_n(\tau_n | h(t))$.
- $p_n(0|h(t))$, $x_n(0|h(t))$ are common knowledge; the belief of the outside firms is singleton, $x(\tau_n|h(t))$, which is the belief that the incumbent's belief is 1.

2.2 Wage Auction

While the worker is hired, he/she generates meeting with a new firm at Poisson rate λ. Although modelled as a random arrival of offers, it can be a new job offer that came across the applicant, which the applicant found it fit, or it can be a referral.

We make a simplifying and strong assumption on the wage determination process. We assume that, once the new firm contacts the worker, the two firms (incumbent and poacher) compete for the future service of the worker through a *common value first-price sealed-bid auction*. Furthermore, we assume that, by initiating this auction, the worker fully commits to accept the result of the auction, by shifting to whichever firm that offers higher bid.

It is worth mentioning why we need such assumption and why we think it is acceptable in terms of developing our main intuition. There has been multiple efforts to understand the role of job-to-job transitions play in the wage profile of a worker and in the aggregate, most notably, starting from [2] who showed that an equilibrium can support multiple (actually, a continuum) levels of wage when a worker is able to quit and shift to a new job without notifying the previous employer.

In reality, for many skilled jobs such as consulting, academia and high-tech industries, where quality of a worker mostly determines the productivity of a match, it is common for wage to be negotiated. [13] and [4] used the Bertrand Nash equilibrium of Rubinstein-bargaining of offers and counter offers as counter-offer interval goes to zero when they were decomposing the wage variations into firm specific and worker specific components, i.e., identifying the worker/firm types. [11] took into account the random shock and learning component into the model (following the literature of evolving match quality, as in [8], [7]) to characterize the aggregate distribution of job to job transitions and wage dynamics.

There are instances where wages in previous firms are fully observable, but there are also other instances where it is not reasonable, such as when the information has to rely on the worker's report (such as 'previous salary' box in job application, and disclosing W2) and California legislature recently mandated no disclosure policy. In any case a worker can hide some information about the past work history we believe it is appropriate to model the negotiation as a first price auction. We do require the worker's commitment, since, as we show in the discussion, the equilibrium may fully unravel when the worker do not commit to the result of an auction.

We first follow the setting and characterization of auction equilibrium in [5] and elaborate on applying the results to our environment.

Definition 3. Z is the value of the object, X is the private signal of an informed bidder, U is an independent uniform random variable on $[0,1]$. Let $H = E[Z|X]$.

Definition 4. Define the informed bidder's *distributional type* be $T = T(H,U)$, uniformly distributed on $[0,1]$, where

$$T(h,u) = Pr(\{H < h, \text{ or } H = h \text{ and } U < u\}).$$

Let

$$H(t) = \inf\{h|P(H \leq h) > t\}$$

to have $H = H(T)$ almost surely.

We solve for the equilibrium strategies $\beta : [0,1] \to \mathbb{R}_+$ for the informed bidder, and the bid distribution G for uninformed bidder.

Proposition 5 ([5])**.** The equilibrium bid distribution β of the informed bidder is,

$$\beta(t) = E[H(T)|T \leq t] = \frac{1}{t}\int_0^t H(s)ds$$

with $\beta(0) = H(0)$, and $\beta(1) = E[H]$.

The equilibrium bid distribution G of the uninformed bidder is

$$G(b) = Pr(\beta(T) \leq b).$$

Proof. Suppose the informed bidder type is $T = t$. If he bids $\beta(\tau)$, then he wins with probability τ, yielding an expected payoff

$$[H(t) - \beta(\tau)]\tau = \int_0^\tau (H(t) - H(s))\, ds$$

which is maximized at $\tau = t$.

For any uninformed bidder, any bid below $H(0)$ yields zero payoff, while bid greater than $E[H]$ generates negative payoff. Consider a bid $b = \beta(t)$. Its expected payoff is

$$E[Z - \beta(t)|T \leq t]t$$

However, $E[Z - \beta(t)|T \leq t] = E[H(T)|T \leq t] - \beta(t) = 0$. □

In our setting, the informed bidder receives a fully revealing signal about the value of an object (the employee): the firm's value takes on one of two possible numbers, depending on whether the bidder received information (arrival of α) or not. The Bayes rational belief about the firms' information x, corresponds to the distribution of the firm's signal. Formally:

Definition 6. The *symmetric-payoff auction game* of incomplete information at time t, when the outside firms attach beliefs $x = x(t)$ and $p = p(t)$, consists of

- Two firms (bidders): $\{1, 2\}$; Three players: $\{1H, 10, 2\}$.

  Comment: In our good news setting, bidder $1H$ stands for bidder 1 who received good news, bidder 10 for bidder 1 who did not receive any news at the time of the auction.

- Two ex-post valuations: $0 \leq V_0 < V_H$, common for both bidders,

- Signals: H is binary,

$$H = \begin{cases} V_H & \text{with probability } x \\ V_0 & \text{with probability } 1 - x \end{cases}$$

  Comment: The probability x in our model stands for the outside firm's belief about the bidder 1's knowledge about the worker's type.

- Bidder 1 is informed of the realization of the draw, while the bidder 2 is not,

- At time 1, bidders submit bids. The bidder with the highest bid wins and pays the bid to the worker.

Applying the result from [5]:

Proposition 7. The common-value auction game with asymmetric information has a unique Bayesian Nash equilibrium where:

1. The support of the bids is $[V_0, \tilde{V}]$, where $\tilde{V} = E[V] = xV_H + (1 - x)V_0$.

2. The distribution

$$H(t) = \begin{cases} V_0 & t \leq 1 - x \\ V_H & t > 1 - x \end{cases}$$

   implies

$$\beta(t) = \frac{1}{t} \int_0^t H(s)\, ds = \begin{cases} V_0, & t \leq 1 - x \\ \frac{1}{t}\Big((1 - x)V_0 + (t - (1 - x))V_H\Big), & t > 1 - x \end{cases}$$

   and,

$$G(b) = \frac{(1 - x)(V_H - V_0)}{V_H - b}, \quad b > V_0$$

Figure 1 depicts a particular pair of distributions. This result can be equally derived using indifference conditions as in Appendix 5.2. We note that the equilibrium strategies imply that with positive probability there are ex-post instances where bidder 2 regrets winning. This is the well-known "winner's curse" in auctions with common values and asymmetric information. However, in order for the bidder 2 to participate in the auction, and to bid non-trivial bids, there has to be some instances in which bidder 2 wins positive profit, which happens when $1H$ loses the auction.

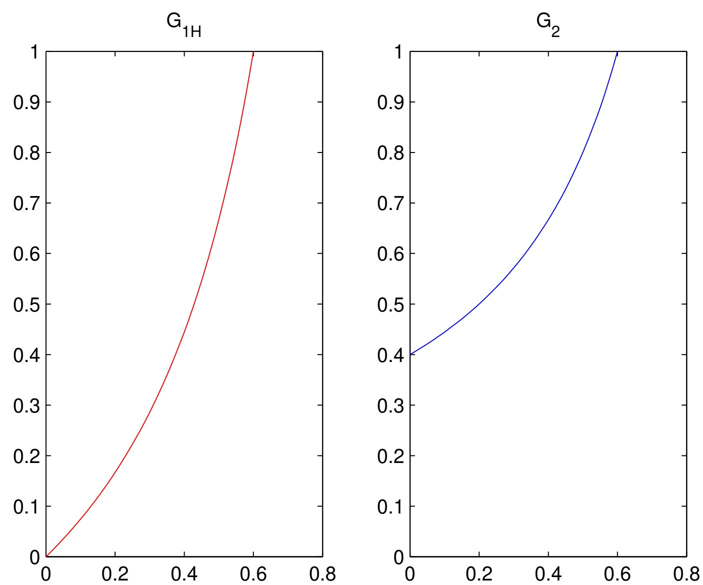

Figure 1: Distribution of Bids G_{1H} and G_2, with $x = 0.6$, $V_H = 1$, $V_0 = 0$

3 Equilibrium

We solve for equilibrium in dynamic environment. Firms discount future at rate $r > 0$. We show how the equilibrium objects, belief and values evolve. Although the underlying mechanism is similar for both models, we solve separately for two structures:

- **(Case 1) Flow payoff following a breakthrough:** When a worker generates a breakthrough, he starts producing flow output $y > 0$ onward, which is immediately observed by the new employer when hired.

- **(Case 2) Poisson arrival of lump-sum payoffs:** The good news payoff arrives intermittently, and every new employment relation restarts the learning process.

3.1 Case 1: Flow Payoff Model

In this case, a breakthrough in one firm is effectively shared by all firms that arrive later. Since no news translates to pessimistic belief about the worker's type, the p_n keeps decreasing with the time in market:

Claim 1.

$$p_n(\tau_n|h(t)) = p(t), \quad \text{given } p(0) = p_0$$

This asserts that the p_n is the function of the starting belief p_0 and time elapsed t only, and is not affected by other elements of the worker's history. Intuitively, if the worker made a breakthrough at some point, all subsequent employers of this worker know that this worker is of good type. Hence, it does not matter *where* the worker has made a breakthrough, but only *whether* the worker has made breakthrough until time t when the firms are updating their beliefs.

Unlike the belief p, the path of x may be discontinuous at the time of job-to-job transition. This is because in equilibrium, workers with no news $(1-x)$ are more likely to transit to another job than the ones who have generated news. However, we can claim that

Claim 2. $x_n(\cdot|h(t))$ is continuous and strictly increasing in $\tau_n \in [0, \infty)$.

Proof. Note that the argument in x_n is the tenure of this worker in a particular firm. For longer tenure τ, we have that there is more chance that the worker has generated good news. Furthermore, since no news workers leave the match faster than the good types, there is selection effect of longer tenure, and the belief is strictly increasing.

Continuity we establish from the continuity of Bayesian updating process and continuity of value functions. □

Claim 3. There is unique continuous, increasing path of public beliefs $x_n(\cdot|h(t))$ for $\tau_n \in [0, \infty)$ attached to a worker staying in this firm.

Proposition 8. Beliefs start from p_0 and $x(0) = 0$. The equilibrium consists of paths of beliefs $\{p(t), \{x_n(\cdot|h(t))\}\}$, paths of values $\{\Pi_1(t|p_0), \Pi_0(t|p_0)\}$, and the bidding strategies $\{G_{1H}(\cdot|t), G_{10}(\cdot|t), G_2(\cdot|t)\}$ such that:

- From $x_n(0|h(t))$, $\tilde{x}(\tau) = x_n(\tau|h(t+\tau))$ evolves according to the ODE:

  $$\tilde{x}'(s) = \Big(\int G_2(v|s+t)\, dG_{1H}(v|s+t)\Big)\lambda\tilde{x}(s)(1-\tilde{x}(s)) + \alpha(1-\tilde{x}(s))p(s+t)$$

  with initial condition

  $$\tilde{x}(0) = \frac{x_{n-1}(\tau_{n-1}|h(t))G_{1H}(v_n|t)}{x_{n-1}(\tau_{n-1}|h(t))G_{1H}(v_n|t) + (1 - x_{n-1}(\tau_{n-1}|h(t)))}$$

where v_n is the winning bid of bidder 2.

- $p(t)$ solves the initial value problem

$$p'(t) = -\alpha p(t)(1 - p(t)), \quad p(0) = p_0$$

- $\Pi_1(t|p_0)$ and $\Pi_0(t|p_0)$ solves the ODE's:

$$\begin{aligned} -\Pi_1'(t|p_0) + (r + \lambda)\Pi_1(t|p_0) &= y + \lambda \int (\Pi_1(t|p_0) - v) G_2(v|t)\, dG_{1H}(v|t) \\ &= y + \lambda(\Pi_1(t|p_0) - \bar{v}) \end{aligned} \tag{1H}$$

with transversality condition $\lim_{t\to\infty} \Pi_1(t|p_0) = \frac{y}{r+\lambda}$; and

$$-\Pi_0'(t|p_0) + (r + \lambda + \alpha p(t))\Pi_0(t|p_0) = \alpha p(t)\Pi_1(t|p_0) \tag{10}$$

- For all $v \in [\Pi_0(t|p_0), \bar{v}]$,

  $G_2(\cdot|t)$ equates the payoff of $1H$:

$$\Big(\Pi_1(t|p_0) - v\Big) G_2(v|t)$$

  while $G_{1H}(\cdot|t)$ equates the payoff of 2:

$$x_{n-1}(\tau_{n-1}|h(t))G_{1H}(v|t)(\tilde{\Pi} - v) + (1 - x_{n-1}(\tau_{n-1}|h(t)))(\tilde{\Pi} - v)$$

  where $\tilde{\Pi}$ starts with initial belief $p(t)$ and

$$x(t) = \frac{x_{n-1}(\tau_{n-1}|h(t))G_{1H}(v|t)}{x_{n-1}(\tau_{n-1}|h(t))G_{1H}(v|t) + (1 - x_{n-1}(\tau_{n-1}|h(t)))},$$

  and is given by

$$x(t)\Pi_1(t) + (1 - x(t))\Pi_0(p(t))$$

Note 1. These are all local conditions imposed governing the evolution of the values at time t. For detailed derivation of the conditions, readers can refer to the Appendix 5.3 where we invoke a limit argument from a discrete model as $dt \to 0$.

Note 2. Using the indifference condition, we write the firm's values as in $(1H')$ which is independent of the actual duration of the match. The values still do depend on the evolution of beliefs, but this is convenient because we can ignore the continuation probability when calculating discounted future payoffs, which is an equilibrium object.

3.2 Case 1: Autonomous Case with Recall Option

To solve for the equilibrium, it is necessary that we keep track of evolution of beliefs and the values at the same time. In this example, we impose an assumption which allows us to solve the beliefs in an autonomous system.

Assumption 2. *Recall option*: The incumbent can buy the worker back from the poacher by paying the value of the worker to the poaching firm, in case the poacher won the worker.

The incumbent is indifferent between paying out the future value of the worker and losing the worker. The poacher still has incentive to bid in the auction because there is positive probability that he might win over the high type worker and is 'reimbursed' by the incumbent. Using this trick, we have that the good news worker kept forever in a firm, while the no-news worker shifting to a new firm.

This assumption is convenient in two aspects. First of all, it pins down the auction outcome, because the high type workers are always kept by the incumbent firm, while the experimenting workers are yielded to the competitor. This means, at the start of a new relation (transfer), the starting belief of the knowledge of productivity in the new relation is

$$x_n(0|h(t)) = \tilde{x}(\tau_n = 0) = 0,$$

since the revealed good workers never leave the incumbent firm. On top of that, since the $1H$ bidder always wins the auction, the evolution of beliefs is

$$x'(t) = \lambda x(t)(1 - x(t)) + \alpha(1 - x(t))p(t).$$

The initial condition and the drift equation implies that the belief is pinned down autonomously without the knowledge of values.

The second reason why the assumption is convenient is because it restores symmetry of values in the auction game. Without the assumption, the starting belief of a new relationship does affect the future force of competition for the worker, and in turn, the profitability of the match. However, with the recall assumption, the poacher only bids to receive monetary benefit of being reimbursed the future value of the current match; therefore, upon winning a high type worker, the firm expects to receive

$$\Pi_1(t|p(t))$$

which is the same value for the $1H$ bidder. If the firm indeed wins the worker, the profit of the new relationship is

$$\Pi_0(0|p(t)),$$

where starting belief is $x(0) = 0$. Since the firm can win the low type worker with the bid

$\Pi_0(t|p(t))$, the expected profit of the firm is

$$(1 - x(t))(\Pi_0(0|p(t)) - \Pi_0(t|p(t))) \tag{1}$$

which has to be same for all the bids that the poacher is randomizing over. In this case, the indifference condition is given by

$$xG_{1H}(v|t)(\Pi_1(t|p(t)) - v) + (1 - x)(\Pi_0(0|p(t)) - v) = (1 - x)(\Pi_0(0|p(t)) - \Pi_0(t|p(t)). \tag{1H}$$

Since the highest bid must yield bidder 2 the profit in (1),

$$x(t)\Pi_1(t|p(t)) + (1 - x(t))\Pi_0(0|p(t)) - \bar{b} = (1 - x(t))(\Pi_0(0|p(t)) - \Pi_0(t|p(t)))$$

the support of the bids is

$$\Big[\Pi_0(t|p(t)),\ x(t)\Pi_1(t|p(t)) + (1 - x(t))\Pi_0(t|p(t))\Big]$$

which is simply, V_0 and $E[H]$ from the incumbent's perspective.

Since the upper bound of the support contains information about how much the incumbent has to pay in order to keep the match with probability 1, this feature allows us to write the incumbent's value also autonomously without requiring knowledge about the worker's value in a new match.

Evolution of Beliefs We have already established that $p(t)$ is only dependent on t and p_0. The unique $p(t)$ in this setting is given by

$$p(t) = \frac{p_0 e^{-\alpha t}}{p_0 e^{-\alpha t} + (1 - p_0)}$$

as in Section 2. Another way to see it using our transition probabilities is that both types of workers (good and bad) who have not generated news up until point t is moving to another firm. This implies that there is no difference in the transition rate for the both types of workers, and the composition of actual good workers in the pool of workers with no news do not diverge from the path of $p(t)$:

$$\begin{aligned} p_n(\tau_n|p(t)) &= \frac{p(t)e^{-(\alpha+\lambda)\tau_n}}{p(t)e^{-(\alpha+\lambda)\tau_n} + (1 - p(t))e^{-\lambda\tau_n}} \\ &= \frac{p_0 e^{-\alpha t} e^{-(\alpha+\lambda)\tau_n}}{p_0 e^{-\alpha t} e^{-(\alpha+\lambda)\tau_n} + (1 - p_0)e^{-\lambda\tau_n}} = p(t + \tau_n) \end{aligned}$$

We now characterize the $x_n(\tau_n|h(t))$. Since only the no-news workers shift to another firm, the market's expectation from observed transition at time t is

$$x_n(\tau_n = 0|h(t)) = 0.$$

Starting from 0, the belief x_n drifts up for two reasons: (1) arrival of news from so far no-news worker, and (2) survival bias, as no-news workers leave the firm with rate λ. Denote by $x(\tau)$ the firms' belief that the worker of tenure τ is a revealed good type. By Bayes rule, this is the fraction of surviving workers in this firm whose arrival of news was observed by the firm. Both the survival probability and the rate of learning affects the Bayesian inference. The probability is given by:[4]

$$\begin{aligned} x(\tau) &= \frac{p(t)\int_0^\tau e^{-(r+\lambda)s} d(1-e^{-\alpha s})}{p(t)\int_0^\tau e^{-(r+\lambda)s}\, d(1-e^{-\alpha s}) + p(t)e^{-(r+\alpha+\lambda)\tau} + (1-p(t))e^{-(r+\lambda)\tau}} \\ &= \frac{p(t)\frac{\alpha}{\alpha+\lambda}(1-e^{-(\alpha+\lambda)\tau})}{p(t)\frac{\alpha}{\alpha+\lambda}(1-e^{-(\alpha+\lambda)\tau}) + e^{-\lambda\tau}(p(t)e^{-\alpha\tau} + (1-p(t)))} \end{aligned}$$

Intuitively, only the subset of good workers who received good news (with probability $1-e^{-\alpha s}$ within duration s) before the first poaching attempt, are the revealed good workers in this firm. It is easily seen in the expression $\frac{\alpha}{\alpha+\lambda}$ that the order of arrival of two independent Poisson processes, rates α and λ, matter for the numerator. The denominator represents the probability of survival in this firm up until tenure τ, either by the arrival of news, or no poaching attempt ($e^{-\lambda\tau}$).

Given the belief path $p(t+\tau)$ for $\tau \geq 0$, $x(\tau)$ is the solution to the non-homogeneous DE with initial condition $x(0) = 0$:

$$x'(\tau) = \lambda x(\tau)(1-x(\tau)) + (1-x(\tau))\alpha p(t+\tau), \quad x(0) = 0$$

The equation admits an intuitive explanation for the two forces that are affecting the drift of x. The first part is the drift for a bad news process with no arrival, reflecting the fact that a worker with no news arrival are poached at rate λ. The second part denotes additional inflow into x from surviving no-news workers, transition happening at rate ($\alpha p(t+\tau)$). We attach a note in Appendix 5.5 verifying that the above equation is the solution to the ODE.

[4]Need to use

$$\begin{aligned} \int_0^\tau e^{-\lambda s}\alpha e^{-\alpha s}\, ds &= \frac{\alpha}{\alpha+\lambda}\int_0^\tau (\alpha+\lambda)e^{-(\alpha+\lambda)s}\, ds \\ &= \frac{\alpha}{\alpha+\lambda}\left(1-e^{-(\alpha+\lambda)\tau}\right) \end{aligned}$$

Value Functions Given the (exogenous) path of beliefs, p is solely the function of calendar time t, and x only possibly 'resets' with the start of a new tenure in a new firm. But once we know the starting belief $x(0)$ of the most recent tenure, τ, the system of ODEs we have to solve are expressed fully in terms of calendar time t:

$$p'(t) = -\alpha p(t)(1 - p(t)), \quad p(0) = p_0,$$

$$x'(t) = \lambda x(t)(1 - x(t)) + (1 - x(t))\alpha p(t), \quad x(t - \tau) = \tilde{x}(0) = x_0.$$

Therefore, the ODEs for values can also be written in terms of calendar time t only. Making use of this notation, the values that we need to solve for satisfies the system of equations:

- $$-\Pi_0'(t) + (r + \lambda + \alpha p(t))\Pi_0(t) = \alpha p(t)\Pi_1(t) \tag{10}$$

- $$-\Pi_1'(t) + (r + \lambda)\Pi_1(t) = y + \lambda \int (\Pi_1(t) - v) G_2(v|t)\, dG_{1H}(v|t) \tag{1H}$$

- Using the fact that $\bar{v} = x(t)\Pi_1(t) + (1 - x(t))\Pi_0(t)$, the second ODE can be rewritten

$$-\Pi_1'(t) + (r + \lambda)\Pi_1(t) = y + \lambda(1 - x(t))(\Pi_1(t) - \Pi_0(t)). \tag{1H'}$$

We characterize the solution of the system of ODEs, using the autonomous structure.

Proposition 9. Fix any time t of the new relationship, where p_0 is given and $x(0) = 0$. The value at time t that the incumbent firm has to pay in order to retain the worker for sure (i.e., upper bound of the bidding support) is given by

$$y \int_t^\infty e^{-(r+\lambda)(z-t)} x(z)\, dz,$$

which is the integral over future path of $x(z)$, $z \in [t, \infty)$.

Proof. Appendix 5.4 □

Although the result is intuitive, proof involves a few simplifying substitutions, which are mathematically tractible, but does not contain much intuition. Readers can refer to Appendix 5.4 for full proof. It turns out that we have to make use of the particular form of the belief drift equation, that cancels the system out nicely. However, we do believe that the intuition will largely carry over to the more general case, for which we establish the existence result below.

The full path of firm profits Π_1, Π_0 are obtained by plugging in this cost term. Note the intuitive explanation of the cost term, which, in the limit as the type of the worker is known

$x(z) = 1$, goes to

$$\frac{y}{r+\lambda}$$

which is the value of the relationship with termination rate $r + \lambda$.

Simulated Path of Wages Below is a simulated path of beliefs x and p for a particular realization of Poisson arrivals, when both λ and α arrives at rate $\frac{1}{3}$.

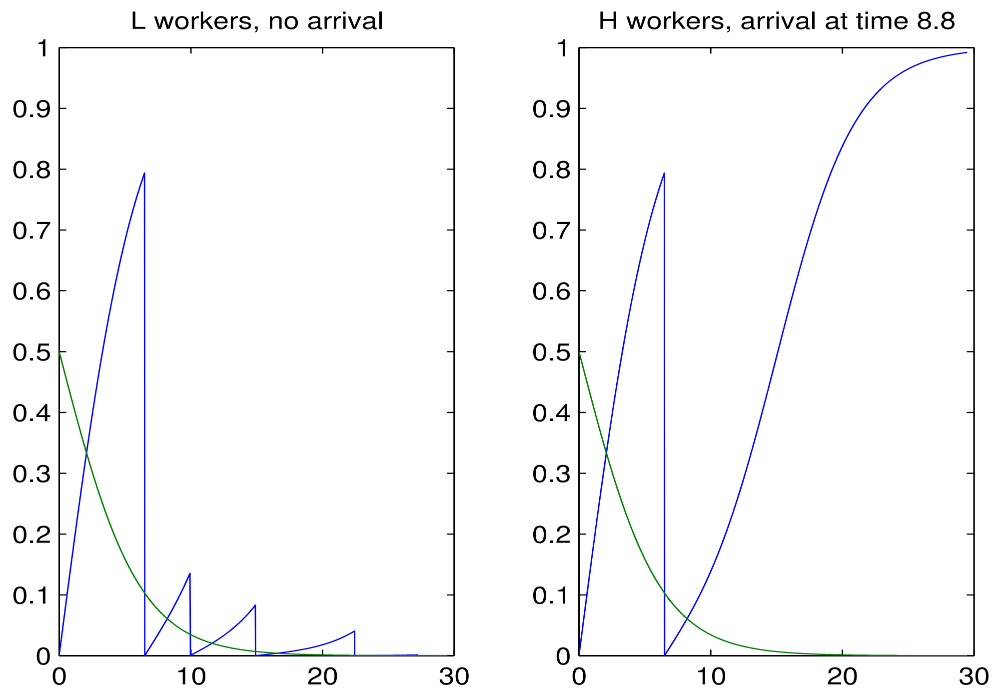

Figure 2: Simulated path of beliefs, x in blue and p in green. Initial belief set at $p_0 = 0.5$. Arrival of transition shocks, λ, shown by the kinks at the left Figure. For H workers, learning shock occured at time 8.8

For these two paths simulated for the length of 30, we see that the λ shocks affect the two types of workers differently. For L type workers, everytime the worker meets another firm, the worker transits to a new firm, which drives down the belief x back to 0. H type workers are not explicitly affected since there is no transition. However, the competitors infer postive news about the worker's type as the worker's tenure increases. Ultimately, the belief reaches the correct one, $x = 1$ for sufficiently long tenure.

We now investigate how the belief path translates to wages paid to the worker. It is useful to recall the retaining cost term, which we wrote as

$$C(t) = y \int_t^{\infty} e^{-(r+\lambda)(s-t)} x(s)\, ds$$

which, starting from any initial belief p_0 about the match and $x(0) = 0$,

$$x(s) = \frac{p_0 \frac{\alpha}{\alpha+\lambda}(1 - e^{-(\alpha+\lambda)s})}{p_0 \frac{\alpha}{\alpha+\lambda}(1 - e^{-(\alpha+\lambda)s}) + e^{-\lambda s}(p_0 e^{-\alpha s} + (1 - p_0))}$$

for the exogenous drift of beliefs we can pin down from the starting belief $x(t)$ and $p(t)$ at instant t. These are calculated from the last observed transition, and is a function of tenure. We solve for this integral and normalize for the termination rate of the match

$$(r + \lambda)C(t)$$

to be the proxy for the constant wage paid to the worker. Note that this is an upper bound of the actual wage paid since actual wage is determined by the outcome of the auction. We graph a corresponding sample wage paths below:

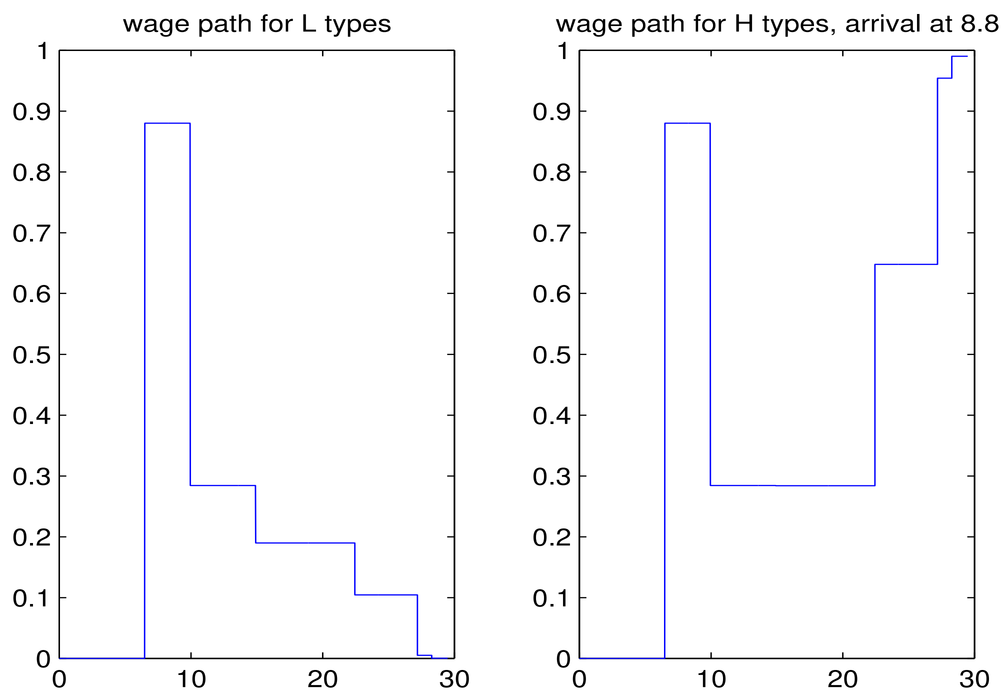

Figure 3: Sample Wage Paths for the Two Types of Workers

The wage paths start to diverge after the arrival of news: for the H type workers, longer duration of stay in the match signals outside firms that the worker is of high quality, and the wage gets to be driven up to the productivity of the match, y. For the L types, although they might initially draw high wage by luck, the wage converges to 0 as the work history drives belief down close to 0.

Dispensing with the Assumption We briefly discuss the consequences of dropping the recall assumption. Without the recall assumption, the value of winning might change because the winning the auction itself confers non-trivial information (with the recall assumption, every transition is an indication that no news arrived) about the type of the worker, which in turn depends on the equilibrium outcome of the auction.

In the symmetric payoff case, the probability that the $1H$ type wins the auction is given by $1 - \frac{1}{2}x$, while 10 type loses the auction for sure. This implies that, assuming symmetric values, Bayes rational inference about the worker's type in the beginning of the new match is:

$$\tilde{x}(0) = \frac{x(\tau|h(t))\big(\frac{1}{2}x(\tau|h(t))\big)}{x(\tau|h(t))\big(\frac{1}{2}x(\tau|h(t))\big) + (1 - x(\tau|h(t)))}$$

It is possible that the change in the starting belief induce the auction payoff to be asymmetric. The fixed point we have to solve for satisfies, the probability of winning, and Bayesian inference derived by from the auction equilibrium, while the values are consistent with the starting belief generated by the Bayesian inference. We later return to the discussion of the existence issue later in this section.

When we assume that the poacher's bid after winning the auction is observed, then we can show that the bids must be distributed over the interval:

$$\Big[\Pi_0(t),\, x(t)\Pi_1(t) + (1 - x(t))\Pi_0(t) - (1 - x(t))(\tilde{\Pi}_0(x(0|h(t)) = 0|p(t)) - \Pi_0(t))\Big].$$

This is due to the indifference condition. The poacher can generate the payoff of

$$(1 - x(t))(\tilde{\Pi}_0(x(0|h(t)) = 0|p(t)) - \Pi_0(t))$$

by bidding $\Pi_0(t)$, in which case the firm will attract only the no-news workers. If the poacher bids the largest possible bid, other firms can infer that the poacher bought off the entire match, whose future expected value is

$$x(t)\Pi_1(t) + (1 - x(t))\Pi_0(t),$$

which is the expected value to the incumbent. Since the poacher has to be indifferent over all bids, it must be that the highest possible bid $\bar{b}$ is given as above. Plugging in the $\bar{b}$ to the incumbent's value function, the ODE of values are not autonomous anymore, because, from the ODE of Π_1:

$$-\Pi_1'(t) + (r+\lambda)\Pi_1(t) = y + \lambda(\Pi_1(t) - \bar{b})$$
$$= y + \lambda\Big((1-x(t))(\Pi_1(t) - \Pi_0(t)) + (1-x(t))(\underbrace{\tilde{\Pi}_0(x(0|h(t)) = 0)}_{*} - \Pi_0(t))\Big)$$

we need information on $(*)$, which is the value of a new match starting from initial belief $x(0|h(t)) = 0$ and $p(t)$. We can no longer solve for the values $\Pi_1(t)$, $\Pi_0(t)$, within the system by manipulation alone.

3.3 Case 2: Lump-Sum Payoff Model

Equilibrium Beliefs If we assume that payoff arrives in lump-sum fashion with arrival rate α, the new employer do not immediately observe if the worker is H type. That is, with the start of the new tenure, x drops down to 0:

$$x_n(0|h(t)) = 0,$$

while the initial belief is now untrivial and depends on the expectation of the auction outcome. For this setting it is convenient to rewrite all beliefs as functions of tenure in this firm, $\tilde{p}$, and $\tilde{x}(\tau) = x_n(\tau|h(t))$, starting from initial belief $\tilde{p}(0) = p(t-\tau_n)$ and $\tilde{x}(0) = 0$:

Definition 10.
$$\tilde{p}(\tau_n) = p_n(\tau_n|h(t)) = p(t), \quad \text{while } \tilde{p}(0) = p(t-\tau_n).$$

Starting from the initial belief $\tilde{p}(0)$, the belief evolves according to:

$$\tilde{p}'(\tau) = -\alpha\tilde{p}(\tau)(1-\tilde{p}(\tau)),$$

$$x'(\tau) = \lambda Pr(win)x(\tau)(1-x(\tau)) + \alpha(1-x(\tau))\tilde{p}(\tau), \quad x(0) = 0$$

Values Given these beliefs, the value equations are derived in a similar manner:

- $$-\Pi_1'(\tau) + (r+\lambda)\Pi_1(\tau) = \alpha y + \lambda \int (\Pi_1(\tau) - v)G_2(v)\, dG_{1H}(v) \tag{1H}$$

- $$-\Pi_0'(\tau) + (r+\lambda+\alpha p(\tau))\Pi_0(\tau) = \alpha p(\tau)(y + \Pi_1(\tau)) \tag{10}$$

Again, we would like to write $(1H)$ in constant exit-rate form. The following assumption is convenient for the purpose:

Assumption 3. When the worker switches to a new firm, the winning bid is disclosed to public.

This is a technical assumption that allows us to easily characterize the auction equilibrium. Given this assumption, the poacher can guarantee the payoff of

$$(1-x(\tau))\big(\Pi_0(p(\tau))-\Pi_0(\tau)\big)$$

by bidding $\Pi_0(\tau)$ and only attracting the workers with no news. We postpone the discussion on the conditions to guarantee this, but under the presumption that a new relationship with starting belief p, $\Pi_0(p)$ is increasing in p, we claim that the maximum bid $\bar{b}$ is when the poacher buys out the entire match, with starting belief

$$\tilde{x}(\tau)=\frac{x(\tau)+(1-x(\tau))p(\tau)}{x(\tau)+(1-x(\tau))}=x(\tau)+(1-x(\tau))p(\tau)$$

while the indifference condition gives us the condition for the bid $\bar{b}$:

$$\Pi_0(\tilde{x}(\tau))-\bar{b}=(1-x(\tau))\big(\Pi_0(p(\tau))-\Pi_0(\tau)\big)$$

hence, plugging in $\bar{b}$:

$$-\Pi_1'(\tau)+(r+\lambda)\Pi_1(\tau)=\alpha y+\lambda\Big((\Pi_1(\tau)-\Pi_0(\tilde{x}(\tau))+(1-x(\tau))\big(\Pi_0(p(\tau))-\Pi_0(\tau)\big)\Big) \quad \text{(1H')}$$

Convenient Expressions Π_0 starting from initial belief $\tilde{p}(0)$ is given by:

$$\Pi_0(0|\tilde{p}(0))=\int_0^\infty e^{-\int_0^s(r+\lambda+\alpha\tilde{p}(z))dz}\alpha\tilde{p}(s)\Big(y+\Pi_1(s|\tilde{p}(0))\Big)\,ds,$$

where $\Pi_1(s|\tilde{p}(0))$ is the value after learning the type of the worker, evaluated over the Bayes rational belief path starting from $\tilde{p}(0)$. We note that Π_0 is the expected future value from transition to the learned state Π_1 and the lump-sum payoff that arrives at rate α in case the worker is indeed $High$ quality.

The expression can be simplified to:

$$\Pi_0(0|\tilde{p}(0))=\alpha\tilde{p}(0)\int_0^\infty e^{-(r+\lambda+\alpha)s}\Big(y+\Pi_1(s)\Big)\,ds.^5$$

Intuitively, Π_0 is the expected value of transition to Π_1 at rate α, with effective discount rate $\lambda+r$, coming from the dissolution of the match at rate λ. The good news drift allows us to replace the effect from change in transition rate over time, $\alpha\tilde{p}(s)$, with the initial probability $\tilde{p}(0)$ and constant rate of transition α.

[5]Due to our good news learning assumption, the rate of transition at tenure τ, $\alpha\tilde{p}(\tau)$, and the evolution of

Now we move on to the value of learning the good type at tenure s, $\Pi_1(s)$. General solution to the ODE (1H):

$$\Pi_1(s) = \frac{\alpha y}{r} - \lambda \int_s^\infty e^{-r(t-s)} \Big(\Pi_0(\tilde{x}(t)) - (1 - x(t))(\Pi_0(p(t)) - \Pi_0(t)) \Big)\, dt$$

That is, Π_1 is the discounted future stream of payoff $\frac{\alpha y}{r}$ minus the cost of retaining the worker by bidding the maximum bid in the support. The cost term sums the value of this relationship in a new match, $\Pi_0(\tilde{x}(s))$, net the poacher's outside option, which lowers the maximum willingness to bid for the relationship $(1 - x(t))(\Pi_0(p(t)) - \Pi_0(t))$. Note that we need to solve for the entire values of $\Pi_0(\cdot)$ simultaneously with the path of beliefs $\tilde{x}(s)$, because it determines the 'market value' of the match.

3.4 Case 2: Stationary Example

In this section we solve for a toy case where the values are stationary. We solve for Markov equilibrium in which the aforementioned two beliefs are state variables.

In this setting, we adopt a more strict interpretation of λ: along with inviting a competitor, the shock also terminates this match. Once the match terminates, there is a new auction for the worker's service until the arrival of the next shock. The auction game is played by informed bidder ('successor firm') and uninformed bidder ('poacher').

The 'successor' firm can be thought of as a different department within the same firm that wants to retain the worker. This can be justified in several ways. In many cases, a worker's resume also contains descriptions of job duties and assignments. Various studies also suggest that one of the key qualifications in climbing up the job ladder is the breadth of experience in a firm.

Effectively, after every λ shock, there is a new auction game between two bidders; one informed and the other uninformed. The symmetric auction result directly applies, and the equilibrium values are given by

Proposition 11. Value of a match with starting belief p, which we denote by $\Pi_0(p)$, is $p\frac{\alpha y}{r+\lambda}$. With the observation of arrival, the value jumps to $\Pi_1 = \frac{\alpha y}{r+\lambda}$.

Informed bidder's value for the worker is either Π_1 with probability x, or $\Pi_0(p)$ with probability $1 - x$. We show that following is the equilibrium of the auction game: **(What**

$\tilde{p}(\tau)$ exactly cancels out in the integral:

$$\begin{aligned}
\alpha\tilde{p}(s)e^{-\int_0^s \alpha\tilde{p}(z)\,dz} &= \alpha\tilde{p}(s)e^{-\alpha s + \int_0^s \alpha(1-\tilde{p}(z))\,dz} \\
&= \alpha\tilde{p}(s)e^{-\alpha s - \int_0^s \frac{\tilde{p}'(z)}{\tilde{p}(z)}\,dz} \\
&= \alpha\tilde{p}(s)e^{-\alpha s}\frac{\tilde{p}(0)}{\tilde{p}(s)} = \alpha\tilde{p}(0)e^{-\alpha s}
\end{aligned}$$

are the steps to show uniqueness?)

Proposition 12. The bids come from the support

$$\big[\Pi_0(p),\, \Pi_0(x+(1-x)p)\big],$$

the distribution of bidder $1H$ bids, G_{1H} solves indifference condition for bidder 2:

$$\Pi_0\big(\tfrac{xG_{1H}(v)+(1-x)p}{xG_{1H}(v)+(1-x)}\big) - v = \frac{xG_{1H}(v)+(1-x)p}{xG_{1H}(v)+(1-x)}\frac{\alpha y}{r+\lambda} - v = 0, \quad \text{for all } v$$

bidder 2's bids G_2 solves the indifference condition for bidder $1H$:

$$G_2(v)\big(\Pi_1 - v\big) = \Pi_1 - \Pi_0(x+(1-x)p), \quad \text{for all } v.$$

The equilibrium bid distributions are differentiable almost everywhere, with no atoms except at the lower bound of the support. We can prove that:

Proposition 13. The probability that $1H$ bidder wins over 2 is

$$1 - \frac{1}{2}x.$$

Proof. Appendix 5.6. □

Hence, the consistent drift of beliefs:

Proposition 14. The pair of beliefs (x, p) as function of tenure solves the following ODE's:

$$x'(t) = \lambda\Big(1 - \frac{1}{2}x(t)\Big)x(t)(1-x(t)) + \alpha(1-x(t))p(t)$$

As a function of tenure t, from a starting belief $p(0)$ and $x(0) = 0$.

$$p(t) = \frac{p(0)e^{-\alpha t}}{p(0)e^{-\alpha t} + (1-p(0))}$$

Simulated Wage Paths Since the path of beliefs is autonomous, an individual's career path can be readily simulated starting from any outcome of an auction, the bids from which governing the starting belief.

Figures 4, 5 depict a simulated path of beliefs and wages. We see that for high quality H workers, the belief gradually converges to his/her true type, 1, while for low quality L workers, the belief converges to 0. However, note that the belief about H worker's type jumps down whenever there is a shift to a new firm, since no news workers are more likely to leave the match (this is a non-conclusive bad news); while the L workers also exhibit upward belief drift between any two transition shocks.

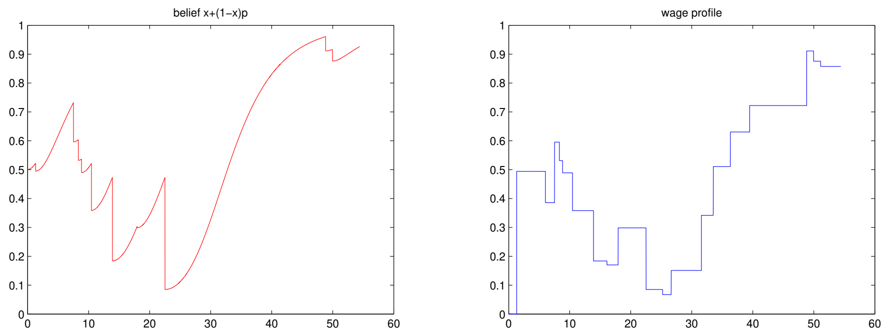

Figure 4: Simulated Belief and Wage Paths for Case 2

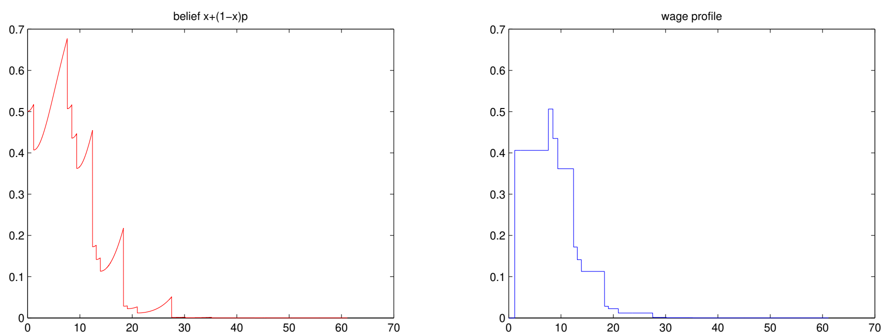

Figure 5: Simulated Belief and Wage Paths for Case 2

3.5 Existence of Equilibrium

Proving existence of equilibrium involves solving for fixed point where the profit functions along belief paths $(\Pi_1(t|p_0), \Pi_0(t|p_0))$ are defined in congruence with the evolution of beliefs (x, p), and outside option $\Pi_0(\cdot)$ with any starting beliefs $p \in [0,1]$. Unlike in Example 1, there are infinitely many differential equations to keep track of simultaneously.

The problem seems daunting at first glance, but we impose an appropriate terminal condition. From the terminal condition, the set of ODEs just is a large set of initial value problems, and the existence of solution can be guaranteed accordingly.

4 Conclusion

5 Appendix

5.1 Good News Drift

Starting from initial probability $p(t) \in [0,1]$, the probability that no news arrives in dt interval is given by:

$$P(H)Pr(\text{no news}|H) + P(L)Pr(\text{no news}|L) = p(t)(1-\alpha dt) + (1-p(t)).$$

Applying Bayes rule, $p(t+dt)$ is given by:

$$p(t+dt) = \frac{p(t)(1-\alpha dt)}{p(t)(1-\alpha dt) + (1-p(t))}.$$

Subtracting $p(t)$ from both sides:

$$\begin{aligned} p(t+dt) - p(t) &= \frac{p(t)(1-p(t))(1-\alpha dt) - p(t)(1-p(t))}{p(t)(1-\alpha dt) + (1-p(t))} \\ &= \frac{p(t)(1-p(t))(-\alpha dt)}{p(t)(1-\alpha dt) + (1-p(t))}. \end{aligned}$$

Dividing both sides by dt and taking limit $dt \to 0$,

$$p'(t) = -\alpha p(t)(1-p(t)).$$

In general, this is true even if match dissolves with some positive rate, as long as the rate for H and L types are identical. Starting from any $p_0 = p(t)$, and using the Bayes' rule, the belief after dt is given by

$$p(t+dt) = \frac{p(t)(1 - rdt - \alpha dt - \lambda dt)}{p(t)(1 - rdt - \alpha dt - \lambda dt) + (1-p(t))(1 - rdt - \lambda dt)}$$

$$p(t+dt)-p(t)=\frac{p(t)(1-p(t))(1-rdt-\alpha dt-\lambda dt)-p(t)(1-p(t))(1-\lambda dt)}{p(t)(1-rdt-\alpha dt-\lambda dt)+(1-p(t))(1-rdt-\lambda dt)}$$

$$p'(t)=-\alpha p(t)(1-p(t)),\quad p(0)=p_0$$

This is true as long as both the high and low types leave at the same rate λ.

5.2 Common Value Auction: Constructive Proof

In this Appendix, we show that in equilibrium:

- 10 submits degenerate bid V_0. $1H$ and 2 submit randomized bids over support $[V_0, \tilde{V}]$, where $\tilde{V} = xV_H + (1-x)V_0$.

- Bidding distributions for the three players are

$$G_{1H}(b)=\frac{1-x}{x}\frac{b-V_0}{\tilde{V}-b}$$

$$G_2(b)=\frac{(1-x)(V_H-V_0)}{(V_H-b)}$$

  G_{10} is degenerate at V_0.

Claim 4. There does not exist a pure strategy NE of this auction game.

Proof. The value of the object at sale is at least V_0.

Any bid $b < V_0$ is not played in equilibrium because any opponent player can make profitable deviation to b' with $b < b' < V_0$.

Suppose bidder 2 always bids $b = V_0$. Then bidder $1H$ has profitable deviation for any bid $b' > V_0$, from which he can bid slightly less, $b' > b'' > V_0$.

Suppose bidder 2 bids $V_0 < b \leq xV_H + (1-x)V_0$. Bidder $1H$'s best response is to bid slightly above b and win for sure. In which case, the bidder 2 expects to gain negative profit $V_0 - b < 0$, and would rather get 0 payoff by bidding V_0.

The bidder 2 never bids above $xV_H + (1-x)V_0$ in equilbrium. Since bidder 10's best response to $b > V_0$ of bidder 2 is to bid below b, bidder 2 always wins over 10 in equilibrium. Hence his maximum willingness to pay for the object is $xV_H + (1-x)V_L$. □

Therefore, if there is an equilibrium, it has to be in mixed strategies, the support is given by the following claim.

Claim 5. In equilibrium, bidder 2's bid distribution G_2 is continuous and strictly increasing over $(V_0, xV_H + (1-x)V_0]$.

Proof. First, note that given the candidate bidder 2's strategy, bidder 10 never puts positive probability on bids above V_0. Therefore, the strategy is best response to bidder $1H$'s. Furthermore, $1H$ and 2 shares the same support. We can also rule out any atom in the interior of the support, otherwise there is profitable deviation. We can also rule out G_2 having atom at $\tilde{V}$ because $1H$ would deviate. □

In equilibrium, $1H$ and 2 mixes over bids to make the opponent indifferent over all bids on the support. That is, G_{1H} solves:

$$xG_{1H}(b)(V_H - b) + (1-x)(V_0 - b) = 0$$

and G_2 solves:

$$G_2(b)(V_H - b) = (V_H - \tilde{V}) = (1-x)(V_H - V_0).$$

When ties are resolved with the toss of a fair coin, the winning probabilities are:

$$\begin{cases} \frac{1}{2}x + (1-x) & \text{for } 1H \\ \frac{1}{2}(1-x) & \text{for } 10 \\ \frac{1}{2} & \text{for } 2 \end{cases}$$

Conditional on type realizations, H or L, probability that bidder 2 wins is

$$Pr(2 \text{ wins}|H) = \frac{1}{2}x, \quad Pr(2 \text{ wins}|0) = x + \frac{1}{2}(1-x)$$

That is, bidder 2's expected payment is

$$\frac{1}{2}x^2 V_H + (x(1-x) + \frac{1}{2}(1-x)^2)V_0$$

Bidder $1H$ expects to pay

$$\frac{1}{2}xV_H + (1-x)V_0$$

Bidder 10: $\frac{1}{2}(1-x)V_0$.

5.3 Equilibrium Conditions for Case 1

Beliefs Time t stands for tenure in the firm. The differential equation is obtained using Bayes rule:

$$x(t+dt) = \frac{x(t)\big(1 - \lambda dt + \lambda dt Pr(1H \text{ wins}|t)\big) + (1-x(t))p(t)\alpha dt}{x(t)\big(1 - \lambda dt + \lambda dt Pr(1H \text{ wins}|t)\big) + (1-x(t))(1 - \lambda dt + \lambda dt Pr(10 \text{ wins}|t))}$$

where

$$Pr(1H \text{ wins}|t) = \int G_2(v|t)\, dG_{1H}(v|t).$$

For simplicity, assume that in case of a tie, 10 worker shifts to the new firm. In which case, $Pr(10 \text{ wins}|t)$ is 0, and

$$x(t+dt)-x(t) = \frac{x(t)(1-x(t))\big(1-\lambda dt+\lambda dt Pr(1H \text{ wins}|t)\big)+(1-x(t))(p(t)\alpha dt - x(t)(1-\lambda dt)\big)}{x(t)\big(1-\lambda dt+\lambda dt Pr(1H \text{ wins}|t)\big)+(1-x(t))(1-\lambda dt)}$$

Dividing by dt and taking $dt \to 0$:

$$x'(t) = \lambda Pr(1H \text{ wins}|t)x(t)(1-x(t)) + \alpha(1-x(t))p(t)$$

Values It is informative to look at the discrete version of the value equations:

$$\Pi_1(t|p_0) = ydt + (1-rdt-\lambda dt)\Pi_1(t+dt|p_0) + \lambda dt \int (\Pi_1(t+dt)-v)G_2(v|t+dt)dG_1(v|t+dt)$$

where the recursive expression $\Pi_1(t+dt)$ takes into account the the location of the future beliefs, which is taken as exogenous by the firm at the time of auction.

Since $\bar{v}$ wins the auction game for sure, substituting the indifference condition:

$$\Pi_1(t|p_0) = ydt + (1-rdt-\lambda dt)\Pi_1(t+dt|p_0) + \lambda dt(\Pi_1(t+dt)-\bar{v})$$

Subtracting both sides by $\Pi_1(t|p_0)$ and dividing by dt:

$$0 = y + \Pi_1'(t|p_0) - (r+\lambda)\Pi_1(t|p_0) + \lambda(\Pi_1(t|p_0)-\bar{v}). \tag{1H'}$$

Since this holds for any t, integrating it for $[s,\infty)$ after mutiplying by $e^{-(r+\lambda)t}$ yields

$$0 = \frac{y}{r+\lambda}e^{-(r+\lambda)s} - e^{-(r+\lambda)t}\Pi_1(t|p_0) + \int_t^\infty e^{-(r+\lambda)s}(\Pi_1(s|p_0)-\bar{v}(s))\,ds.$$

Similarly, for Π_0, the value it derives are from expected value of transition to state Π_1,

$$\Pi_0(t) = (1-r\,dt-\lambda dt-\alpha p(t)dt)\Pi_0(t+dt) + \alpha p(t)\,dt\Pi_1(t+dt),$$

$$0 = \Pi_0'(t|p_0) - (r+\lambda+\alpha p(t))\Pi_0(t|p_0) + \alpha p(t)\Pi_1(t|p_0) \tag{10}$$

5.4 Proof of Proposition 9

Define

$$D(t) = \Pi_1(t) - \Pi_0(t).$$

The system of profit equations can be subtracted to yield the following ODE in terms of D only:

$$-D'(t) + (r + \lambda)D(t) = y + (\lambda(1 - x(t)) - \alpha p(t))D(t)$$

$$-D'(t) + (r + \lambda x(t) + \alpha p(t))D(t) = y$$

Plugging in the drift equation (This is where the recall assumption comes in handy, in terms of math)

$$-D'(t) + (r + \frac{x'(t)}{1 - x(t)})D(t) = y$$

$$-\frac{d}{dt}e^{-rt}(1 - x(t))D(t) = e^{-rt}(1 - x(t))y$$

Integrating

$$e^{-rt}(1 - x(t))D(t) = \int_t^\infty e^{-rs}(1 - x(s))y\, ds$$

Plugging this information into $(1H)$:

$$e^{-(r+\lambda)t}\Pi_1(t) = \frac{y}{r + \lambda}e^{-(r+\lambda)t} + \lambda \int_t^\infty e^{-(r+\lambda)s} \int_s^\infty e^{-r(z-s)}(1 - x(z))y\, dz\, ds$$

Changing the order of integration:

$$e^{-(r+\lambda)t}\Pi_1(t) = \frac{y}{r + \lambda}e^{-(r+\lambda)t} + \int_t^\infty \left(e^{-\lambda t} - e^{-\lambda z}\right)e^{-rz}(1 - x(z))y\, dz - \int_t^\infty e^{-r(s-t)}(1 - x(s))y\, ds$$

$$\Pi_1(t) = \frac{y}{r + \lambda} + \int_t^\infty \left(1 - e^{-\lambda(z-t)}\right)e^{-r(z-t)}(1 - x(z))y\, dz$$

Use this to back out the $C(t) = x(t)\Pi_1(t) + (1 - x(t))\Pi_0(t) = \Pi_1(t) - (1 - x(t))D(t)$:

$$\begin{aligned}
C(t) &= \Pi_1(t) - (1 - x(t))D(t) \\
&= \frac{y}{r + \lambda} + \int_t^\infty \left(1 - e^{-\lambda(z-t)}\right)e^{-r(z-t)}(1 - x(z))y\, dz - \int_t^\infty e^{-r(s-t)}(1 - x(s))y\, ds \\
&= \frac{y}{r + \lambda} - \int_t^\infty e^{-(r+\lambda)(z-t)}(1 - x(z))y\, dz \\
&= y \int_t^\infty e^{-(r+\lambda)(z-t)}x(z)\, dz
\end{aligned}$$

5.5 Verifying the Solution x in Case 1 Recall Example

This is a special case where $1H$ never loses the worker. Plugging in $Pr(1H \text{ wins}|t) = 1$ in the equation, we obtain the ODE. To verify that the solution to the ODE is indeed as above, we check our algebra by plugging in the expression for x into the differential equation:

$$x'(t) = \frac{p_0 \alpha e^{-(\alpha+\lambda)t}}{A(t)} - \frac{p_0 \frac{\alpha}{\alpha+\lambda}(1 - e^{-(\alpha+\lambda)t}) A'(t)}{A(t)^2},$$

where

$$A(t) = p_0 \frac{\alpha}{\alpha+\lambda}(1 - e^{-(\alpha+\lambda)t}) + e^{-\lambda t}(p_0 e^{-\alpha t} + (1 - p_0)),$$

and

$$\begin{aligned} A'(t) &= p_0 \alpha e^{-(\alpha+\lambda)t} - p_0(\alpha+\lambda)e^{-(\alpha+\lambda)t} - \lambda(1-p_0)e^{-\lambda t} \\ &= -\lambda p_0 e^{-(\alpha+\lambda)T} - \lambda(1-p_0)e^{-\lambda t}. \end{aligned}$$

Meanwhile,

$$1 - x(t) = \frac{p_0 e^{-(\alpha+\lambda)t} + (1-p_0)e^{-\lambda t}}{A(t)},$$

hence,

$$\begin{aligned} \frac{x'(t)}{1-x(t)} &= \frac{p_0 \alpha e^{-(\alpha+\lambda)t}}{p_0 e^{-(\alpha+\lambda)t} + (1-p_0)e^{-\lambda t}} - \frac{p_0 \frac{\alpha}{\alpha+\lambda}(1 - e^{-(\alpha+\lambda)t})(-\lambda)}{A(t)} \\ &= \alpha p(t) + \lambda x(t) \end{aligned}$$

5.6 Equilibrium Outcomes, Case 2 Example

As before, the G_2 is given by

$$G_2(v) = \frac{\Pi_1 - \Pi_0(x + (1-x)p)}{\Pi_1 - v} = \frac{(1-x)(1-p)\frac{\alpha y}{r+\lambda}}{\frac{\alpha y}{r+\lambda} - v}$$

with atom $1 - x$ at the lower bound, $p\frac{\alpha y}{r+\lambda}$. G_{1H}, on the other hand:

$$G_{1H}(v) = \frac{1-x}{x}\Big(\frac{1-p}{1 - v\frac{r+\lambda}{\alpha y}} - 1\Big)$$

Since the value and beliefs are one-to-one, it is convenient to write in terms of beliefs $\tilde{p} = \frac{r+\lambda}{\alpha y} v$ and calculate the probability that the $1H$ type wins over 2, given by:

$$\int_p^{x+(1-x)p} G_2(\tilde{p})\, dG_{1H}(\tilde{p}) = \int_p^{x+(1-x)p} \frac{(1-x)(1-p)}{1-\tilde{p}} \frac{1-x}{x} \frac{(1-p)}{(1-\tilde{p})^2}\, d\tilde{p}$$

After some algebra,[6] we get the probability $\frac{1}{2}(2-x) = 1 - \frac{1}{2}x$.

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[6] The antiderivative of $\frac{1}{(1-\tilde{p})^3}$ being $\frac{1}{2(1-\tilde{p})^2}$, the definite integral that is to be multiplied by $\frac{(1-x)^2(1-p)^2}{x}$ is

$$\frac{1}{2}\frac{1}{(1-\tilde{p})^2}\Big|_p^{x+(1-x)p} = \frac{1}{2}\Big(\frac{1}{(1-x)^2(1-p)^2} - \frac{1}{(1-p)^2}\Big) = \frac{1}{2}\Big(\frac{1-(1-x)^2}{(1-x)^2(1-p)^2}\Big) = \frac{1}{2}\frac{x(2-x)}{(1-x)^2(1-p)^2}$$

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