Measuring Higher-Order Uncertainty: Evidence from a Consensus Pricing Service*

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Abstract

This paper estimates a structural model of learning from prices. The aim is to quantify valuation uncertainty among market makers in the over-the-counter market for S&P 500 index options. The analysis is based on a unique dataset of price estimates that major financial institutions provide to a consensus pricing service for over-the-counter derivatives. We consider two dimensions of uncertainty: uncertainty about fundamental asset values and strategic uncertainty about competitors' valuations. We construct novel empirical measures of fundamental and strategic uncertainty based on model-implied posterior beliefs of market participants. We show that both dimensions of valuation uncertainty vary substantially across the different segments of the S&P 500 index options market. Furthermore, we find that the consensus pricing service helps to significantly reduce strategic valuation uncertainty among market participants.

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1 Introduction

Many important financial markets suffer from a lack of publicly available price data. Often intransparent market structures are the root cause of this problem. Most financial derivatives, for example, are traded over-the-counter, that is directly between the two trading partners without the intermediation of an exchange. In such over-the-counter markets transaction prices tend to be proprietary information of the trading partners and are typically not shared with other market participants. But even for markets that provide good price transparency in normal times, data availability can become problematic during periods of stress when transaction volumes dry up. Lowenstein (2000) provides a vivid account of such a situation in the bond market at the height of the LTCM crisis on August 31, 1998:

> "It was as if a bomb had hit; traders looked at their screens, and the screens stared blankly back. [...] So few issues traded, you had to guess where they were." (p.159)

Clearly, the lack of reliable price signals can increase market participants' uncertainty about asset values as well as their uncertainty about other market participants' trading intentions. As a market response to such data availability issues, information aggregation services have sprung up in many of the most affected financial markets. These services are typically referred to as pricing or benchmarking services. They collect price estimates from market participants and aggregate these price data into a so-called consensus prices. The consensus price is intended to reflect the current state of the market. A well-known example is London Interbank Offered Rate (LIBOR), an interest rate benchmark that aims to provide consensus interest rates for the unsecured interbank market, a major over-the-counter financial market with no publicly observed transaction prices.

In this paper we study the ability of such pricing services to reduce market participants' uncertainty about asset values and competitors' behaviour. The analysis is based on a unique dataset of individual price estimates that major broker-dealers[1] have provided to a consensus pricing service that operates in the market for over-the-counter (OTC) financial derivatives. This market is a central piece of financial market infrastructure ensuring efficient risk transfers within the financial system. To evaluate the ability of the service to reduce uncertainty, we construct empirical measures of uncertainty based on market participants' first and higher-order posterior

[1]Broker-dealers are the main market makers in the OTC derivatives market. They stand ready to satisfy client demand for derivatives and manage the risk from these positions by either hedging it with other instruments or entering an offsetting trade with a third counterparty.

beliefs about asset values. These beliefs are obtained from the structural estimation of a model of learning from consensus prices. In particular, we consider two dimensions of uncertainty which we refer to as fundamental uncertainty and strategic uncertainty. Fundamental uncertainty concerns market participants' uncertainty about fundamental asset values. A typical source of fundamental uncertainty for broker-dealers is, for example, fluctuations in aggregate client demand for hedging the risk in a specific underlying asset. Strategic uncertainty, in turn, refers to market participants' uncertainty about the valuations of their competitors. Strategic uncertainty matters when coordination motives play an important role. In financial markets, coordination is particularly important when it comes to participation decisions as often "liquidity begets liquidity" – more potential trading partners increase one's own likelihood to become active in the market creating a virtuous circle. In the presence of coordination motives, individual trading decisions no longer only depend on beliefs about some exogenous payoff process but also on beliefs about competitors' actions. Welfare in the presence of such non-pecuniary externalities will, consequently, depend on both fundamental and strategic uncertainty (e.g. Angeletos and Pavan (2007)).

The analysis in this paper is based on a proprietary dataset of S&P500 index option consensus prices collected by IHS Markit's Totem service, a major consensus pricing service in the OTC derivatives market. S&P500 index options are the central derivatives product for the equity derivatives market. They are an important source of information about market participants' risk preferences and beliefs about future stock market volatility. We have access to a panel of price estimates that major broker-dealers have provided to the Totem service. To our knowledge, this paper is the first to analyse individual submission data to a consensus pricing or benchmarking service. To evaluate the ability of the consensus pricing service to reduce their valuation uncertainty, we develop a structural model of learning from publicly observable consensus prices, and private signals about time-varying fundamental asset values. To solve the model, we apply we adopt a solution algorithm developed in Nimark (2017) to tackle the well-known technical difficulty of filtering an endogenous signal, in our case the consensus price.[2] We estimate this model using the panel

[2]As first pointed out by Townsend (1983), learning from such endogenous variables has the well-known consequence of potentially infinite state spaces: agents have to form higher order beliefs in order to extract the relevant information from prices. Some progress has recently been made to address such problems, either by showing that in certain cases the original problem can be approximated arbitrary well with a finite state space (Huo and Takayama (2015), Huo and Pedroni (2017), Nimark (2017); see Sargent (1991) for earlier work using similar ideas) or by working with frequency domain techniques (Kasa (2000), Kasa, Walker, and Whiteman (2014), Rondina and

of price estimates and the consensus price feedback. Our measures of fundamental and strategic uncertainty are based on broker-dealers' posterior beliefs about fundamental asset value implied by the model. To quantify fundamental uncertainty, we construct posterior (or credible) intervals centred around the mean of an individual broker-dealer's model-implied beliefs about fundamental values. In our model, the notion of fundamental asset values corresponds to the option price (expressed as Black-Scholes implied volatilities) that would prevail in a perfectly competitive market without informational frictions. To measure strategic uncertainty, we construct posterior intervals centred around the mean of an individual broker-dealer's model-implied beliefs about his competitors' average valuations. To gauge the importance of the consensus price feedback for reducing valuation uncertainty, we perform a counterfactual experiment within our model. We eliminate the consensus price feedback and study how posterior beliefs are affected by this change in broker-dealers' information sets.

We find that valuation uncertainty, both fundamental and strategic, varies substantially across the different segments of the S&P500 index options market. The 95% posterior intervals measuring a broker-dealer's fundamental uncertainty can be as wide as 10 volatility points for deep out-of-the-money call options with short times-to-expiration. They tend to be below 1 volatility point for option contracts with strike price close to the current index level and out-of-the-money options with longer times-to-expiration. The posterior intervals for strategic uncertainty show similar qualitative patterns. Strategic uncertainty is highest for option contracts with strike prices that correspond to extreme index moves. We find that the informational value of the consensus price feedback for broker-dealers is most valuable for option contracts with extreme strike prices. It is particularly important for reducing strategic uncertainty. The consensus price provides a commonly shared signal that helps to reduce uncertainty about the location of the current average market valuation. The counterfactual elimination of the consensus price feedback leads to a significant increase in strategic uncertainty among broker-dealers. Again, we find this increase to be particularly strong for option contracts with extreme strike prices.

This paper has two main contributions. The first contribution is methodological. We construct empirical measures of valuation uncertainty based on market participants' posterior beliefs. These beliefs derive from a structural model of learning from prices. The nature of the consensus price data is ideally suited for this exercise. The data consist of individual price estimates by market participants and the aggregate price

Walker (2014)).

feedback they obtain from the service. This enables us to estimate Bayesian updating dynamics directly at the individual level. We can study how market participants extract information about asset values from prices and how, in turn, learning affects the dynamics of asset prices. We also obtain model-implied estimates of market participants' higher-order beliefs, which allow us to develop novel empirical measures of strategic uncertainty. There is a large literature on learning about asset values in the presence of informational frictions.[3] Data availability issues pose a serious challenge for empirical work on the topic. An early empirical contribution studying learning in financial markets is Biais, Hillion, and Spatt (1999). Building on theoretical insights into the speed of learning in financial markets by Vives (1993, 1995), the authors study the informational content of pre-opening prices at the Paris Bourse and quantify the speed of learning by traders. The study uses reduced form regression techniques to identify the informational content of these indicative prices which are similar in nature to consensus prices. Additionally, it provides an indirect measure of valuation uncertainty based on the variance of residuals from reduced form regressions. This differs from the analysis of valuation uncertainty in this paper which is based on model-implied posterior beliefs of market participants. A set of more recent papers studies the informational content of survey forecasts. While survey forecasts focus on macroeconomic aggregates, most importantly GDP growth and inflation, they bear some similarity to consensus prices. Forecasters provide a series of best estimates for a well defined outcome. And, similar to consensus prices, survey forecasts tend to have non-trivial cross-sectional dispersion which can be used to estimate the precision of forecasters' information. A study closely related to our paper using survey forecasts is Coibion and Gorodnichenko (2012). While the aim of their paper is to use inflation forecasts to differentiate between sticky price models and models of sluggish price adjustment due to informational frictions, as part of this exercise the authors develop a structural model of informational frictions based on Woodford (2003). They estimate this model exploiting both the cross-sectional dispersion of forecasts and their dynamics. Barillas and Nimark (2017) and Struby (2016) make use of the information contained in the cross-sectional dispersion of survey forecasts to estimate an affine factor model of the interest rate term structure. Their models also exploit the cross-sectional dispersion to identify the precision of signals that market participants receive.

The second contribution of the paper is an evaluation of the informational properties of the consensus pricing mechanism itself. This paper is to our knowledge

[3] For summaries of the literature on learning and information in financial markets see, for example, Chamley (2004) and Veldkamp (2011).

the first to empirically study the importance of consensus prices as an information aggregation mechanism. As such, the results reported in this paper can inform the ongoing regulatory debate about the usefulness of consensus pricing services as information aggregation devices. In the wake of the LIBOR scandal of 2012[4] the value of consensus prices has come under close scrutiny. There has been an ongoing regulatory effort to switch important financial benchmarks from using consensus prices to using transaction prices and firm quotes.[5] It is, however, unclear whether a complete switch, if at all feasible, is also desirable. Particularly during episodes of extreme market stress, transaction- or quote-based price data might become unavailable due to market illiquidity. In such situations, consensus prices will often be the only available source of valuation data. But these are exactly the periods during which pricing information can be most valuable. It is thus crucially important to understand whether and how this mechanisms works in practice. A recent literature has studied the informational value of benchmarks in search markets. Duffie and Stein (2015) provide a summary of the debate with a focus on interest rate benchmarks in the wake of the LIBOR scandal. Duffie, Dworczak, and Zhu (2017) build a theoretical model in which they show how benchmarks can help reduce informational asymmetries in search markets and thereby improve allocational efficiency. Unlike benchmarks, consensus prices are not used as reference values in financial contracts such as interest rate swaps. However, their construction is analogous to benchmarks; market participants are asked to provide their best estimates for the "going market price" of a given financial instrument. This paper also contributes to the discussion on the impact of price transparency on market behavior in dealer markets. A large part of this literature has focused on the US corporate bond market, in particular the impact of the introduction of mandatory post-trade reporting via the trade reporting and compliance engine (TRACE) on trading costs (Bessembinder, Maxwell, and Venkataraman (2006), Edwards, Harris, and Piwowar (2007), Goldstein, Hotchkiss, and Sirri (2007); see Bessembinder and Maxwell (2008) for a summary). These studies find that increased post-trade transparency led to a significant drop in trading costs in the US corporate bond market. A key mechanism appears to have been the reduction in big dealers' informational advantage over their clients.

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[4]The LIBOR scandal refers to the attempted manipulation of the London Interbank Offered Rate (LIBOR) by financial institutions contributing to the benchmarking service for unsecured interbank lending rates.

[5]For an overview of the current regulatory debate see Wheatley (2012), IOSCO (2013), Financial Stability Board (2014).

The plan of the paper is as follows. Section 2 explains the Totem consensus pricing service and provides summary statistics for our data. Section 3 develops a theoretical model of consensus pricing. Section 4 explains the estimation of this model and presents the results. Section 5 concludes.

2 Data: The Totem Service

IHS Markit's Totem service is a leading industry source for asset valuations and price verification data in the OTC derivatives markets. The origins of the service date back to the 1998 Russian financial crisis when a range of important derivatives markets became illiquid following the catastrophic losses suffered by a large hedge fund, LTCM. Totem was created to address the demand for reliable price information by financial institutions who needed to mark their books and manage their risk exposures. Nowadays, most broker-dealers that participate in the market for OTC derivatives subscribe and contribute to the Totem service. Totem offers a wide range of pricing services covering most asset classes and derivatives types. Totem subscribers can choose the services they want to contribute to and pay a subscription fee per service.

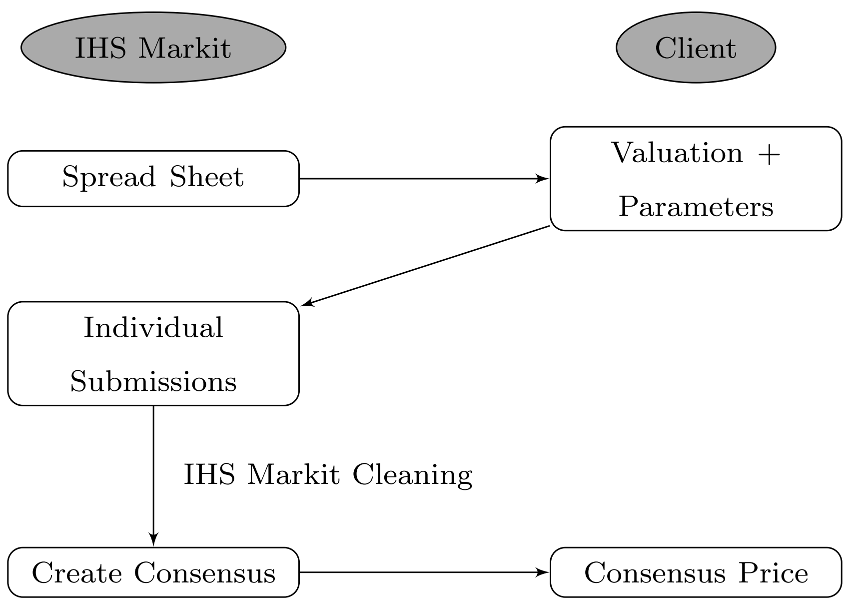

Figure 1: Totem submission process

Totem participants provide estimates of the mid-market value for the derivatives contracts that are included in the services they have subscribed to. Totem then aggregates these estimates, and returns an aggregate "consensus price" for each contract to the subscribers. This consensus pricing process operates at regular intervals, typically monthly. Figure 1 gives a schematic description of the process. For each derivatives service, Totem provides submitters with a spreadsheet into which they input the relevant prices. Submitters provide additional information in this spreadsheet, such as their discount factor and the spot price used in calculating their prices. At IHS Markit, these data are then "cleaned" after which submitters whose prices have been accepted on the given submission date receive the consensus data.[6]

2.1 S&P 500 index option consensus prices

In the empirical analysis of the paper we focus on Totem's pricing service for European S&P500 index options. This choice is motivated by two observations. Firstly, S&P500 index options are the central derivatives product for the equity derivatives market. They contain important information about market participants' risk attitudes and their beliefs about future index moves. S&P500 option prices are a key input for financial institutions' pricing models that are employed to value a range of more exotic derivatives products. Secondly, while S&P500 index options with strike prices close to the current index value and times-to-expiration of less than one year are regularly traded on centralised exchanges, options with more extreme strike prices and longer terms are primarily traded over-the-counter.[7] This variation in the dominant market structure for S&P500 index options trading provides an ideal setting to study the importance of consensus prices as a source of valuation information under varying levels of market opacity.

We have access to the full history of Totem participants' midquote submissions for European put and call options on the S&P500 index. The individual institutions are anonymised, but we can track each institution's submissions over time and across contracts. Our baseline sample consists of option contracts with moneyness between 60 and 200 and times-to-expiration between six months and seven years.[8] The sample period is January 2002 to February 2015.[9] In total we consider 68 distinct option

[6]In the cleaning process, midquotes are checked for basic arbitrage violations and attempts to manipulate the consensus price. Submissions are rejected if they meet these criteria and the excluded submitter does not receive any consensus data for the affected submission date. This mechanism is meant to incentivize submitters to provide their best estimate of the current midquote. For a more detailed description see the Appendix 6.6.

[7]Figure 6 in the Appendix displays the percentage of days in a month that a particular S&P500 index option is traded on US options exchanges.

[8]The moneyness definition used by Totem is the strike price of the option, K, divided by the spot price of the underlying index, S, times 100, that is $(K/S) \times 100$. All options in Totem are at-the-money (ATM) or out-of-the-money (OTM) options. Options with moneyness smaller than 100 are put options, options with moneyness greater than 100 are call options. For moneyness 100, Totem provides both put and call option prices.

[9]Tables 1 and 2 in the Appendix give an overview of the coverage of the options available to us,

contracts. Over the sample period, the average number of submitters per option contract is 30.[10]

To provide a sense of the cross-sectional dispersion in Totem submitters' option valuations, Figure 2 depicts cross-sectional standard deviations of price submissions across the volatility surface. The dispersion in submitters' prices (expressed in terms of implied volatilities) attains its highest level for short-dated OTM options with extreme strike prices. This is true for both OTM call and put options. There appears to be more agreement on the value of long-dated options. Figure 8 in the Appendix compares the range of Totem price submissions to the bid-ask spread of exchange-traded prices obtained from OptionMetrics for a European put option on the S&P500 with a time-to-maturity of 1 month and moneyness 95. The range in the midquote quote dispersion is on the same level as the bid-ask spread. This indicates that the cross-sectional differences in Totem submissions are economically meaningful and do not move proportionally to the bid-ask spread observed on option exchanges.

their corresponding sample period, and the number of submitters per contract. Initially, contracts with moneyness between 80 and 120 and terms varying from 6 to 60 months were introduced in 1998. Later, as demand for more extreme contracts and a finer grid increased, additional standardized contracts were added to the service.

[10]See Table 2 in the Appendix.

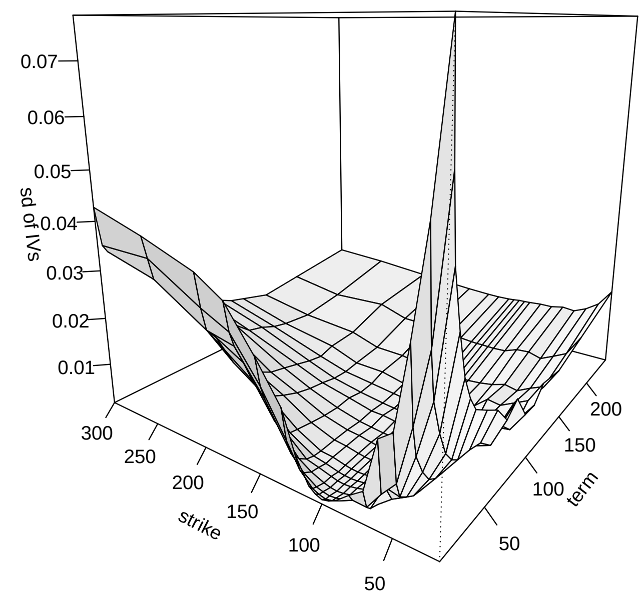

Figure 2: Cross-sectional dispersion of IVs (S&P500)

This figure depicts the monthly time-series average of the cross sectional standard deviation of Totem midquote submissions for S&P500 options. The midquote estimates are provided by large broker-dealers who submit to IHS Markit's totem service. The midquotes are expressed as implied volatilities. The axis labelled "term" indicates the time-to-expiration of the option contract in months. The axis labelled "strike" indicates the moneyness of the option contract. Moneyness is defined as the strike price divided by the spot price multiplied by 100. All contracts are out-of-money or at-the-money contracts. The sample period is December 2002 to February 2015.

3 A Model of Consensus Pricing

We now develop a stylized model of consensus pricing that captures the most important features of the consensus pricing process. The main technical challenge in solving the model is the learning problem faced by the consensus price submitters. The participating institutions have to extract information about a latent fundamental asset values from an endogenous signal, the consensus price. We give a detailed description of the algorithm employed to solve this filtering problem. The section concludes with the derivation of a welfare criterion to evaluate the welfare properties of the information feedback provided by the consensus pricing service.

3.1 Basic Setup

A large number of financial institutions, modelled as a continuum indexed by $i \in [0, 1]$, participate in a consensus pricing service. At discrete submission dates, indexed by t, each institution submits its best estimate for the current value of a latent stochastic process (θ_t), the fundamental value of an asset, to the service. The stochastic process itself evolves according to

$$\theta_t = (1 - \rho)\bar{\theta} + \rho\, \theta_{t-1} + \sigma_u\, u_t \tag{1}$$

with $-1 < \rho < 1$ and where the innovations u_t are independent across time and standard normally distributed, $u_t \sim N(0, 1)$.[11]

At each subsequent submission date t, submitters observe two signals. Each institution receives a noisy private signal $s_{i,t}$ about θ_t,

$$s_{i,t} = \theta_t + \sigma_\eta\, \eta_{i,t},$$

where $\eta_{i,t} \sim N(0, 1)$.[12] Additionally, each institution can observe the consensus price p_t, which is the average of all institutions' previous period's expectations about θ_{t-1}

[11] We do not explicitly model the economic forces responsible for the variation in fundamental values. Our preferred interpretation is based on demand-based option pricing models (see Gârleanu, Pedersen, and Poteshman (2009)). Changes in fundamental values derive from time-varying client demand that is satisfied by risk-averse broker-dealer. Under this interpretation, u_t is an aggregate demand shock for options with a specific strike price and maturity combination. Appendix 6.2 provides are more detailed analysis of this interpretation.

[12] Under a demand-based interpretation of fundamental values, this signal could derive from bilateral trading activity institution i engages in with clients. Higher activity signals stronger aggregate demand for a given option contract.

plus an aggregate noise shock $\varepsilon_t \sim N(0,1)$,

$$p_t = \int_0^1 \mathbb{E}_{i,t-1}\left(\theta_{t-1}\right) \, di + \sigma_\varepsilon \, \varepsilon_t. \tag{2}$$

All shocks (u_t), (ε_t), and $(\eta_{i,t})$ are uncorrelated across institutions and time.

Institution i's information set in period t, denoted $\Omega_{i,t}$, thus consists of the (infinite) history of public and private signals that i has observed up to period t, that is

$$\Omega_{i,t} = \{s_{i,t}, p_t, \Omega_{i,t-1}\}.$$

3.2 Learning from Consensus Prices

In order to characterise institution i's submission to the consensus pricing service, we need to calculate the institution's best estimate of θ_t, given by $\mathbb{E}\left(\theta_t | \Omega_{i,t}\right)$, for all periods t. Its information set $\Omega_{i,t}$, however, depends on all other institutions' submissions via the consensus price process (p_t). The information set is endogenous, as (p_t) is both an input and and output of the joint learning process of the consensus pricing participants. This endogeneity causes well-known technical difficulties and has spawned a large literature starting with Townsend (1983). We adopt an iterative algorithm developed in Nimark (2017) to solve filtering problem with endogenous signals.

The algorithm works as follows:

1. Start with any covariance-stationary process (p_t^0) that lies in the space spanned by linear combinations of current and past aggregate shocks (u_t) and (ε_t).

2. This consensus price process (p_t^0) yields information sets for all i and t defined recursively by $\Omega_{i,t}^0 = \{s_{i,t}, p_{t-1}^0, \Omega_{i,t-1}^0\}$.

3. Given information set $\Omega_{i,t}^0$, each institution i can compute the conditional expectation $\mathbb{E}(\theta_t | \Omega_{i,t}^0)$ for each period t.

4. Averaging the expectations across submitters yields a new consensus price process

$$p_t^1 = \int_0^1 \mathbb{E}(\theta_{t-1} | \Omega_{i,t-1}^0) \, di + \sigma_\varepsilon \, \varepsilon_t \;\text{ for all } t.$$

5. If the distance (in m.s.e) between (p_t^0) and (p_t^1) is smaller than some pre-specified stopping criterion, stop. Otherwise, go to step 2 using (p_t^1) as the new consensus price process and so on.

Nimark (2017) shows that for any initial choice of (p_t^0) the sequence of price processes $\{(p_t^n)\}_n$ converges (in m.s.e.) to a unique limit process (p_t) that is the solution of the original filtering problem. The proof relies on the fact that the integral in step 4 is a contraction on the space of covariance-stationary price processes. This allows the calculation of bounds for the approximation error when stopping the algorithm after a finite number of steps. A smart choice of the initial process (p_t^0) allows the problem to be solved by a sequential application of the Kalman filter. Appendix 6.1 provides a detailed description of the solution algorithm applied to the above consensus pricing model.

If the algorithm stops after k steps, the dynamics of the consensus price are (approximately) characterized by a $k+1$ dimensional system. The state space of this system is given by the average expectations about the fundamental θ_t up to level k, where these higher-order expectations are defined recursively as

$$\theta_t^{(0)} \equiv \theta_t \text{ and } \theta_t^{(n)} = \int_0^1 \theta_{i,t}^{(n)}\, di \text{ for all } n \geq 1,$$

where

$$\theta_{i,t}^{(n)} = \mathbb{E}\left(\theta_t^{(n-1)} | \Omega_{i,t}\right).$$

The system has Markovian dynamics

$$\theta_t^{(0:k)} = (I - M)\,\bar{\theta} + M\,\theta_{t-1}^{(0:k)} + N\,w_t, \tag{3}$$

where $\theta_t^{(0:k)} = (\theta_t, \theta_t^{(1)}, ..., \theta_t^{(k)})^\intercal$, $w_t = (u_t, \varepsilon_t)^\intercal$ and the matrices M and N are outputs of the above described solution algorithm. This determines the dynamics of the consensus price, which is given by

$$p_t = \theta_{t-1}^{(1)} + \sigma_\varepsilon\,\varepsilon_t.$$

Given the combination of linear dynamics of the state space and normally distributed shocks, institution i's expectations $\mathbb{E}(\theta_t|\Omega_{i,t})$ can then be calculated with standard Kalman filtering techniques. Posterior beliefs have a normal distribution with a constant covariance matrix Σ,

$$\theta_t^{(0:k)} | \Omega_{i,t} \sim N\left(\theta_{i,t}^{(1:(k+1))}, \Sigma\right) \text{ and}$$

$$\theta_{i,t}^{(1:(k+1))} = (I-M)\bar{\theta} + M\,\theta_{i,t-1}^{(1:(k+1))} + K\left[\begin{pmatrix} s_{i,t} \\ p_t \end{pmatrix} - D\,M\,\theta_{i,t-1}^{(1:(k+1))}\right], \qquad (4)$$

where K is the Kalman gain.[13] This then characterizes institution i's submission process ($\theta_{i,t}^{(1)}$) to the consensus pricing service.

3.3 Welfare analysis

To evaluate the welfare consequences of different informational settings, we employ a "reduced-form" model of market behaviour. In particular, we assume that in every period each institution $i \in [0,1]$ undertakes an payoff-relevant action $a_{i,t}$. Institution i's expected per-period payoff given information set $\Omega_{i,t}$, is

$$-\mathbb{E}\left[(\theta_t - a_{i,t})^2|\Omega_{i,t}\right] - \left(\frac{\beta}{1-\beta}\right)\mathbb{E}\left[(\bar{a}_t - a_{i,t})^2|\Omega_{i,t}\right],$$

where $-1 < \beta < 1$ and $\bar{a}_t$ is the average action of competitor institutions,

$$\bar{a}_t = \int_0^1 a_{i,t}\,di.$$

Each institution cares both about being close to the current fundamental, θ_t, and the relative position of its action to the average action of its competitors. To fix ideas, we think of $a_{i,t}$ as an ask price institution i quotes to its clients in period t.[14] Note that, unless $\beta = 0$, $a_{i,t}$ does not correspond to i's submission to the consensus pricing service. β introduces a coordination motive into market participants' behaviour. If $\beta > 0$, their actions are strategic complements, if $\beta < 0$ their actions a strategic substitutes.

Institution i's' optimal action is given by

$$a_{i,t} = (1-\beta)\theta_{i,t}^{(1)} + \mathbb{E}\left(\theta_t|\Omega_{i,t}\right) + \beta\,\mathbb{E}\left(\bar{a}_t|\Omega_{i,t}\right) = (1-\beta)\sum_{k=0}^{\infty}\beta^k\,\theta_{i,t}^{(k+1)},$$

where the last equality follows from iteratively substituting out $\bar{a}_t$.[15] The expected

[13]See Appendix 6.1 for derivations.

[14]Dealers tend to be net suppliers of S&P500 index options (see, e.g., Gârleanu, Pedersen, and Poteshman (2009)), hence the focus on ask rather than bid price.

[15]For details on this "forward solution" in terms of higher-order expecations see, for example, Morris and Shin (2002) or Woodford (2003).

per-period payoff is then

$$U(\Omega_{i,t}) = -\mathbb{E}\left[(1-\beta)\sum_{k=0}^{\infty}\beta^k\left(\theta_t - \theta_{i,t}^{(k+1)}\right)^2 |\Omega_{i,t}\right] \\ - \left(\frac{\beta}{1-\beta}\right)\mathbb{E}\left[(1-\beta)\sum_{k=0}^{\infty}\beta^k\ \left(\theta_t^{(k+1)} - \theta_{i,t}^{(k+1)}\right)^2 |\Omega_{i,t}\right].$$

It can be shown that that for a "small" β ex-ante expected per-period payoff is of the form

$$\mathbb{E}\left[U(\Omega_{i,t})\right] = -(1+\beta)\Sigma_{11} - 2\beta\,\Sigma_{22} + 2\beta\,\Sigma_{12} + O(\beta^2),$$

where Σ_{nm} is element (n, m) of the covariance matrix of submitter i's posterior beliefs[16] and $O(\beta^2)$ indicates terms of order β^2 and higher.[17]

For sufficiently small β, the ex-ante expected per-period payoff is dominated by the variances of first and second order posterior beliefs and their covariance. When evaluating the welfare consequences of the consensus pricing service in section 4, we will concentrate on these three moments. In particular, we will consider how these moments change in a counterfactual informational setting where market participants do not have access to the consensus price information. While increases in the variance of first order beliefs decrease welfare, the impact of the variance of second order beliefs and the covariance between first and second order beliefs depends on the sign of β. If actions are strategic complements ($\beta > 0$), a higher variance of second order beliefs and a lower correlation between first and second order beliefs decreases welfare as it makes the trade-off between being close to the fundamental and being close to the average action more severe. However, when actions are strategic substitutes ($\beta < 0$), institutions attempt to differentiate themselves from their competitors' actions. In this case, a higher variance of second order beliefs and a negative correlation between first and second order beliefs are desirable.

[16]Note that the covariance matrix of posterior beliefs, Σ, does not depend on the realization of $\Omega_{i,t}$ and is therefore known ex-ante.

[17]See Appendix 6.3.

4 Empirical Model and Results

4.1 Estimation

We estimate the parameters of the model presented in section 3, namely $\Phi = \{\rho, \bar{\theta}, \sigma_u, \sigma_\varepsilon, \sigma_\eta\}$, by maximum likelihood separately for each options contract. For a given contract, that is a given time-to-expiration, moneyness, and option type (put or call), our data consists of two time series. First, the time series of submissions by the institutions participating in the Totem consensus pricing service for the specified contract; second, the consensus price time series. Let S be the total number of institutions that have submitted to Totem over the course of our sample period and let $\iota_t \subset \{1, 2, .., S\}$ be the set of institutions active in t. The time series of submissions is given by $(\mathbf{m}_t)_{t=1}^T$, where $\mathbf{m}_t = (m_{j,t})_{j \in \iota_t}$ is a $|\iota_t|$-dimensional vector consisting of the individual period t consensus price submissions. Our data set for a given contract, $(\mathbf{y})_{t=1}^T$, then consists of the time-series of institutions' price submissions for this contract and the corresponding consensus price , i.e. $\mathbf{y}_t = (p_t \, , \mathbf{m}_t)^\intercal$.[18]

To estimate the model for a given contract, we assume that the observed consensus price of period $t-1$ is equal to the average first order belief of period $t-1$ plus aggregate noise, that is

$$p_t = \theta_{t-1}^{(1)} + \sigma_\varepsilon \, \varepsilon_t$$

The latent state space has the dynamics derived in the previous section, namely

$$\theta_t^{(0:k)} = [I - M(\Phi)] \, \bar{\theta} + M(\Phi) \, \theta_{t-1}^{(0:k)} + N(\Phi) \, [u_t \; \varepsilon_t]^\intercal,$$

where $M(\Phi)$ and $N(\Phi)$ are obtained employing the previously explained solution algorithm for a given parameter vector Φ and

$$\begin{pmatrix} u_t \\ \varepsilon_t \end{pmatrix} \sim N\left(\mathbf{0}, I_2\right).$$

We assume that individual price submissions are institution i's best estimate of θ_t plus uncorrelated measurement error, i.e.

$$m_{i,t} = \theta_{i,t}^{(1)} + \sigma_\psi \, \psi_{i,t} \;\; \text{with} \;\; \psi_{i,t} \sim N(0, 1),$$

[18] To be precise, $m_{j,t}$ is the natural logarithm of the Black-Scholes implied volatility of submitter j's time t price submission, and p_t is the natural logarithm of the consensus Black-Scholes implied volatility calculated by Totem for the corresponding contract.

where institution i's beliefs are treated as latent variables that have the previously derived dynamics

$$\theta_{i,t}^{(1:(k+1))} = [I - M(\Phi)]\bar{\theta} + M(\Phi)\,\theta_{i,t-1}^{(1:(k+1))} + K(\Phi)\left[\begin{pmatrix} s_{i,t} \\ p_t \end{pmatrix} - D(\Phi)\,M(\Phi)\,\theta_{i,t-1}^{(1:(k+1))}\right].$$

We treat institution i's private signal,

$$s_{i,t} = \theta_t + \sigma_\eta\,\eta_{i,t} \;\text{ with } \eta_{i,t} \sim N(0,1),$$

as a latent variable that is not observed by the econometrician. All shocks $\eta_{i,t}$ are uncorrelated across submitting institutions and time.

Given the linearity of the above system and the joint normality of all shocks the likelihood function for the observed data $(\mathbf{y})_{t=1}^{T}$ can be derived using the Kalman filter. We obtain maximum likelihood estimates for the parameter vector Φ by maximising the log-likelihood function numerically. Appendix 6.4 provides a detailed derivation of the filter for the above model and the estimation procedure.

4.2 Measuring uncertainty

Having obtained an estimate of the parameter vector Φ for a given contract, we can calculate the corresponding covariance matrix $\Sigma(\Phi)$ of submitters' posterior beliefs, where

$$\theta_t^{(0:k)}|\Omega_{i,t} \sim N\left(\theta_{i,t}^{(1:(k+1))}, \Sigma(\Phi)\right).$$

The variance of posterior beliefs concerning the current value of the fundamental, θ_t, is given by[19]

$$\Sigma_{11} = Var\left(\theta_t^{(0)}|\Omega_{i,t}\right) = \mathbb{E}\left[\left(\theta_t - \theta_{i,t}^{(1)}\right)^2 |\Omega_{i,t}\right].$$

This posterior variance is a natural measure of submitters' uncertainty about the fundamental value. We refer to this uncertainty as *fundamental uncertainty*. A convenient way to display this uncertainty is via posterior intervals. As posterior beliefs are normally distributed, an individual submitter's $x\%$ posterior interval for the fundamental value θ_t is

$$\left[\theta_{i,t}^{(1)} - \alpha_x\sqrt{\Sigma_{11}}\,,\,\theta_{i,t}^{(1)} + \alpha_x\sqrt{\Sigma_{11}}\right], \tag{5}$$

[19] To ease notation we drop the explicit dependence on the parameter vector Φ from here onwards.

where α_x is the critical value of the standard normal distribution such that $\Phi(\alpha_x) = x/2$.

Knowing submitters' higher order beliefs allows us to go beyond measures of fundamental uncertainty. We can also measure how uncertain a given submitter i is about the mean valuations of his peers using the variance of his second order beliefs, that is

$$\Sigma_{22} = Var\left(\theta_t^{(1)}|\Omega_{i,t}\right) = \mathbb{E}\left[\left(\theta_t^{(1)} - \theta_{i,t}^{(2)}\right)^2|\Omega_{i,t}\right].$$

Σ_{22} is a measure of *strategic uncertainty*, that is uncertainty about the relative position of the average market participant.[20] The more uncertain a submitter is about the average position of his co-submitters, the higher is Σ_{22}. If the average valuation is common knowledge, then this variance is zero. This would, for example, be the case if the only informative signal was the consensus price which is a public signal and, consequently, common to all submitters. Again, as beliefs are normally distributed we can express this strategic uncertainty in terms of a submitter's posterior interval centred around the mean of his current second order beliefs $\theta_{i,t}^{(2)}$,

$$\left[\theta_{i,t}^{(2)} - \alpha_x\sqrt{\Sigma_{22}}\,,\,\theta_{i,t}^{(2)} + \alpha_x\sqrt{\Sigma_{22}}\right]. \tag{6}$$

Figure 3 displays 95% posterior intervals for fundamental and strategic uncertainty based on expressions (5) and (6). Here, the confidence intervals are centered around the estimate of the unconditional expectation for first and second order beliefs, that is $\bar{\theta}$. The top panels in Figure 3 display option contracts with a fixed time-to-expiration of 12 months, the bottom panels in Figure 3 shows option contracts with a time-to-expiration of 5 years. Strike prices range from a moneyness of 60 to 200.[21] The plots on the left show 95% posterior intervals around $\bar{\theta}$ for submitters' first order beliefs, which we refer to as fundamental uncertainty. The plots on the right display 95% posterior intervals for submitters' second order beliefs centred around the same mean, our measure of strategic uncertainty.

[20] This notion is not to be confused with informational asymmetries - in our model all submitters are symmetrically informed about asset values.

[21] For terms of 6 months and one year we exclude contracts with a moneyness of 200 from the analysis because of numerical precision issues. For these terms, moneyness 200 is an extreme strike price for a call option. Option prices will be close to zero and crucially depend on the numerical precision used by Totem submitters when reporting prices. Additionally, the inversion of the prices to Black-Scholes implied volatilities can become numerically unstable.

As the unconditional mean of the fundamental values, $\bar{\theta}$, are expressed in terms of Black-Scholes implied volatilities, both figures show the well-known "smirk" of the implied volatility curve. Out-of-the money (OTM) put options (monyeness below 100) tend to be relatively more expensive than at-the-money (ATM) put option or out-of-the-money (OTM) call options (moneyness above 100).[22] Figure 3 shows that for option contracts that are deeper out-of-the-money (further away from moneyness 100), fundamental and strategic valuation uncertainty is also higher - the posterior intervals are wider. Posterior intervals for deep OTM put options with moneyness 60 and time-to-expiration of 12 months are on the order of five volatility points. This contrasts with posterior intervals well below one volatility point for ATM options with the same time-to-expiration. This variation in valuation uncertainty across the volatility curve is intuitive: option contracts closer to ATM tend to trade more often than deep OTM options. Particulary deep OTM call options with short times-to-expirations tend to be very illiquid. The absence of trading is expected to increase both fundamental uncertainty and the uncertainty about other market participants' valuation when compared to the more liquidly traded options close to ATM.

While strategic uncertainty is visibly smaller than fundamental uncertainty for deep OTM put and call options with time-to-expiration of 12 months, this effect is much less pronounced for option contracts with longer times-to-expiration and closer to ATM. This is confirmed in Figure 4 which plots term structure of fundamental and strategic uncertainty. The top panels in Figure 4 show the term structure for OTM put options with moneyness 60 in black and, for comparison, the corresponding term structure for ATM put options. The bottom panels in Figure 4 provide the same comparison for OTM call options with moneyness 150. Again, plots on the left display fundamental uncertainty, plots on the right strategic uncertainty.

[22] Cont and Da Fonseca (2002) provide a detailed analysis of the dynamic behavior of the implied volatility surface of S&P500 index options.

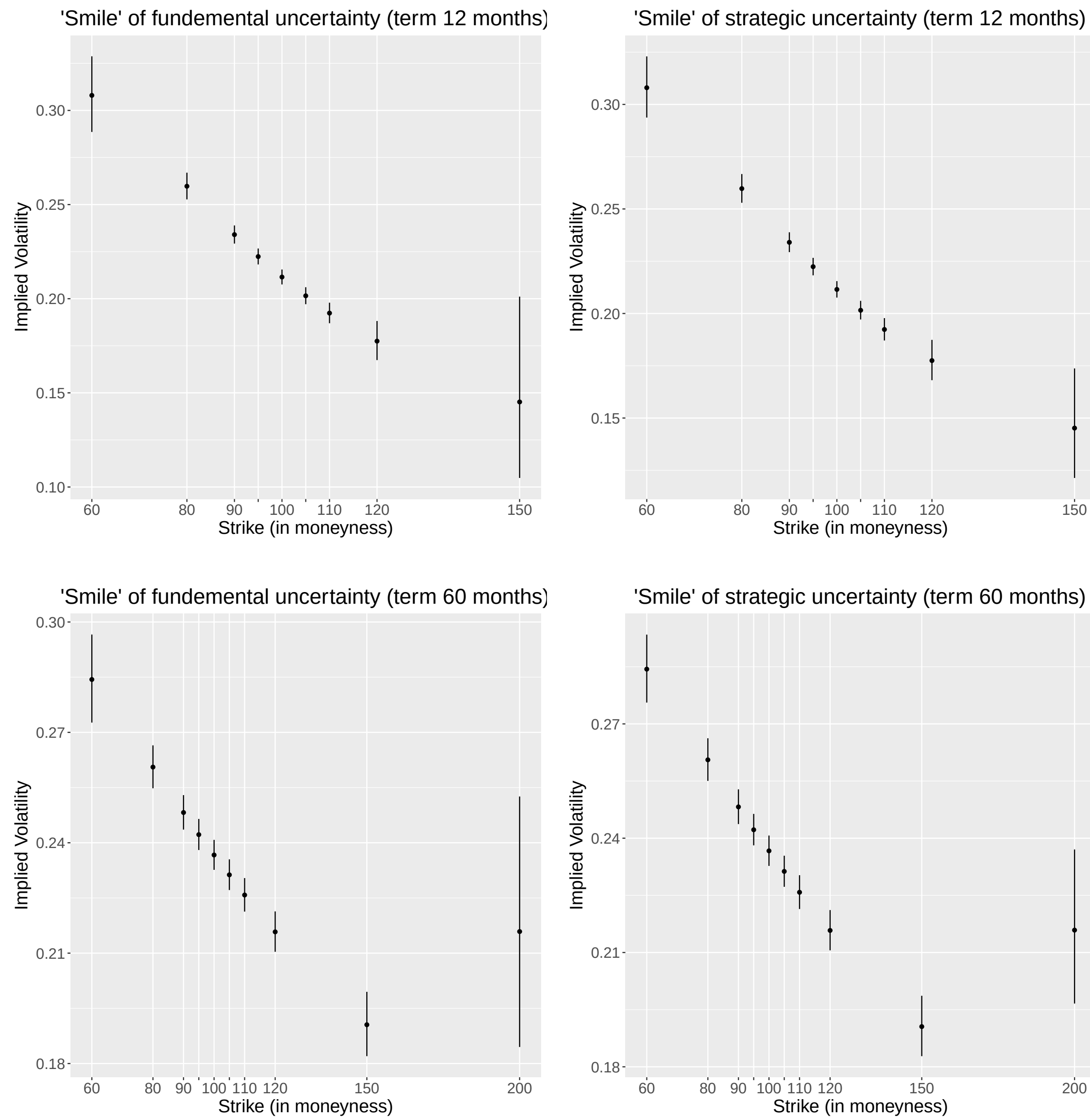

Figure 3: These figures present the variance of submitters' posterior beliefs expressed in terms of posterior intervals centred around the unconditional mean of implied volatilities, $\bar{\theta}$. The horizontal axis denotes the moneyness of the option contracts. The black dots in both figures represent the exponentiated point estimates of $\bar{\theta}$. The *left* figures depict the posterior intervals for first order beliefs as given in (5). The figures on the *right* display the posterior intervals for second order beliefs as in (6). The *top* figures consider the option contracts with a **time-to-expiration** of **12 months**. The *bottom* figures consider the option contracts with a **time-to-expiration** of **60 months**. The sample period is December 2002 to February 2015 for the option contracts on the S&P500 index. Data provided by IHS Markit's Totem service.

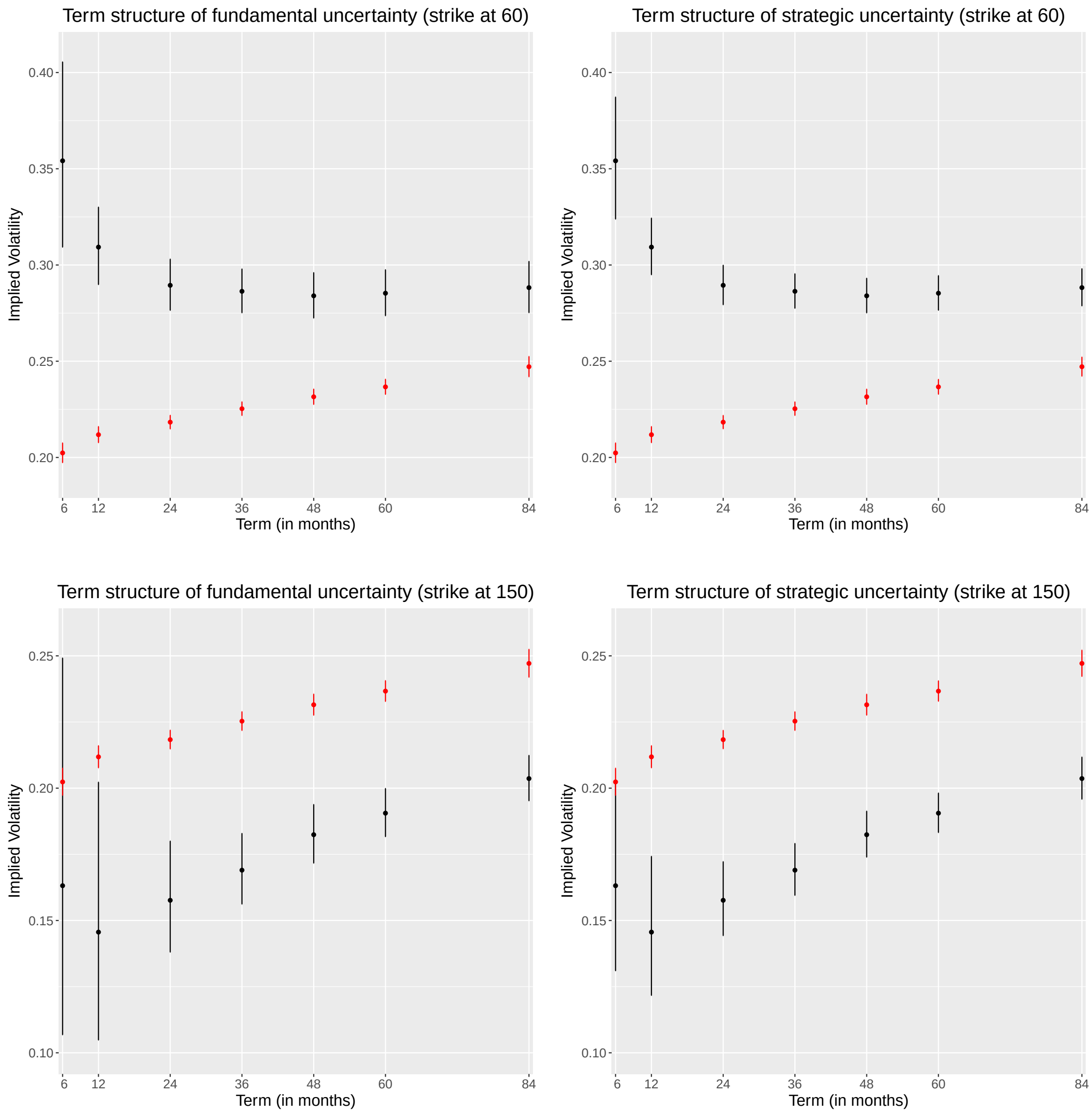

Figure 4: These figures present **term structures** of the variance of submitters' posterior beliefs expressed in terms of posterior intervals centred around the unconditional mean of implied volatilities, $\bar{\theta}$. The horizontal axis denotes time-to-expiration in months. The black dots in both figures represent the exponentiated point estimates of $\bar{\theta}$. The *left* figures depict the term structures of posterior intervals for first order beliefs as given in (5). The figures on the *right* display the term structures of the posterior intervals for second order beliefs as given in (6) with the same color scheme. The *top* figures depict the term structures for **call options** with **monyeness 150 (black)** and **monyeness 100 (red)**. The sample period is December 2002 to February 2015 for the option contracts on the S&P500 index. Data provided by IHS Markit's Totem service.

4.3 The informational value of consensus prices

Next, we measure the importance of the consensus price feedback for reducing subscribers' fundamental and strategic uncertainty. To do so, we ask the counterfactual question,"By how much would an institution's posterior uncertainty increase if it loses access to the consensus price keeping all else equal?"[23] More precisely, we calculate the percentage increase in the posterior intervals for fundamental and strategic uncertainty specified in (5) and (6) that would result if submitters did not have access to the consensus price feedback provided by the Totem service.

Denote by $\hat{\Sigma}$ the covariance matrix of institution i's posterior beliefs under this counterfactual information set, namely $\hat{\Omega}_{i,t} = \{s_{i,t}, \hat{\Omega}_{i,t-1}\}$. This covariance matrix can be obtained by solving a standard single-agent learning problem using the parameters $\{\rho, \sigma_u, \sigma_\eta\}$ obtained in the estimation.[24] The length of the posterior interval for the first order belief, specified in (5), is $2\alpha_x\sqrt{\Sigma_{11}}$. The length of the counterfactual posterior interval for first order beliefs is $2\alpha_x\sqrt{\hat{\Sigma}_{11}}$. Thus, the percentage increase in the posterior interval for first-order beliefs cause by a loss of access to the consensus price feedback for a given contract is

$$\Delta_1 \equiv \left(\frac{\hat{\Sigma}_{11}}{\Sigma_{11}}\right)^{1/2} - 1. \tag{7}$$

Similarly, the percentage increase in posterior intervals for second order beliefs caused by a loss of access to the consensus price feedback for the contract under consideration is given by

$$\Delta_2 \equiv \left(\frac{\hat{\Sigma}_{22}}{\Sigma_{22}}\right)^{1/2} - 1. \tag{8}$$

Figure 5 displays Δ_1 and Δ_2 across different levels of moneyness for options with a term of one year and five years respectively. The figures reveal that the increase in valuation uncertainty that would result from a loss of the consensus price feedback is smallest for option contracts with strike price close to the current index value. These are options contracts that frequently trade on-exchange (see Figure 6) and

[23] "Keeping all else equal" is arguably a strong assumption. In particular, this assumes that the institution does not change its information acquisition strategy and hence, the precision of the private signal as parameterized by σ_η remains unchanged.

[24] It can, equivalently, be obtained within our framework as the limit of Σ as the variance of the noise in the consensus price, σ_ε, goes to infinity.

we expect submitters to have sufficiently precise alternative informational source for valuing these options. Within our model, this implies a sufficiently precise private signal, i.e. a small σ_η. For contracts with more extreme strike prices and terms, losing access to the consensus service feedback induces a relatively larger increase in valuation uncertainty. This pattern is present for both fundamental and strategic uncertainty. We conjecture that the lack of trade information in the more illiquid parts of the options market makes the consensus price feedback a more important sources of information.

Subscribers' increase in fundamental uncertainty, as measured by percentage increase in the posterior intervals of first order beliefs, is between 0.9% to 3.3% for the more extreme contract, e.g contracts with moneyness of 60, 150 and 200. The increase in strategic uncertainty, as measured by posterior intervals for second order beliefs, for the same contracts ranges from 7.0% to 20.2%. The relative larger increase in strategic uncertainty in comparison to fundamental uncertainty points to the importance of the consensus price to learn about the relative position of other market participants. Given the scarcity of publicly available trading data for such extreme contracts, the ability of the the consensus price to significantly reduce strategic uncertainty is both intuitive and important.

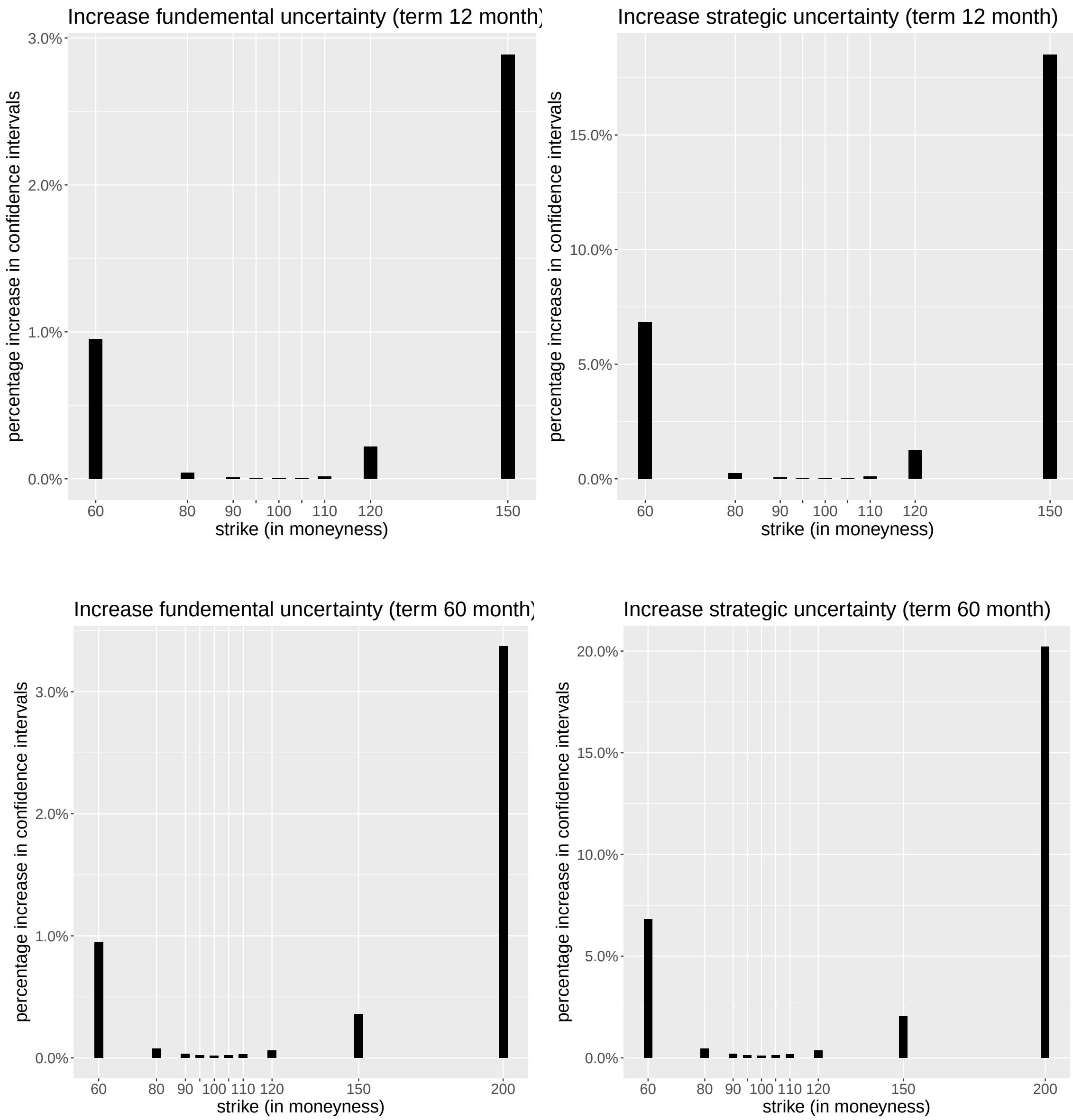

Figure 5: These figures present the results for the increase in valuation uncertainty caused by a loss of the consensus price feedback, given by Δ_1 and Δ_2 in (7) and (8). Here Δ_1 is the percentage increase in fundamental uncertainty and Δ_2 is the percentage increase in strategic uncertainty. The horizontal axis denotes the moneyness of the options under consideration. In the left figures we depict Δ_1 and in the right figures we depict Δ_2. The top figures depict contracts with a time-to-expiration of **12 months** and the bottom figures depict contracts with a time-to-expiration of **60 months**. The sample period December 2002 to February 2015.

5 Conclusion

In this paper we provide empirical evidence on the informational value of consensus prices for the valuation of S&P500 index options that are predominately traded in the over-the-counter market. This evidence is based on a structural model of learning from prices. The estimation is based a proprietary panel of price estimates that large broker-dealers have provided to a consensus pricing service for OTC derivatives. The structural model allows us to address two questions. First, how large is the valuation uncertainty of broker-dealers participating in the OTC market for S&P500 index options? Here, we consider two dimensions of uncertainty: a dealer's uncertainty about fundamental option values and uncertainty about its option valuations in relation to other market participants' valuations? Second, we ask the question of how useful the consensus price feedback is for market participants to reduce valuation uncertainty?

We find both fundamental and strategic valuation uncertainty to be pervasive across the volatility surface of S&P500 options that are traded in the OTC market. We find higher uncertainty for option contracts with strike prices that correspond to more extreme moves in the S&P500 index. Broker-dealers do not appear rely heavily on the consensus price feedback to reduce fundamental uncertainty. The consensus price feedback is found to be most important for reducing strategic uncertainty, and particularly so for extreme option contracts. This result is consistent with the scarcity of shared valuation information and, typically, heavy reliance on proprietary pricing models for such extreme contracts. It stresses the importance of publicly observable valuation data, such as benchmarks, to establish a shared understanding of market conditions in OTC markets. Such a shared understanding will be particularly valuable during episodes of market stress where high levels of strategic uncertainty might cause derivatives markets to freeze up.

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6 Appendix

6.1 Solution algorithm

Here we show how to apply the algorithm developed in Nimark (2017) to solve the consensus pricing problem of section 3. We adopt the following standard notation for higher-order beliefs, defining recursively[25]

$$\theta_t^{(0)} = \theta_t - \bar{\theta},$$
$$\theta_{i,t}^{(k+1)} = \mathbb{E}\left(\theta_t^{(k)} | \Omega_{i,t}\right) \quad \text{and} \quad \theta_t^{(k+1)} = \int_0^1 \theta_{i,t}^{(k+1)}\, di \text{ for all } k \geq 0.$$

We denote institution i's hierarchy of beliefs up to order k by

$$\theta_{i,t}^{(1:k)} = \left(\theta_{i,t}^{(1)}, ..., \theta_{i,t}^{(k)}\right)^{\mathsf{T}}$$

and for the hierarchy of average beliefs up to order k, including the fundamental value $\theta_t^{(0)}$ as first element,

$$\theta_t^{(0:k)} = \left(\theta_t^{(0)}, \theta_t^{(1)}, ..., \theta_t^{(k)}\right)^{\mathsf{T}}.$$

The solution procedure proceeds recursively. It starts with a fixed order of beliefs $k \geq 0$ and postulates that the dynamics of average beliefs $\theta_t^{(0:k)}$ are given by the VAR(1)

$$\theta_t^{(0:k)} = M_k\, \theta_{t-1}^{(0:k)} + N_k\, w_t \tag{9}$$

with $w_t = (u_t, \varepsilon_t)^{\mathsf{T}}$ and $\theta_t^{(n)} = \theta_t^{(k)}$ for all $n \geq k$.

Institution i's signal can be expressed in terms of current and past average beliefs, $\theta_t^{(0:k)}$ and $\theta_{t-1}^{(0:k)}$, and the period t shocks w_t and $n_{i,t}$. The (demeaned)[26] private signal can be written as

$$\hat{s}_{i,t} = e_1^{\mathsf{T}}\, \theta_t^{(0:k)} + \sigma_\eta\, \eta_{i,t}$$

where e_j denotes a column vector of conformable length with a 1 in position j, all other elements being 0. Similarly, we can express the (demeaned) consensus price $\hat{p}_t$ as

$$\hat{p}_t = \theta_{t-1}^{(1)} + \sigma_\varepsilon\, \varepsilon_t = e_2^{\mathsf{T}}\, \theta_{t-1}^{(0:k)} + \sigma_\varepsilon\, \varepsilon_t.$$

[25] Here we express the fundamental in terms of deviations from its mean $\bar{\theta}$ to lighten notation. Defining $\theta_t^{(0)} = \theta_t$ and carrying forward a constant or deterministically changing mean would not cause any conceptual difficulties.

[26] We define $\hat{z}_{i,t} = z_{i,t} - \bar{\theta}$ as deviations from the mean $\bar{\theta}$ for a generic random variable $z_{i,t}$.

Denote the vector of signals by $y_{i,t} = (\hat{s}_{i,t}, \hat{p}_t)^\intercal$. We can now express the signals in terms of current average beliefs and shocks,

$$y_{i,t} = D_{k,1}\,\theta_t^{(0:k)} + D_{k,2}\,\theta_{t-1}^{(0:k)} + R_w\,w_t + R_\eta\,\eta_{i,t} \tag{10}$$

where

$$D_{k,1} = \begin{bmatrix} e_1^\intercal \\ 0_{k+1}^\intercal \end{bmatrix} \;,\; D_{k,2} = \begin{bmatrix} 0_{k+1}^\intercal \\ e_2^\intercal \end{bmatrix} \;,\; R_\eta = \begin{bmatrix} \sigma_\eta \\ 0 \end{bmatrix} \text{ and } R_w = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_\varepsilon \end{bmatrix}.$$

We thus obtain a state space representation of the system from the perspective of institution i. Equation (9) describes the dynamics of the latent state variable $\theta_t^{(0:k)}$, equation (10) is the observation equation that provides the link between the current state and i's signals. Using a Kalman filter that allows for lagged state variables (see Nimark (2015)) allows us to express institution i's beliefs conditional on the information contained in $\Omega_{i,t}$ as

$$\theta_{i,t}^{(1:k+1)} = M_k\,\theta_{i,t-1}^{(1:k+1)} + K_k\left[y_{i,t} - D_{1,k}\,M_k\,\theta_{i,t-1}^{(1:k+1)} - D_{2,k}\,\theta_{i,t-1}^{(1:k+1)}\right], \tag{11}$$

where K_k is the (stationary) Kalman gain. Substituting out the signal vector in terms of current state and shocks, this can equivalently be written as

$$\begin{aligned}\theta_{i,t}^{(1:k+1)} &= \left[M_k - K_k(D_{1,k}M_k + D_{2,k})\right]\theta_{i,t-1}^{(1:k+1)} \\ &\quad + K_k(D_{1,k}M_k + D_{2,k})\theta_{t-1}^{(0:k)} + K_k(D_{1,k}N_k + R_w)w_t + K_k R_\eta\,\eta_{i,t}.\end{aligned}$$

Averaging this expression across all submitters, assuming that by a law of large numbers $\int_0^1 \eta_{i,t}\,di = 0$, average beliefs are then given by

$$\begin{aligned}\theta_t^{(1:k+1)} &= \left[M_k - K_k(D_{1,k}M_k + D_{2,k})\right]\theta_{t-1}^{(1:k+1)} \\ &\quad + K_k(D_{1,k}M_k + D_{2,k})\theta_{t-1}^{(0:k)} + K_k(D_{1,k}N_k + R_w)w_t.\end{aligned}$$

Combined with the fact that $\theta_t^{(0)} = \rho\,\theta_{t-1}^{(0)} + \sigma_u\,u_t$ we now obtain a new law of motion for the state

$$\theta_t^{(0:k+1)} = M_{k+1}\,\theta_{t-1}^{(0:k+1)} + N_{k+1}\,w_t$$

with

$$M_{k+1} = \begin{bmatrix} \rho\,e_1^\intercal & 0 \\ K_k(D_{1,k}M_k + D_{2,k}) & 0_{k\times1} \end{bmatrix} + \begin{bmatrix} 0 & 0_{1\times k} \\ 0_{k\times1} & M_k - K_k(D_{1,k}M_k + D_{2,k}) \end{bmatrix} \tag{12}$$

and

$$N_{k+1} = \begin{bmatrix} \sigma_u \, e_1^\intercal \\ K_k(D_{1,k} N_k + R_w) \end{bmatrix}. \tag{13}$$

Note, however, that now the state space has increased by one dimension from $k+1$ to $k+2$. This is a consequence of the well-known infinite regress problem when filtering endogenous signals. When filtering average beliefs of order k, institutions have to form beliefs about average beliefs of order k. But this implies that equilibrium dynamics will be influenced by average beliefs of order $k+1$, and so on for all orders $k \geq 0$.

In practice, the solution algorithm works as follows. We initialise the iteration at $k = 0$ with $M_0 = \rho$ and $N_0 = \sigma_u$ which implies that $\theta_t^{(1)} = \theta_t^{(0)}$ for all t. Consequently, the consensus price of the first iteration is given by[27]

$$\hat{p}_t^{[1]} = \theta_{t-1}^{(0)} + \sigma_\varepsilon \, \varepsilon_t.$$

This will yield a Kalman gain K_0 (here a two-dimensional vector) which can then be used to obtain M_1 and N_1 via equations (12) and (13) and so on until convergence of the process $\hat{p}_t^{[n]}$ has been achieved according to a pre-specified convergence criteria after n steps. The highest order belief that is not trivially defined by lower order beliefs is then of order n.

[27] Superscripts in square brackets denote iterations of the algorithm.

6.2 Demand-based option pricing

Suppose the underlying is geometric Brownian motion, i.e.

$$dS_t = \mu \, S_t \, dt + \sigma \, S_t \, dW_t.$$

In the absence of demand pressure, each option has a constant Black-Scholes implied volatility of σ, that is the volatility surface is flat in all periods.

Demand pressure paired with market incompleteness moves option IVs away from the Black-Scholes IVs. In particular, let us assume that the only source of friction is the inability to hedge options continuously. Then, Gârleanu, Pedersen, and Poteshman (2009) show that the Black-Scholes IV for option i changes with a change in demand for option j according to

$$\frac{\partial \sigma_t^i}{\partial d_t^j} = \frac{\gamma \, r \, Var_t \left((\Delta S)^2 \right)}{4} \, \frac{f_{SS}^i}{\nu^i} \, f_{SS}^j + o\left(\Delta_t^2 \right)$$

where f^i is the Black-Scholes price of option i, f_{SS}^i is option i's Black-Scholes gamma and ν^i is option i's Black-Scholes vega. Δ_t is the time interval between two re-hedging opportunities, the only source of friction in this model.

$$\Delta S = S_{t+\Delta_t} - S_t \approx \mu \, S_t \, \Delta_t + \sigma \sqrt{\Delta_t} \, S_t \, \varepsilon,$$

where $\varepsilon \sim N(0,1)$. We thus have

$$Var_t \left((\Delta S)^2 \right) = Var_t \left(\sigma^2 \, \Delta_t \, S_t^2 \, \varepsilon^2 + 2\mu \, \sigma \, S_t^2 \, \Delta_t^{3/2} \, \varepsilon \right) = 2\sigma^4 \, S_t^4 \, \Delta_t^2 + o\left(\Delta_t^2 \right),$$

$$f_{SS}^i = \frac{\phi(d_1)}{\sigma \, S_t \sqrt{\tau}},$$

and

$$\nu^i = S_t \sqrt{\tau} \, \phi(d_1),$$

where

$$d_1 = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

It follows that the change in BS IV for option i induced by a marginal change in demand for this option is

$$\frac{\partial \sigma_t^i}{\partial d_t^i} = \frac{1}{2} \, \gamma \, r \left(\frac{\sigma \, \Delta_t}{\sqrt{\tau}} \right)^2 \frac{\phi(d_1)}{\sqrt{\tau}} \, S_t + o\left(\Delta_t^2 \right).$$

d_t^i in Gârleanu, Pedersen, and Poteshman (2009) is client demand for option i in units of options. Define the corresponding dollar demand for option i as

$$\hat{d}_t^i = d_t\, p_t^i = \kappa^i\, S_t\, d_t$$

where κ^i is a function of moneyness K/S, σ, and τ only. It follows that

$$\frac{\partial \sigma_t^i}{\partial \hat{d}_t^i} \approx \frac{1}{2}\,\gamma\, r \left(\frac{\sigma\,\Delta_t}{\sqrt{\tau}}\right)^2 \frac{\phi(d_1)}{\kappa^i\,\sqrt{\tau}}.$$

Also note that

$$\frac{\partial \log\, \sigma_t^i}{\partial \hat{d}_t^i} = \frac{\partial \sigma_t^i}{\partial \hat{d}_t^i}\frac{1}{\sigma_t^i},$$

which, for σ_t^i close to its long-run mean σ^i, implies that

$$\frac{\partial \log\, \sigma_t^i}{\partial \hat{d}_t^i} \approx \frac{1}{2}\,\gamma\, r \left(\frac{\sigma\,\Delta_t}{\sqrt{\tau}}\right)^2 \frac{\phi(d_1)}{\kappa^i\,\sigma^i\sqrt{\tau}} \equiv \lambda^i.$$

Assume that the influence of $\hat{d}_t^j$ on BS IV for option i is of second order importance for all $j \neq i$. Let $\bar{d}^i$ be the mean of $\hat{d}_t^i$, then for σ_t^i close to σ^i and $\hat{d}_t^i$ close to $\bar{d}^i$ we approximately have

$$\log \sigma_t^i = \log \sigma^i + \lambda^i \left(\hat{d}_t^i - \bar{d}^i\right) = \left(\log \sigma^i - \lambda^i \bar{d}^i\right) + \lambda^i\, \hat{d}_t^i.$$

Now suppose dollar demand for option i, $\hat{d}_t^i$, follows the AR1 process

$$\hat{d}_t^i = (1-\rho^i)\bar{d}^i + \rho^i\, \hat{d}_{t-1}^i + e_t^i.$$

Substituting into the previous expression yields an AR1 process for log IV that is driven by the demand shock e_t^i:

$$\log \sigma_t^i = (1-\rho^i)\log \sigma^i + \rho^i\, \log \sigma_{t-1}^i + \lambda^i\, e_t^i.$$

6.3 Welfare Analysis

$$u\left(a_{i,t},\theta_t,a_t\right) = -\left(\theta_t - a_{i,t}\right)^2 - \lambda\left(a_t - a_{i,t}\right)^2$$

where $\lambda > -1/2$ and $a_t = \int a_{i,t}\, di$ is the average action in t. A first order condition wrt $a_{i,t}$ yields the optimal action for i in period t, namely

$$a_{i,t} = (1-\beta)\,\mathbb{E}_{i,t}\left(\theta_t\right) + \beta\,\mathbb{E}_{i,t}\left(a_t\right),$$

where $\beta = \lambda/(1+\lambda)$. As $\lambda > -1/2$ we have $|\beta| < 1$ and we can solve the above expression "forward" iteratively substituting out a_t (see Morris and Shin (2002) or Woodford (2003)) to get

$$a_{i,t} = (1-\beta)\sum_{k=0}^{\infty}\beta^k\,\theta_{i,t}^{(k+1)}$$

and

$$a_t = (1-\beta)\sum_{k=0}^{\infty}\beta^k\,\theta_t^{(k+1)}.$$

We substitute these expressions for the optimal action back into the utility function to obtain

$$-\left[(1-\beta)\sum_{k=0}^{\infty}\beta^k\left(\theta_t - \theta_{i,t}^{(k+1)}\right)\right]^2 - \left(\frac{\beta}{1-\beta}\right)\left[(1-\beta)\sum_{k=0}^{\infty}\beta^k\left(\theta_t^{(k+1)} - \theta_{i,t}^{(k+1)}\right)\right]^2.$$

Now note that

$$\mathbb{E}_{i,t}\left[\sum_{k=0}^{\infty}\beta^k\left(\theta_t - \theta_{i,t}^{(k+1)}\right)\right]^2 = \left(\frac{1}{1-\beta}\right)^2\mathbb{E}_{i,t}\left(\theta_t - \theta_{i,t}^{(1)}\right)^2 +$$
$$\sum_{k=1}^{\infty}\beta^{2k}\left(\theta_{i,t}^{(1)} - \theta_{i,t}^{(k+1)}\right)^2 + 2\sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty}\beta^{k+l}\left(\theta_{i,t}^{(1)} - \theta_{i,t}^{(k+1)}\right)\left(\theta_{i,t}^{(1)} - \theta_{i,t}^{(l+1)}\right),$$

and

$$\mathbb{E}_{i,t}\left[\sum_{k=0}^{\infty}\beta^k\left(\theta_t^{(k+1)} - \theta_{i,t}^{(k+1)}\right)\right]^2 = \sum_{k=1}^{\infty}\beta^{2(k-1)}\,\mathbb{E}_{i,t}\left(\theta_t^{(k)} - \theta_{i,t}^{(k+1)}\right)^2 +$$
$$2\sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty}\beta^{k+l-2}\,\mathbb{E}_{i,t}\left(\theta_t^{(k)} - \theta_{i,t}^{(k+1)}\right)\left(\theta_t^{(l)} - \theta_{i,t}^{(l+1)}\right) +$$
$$\sum_{k=1}^{\infty}\beta^{2(k-1)}\left(\theta_{i,t}^{(k+1)} - \theta_{i,t}^{(k)}\right)^2 + 2\sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty}\beta^{k+l-2}\left(\theta_{i,t}^{(k+1)} - \theta_{i,t}^{(k)}\right)\left(\theta_{i,t}^{(l+1)} - \theta_{i,t}^{(l)}\right).$$

Expected utility in period t can then be expressed as

$$U(\Omega_{i,t}) \equiv \max_{a_{i,t}} \mathbb{E}\left[u(a_{i,t}, \theta_t, a_t)|\Omega_{i,t}\right] =$$
$$- Var(\theta_t|\Omega_{i,t}) - \beta\,(1-\beta)\sum_{k=1}^{\infty}\beta^{2(k-1)}\,Var\left(\theta_t^{(k)}|\Omega_{i,t}\right) -$$
$$2\beta\,(1-\beta)\sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty}\beta^{k+l-2}\,Cov\left(\theta_t^{(k)},\theta_t^{(l)}|\Omega_{i,t}\right) - G(\Omega_{i,t})$$

where

$$G(\Omega_{i,t}) = (1-\beta)^2\sum_{k=1}^{\infty}\beta^{2k}\,\left[\mathbb{E}\left(\theta_t - \theta_t^{(k)}|\Omega_{i,t}\right)\right]^2 +$$
$$+ 2(1-\beta)^2\sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty}\beta^{k+l}\,\mathbb{E}\left(\theta_t - \theta_t^{(k)}|\Omega_{i,t}\right)\mathbb{E}\left(\theta_t - \theta_t^{(l)}|\Omega_{i,t}\right) +$$
$$\beta\,(1-\beta)\sum_{k=1}^{\infty}\beta^{2(k-1)}\,\left[\mathbb{E}\left(\theta_t^{(k)} - \theta_t^{(k-1)}|\Omega_{i,t}\right)\right]^2 +$$
$$2\beta\,(1-\beta)\sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty}\beta^{k+l-2}\,\mathbb{E}\left(\theta_t^{(k)} - \theta_t^{(k-1)}|\Omega_{i,t}\right)\mathbb{E}\left(\theta_t^{(l)} - \theta_t^{(l-1)}|\Omega_{i,t}\right).$$

We now calculate the ex-ante expectation of steady-state utility under a common prior. The steady state covariance matrix is constant. Let $\Sigma_{k+1,l+1}$ denote submitter i's steady-state covariance between $\theta_t^{(k)}$ and $\theta_t^{(l)}$ for all $k, l \geq 0$. Furthermore, under the common prior assumption we have $\mathbb{E}\left[\mathbb{E}(\theta_t^{(k)} - \theta_t^{(l)}|\Omega_{i,t})\right] = \mathbb{E}(\theta_t^{(k)} - \theta_t^{(l)}) = 0$. The ex-ante expectation of steady-state utility is then given by

$$U = -\Sigma_{1,1} - \beta\,(1-\beta)\sum_{k=1}^{\infty}\beta^{2(k-1)}\,\Sigma_{k+1,k+1}$$
$$- 2\beta\,(1-\beta)\sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty}\beta^{k+l-2}\,\Sigma_{k+1,l+1} - G.$$

where

$$
\begin{aligned}
G = (1-\beta)^2 \sum_{k=1}^{\infty} \beta^{2k} \left(\Sigma_{1,1} - 2\Sigma_{1,k+1} + \Sigma_{k+1,k+1}\right) + \\
+ 2(1-\beta)^2 \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} \beta^{k+l} (\Sigma_{1,1} - \Sigma_{1,k+1} - \Sigma_{1,l+1} + \Sigma_{k+1,l+1}) + \\
\beta\,(1-\beta) \sum_{k=1}^{\infty} \beta^{2(k-1)} (\Sigma_{k+1,k+1} - 2\Sigma_{k+1,k} + \Sigma_{k,k}) + \\
2\beta\,(1-\beta) \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} \beta^{k+l-2} (\Sigma_{k+1,l+1} - \Sigma_{k+1,l} - \Sigma_{k,l+1} + \Sigma_{k,l}).
\end{aligned}
$$

Assuming $|\beta|$ is "small" and ignoring terms of order $O(\beta^2)$, ex-ante expected steady-state utility is approximately

$$
U \approx -(1+\beta)\Sigma_{1,1} - 2\beta\,\Sigma_{2,2} + 2\beta\,\Sigma_{1,2}.
$$

6.4 Kalman Filter for Estimation

We estimate the parameters of the model presented in section 3, namely $\Phi = \{\rho, \bar{\theta}, \sigma_u, \sigma_\varepsilon, \sigma_\eta\}$, by maximum likelihood separately for each options contract. For a given contract, that is fixed time-to-maturity, moneyness, and option type (put or call), our data consist of two series. First, the time-series of submissions by the institutions participating in the Totem consensus pricing service for the specified contract; second, the consensus price time-series. If an institution does not submit a price in t, we treat this as a missing value. However, it is assumed that this institution received both the consensus price and the private signal about the fundamental in that period. As we will use the Kalman filter to derive the likelihood function, the treatment of such missing values is straightforward.[28] Let S be the total number of institutions that have submitted to Totem over the course of our sample and let $\iota_t \subset \{1, 2, .., S\}$ be the set of institutions active in t. Our sample of submissions is then given by $(\mathbf{m}_t)_{t=1}^T$, where $\mathbf{m}_t = (m_{j,t})_{j \in \iota_t}$ is a $|\iota_t|$-dimensional vector consisting of the individual period t consensus price submissions. We assume that consensus price submissions are institution i's best estimate of θ_t plus uncorrelated measurement error, i.e.

$$m_{i,t} = \bar{\theta} + \theta_{i,t}^{(1)} + \sigma_\psi \, \psi_{i,t} \text{ with } \psi_{i,t} \sim N(0, 1). \tag{14}$$

Following our model, we assume that the consensus price of period $t-1$, which we call p_t, equals the average first order belief of period $t-1$ plus aggregate noise, that is

$$p_t = \bar{\theta} + \theta_{t-1}^{(1)} + \sigma_\varepsilon \, \varepsilon_t.$$

Our data set for a given contract, $(\mathbf{y})_{t=1}^T$, then consists of the time-series of institutions' price submissions for this contract and the corresponding consensus price, i.e. $\mathbf{y}_t = (p_t \,, \mathbf{m}_t)^\intercal$.[29]

To estimate the model, we fix the maximum order of beliefs at $\bar{k} = 4$ and assume that the system has reached its stationary limit.[30] Average beliefs then evolve according to (9), namely

$$\theta_t^{(0:\bar{k})} = M_{\bar{k}} \, \theta_{t-1}^{(0:\bar{k})} + N_{\bar{k}} \, w_t,$$

[28]For the treatment of missing values in the Kalman filter see, for example, Durbin and Koopman (2012), Chapter 4.10.

[29]To be precise, $m_{j,t}$ is the natural logarithm of the Black-Scholes implied volatility of submitter j's time t price submission, and p_t is the natural logarithm of the consensus Black-Scholes implied volatility calculated by Totem for the corresponding contract.

[30]Allowing $\bar{k}$ greater than 4 does not change the estimates noticeably.

where $M_{\bar{k}}$ and $N_{\bar{k}}$ are functions of the parameters Φ defined recursively by equations (12) and (13) and $w_t = (u_t\,,\,\varepsilon_t)^\intercal \sim N(\mathbf{0}_2, I_2)$.[31] The dynamics of institutions i's conditional beliefs $\theta_{i,t}^{(1:\bar{k})}$ can be expressed in terms of deviations from average beliefs, $x_{i,t}^{(1:\bar{k})} \equiv \theta_{i,t}^{(1:\bar{k})} - \theta_t^{(1:\bar{k})}$, as

$$x_{i,t}^{(1:\bar{k})} = Q_{\bar{k}}\, x_{i,t-1}^{(1:\bar{k})} + V_{\bar{k}}\, \eta_{i,t},$$

where

$$Q_{\bar{k}} = [I_{\bar{k}} - K_{\bar{k}}\, D_{\bar{k}}]\, M_{\bar{k}} \ \text{ and } \ V_{\bar{k}} = K_{\bar{k}}\, R_\eta.$$

$K_{\bar{k}}$ is the stationary Kalman gain, $D_{\bar{k}}$ and R_η are defined in (**??**) and $\eta_{i,t} \sim N(0,1)$.

Given the linearity of the above system and the assumed normality of shocks the likelihood function for the observed data $(\mathbf{y})_{t=1}^T$ with $\mathbf{y}_t = (p_t\,,\mathbf{m}_t)^\intercal$ can be derived using the Kalman filter. We define $\alpha_t = (\theta_t^{(0:\bar{k})}\,, x_{1,t}^{(1:\bar{k})}\,, ..., x_{S,t}^{(1:\bar{k})}\,, u_t\,, \varepsilon_t)^\intercal$ to be the state of the system in t.

The *transition equation* of the system in state space form is then given by

$$\alpha_t = T\alpha_{t-1} + R\,\epsilon_t$$

where

$$T = \begin{pmatrix} M_{\bar{k}}\,,\, 0_{\bar{k}+1\times S\bar{k}+2} \\ 0_{S\,\bar{k}\times\bar{k}+1}\,,\, I_S \otimes Q_{\bar{k}}\,,\, 0_{S\,\bar{k}\times 2} \\ 0_{2\times\bar{k}+1+S\bar{k}+2} \end{pmatrix}\ ,\ R = \begin{pmatrix} N_{\bar{k}}\,,\, 0_{\bar{k}+1\times S} \\ 0_{S\bar{k}\times 2}\,,\, I_S \otimes \sigma_\eta\, V_{\bar{k}} \\ I_2\,,\, 0_{2\times S} \end{pmatrix}$$

and $\epsilon_t = (u_t\,, \varepsilon_t\,, \eta_{1,t}\,, ..., \eta_{S,t})^\intercal \sim N\left(\mathbf{0}_{2+S}, I_{2+S}\right)$.

We now derive the *observation equation* for the system given by

$$\mathbf{y}_t = c_t + Z_t\,\alpha_t + \phi_t.$$

First note that the consensus price p_t can be expressed in terms of the current state vector α_t as

$$p_t = \bar{\theta} + \theta_{t-1}^{(1)} + \sigma_\varepsilon\,\varepsilon_t = \bar{\theta} + e_2^\intercal\, M_{\bar{k}}^{-1}\,\theta_t^{(0:\bar{k})} - e_2^\intercal\, M_{\bar{k}}^{-1}\, N_{\bar{k}}\, w_t + \sigma_\varepsilon\,\varepsilon_t.$$

Next, note that we can write institution i's submission $m_{i,t}$ as

$$m_{i,t} = \bar{\theta} + \theta_{i,t}^{(1)} + \sigma_\psi\,\psi_{i,t} = \bar{\theta} + \theta_t^{(1)} + x_{i,t}^{(1)} + \sigma_\psi\,\psi_{i,t}.$$

[31]We use $0_{n\times m}$ to denote a $n\times m$ matrix of zeros, 1_n is a (column) vector containing n ones, and I_n is an n-dimensional identity matrix.

The above derivations allow us to write c_t and Z_t in terms of the parameters of the model. We start by defining an auxiliary matrix J_t that allows us to deal with missing submissions by some institutions in period t. Recall that $\iota_t \subset \{1, 2, .., S\}$ is the set of institutions submitting in t. Let $\iota_{k,t}$ designate the k-th element of the index ι_t. J_t is a $(|\iota_t| \times S)$ matrix whose k-th row has a 1 in position $\iota_{k,t}$ and zeros otherwise.

We thus have

$$c_t = J_t \, 1_{S+1} \, \bar{\theta} \;\text{ and}$$

$$\phi_t = \begin{pmatrix} 0 \\ \sigma_\psi \, J_t \, (\psi_{1,t}, ..., \psi_{N,t})^\intercal \end{pmatrix} \;\text{ with }\; \Gamma_t = \mathbb{E}\left(\phi_t \, \phi_t^\intercal\right) = \begin{pmatrix} 0 & \mathbf{0}_{|\iota_t|}^\intercal \\ \mathbf{0}_{|\iota_t|} & \sigma_\psi^2 \, I_{|\iota_t|} \end{pmatrix}.$$

Furthermore, we have $Z_t = J_t \, Z$ where

$$Z = \begin{pmatrix} e_2^\intercal \, M_{\bar{k}}^{-1} \,,\, 0_{1\times S\bar{k}} \,,\, e_2^\intercal \, M_{\bar{k}}^{-1} \, N_{\bar{k}} + (0 \; \sigma_\varepsilon) \\ 0 \,,\, 1 \,,\, e_1^\intercal \\ 0 \,,\, 1 \,,\, e_{\bar{k}+1}^\intercal \\ \vdots \\ 0 \,,\, 1 \,,\, e_{(S-1)\bar{k}+1}^\intercal \end{pmatrix}.$$

Given a prior for the state of the system at $t = 1$, $\alpha_1 \sim N(\mathbf{a}_1, P_1)$, we can now apply the usual Kalman filter recursion to derive the likelihood function for our data $(\mathbf{y}_t)_{t=1}^T$ given the parameter vector Φ denoted $L\left((\mathbf{y}_t)_{t=1}^T \mid \Phi\right)$. We obtain maximum likelihood estimates for Φ by maximising the corresponding log-likelihood function numerically.

6.5 Additional figures

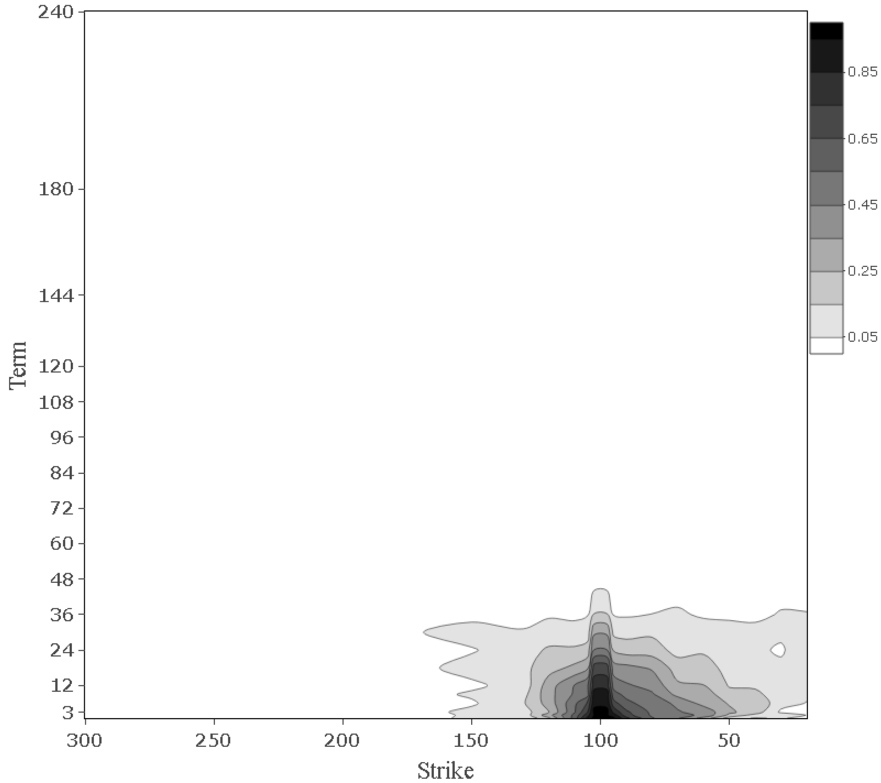

Figure 6: IHS Markit Totem surface vs on-exchange option trade

This figure presents the average percentage of days in a month options on the S&P500 index are traded between 1998 and 2015. Days where the total volume is 10 or higher are included as trading days. The data includes both ITM and OTM option contracts. The trading activity is depicted against the surface constructed from all the options that are available in the Totem service on February 2015. Due to the coarse grid of the options reported to the Totem service, exchange traded contracts in the proximity of a Totem contract are aggregated to one point. Proximity is defined here as less than half the distance to the next totem contract by term and moneyness. For a particular submission date we sum the days of trade from the day after the previous submission date till the current submission date. The data is provided by OptionMetrics.

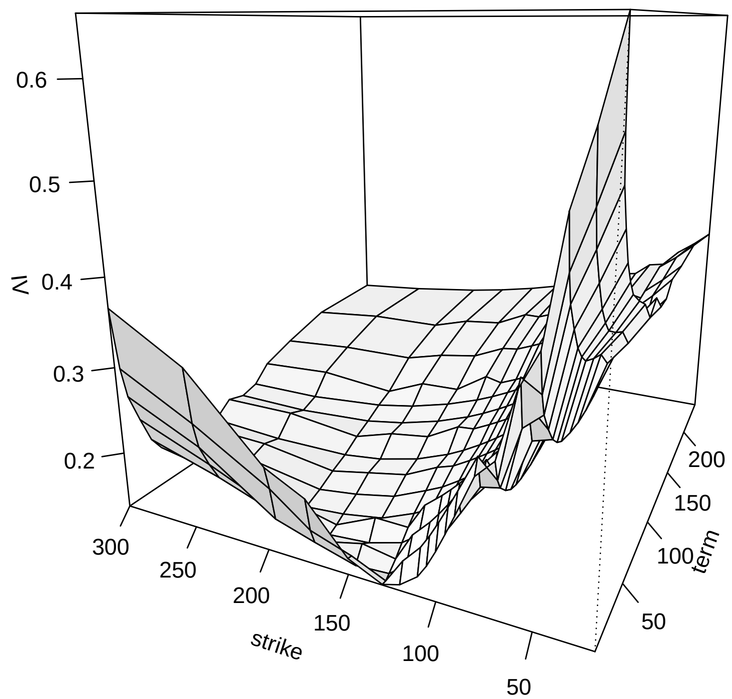

Figure 7: Average consensus IV (S&P 500)

This figure depicts the time-series average of the consensus price derived from the submitted midquotes. These midquotes are submitted by large broker dealers to IHS Markit's Totem service. The midquote estimates are expressed as implied volatilities. The axis labeled term indicates the time-to-maturity of the option contract in months. The axis labeled strike indicates the moneyness of the option contract. Moneyness is defined as the strike price devised by the spot price multiplied by 100. All contracts are out-of-money contracts, except contracts with the moneyness 100 which are at-the-money. The data sample is from December 2002 to February 2015 for the option contracts on the S&P 500 index.

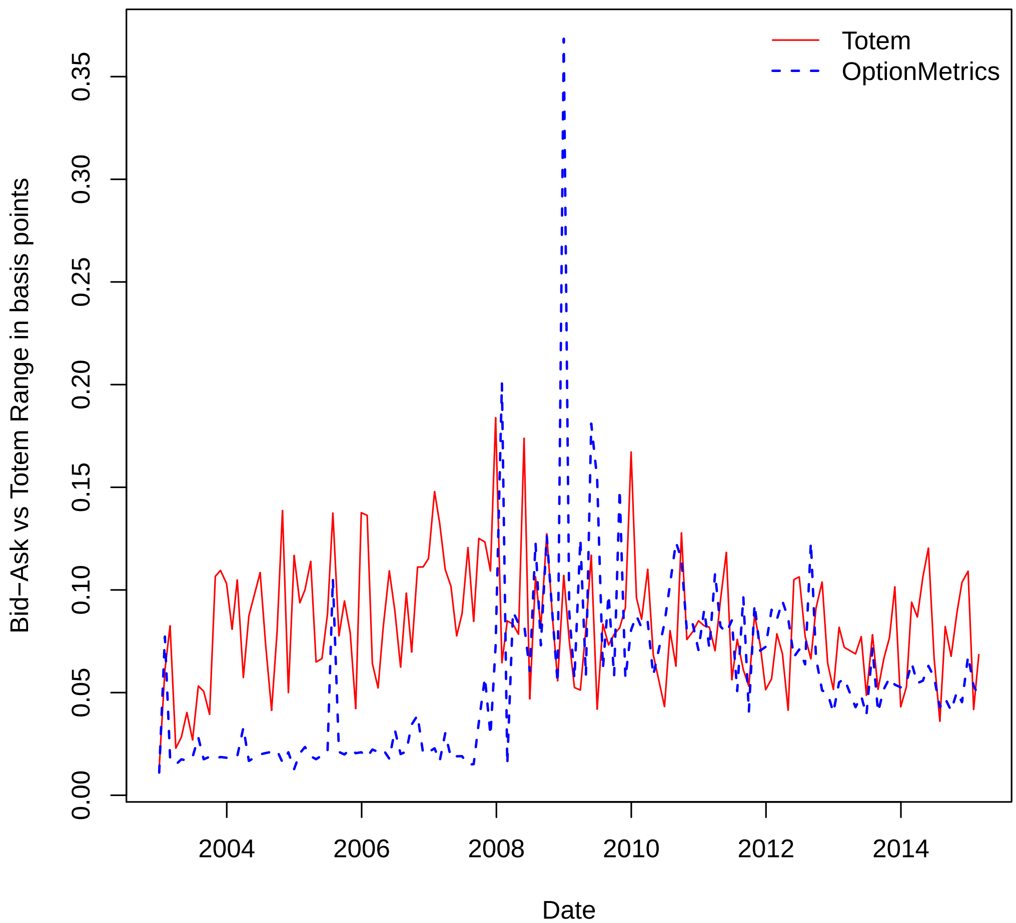

Figure 8: Bid-Ask spread vs Submission Range IV (S&P 500)

The figure above displays the difference between the range of the price submissions to the Totem service and bid-ask spread on traded options from OptionMetrics. The bid-ask spread is given by the best difference between the best closing bid price and best closing ask price across all US option exchanges. The options in the Totem service are matched to the traded options in the OptionMetrics database. On a given Totem valuation date we match OptionMetrics option contracts that are a close proxy for the totem option contracts. We search for contracts with a $\pm$ 10 days-to-maturity and a ± 1 moneyness value. When multilple options match the criteria an average is taken of their bid-ask spread.

6.6 Valuation submission process

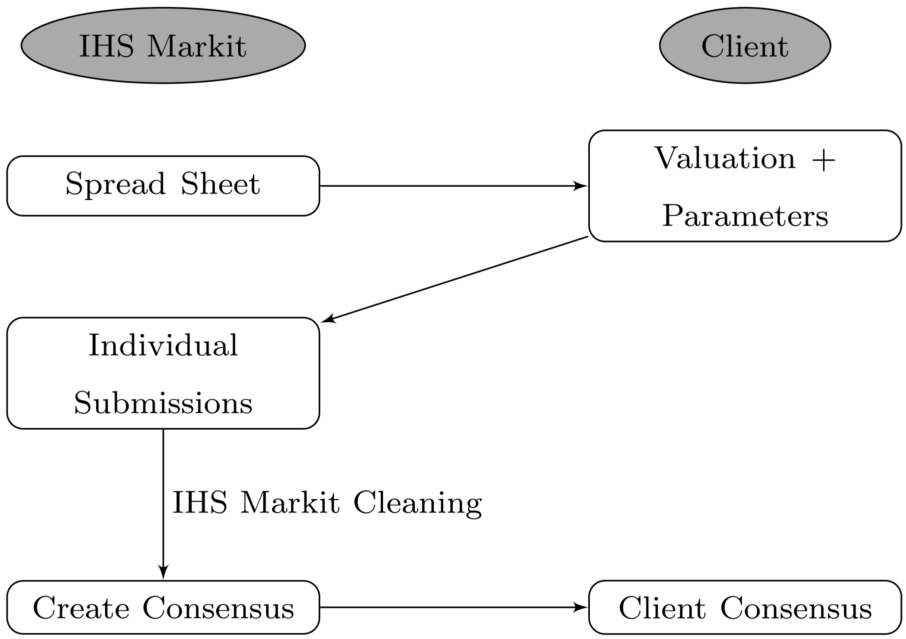

Figure 9: Diagram: Submission process

Figure 9 depicts a diagram of the submission process to IHS Markit's Totem service for plain vanilla index options.[32] Totem issues at the end of each month a spread sheet to $N_{K,T}$ submitters. Here K is the moneyness of the contract defined as the strike price divided by the spot price multiplied by 100 and T is the time-to-maturity of the contract in months. participating submitters are required to submit their mid-price estimate for a range of put option with a moneyness between 80 and 100 and a range of call option with a moneyness ranging from 100 to 120 with a time to

[32]Data provided by IHS Markit^{TM} - Nothing in this publication is sponsored, endorsed, sold or promoted by IHS Markit or its affiliates. Neither IHS Markit nor its affiliates make any representations or warranties, express or implied, to you or any other person regarding the advisability of investing in the financial products described in this report or as to the results obtained from the use of the IHS Markit Data. Neither IHS Markit nor any of its affiliates have any obligation or liability in connection with the operation, marketing, trading or sale of any financial product described in this report or use of the IHS Markit Data. IHS Markit and its affiliates shall not be liable (whether in negligence or otherwise) to any person for any error in the IHS Markit Data and shall not be under any obligation to advise any person of any error therein.

maturity of 6 months. Submitters which want to submit to any other contracts with a different maturity or/and different moneyness are required to submit to all the available terms and strikes which lie in between the required contracts and the additionally demanded contracts.

Submitter i submits its mid-price estimate for different out of the money put and call options, $P_t^i(p, K, T)$ and $P_t^i(c, K, T)$ respectively. The inputs which are required in addition to the mid-price estimates are:

- Their discount factor $\beta_t^i(T)$.
- Reference level R_t^i (This is the price of a futures contract with maturity date closest to the valuation date, i.e t.)
- Implied spot level $S_t^i(K, T)$ (Implied level of the underlying index of the futures contract)

There are strict instructions on the timing of the valuation of the contract and the reference level used. To address any issues which might still arise with respect to valuation timing and the effect it could have on the comparability of the prices, the various prices are aligned according to the following mechanism.

1. $\text{Basis}_i = {R^i}_t - S_t^i(K = 100, T = 6)$
2. $S_t^{i*}(K, T) = \underset{i}{\text{mode}}[{R^i}_t] - \text{Basis}_i$
3. Remove from $S_t^{i*}(K, T)$ the lowest, highest and erroneous adjusted spot levels.
4. $\bar{S}_t(K, T) = \underset{i}{\text{mode}}[{R^i}_t] - \frac{1}{N^*(K,T)} \sum_{i=1}^{N^*(K,T)} S_t^i(K, T)$

This consensus implied spot from the at-the-money 6 month option is used for all other combinations of K and T. The submitted prices are restated in terms of $\bar{S}_t(K, T)$, giving: $\widehat{p}_t^i(\{c, p\}, K, T) = {}^{P_t^i(\{c,p\},K,T)}/_{\bar{S}_t(K,T)}$.

Given the submitted quantities a security analyst calculates various implied quantities to validate the individual submissions. The security analyst utilizes put-call parity on ATM options to retrieve the relative forward, i.e

$$f_t^i(K, T) = \frac{\widehat{p}_t^i(c, K, T) - \widehat{p}_t^i(p, K, T)}{\beta_t^i(K, T)} + 1$$

The above inputs are then used in the Black and Scholes model,

$$\widehat{p}_t^i(c, K, T) = \beta_t^i(K, T)\left[f_t^i(K, T) N(d_1) - KN(d_2)\right]$$

$$d_1 = \frac{\ln\left(\frac{f}{K}\right) + \left(\frac{\sigma^2}{2}\right)\Delta T_t}{\sigma\sqrt{\Delta T_t}} \quad \text{, where} \quad \Delta T_t = \frac{\text{days(T)}}{365.25}$$

$$d_2 = d_1 - \sigma\sqrt{\Delta T_t}$$

to back-out the implied volatility, $\sigma_i(K, T)$.

When reviewing submissions security analysts compare the implied volatilities against other submitted prices and market conditions by taking the following points into consideration:

- The number of contributors
- Market activity & news
- Frequency of change of variables
- Market conventions
- In a one way market, is the concept of a mid-market price clearly understood?
- The distribution and spread of contributed data

In addition to these criteria security analyst also visually inspect the ATM implied volatility term structure and the implied volatility along the moneyness for a given term.[33] After the vetting process the security analyst proceeds to the aggregation of the individual submissions into the consensus data.

Given the Black and Scholes model they back out $\sigma_i(K, T)$ and aggregate it into the consensus implied volatility.

[33] Also referred to as the skew or smile.

$$\bar{\sigma}(K,T) = \frac{1}{n_{(K,T)} - n^r} \sum_{i=1}^{n_{K,T}-n^r} \sigma_i(K,T)$$

Here n^r are the number of excluded prices. The exclusions consist of the lowest, highest and rejected prices. The highest and lowest acceptable $\sigma_i(K,T)$ are consistent and reasonable IV's, but are excluded to safeguard the stability of the consensus IV.[34] The same process takes place for the submitted prices.

The submitters of which the pricing information is not rejected receive from the security analyst the consensus information.[35] The consensus data includes the average, standard deviation, skewness and kurtosis of the distribution of acceptable prices and implied volatility. They also include the number of submitters to the consensus data.

[34] If the number of acceptable prices is 6 or below the highest and lost submissions are included in the calculations.

[35] The time between submission of the mid-price estimate and receiving the consensus price back normally takes less than half a day.

6.7 Data

Table 1: Available data for plain vanilla option on the S&P 500

	moneyness																		
term	20	30	40	50	60	70	80	90	95	100	105	110	120	130	150	175	200	250	300
1	08/09	08/09	08/15	13/15	07/15	13/15	02/15	02/15	02/15	02/15	02/15	02/15	02/15	09/15	07/15	09/15	07/15	-	-
2	08/09	08/09	08/15	13/15	08/15	13/15	08/15	08/15	08/15	08/15	08/15	08/15	08/15	09/15	08/15	09/15	08/15	-	-
3	08/09	08/09	08/15	13/15	07/15	13/15	02/15	02/15	02/15	02/15	02/15	02/15	02/15	09/15	07/15	09/15	07/15	-	-
6	08/15	08/15	08/15	13/15	98/15	13/15	98/15	98/15	98/15	98/15	98/15	98/15	98/15	09/15	99/15	09/15	02/15	08/15	08/15
9	08/15	08/15	08/15	13/15	07/15	13/15	07/15	07/15	07/15	07/15	07/15	07/15	07/15	09/15	07/15	09/15	07/15	08/15	08/15
12	08/15	08/15	08/15	13/15	98/15	13/15	98/15	98/15	98/15	98/15	98/15	98/15	98/15	09/15	99/15	09/15	02/15	08/15	08/15
18	08/15	08/15	08/15	13/15	07/15	13/15	07/15	07/15	07/15	07/15	07/15	07/15	07/15	09/15	07/15	09/15	07/15	08/15	08/15
24	08/15	08/15	08/15	13/15	98/15	13/15	98/15	98/15	98/15	98/15	98/15	98/15	98/15	09/15	99/15	09/15	02/15	08/15	08/15
30	08/15	08/15	08/15	13/15	07/15	13/15	07/15	07/15	07/15	07/15	07/15	07/15	07/15	09/15	07/15	09/15	07/15	08/15	08/15
36	08/15	08/15	08/15	13/15	98/15	13/15	98/15	98/15	98/15	98/15	98/15	98/15	98/15	09/15	99/15	09/15	02/15	08/15	08/15
48	08/15	08/15	08/15	13/15	98/15	13/15	98/15	98/15	98/15	98/15	98/15	98/15	98/15	09/15	99/15	09/15	02/15	08/15	08/15
60	08/15	08/15	08/15	13/15	98/15	13/15	98/15	98/15	98/15	98/15	98/15	98/15	98/15	09/15	99/15	09/15	02/15	08/15	08/15
72	10/15	10/15	10/15	13/15	10/15	13/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15
84	08/15	08/15	08/15	13/15	99/15	13/15	99/15	99/15	99/15	99/15	99/15	99/15	99/15	09/15	99/15	09/15	02/15	08/15	08/15
96	10/15	10/15	10/15	13/15	10/15	13/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15
108	10/15	10/15	10/15	13/15	10/15	13/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15	10/15
120	08/15	08/15	08/15	13/15	99/15	13/15	99/15	99/15	99/15	99/15	99/15	99/15	99/15	09/15	99/15	09/15	03/15	08/15	08/15
144	08/15	08/15	08/15	13/15	05/15	13/15	05/15	05/15	05/15	05/15	05/15	05/15	05/15	09/15	05/15	09/15	05/15	08/15	08/15
180	08/15	08/15	08/15	13/15	05/15	13/15	05/15	05/15	05/15	05/15	05/15	05/15	05/15	09/15	05/15	09/15	05/15	08/15	08/15
240	11/15	11/15	11/15	13/15	11/15	13/15	11/15	11/15	11/15	11/15	11/15	11/15	11/15	11/15	11/15	11/15	11/15	11/15	11/15

This table gives the coverage of the data for the specific contracts on the S&P 500 Index. The table reports the start and end year that a contract covers.

Table 2: Average number of submitters for plain vanilla option on the S&P 500

	moneyness																		
term	20	30	40	50	60	70	80	90	95	100	105	110	120	130	150	175	200	250	300
1	14	14	17	18	18	21	21	21	21	21	21	21	21	18	18	18	17	-	-
2	14	14	17	18	18	21	24	24	24	24	24	24	24	18	18	18	17	-	-
3	14	14	17	19	18	21	21	21	21	21	21	21	21	19	18	18	18	-	-
6	17	17	21	24	27	30	31	31	31	31	31	31	31	29	27	24	19	13	12
9	16	16	21	24	26	30	29	29	29	29	29	29	29	29	26	24	22	13	12
12	16	16	21	24	27	30	30	30	30	30	30	30	30	28	27	23	19	13	12
18	16	16	21	24	25	29	28	28	28	28	28	28	28	28	25	23	21	13	12
24	16	16	21	24	27	29	30	30	30	30	30	30	30	28	26	23	19	13	12
30	16	16	21	23	25	29	27	27	27	27	27	27	27	28	25	23	20	13	12
36	16	16	20	23	26	28	29	29	29	29	29	29	29	27	26	23	18	13	11
48	16	16	20	22	26	27	29	29	29	29	29	29	29	26	25	22	18	12	11
60	16	16	20	22	25	26	28	28	28	28	28	28	28	26	25	22	18	12	11
72	16	16	19	21	25	24	25	25	25	25	25	25	25	24	24	21	20	13	12
84	15	15	18	20	24	24	25	25	25	25	25	25	25	24	23	21	17	12	11
96	15	15	18	19	22	23	23	23	23	23	23	23	23	22	22	19	19	12	11
108	14	14	17	17	21	20	22	22	22	22	22	22	22	21	21	18	18	12	11
120	13	13	16	17	20	20	21	21	21	21	21	21	21	21	20	17	15	11	10
144	11	11	13	13	13	14	13	13	13	13	13	13	13	14	13	14	12	9	8
180	10	10	11	12	11	12	11	11	11	11	11	11	11	12	11	12	11	8	8
240	5	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	5	5

This table provides the average number of submitters for the specific options on the S&P 500 Index. These are the accepted prices per contract for the dates that the contract is polled. In our analysis we ignore submissions with a price of 0. The data sample is from December 2002 till February 2015.