Collateral and Reputation in a Model of Strategic Defaults*

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Abstract

This paper builds a finite-horizon model to study the role of physical collateral in a model of strategic defaults, when the borrower can develop reputation for being honest. Asset ownership increases attractiveness of the reputational channel: the borrower who would prefer to remain in autarky in the absence of the asset applies for collateralized debt. Pledging the asset as collateral facilitates reputation building, which is especially successful at the times of asset price drops, because these are the times when default is most tempting. The model sheds some light on the co-movement of defaults and the household's financial and non-financial income.

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1 Introduction

The length and severeness of the U.S. 2007-2009 mortgage crisis drew attention of both academic community and wide public to the causes of why the defaults were so widespread, and to the damage inflicted on the economy as a whole. It was suggested that *strategic defaults*[1] constituted a significant fraction of all loan delinquencies.[2] Furthermore, the collapse in house prices and the associated reduction in home equity for the borrowers were one of the primary reasons for the households to default on their mortgage. Even though there exists ample evidence that the drop in home equity and the decision to default are tightly connected,[3] there is still a gap in the theoretical literature which would link the borrower's reputation for repayment, on the one hand, and the use of collateral for lending, on the other. The present paper tries to fill in this gap.

We focus on the interaction between extrinsic and intrinsic incentives to repay. It is shown that there exists *complementarity* between developing a reputation for being honest and pledging the asset as collateral: other things being equal, the potential borrower would not wish to take a loan if either (i) he can develop a reputation, but cannot collateralize the loan, or (ii) can use collateral but cannot build reputation. At the same time, the borrower would be willing to take a loan when *both* channels are present.

This result is non-trivial, since common sense would suggest that reputation and collateral are *substitutes* to each other: the borrower would either use his goodwill or physical collateral in the loan contract. Complementarity arises for the following reason: it is precisely when house prices fall that the borrower finds it most tempting to default on the loan – and therefore, debt repayment signals to the lenders that the borrower is of the 'honest' type, which in turn would allow borrowing on more favorable terms in the future. Hence, this reputational channel acts as an *insurance against downside risks* associated with asset price drops.

In addition, we explore how the option to sell the asset interacts with the option to default. The borrower would prefer to sell the asset whenever the ratio of its price relative to the borrower's non-financial income is either very low or very high, and the borrower would prefer to borrow against the asset if this ratio is intermediate.

This paper builds a setup in which the long-living borrower in each period applies for the loan from a group of short-lived lenders. Each lender observes whether the borrower has defaulted in the past and updates his posterior that the borrower is an 'honest' type. The model demonstrates that asset ownership can significantly lever up the borrower's scope of reputation building: he is more eager to choose to borrow (as opposed to staying in *autarky*), as long as the ratio of the asset price to his non-financial income is neither too high nor too low. The opportunity to pledge the asset, in the form of physical collateral, offers additional credibility that is valuable, because it increases the possibility to develop

[1] The findings of Fay et al. (2002) suggest that an important component affecting the household's decision to file for bankruptcy is financial benefits of default. According to Edmans (2010), strategic defaults constitute around 30% of all housing delinquencies.

[2] See Zingales (2010).

[3] The good overview of the empirical evidence is provided by Mian and Sufi (2015).

his reputation: in the future, the asset price might drop, and if *despite this drop*, the borrower still decides to repay, this acts as a *signal* to the lenders that the borrower is 'honest'. In that sense, physical and reputational collateral tend to be *complementary* to each other: the asset becomes valuable both for the option to resell it in the future and for the option to default.

Reputation is introduced along the lines of Kreps and Wilson (1982) and Milgrom and Roberts (1982). Each lender starts with a given prior probability that the borrower is 'honest' (meaning that he faces infinitely high cost of default, and thus will repay the loan under all circumstamces). That way, the 'fully rational' borrower who *is* capable of default (the 'strategic' type) may find it worth *mimicking* the behavior of the 'honest' type via current repayment in order to benefit from default on a loan in the future.

The main body of the paper develops a three-period example in which the individual borrower faces an overlapping generations of short-lived, perfectly competitive lenders. At each date, the borrower can offer a non-contingent debt contract. Each lender observes the proposed contract and the borrower's credit history; based on that information, the lender tries to infer the borrower's type and decides whether to accept the loan or reject it. Debt contracts can also be collateralized by the asset. Finally, the price of the asset is assumed to fluctuate for exogenous reasons.

The main results of the paper are stated in Proposition 4 and Theorem 1. They characterize equilibrium, in which at the initial date, the borrower prefers to sell the asset at very high and very low prices, but chooses to keep the asset and borrow against it for the intermediate price range. In the continuation game when there was borrowing at the initial date, the strategic borrower (i) defaults on the loan with certainty if the asset price turns out to be very low, (ii) repays the debt with certainty and sells the asset if the price turns out to be high, and (iii) randomizes between default and repayment whenever the price lies in between the two thresholds. It is shown that, as long as the borrower is sufficiently impatient, this result holds even in the limiting case as the probability that the borrower is of the 'honest' type tends to zero.

This equilibrium structure points to the existence *complementarity* between the external and internal enforcement mechanisms for repayment: at some price levels, the borrower can gain more by establishing reputation for honesty whenever he can also collateralize the loan contract. In this sense, reputational concerns complement physical collateral, and vice versa.

1.1 Literature review

Our paper delivers some insights which fit within the empirical literature of the 2007-2009 housing bubble. Mian and Sufi (2009) suggest that credit expansion of 2002 to 2005 was accompanied by a sharp *decline* in relative income growth, in contrast to their positive comovement in the preceding decade. The analysis of the benchmark case developed in section 3.3 suggests that the reputational channel does not work, and thus the complementarity between collateral and reputation starts to play a role, once income growth slows down: this is the time when the borrowers need the collateral in order to be willing to continue reputation building. By contrast, the reputational channel works on its own at times of high income growth.

Guiso et al. (2013) establish that one of the main contributing factors to the willingness to walk away from the mortgage is the relative equity shortfall (as a percentage of the value of the house): a one standard deviation increase in this measure is likely to raise the probability of strategic default by a quarter, this impact being reduced for those who lived in their house for more than 5 years. In our setup, the borrower would wish to back the loan by the asset when the reputation is initially low; on the other hand, honest borrowers with high reputation can successfully separate themselves from the 'strategic' types when the reputation is initially high. Although the horizon is kept fixed, to the extent that the borrower's reputation positively correlates with the time that since the house was bought, our model appears to be consistent with this result.

Exploring the key drivers of mortgage defaults, Elul et al. (2010) have pointed to the borrower's *illiquidity* as one of the main factors, along with negative equity. In our framework, the borrower's illiquidity can be proxied his non-financial income.[4] The result of section 4 imply that lower non-financial income in the future is likely to narrow down the range for asset prices at which the strategic type will randomize between default and repayment, thereby reducing the scope for reputation building and increasing the threshold below which default will be certain.

Focussing on the demand-side explanations, Dell'Ariccia et al. (2012) tested the relationship between borrowing constraints and credit expansion during the U.S. housing boom. They showed that the areas with faster credit growth were facing smaller loan denial rates, subsequently experiencing a spike in delinquency rates, the effect complementary to the supply-side channel of increased securitization and the rise in housing prices. In the context of the present paper, the main drivers for increased demand for borrowing are heavy discounting and the prospective rise in future income. The findings of Theorem 1 suggest that both the decline in the discount factor and the increase in the borrower's future income tend to expand the range of parameters (the lenders' prior π_0) for which the reputational channel works.

Some of the insights of the our paper are applicable to secondary loan markets in general. Chari et al. (2014) develop a dynamic adverse selection model showing that a small reduction in collateral values tends to generate sharp collapses in issuance of new secondary loans, with the similar impact on incentives to develop reputation. However, in their story, reputation is related to the quality of the borrowers' projects, which determines the *ability* to repay the loan. The strategic defaults' story outlined in our work proposes a complementary explanation which drives the qualitatively similar dynamics.

The results of Garmaise and Natividad (forthcoming) explore the relationship between worsening of the credit rating and the future access to credit. They found that

[4] In their study, Elul et al. (2010) used credit card utilization rates as a proxy for illiquidity. Additionally, they show that county-level unemployment shocks also contribute to the default risk; the same applies for those having a second mortgage.

for the delinquent borrowers, the downgrade accounts from 25% to 65% reduction in subsequent financing. Although our setup cannot properly account for the downgrades,[5] it lends some indirect support for these findings. The reputational channel creates an endogenous intertemporal link between the borrowing constraints: any factor that (i) facilitates future borrowing and (ii) makes it more desirable at the same time facilitates current lending, since the borrower's incentive to repay the loan is positively affected by the subsequent access to credit.

The paper is organized as follows. Section 2 describes the basic framework. Section 3 studies several benchmark cases. Section 4 focuses on the interplay between extrinsic and intrinsic repayment incentives; it delivers the main result of the paper. Section 5 briefly discusses implications of the model and provides extensions to infinite horizon and to overlapping generations' versions of the setup. Section 6 concludes.

All the proofs are relegated to the Appendix.

2 The model

2.1 Agents

Consider an individual who lives for $(T+1)$ periods $0, 1, 2, \ldots, T$. His instantaneous utility of consumption is given by $u(c_t)$, with $u' > 0$ and $u'' \leq 0$. Moreover, the individual discounts the future, so that his overall lifetime utility equals

$$U(\{c_t\}_{t=0}^{T}) = \sum_{t=0}^{T} \beta^t u(c_t), \quad \text{with } \beta < 1. \tag{2.1}$$

The individual receives deterministic and publicly observed income stream $\{y_t\}_{t=0}^{T}$. Since he is assumed to start with zero debt, and therefore without loss of generality, we take the date-0 income to be zero: $y_0 = 0$. In what follows, y_t will be referred to as date-t *non-financial* income.

2.2 Asset

The individual starts with holding one unit of non-divisible physical asset which delivers him the per-period dividend $x \geq 0$ in monetary terms. At each date $t \in \{0, \ldots, T\}$, the asset can be sold at a price p_t.

It is assumed that the dividend income x accrues specifically to the individual in question and is lost whenever the asset is transferred to a third party.[6]

[5] Due to the extremely conservative borrowing on behalf of the 'honest' type (the one who is assumed to face infinite costs of default), non-repayment completely reveals the borrower's type, which in turn implies that there can be no *partial* downgrade in equilibrium.

[6] In order to relate the price process $\{p_t\}$ to the asset's fundamental value, we can think of the asset as delivering a stochastic dividend p_2 at the end of date T. At each date t, there is an arrival of new information

The asset price follows the stochastic process satisfying the martingale property: for each $t = 0, 1, \ldots, T-1$,

$$p_{t+1}|p_t \sim F(\cdot|p_t), \quad \text{with} \int \widetilde{p}\mathrm{d}F(\widetilde{p}|p_t) \quad \text{and } p_0 > 0 \text{ given.} \tag{2.2}$$

2.3 Lending contracts

At each date t, the individual (henceforth "the borrower") faces the group of perfectly competitive lenders who have deep pockets and whose discount factor is normalized to one. Each date-t lender lives for two consecutive periods: he gives out the loan at date t and receives repayment at date $t+1$.

The borrower approaches one lender at random and offers him the contract $\mathcal{B}_t$. The date-t contract specifies (i) the amount that the individual can borrow today, b_t; (ii) the repayment on the debt b_t to be made to the lender at the subsequent date – R_{t+1}, and (iii) in case if the borrower still holds the asset, the loan contract specifies whether the borrower would want to pledge the asset as a collateral, $\kappa_t \in \{0, 1\}$, where $\kappa_t = 1$ stands for collateralized loan.

If the borrower pledges the asset as collateral, then in case of default in period t (on the loan b_{t-1} taken at the preceding date), the lender expropriates the asset and resells it at a price p_t to a third party.

Perfect competition on the part of lenders implies that the borrower has full bargaining power. Once the borrower has offered the contract $\mathcal{B}_t$, the lender decides whether to accept or reject the contract; in case of indifference, the lender can randomize between accepting and rejecting, with the acceptance probability denoted by α_t.[7] It is assumed that, if the borrower has been turned down, he cannot approach another lender; likewise, he cannot approach multiple lenders simultaneously.

Importantly, we assume that the borrower *cannot commit* to the lending contract $\mathcal{B}_t$ offered to the lender at date t before date t actually arrives. In particular, the repayment R_t on the date-$(t-1)$ cannot be made contingent on b_t, the loan that the borrower would be able to get at date t.

The date-t *public history* h^t includes the borrower's past and prospective income stream $\{y_t\}_{t=0}^{T}$, the history of asset prices up to date t, $\{p_0, \ldots, p_t\}$, the amount of borrower's asset holdings $a_t \in \{0, 1\}$ (that is, whether the asset has been sold or not), and the borrower's *credit history* indicating whether the borrower has ever defaulted in the past. It is assumed that the lender's acceptance decision $\alpha(h^t, \mathcal{B}_t)$ can be made contingent on the date-t public history and the contract $\mathcal{B}_t$ with which the borrower approaches.

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regarding this dividend that gets encoded in the current asset price p_t.

Alternatively, as in the search literature, the agent's private value for holding the asset can be motivated by his hedging needs: in case if short-selling would require the individual to search for the asset lender and/or bargain over the lending fee, the borrower may attribute a higher valuation for the given stream of dividends than the lenders. See Duffie et al. (2002, 2007).

[7] This assumption is important for the construction of equilibrium in section 4. See the discussion that follows Proposition 4.

2.4 Reputation

At the start of date 0, the lenders hold a prior $\pi_0 \in [0,1]$ that the borrower is of the 'non-strategic', or 'honest', type. In terms of modelling, one can think of this type as the one who suffers an infinite disutility from default.[8] With complementary probability $1 - \pi_0$, the lenders believe that the borrower's type is 'strategic'. This implies that he does not face any costs of default, and hence will default on the debt whenever he finds it profitable to do so.

At each date t, lenders update their beliefs to

$$\pi_t = \pi(h^t; \mathcal{B}_t).$$

using Bayes' rule.

Date-t lenders can draw inference about the borrower's type upon observing the contract $\mathcal{B}_t$ that he proposes, which raises the possibility of signalling by contracts.

2.5 Timing

The timing within period t is as follows.

First, the borrower observes the current asset price p_t and decides whether to repay the existing debt R_t or default.

Second, conditional on repayment, the borrower obtains the per-period dividend income x and decides whether to sell or to keep the asset.

Finally, the borrower approaches the lenders with the loan contract $\mathcal{B}_t = (b_t, R_{t+1}, \kappa_t)$. The lender accepts (with probability α) or rejects (with probability $1 - \alpha$) the contract, and the borrower consumes the remaining amount.

The borrower's date-t consumption is given by

$$c_t = y_t + (1 - d_t)\left[(x + p_t s_t)a_t - R_t + \alpha(h^t)b_t\right], \tag{2.3}$$

where $d_t \in \{0,1\}$ is the default decision, $a_t \in \{0,1\}$ is the beginning-of-period asset holdings and $s_t \in \{0,1\}$, with $s_t \leq a_t$, is the selling decision.

The consumption is required to be non-negative: $c_t \geq 0$.[9]

2.6 An example

This section puts forth a leading example which will be used throughout the analysis in sections 3 and 4.

[8] It is only in this sense in which he differs from the borrower whom we refer to as 'strategic'. In particular, the 'honest' borrower's behavior is not postulated ad hoc, but derived from his optimization, *given* the additional restriction that default on the debt is not an option for him.

[9] As we will see, this condition imposes an implicit restriction on the amount that can be borrowed: since the 'honest' type faces infinite costs when he is *forced* into default, this type would prefer to borrow only to the extent that he can be certain to have non-negative consumption without defaulting under *all possible circumstances*.

First, we take $T = 2$, so that there are three dates $t = 0, 1, 2$.
Second, individual's utility is *linear* in consumption:

$$u(c_t) = c_t.$$

Third, the conditional distribution of date-t asset price is uniform:

$$\text{for } t = 0, 1: \quad p_{t+1} | p_t \sim \mathcal{U}[0, 2p_t], \quad \text{with } p_0 > 0 \text{ given.}$$

3 Benchmark cases

3.1 Autarky and full commitment

We start by considering two benchmark cases: the *autarkic* case, when the individual does not borrow (either because he is unable or because he does not want to), and the case of *full commitment*, when it is commonly known that the individual's type is 'non-strategic', implying that he will repay the loan with probability one. In line with the general setup, it is assumed that the borrower who is *forced* into default incurs an infinite loss, implying that he can borrow only against the non-stochastic component of his future income, which includes both the non-financial income stream and the dividend.

Our primary interest in this exercise concerns the way in which the borrowing decision interacts with the decision to sell the asset. The next proposition provides full characterization of the individual's optimal selling strategy and the borrower's utility:

Proposition 1. *In case of autarky, the individual chooses to sell the asset at date t if and only if the asset price exceeds the threshold: $p_t > p_t^a$, where the threshold is equal to*

$$p_t^a = \begin{cases} \frac{\beta x}{2(1-\beta)} \left(1 + \sqrt{1+\beta}\right) & \text{if } t = 0, \\ \frac{\beta x}{1-\beta} & \text{if } t = 1, \\ 0 & \text{if } t = 2. \end{cases} \tag{3.1}$$

In case of full commitment, the individual chooses to sell the asset if and only if $p_t > p_t^c$, where

$$p_t^c = \frac{p_t^a}{\beta} > p_t^a \quad \text{for both } t = 0, 1. \tag{3.2}$$

Proof. See Appendix A.1. □

For any period t, and for either regime ('autarky' and 'commitment', respectively), the individual's selling strategy is characterized by the threshold price (p_t^a or p_t^c), such that the individual chooses to sell the asset if and only if the period-t asset price exceeds the threshold.

The threshold is pinned down by the agent's indifference condition. This condition equates the payoff that the agent gets from selling the asset today (which is equal to

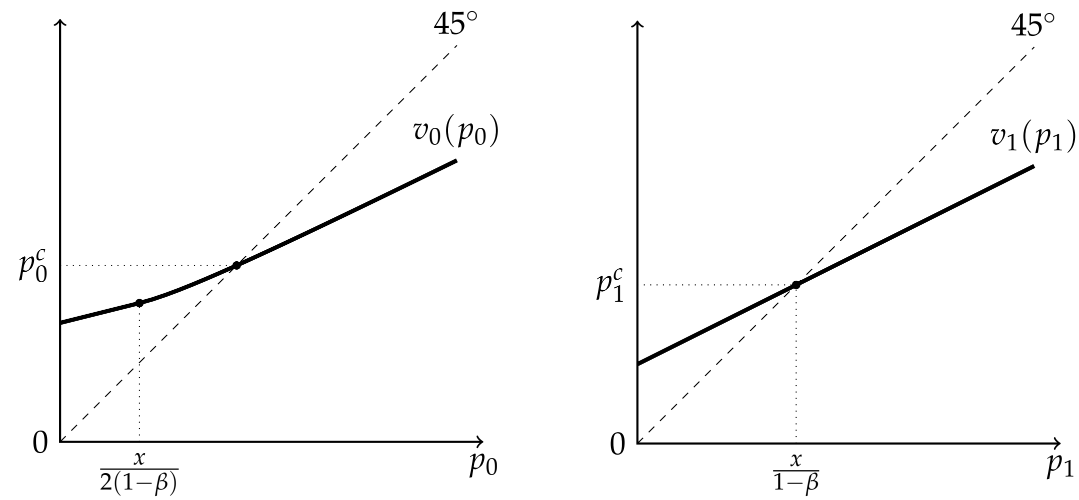


Figure 1: Determination of the thresholds p_0^c and p_1^c.

p_t) to the option value of keeping the asset for one more period, which is denoted by $v(p_t)$. For the autarkic regime, the option value consists of the discounted future-period dividend (βx) and the expectation of the *maximum* between the payoff the agent would get from selling the asset on the next date and the option to keep it. For the case of full commitment, keeping the asset allows the individual to borrow the amount x, effectively bringing one period forward the next period's dividend.

So, for $r \in \{a, c\}$, the option value of keeping the asset in period $t \in \{0, 1\}$ is recursively defined as follows:[10]

$$v^r(p_t) = \begin{cases} \beta\left(x + \mathbb{E}\Big[\max\left\{p_{t+1}, v^r(p_{t+1})\right\}|p_t\Big]\right) & \text{if } r = a, \\ x + \beta\mathbb{E}\Big[\max\left\{p_{t+1}, v^r(p_{t+1})\right\}|p_t\Big] & \text{if } r = c. \end{cases} \tag{3.3}$$

For either regime $r \in \{a, c\}$, the sequence of thresholds $\{p_t^r\}$ is pinned down by the indifference condition

$$p_t^r = v^r(p_t^r).$$

It can be easily shown that for any $t \in \{0, 1\}$, the option value of keeping the asset $v(p_t)$ is continuous and strictly increasing in p_t, that $v(0) > 0$, and that the slope of v is bounded above by the discount factor: $v'(\cdot) \leq \beta < 1$. Hence, the above equation has exactly one solution, and we have $p_t \lesseqgtr v(p_t)$ iff $p_t \lesseqgtr p_t^r$. Furthermore, from (3.3) one can easily see that the difference in option values is given by

$$\begin{aligned} \Delta v(p_t) &\equiv v^c(p_t) - v^a(p_t) \\ &= (1-\beta)x + \beta\left(\mathbb{E}\Big[\max\left\{p_{t+1}, v^c(p_{t+1})\right\}|p_t\Big] - \mathbb{E}\Big[\max\left\{p_{t+1}, v^a(p_{t+1})\right\}|p_t\Big]\right), \end{aligned}$$

[10]For the last date $t = 2$, we have $v(p_2) = 0$, and hence the agent is willing to sell the asset at *any* price (provided that he did not do so before). Evidently, this recursive characterization in (3.3) would remain valid for any finite horizon T.

which translates into the difference between the thresholds, $p_t^c - p_t^a$.

Figure 1 plots the graphs $v(p_t)$ for $t \in \{0, 1\}$, illustrating the way in which threshold prices for the full commitment regime $r = c$ are determined.[11] Both thresholds are increasing in both x and β: the agent is more eager to sell the asset now rather than keep it until the next period either in case of heavy discounting (low β) or in case of negligible dividends (low x). The discrepancy between the two thresholds, $\Delta p^a \equiv p_0^a - p_1^a$, also increases in both β and x.

Next, since $p_t^c > p_t^a$, other things equal, the committed agent will be more eager to *keep* the asset, since it raises his borrowing capacity and allows to consume the future dividends earlier.

When comparing the benefits of keeping the asset in period 0 of the committed borrower to those of the 'autarkic' borrower, one can notice two forces at work. On the one hand, the asset is more valuable to the committed borrower because it raises his borrowing capacity by bringing forward future-period dividends. On the other hand, the option to sell the asset in period 1 is more valuable to the borrower in case of autarky, because he sells the asset for a larger range of realizations of p_1, that is, $p_1^c > p_1^a$. It turns out that the first effect always dominates. In case if the agent is very impatient (β close to zero), the discrepancy between p_t^a and p_t^c is especially pronounced.

At the opposite extreme, in the limit when $\beta \to 1$, we have $p_t^a \to p_t^c$, and the asset is almost never sold – except for the cases when p_0 happens to be very large. Finally, notice that when the asset yields no dividend ($x = 0$), we have $p_t^a = p_t^c = 0$: in both cases, the agents would be willing to sell the asset right away, in period 0.

3.2 Borrowing against asset

Now consider the case when borrowing can be made only against the asset. That is to say, the lenders know with certainty that the borrower is of the 'strategic' type, so that $\pi_0 = 0$. Since loan repayment cannot be enforced by reputation, the lenders will accept only the collateralized loan contracts – those with $\kappa_t = 1$. (The only feasible loan contract without collateral is the 'autarkic' contract $\mathcal{B}_t = (0, 0, 0)$.) Hence, the agent who has sold the asset can no longer borrow.

The next proposition characterizes the individual's optimal selling, borrowing and default decisions.

Proposition 2. *At date* 0*, the individual sells the asset if and only if the asset price exceeds the threshold:* $p_0 \geq \hat{p}_0$*, where*

$$\hat{p}_0 = \tfrac{2x}{1-\beta^2}. \tag{3.4}$$

Otherwise, an individual keeps the asset and offers the contract

$$\left(\hat{b}_0, \hat{R}_1\right) = (2x, 2x). \tag{3.5}$$

[11]Similar graph can be drawn for the case of autarky ($r = a$). Explicit expressions for option values $v(p_t)$ can be found in the Appendix.

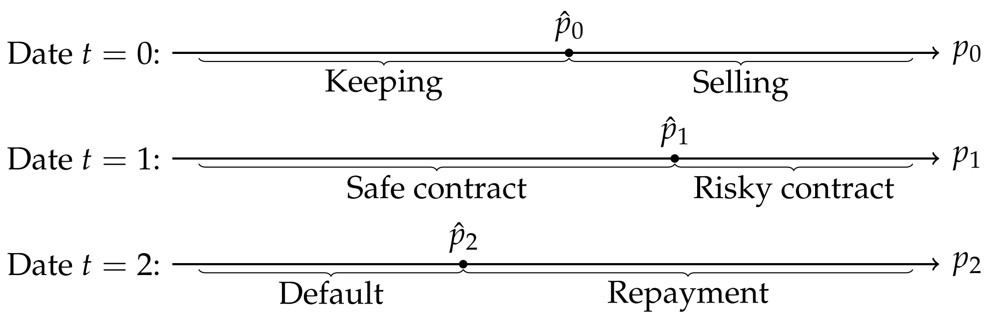


Figure 2: The structure of equilibrium with collateral only.

At date 1*, the individual who kept the asset (and borrowed against it) repays his loan and offers the loan contract*

$$(\hat{b}_1, \hat{R}_2) = \begin{cases} (x, x) & \text{if } p_1 < \hat{p}_1, \\ \left(p_1 + \frac{1-2\beta}{p_1} \left(\frac{x}{2(1-\beta)} \right)^2, 2p_1 - \frac{\beta x}{1-\beta} \right) & \text{if } p_1 \geq \hat{p}_1, \end{cases} \tag{3.6}$$

where

$$\hat{p}_1 = \frac{x}{2(1-\beta)}. \tag{3.7}$$

At date 2*, the individual defaults on the date-*1 *loan whenever the asset price falls below the threshold:* $p_2 < \hat{p}_2$*, where*

$$\hat{p}_2 = R_2 - x. \tag{3.8}$$

Proof. See Appendix A.2. □

The structure of equilibrium is illustrated on Figure 2.

At this point it should be observed that the individual who borrowed at date 0 never defaults on the loan at date 1 and never sells the asset at dates 1 and 2.

To gain intuition for the last claim, notice that selling the asset is weakly dominated by borrowing against it. This result does not depend on the specificities of the linear example and holds quite generally: there exists a lending contract (b_t, R_{t+1}) that can replicate the payoff from selling the asset: just set $b_t = \mathbb{E}[p_{t+1}|p_t]$ and make R_{t+1} infinitely large, so that at $t+1$ the agent defaults under all circumstances. Due to the martingale property of the price process, we have $\mathbb{E}[p_{t+1}|p_t] = p_t$, so that the sum which the agent can raise through borrowing can be made as large as the sum from selling the asset.

At date 0, depending on the realization of p_0, the borrower would either prefer to borrow the sum $\hat{b}_1 = 2x$, which will be repaid with probability one at date 1, or else, if p_0 is large enough, to borrow the sum

$$\hat{b}_0 = 2p_0 + x + \tfrac{x^2}{8p_0(1-\beta)},$$

an amount on which he will default for sure at $t = 1$. Such a risky loan contract is, in fact, equivalent to selling the asset, which is the interpretation given in the proposition.

Next , it should be noted that the bang-bang nature of the solution is due to linearity of the utility function. For concave utility, one would anticipate that in each period, the range of asset prices can be partitioned into the three regions: the agent would engage in risky borrowing for very large prices, would keep the asset and borrow safely (that is, borrow the amount that will be repaid on the next date under all circumstances) for the intermediate range of asset prices, and would default on the existing debt if the asset price happens to fall below the specified threshold.

The selling threshold $\hat{p}_0$ can be compared with p_0^c, the committed borrower's threshold: direct comparison of (3.2) and (3.4) reveals that

$$\hat{p}_0 > p_0^c$$

if and only if

$$\beta \leq \hat{\beta} \in (0,1),$$

where $\hat{\beta}$ is the unique solution to $\beta^3 + 2\beta^2 + 9\beta - 8 = 0$.

So, a relatively impatient individua will be more eager to *keep* the asset when he can borrow against it, as compared to the case when he can is known to be non-strategic, and hence can fully commit to repay without the need for the borrowing to be backed by the asset. Intuitively, with the increase in β, two opposing forces are at work: on the one hand, the strategic borrower would benefit more from the option to borrow against the asset and default on date 2; on the other hand, he would realize this option for a smaller range of realizations of p_2. For low β, the first effect dominates, making the strategic borrower less willing to sell the asset at date 1, whereas for high β, the second effect dominates.

To summarize the discussion, for the case when the individual is known to be strategic and can borrow against the asset, the agent sells the asset at date 0 if its price happens to be larger than the threshold; otherwise, he borrows safely and keeps the asset until date 1, in which case he borrows against the asset, and defaults at date 2 if the asset price drops below the threshold.

In this version of the model, defaults would be observed only at date 2, and only in case of substantial increase in the asset price from p_0 to p_1 and the subsequent decrease from p_1 to p_2, that is, when the asset price follows the hump-shaped pattern. To the extent that such a pattern in $\{p_t\}$ can be attributed to bubbles,[12] the model highlights one channel through which a burst of the bubble triggers default.

3.3 Borrowing against reputation

In this section, we consider the case when the individual does not hold the asset, and thus any loan contract $\mathcal{B}_t$ that he may offer to the lenders has $\kappa_t = 0$. At the same time, the lenders have the positive prior $\pi_0 > 0$ that the borrower is of the committed type.

Since there is no asset, the lenders cannot recover anything if there was a default, and so in period t they will be willing to lend positive amount only to the borrower

[12]Since the price process is postulated exogenously, this setup remains silent about the possible cause of the rise in the asset price above the fundamental in the first place.

who has not lost his reputation so far. This implies, in particular, that there cannot be signalling through contracts: in each period t, provided that there was no default in the past, the two types of borrowers *pool* on the contracts, with the strategic type mimicking the committed type: if in equilibrium in question the committed type ends up offering the autarkic contract $\mathcal{B}_t = (0,0,0)$, the strategic type can do no better than that, because any other contract would reveal his type, and hence will be rejected.

Since pledging the asset as collateral is not possible ($\kappa_t = 0$), to economize on notation, we will refer to the date-t typical loan contract as (b_t, R_{t+1}).

We consider the class of equilibria with the following structure:

1. At date 0, the borrowers pool on the contract (b_0, R_1);
2. At date $t = 1$, the strategic borrower defaults on the loan (b_0, R_1) with probability $\delta_1 \in [0,1]$;
3. Conditional on repayment at date 1, the borrowers pool on the contract (b_1, R_2);
4. At date $t = 2$, the strategic borrower defaults with probability one on (b_1, R_2).

The next lemma characterizes all possible equilibria depending on the parameters.

Lemma 1. *There exist several classes of signalling equilibria:*

I. If $\pi_0 \leq \beta^2$, the only equilibrium is

$$b_t^* = R_{t+1}^* = 0, \quad \textit{for } t = 0,1,$$

which corresponds to autarky.

II. If $\beta^2 \leq \pi_0 < \beta$, then in addition to the autarkic equilibrium, there exists two other types of equilibria:

(i) Any sequence of contracts $\{(b_0^*, R_1^*), (b_1^*, R_2^*)\}$ satisfying

$$b_0^* = \pi_0 R_1^*, \quad b_1^* = R_2^* \quad \textit{and} \quad R_1^* \leq \min\left\{\tfrac{\beta(1-\beta)}{\beta-\pi_0} b_1^*, y_1 + b_1^*\right\}$$

constitutes an equilibrium in which the strategic borrower defaults with probability one at both dates 1 and 2.

(ii) Any sequence of contracts $\{(b_0^*, R_1^*), (b_1^*, R_2^*)\}$ satisfying

$$b_0^* = \pi_0 R_2^*, \quad b_1^* = R_1^* \quad \textit{and} \quad R_2^* \leq y_1 + y_2$$

constitutes an equilibrium in which the strategic borrower defaults with probability

$$\delta_1^*(b_1^*, R_2^*; \pi_0) = 1 - \tfrac{\pi_0}{1-\pi_0}\left(\tfrac{R_2^*}{b_1^*} - 1\right)$$

at date 1 and defaults with probability one at date 2.

III. *If $\pi_0 \geq \beta$, then in addition to the equilibria described in cases (I) and (II), there exists another class of equilibria with the sequence of contracts $\{(b_0^*, R_1^*), (b_1^*, R_2^*)\}$ satisfying*

$$b_0^* = R_1^*, \quad b_1^* = \pi_0 R_2^*, \quad R_1^* \leq y_1 + b_1^* \quad \textit{and} \quad R_2^* \leq y_2,$$

This corresponds to an equilibrium in which the strategic borrower never defaults at date 1 *(and always defaults at date 2).*

Proof. See Appendix A.3. □

The equilibrium set expands as π_0 increases.

When the lenders' prior is very low ($\pi_0 < \beta^2$), the committed type of borrower will not find it optimal to ask for a loan at either date 0 or 1, so the only equilibrium will be autarky. For intermediate values of π_0 (when $\beta^2 \leq \pi_0 < \beta$), there exist equilibria in which borrowing occurs on date 0, then on date 1 the committed borrower repays the loan, successfully separates from the strategic type and keeps on borrowing at a risk-free rate, while the strategic type defaults on the date-0 loan with probability one.

In addition, there exist equilibria in which the strategic type randomizes between defaulting and repaying at date 1; in this class of equilibria, the committed type can only partially separate himself from the strategic type by repaying the date-1 loan. Nevertheless, the lenders' posterior π_1 following loan repayment is sufficiently high to induce the committed type to ask for positive borrowing at the end of date 1.

Finally, when π_0 is very large, there also exist equilibria in which at date 1, the strategic type repays the date-0 loan (b_0, R_1) with probability one, thus pooling with the committed type. This requires a large prior ($\pi_0 \geq \beta$), because the committed type should be willing to borrow at date 1 despite the fact that repayment of the date-0 loan did not bring any 'reputational reward' for him (since we have $\pi_1 = \pi_0$ due to pooling).

All of those equilibria are supported by out-of-equilibrium beliefs, according to which the lenders attribute any deviation from the pooling contract to the strategic type. This restriction passes the refinement of Cho and Kreps (1987). To see this, first consider the borrower who deviates from the equilibrium contract at date 1. Now, the lenders know that if it were the strategic type, he would default with certainty at date 2 – so, no one will provide the borrower the contract (b_1, R_2) on the next date *if the deviation from the equilibrium contract offered at the preceding date 0 has revealed that his type is strategic*. And if it happens so that at date 1, the borrower cannot repay the previous-period debt R_1 unless he can borrow b_1, the committed type will be *less* willing to deviate from the equilibrium contract (b_1, R_2), because this would force him into default on the previous-period loan, from which he would suffer an infinite utility loss. So, since the strategic borrower looses less by deviating from the equilibrium contract at date 1, the lenders' update upon observing the deviation passes the criterion.

On the other hand, at date 0, both types find it *equally* unattractive to deviate to a contract revealing to the lenders that the borrower is of the strategic type: knowing the borrower's type, the date-1 lenders would deny him the loan. In turn, this would imply that any positive repayment R_1 would induce default with probability one by the strategic type at date 1. Coupled with the date-0 lenders' zero-profit condition, this implies that the only contract to which the borrower can deviate *and* which will be accepted by the lenders is $(b_0, R_1) = (0, 0)$ – which is equally unattractive to both types.[13]

We should note that, although there exist many equilibria for $\pi_0 \geq \beta^2$, all of them can be *Pareto ranked*, because both types of borrower's utility increases in the amount borrowed in any given period, b_t. We will focus our analysis on the Pareto-dominant equilibria. At each date, the borrower proposes the contract which maximizes the committed type's expected utility. Depending on the value of π_0, β and the income stream (y_1, y_2), three possible outcomes can occur.

Proposition 3. *Define the income growth rate* $g \triangleq \frac{y_2 - y_1}{y_1}$ *and the critical values for the prior*

$$\widetilde{\pi}_0 = \frac{1}{2}\left(1 + \beta + (1-\beta)\sqrt{1 - \frac{4}{1+g}}\right) \quad \text{and} \quad \pi_0^* = \frac{1}{2+g}\beta + \frac{1+g}{2+g}\beta^2. \tag{3.9}$$

Lending contracts proposed in the Pareto dominant equilibrium can be of three different types depending on the values of the parameters:

1. *If*
$$g > 3 \quad \text{and} \quad \pi_0 \in [\beta, \widetilde{\pi}_0],$$
the equilibrium lending contracts are given by
$$\left\{(b_0^*, R_1^*), (b_1^*, R_2^*)\right\} = \left\{(y_1, y_1), (\pi_0 y_2, y_2)\right\}.$$

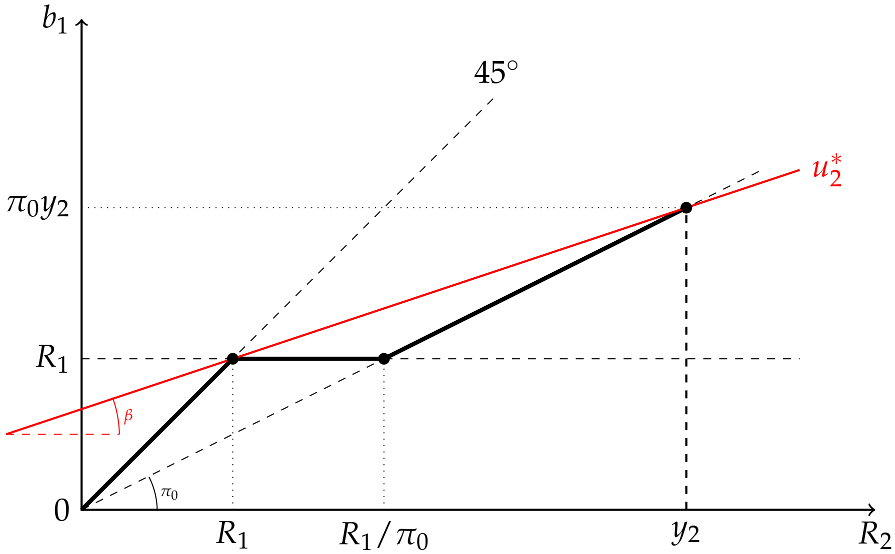
2. *If either*
$$g > 3 \quad \text{and} \quad \pi_0 \in (\widetilde{\pi}_0, 1]$$
or
$$g \leq 3 \quad \text{and} \quad \pi_0 \geq \pi_0^*,$$
the equilibrium lending contracts are given by
$$\left\{(b_0^*, R_1^*), (b_1^*, R_2^*)\right\} = \left\{(\pi_0(y_1 + y_2), y_1 + y_2), (y_2, y_2)\right\}.$$

3. *In all other cases, the equilibrium lending contracts are given by*
$$\left\{(b_0^*, R_1^*), (b_1^*, R_2^*)\right\} = \left\{(0, 0), (0, 0)\right\}.$$

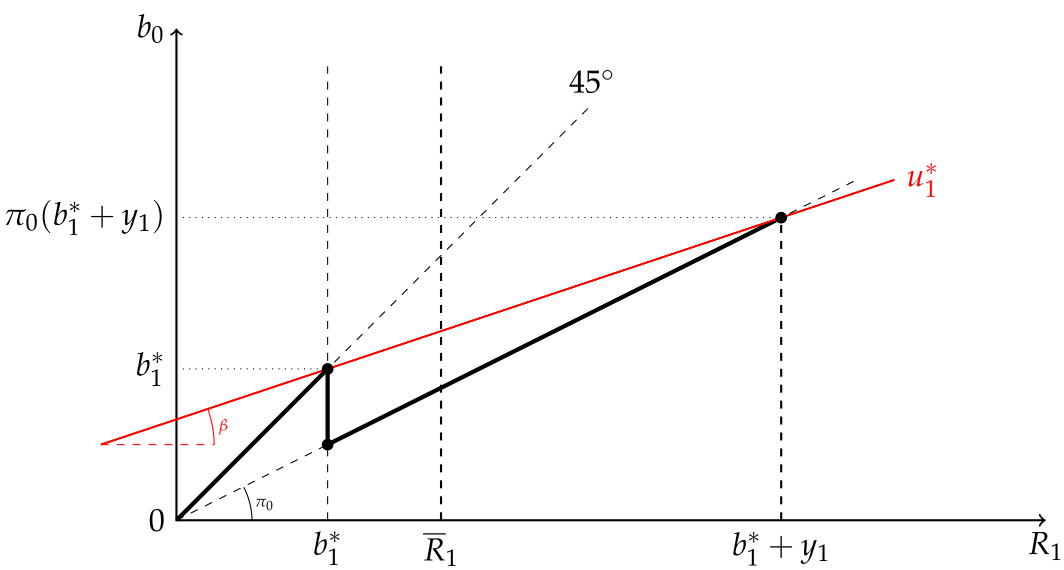
Proof. See Appendix A.4. □

Figure 3(a) illustrates the set of date-1 lending contracts that satisfy the lenders' zero-profit condition. The additional restriction $R_2 \leq y_2$ is the *feasibility* condition: the committed type should not be forced into default at date 2, and hence his income y_2 should not be smaller than the amount promised for repayment, R_2.

[13] A special case of this unravelling argument would be used to prove that only autarkic outcome is feasible for the case when the borrower is non-strategic ($\pi_0 = 0$).



(a) Feasible date-1 loan contracts, for a given R_1.



(b) Feasible date-0 loan contracts, for a given b_1^*.

Figure 3: The set of feasible lending contracts at $t = 1, 2$.

To see how this graph is constructed, first suppose that at date 1, the borrower proposes the contract (b_1, R_2) such that the amount borrowed at this date is less than the repayment on the previous-period loan: $b_1 < R_1$. In order to be able to borrow, the individual has to repay the previous-period debt, R_1. The strategic borrower could, in principle, repay R_1, borrow b_1 and default on this loan on the next date. However, this will bring him the net gain of $b_1 - R_1 < 0$. Therefore, for $b_1 < R_1$, the strategic borrower will be better off *defaulting* on R_1. So, observing the borrower who has repaid the loan R_1, the date-1 lenders infer that the borrower is 'honest', and thus will allow him to borrow at a risk-free rate ($b_1 = R_2$). Therefore, for $b_1 < R_1$, the segment of the zero-profit condition coincides with the 45 degree line starting from the origin.

Suppose instead that the borrower were to pick the loan contract for which $b_1 > R_1$. Now by repaying the previous-period debt R_1, borrowing b_1 and defaulting on this debt at date 2, the strategic borrower would get a net gain of $b_1 - R_1 > 0$. Therefore, he will strictly prefer to repay R_1. But in that case, the date-1 lender who sees that the borrower has repaid R_1 and is now asking for a greater loan $b_1 > R_1$ will not learn anything from the fact of repayment. His posterior belief that the lender is of the committed type will stay unchanged ($\pi_1 = \pi_0$), so the upper segment of the zero-profit condition corresponds to the line that starts at the origin and that has the slope of π_0.

Finally, when $b_1 = R_1$, the contract allows exact roll-over of the previous-period debt, and the strategic borrower is indifferent between defaulting and repaying, since his net gain from repayment is zero ($b_1 - R_1 = 0$). As we move along the segment and R_2 increases from R_1 to R_1/π_0, the probability with which the strategic type defaults on R_1 gradually decreases from one to zero.

Since the strategic type always mimics the committed type, in Pareto-dominant equilibrium, the borrower picks the contract that is optimal for the committed type. The red line represents the committed borrower's indifference curve. The contract (b_1, R_2) brings him the net increase in utility of $b_1 - \beta R_2$ – therefore, the committed type's indifference curve is represented by the straight line with the positive slope equal to β. The borrower's utility increases in the north-west direction. Depending on the values of β and π_0, two generic cases for the borrower's optima might arise: if β is relatively high and π_0 is relatively low, he would pick the contract $(b_1, R_2) = (R_1, R_1)$, whereas if β is relatively low and π_0 is relatively high, he would prefer the contract $(b_1, R_2) = (\pi_0 y_2, y_2)$. The red line on the graph shows the knife-edge case when the committed borrower is exactly indifferent between the two contracts.

Figure 3(b) represents the set of date-0 feasible contracts satisfying the lenders' zero-profit condition, for a given anticipated continuation contract with the amount of borrowing b_1^* at date 1. The borrower faces a trade-off between applying for the small loan at a low interest rate and applying on a large loan at a high interest rate. First, suppose he were to offer a loan contract (b_0, R_1) with $R_1 < b_1^*$. In that case, both types of borrower will repay the loan at date 1 – therefore, from the perspective of the date-0 lender, this loan is safe: we have $b_0 = R_1$, and the zero-profit condition coincides with the 45 degree line. Next, suppose that $R_1 > b_1^*$. In that case, the date-0 lenders will expect the strategic borrower to default on the date-0 debt, and from the lenders' zero-profit condition, we will have $b_0 = \pi_0 R_1$. This corresponds to the line segment with the slope π_0. As we move downward along the vertical segment, the strategic borrower's default probability gradually increases from zero to one.

Once again, the figure illustrates the knife-edge case when the borrower is indifferent between the two options.

Let us briefly summarize the main takeaway from this benchmark case. First, when $\pi_0 < \beta^2$, the committed borrower will choose not to borrow even if repayment of the loan at date 1 could separate him from the strategic type – in that case, the only equilibrium will correspond to autarky. Second, if the income growth is very large, the strategic type can pool with the committed type at date 1 and default on a large debt at date 2 (the amount of debt that will not be repaid by the strategic type is equal to y_2). However, this outcome is possible only when the lenders' prior π_0 is intermediate. Third, if the income growth is not too large while the lenders' prior π_0 is high enough, the committed type can separate from the strategic type at date 1 and continue borrowing.

The next section will illustrate how the option to pledge physical collateral can expand the working of the reputation mechanism.

4 Physical collateral and reputation

In this section, we introduce the possibility for the borrowers *both* to pledge physical collateral and to build reputation: we consider the case when $\kappa_t \in \{0, 1\}$ and $\pi_0 > 0$. Throughout the analysis we will impose the restriction that $\pi_0 < \pi_0^*$. As we established in Proposition 3, in the absence of the possibility to pledge the asset, reputation will not work: in both periods, the committed type will find it optimal to choose autarky.

In this section, we show that the option to pledge the asset as collateral increases the value of the asset when the borrower can develop reputation. To isolate this effect, we confine ourselves to the case when the asset yields no dividend stream: $x = 0$, implying that the asset is valuable only due to the option of future resale. However, the analysis in Section 3.2 suggests that the agent will prefer to sell this asset in period 0 at any price.

We consider an equilibrium with the following structure:

1. At date $t = 0$, there exist two threshold $\underline{p}_0$ and $\overline{p}_0$, such that
   (a) Both types keep the asset and pledge it as collateral $p_0 \in [\underline{p}_0, \overline{p}_0]$;
   (b) Both types sell the asset when either $p_0 < \underline{p}_0$ or $p_0 > \overline{p}_0$.
2. For the given date-0 loan contract (b_0, R_1), in the date-1 continuation game when both types keep the asset, there exist two thresholds, $\underline{p}_1$ and $\overline{p}_1$, such that
   (a) Strategic type repays with probability one for $p > \overline{p}_1$;
   (b) Strategic type randomizes between default and repayment for $p_1 \in [\underline{p}_1, \overline{p}_1]$;
   (c) Strategic type defaults with probability one for $p_1 < \underline{p}_1$.

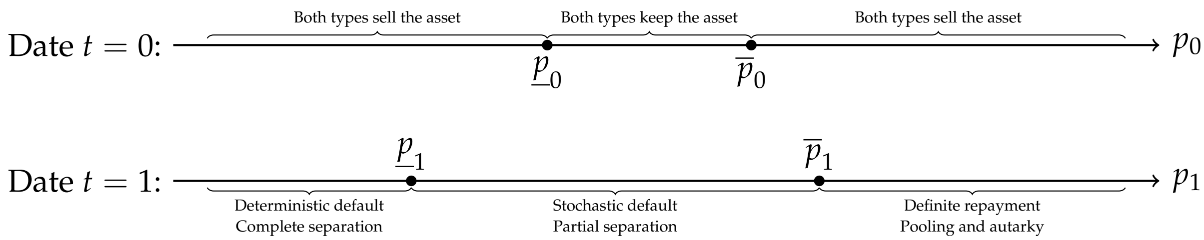


Figure 4: The structure of equilibrium with both collateral and reputation.

Notice that the types pool on their date-0 selling decisions.[14]

On the other hand, at date 2, the two types of borrowers pool for some realizations of p_1 and separate for other realizations: there is complete separation for $p_1 < \underline{p}_1$, partial separation for $\underline{p}_1 \leq p_1 \leq \overline{p}_1$, and pooling for $p_1 > \overline{p}_1$.

The structure of equilibrium is schematically illustrated on Figure 4.

There are several novel elements in comparison to the two benchmark cases analyzed in Sections 3.2 and 3.3. First, for the bottom tail of asset prices, at the interim date $t = 1$, the types can be separated via default. Second, even in the Pareto-dominant equilibrium, the strategic type can randomize between default and repayment for intermediate price range.

Construction of this equilibrium proceeds as follows. We start by analyzing continuation game at date 1 that follows borrowing at date 0 and repaying the debt at date 1. For a given lenders' posterior π_1, we characterize the optimal contract (b_1^*, R_2^*) that is proposed by the committed type. Second, we compare the committed type's utility from selling the asset to the one from keeping it (in order to borrow until date 2), and we show that selling the asset always dominates keeping it. Third, we derive the lenders' posterior π_1 from the strategic borrower's indifference condition, and subsequently determine the default probability $\delta_1^*(p_1; R_1)$ and the two thresholds $\underline{p}_1$ and $\overline{p}_1$.

Then we proceed to characterizing optimal loan contract (b_0, R_1). We show that, as in Section 3.2, the borrower's problem has bang-bang solution: the best non-autarkic debt contract is such that the strategic type defaults with probability one at date 1.

Finally, we characterize the borrower's selling decision at date 0. We show that both types of borrower keep the asset for $p_0 \in [\underline{p}_0, \overline{p}_0]$.

[14] To see that this is the only possible outcome, by contradiction suppose that for some price p_0, one type of borrower sells the asset while the other keeps it. If it is the committed borrower who sells the asset, whereas the strategic borrower keeps it, then the mere fact of offering the contract reveals that he is strategic. We know that the maximal amount that the strategic borrower will be able to borrow at date 0 is equal to $\mathbb{E}[p_1|p_0] = p_0$, the payoff which he would get by selling the asset.

Alternatively, suppose that at a price p_0, only the committed type keeps the asset. Since the strategic type has an option to default on the loan at date 1, for him the value of the asset is at least as large as the one for the committed type – therefore, he will prefer to deviate and pool with the committed type.

Therefore, the selling decision can not separate the two types.

4.1 Borrowing and default decisions at date 1

First, take the continuation game when the borrower has kept the asset and repaid the date-0 debt. The next lemma shows that, irrespective of the realization of p_1, the borrowers always finds it optimal to sell the asset at $t = 1$.

Lemma 2. *At date $t = 1$, the borrower always chooses to sell the asset.*

Proof. See Appendix A.5. □

Intuitively, the asset which yields no dividends ($x = 0$) is valuable to the borrower only to the extent that it enables him to lever up his reputation: when the realization of p_1 is exceptionally low, the strategic type finds it most tempting to default, and thus repayment of the previous-period loan allows the committed type to separate from the strategic type.

The strategic type will indeed find it optimal to default when the price is low and to repay when the price is high. The next proposition gives a formal characterization of the default decision.

Proposition 4. *Given R_1, the strategic type always defaults when $p_1 < \underline{p}_1$, always repays when $p_1 > \overline{p}_1$, and randomizes between defaulting and repaying whenever $p_1 \in [\underline{p}_1, \overline{p}_1]$, where*

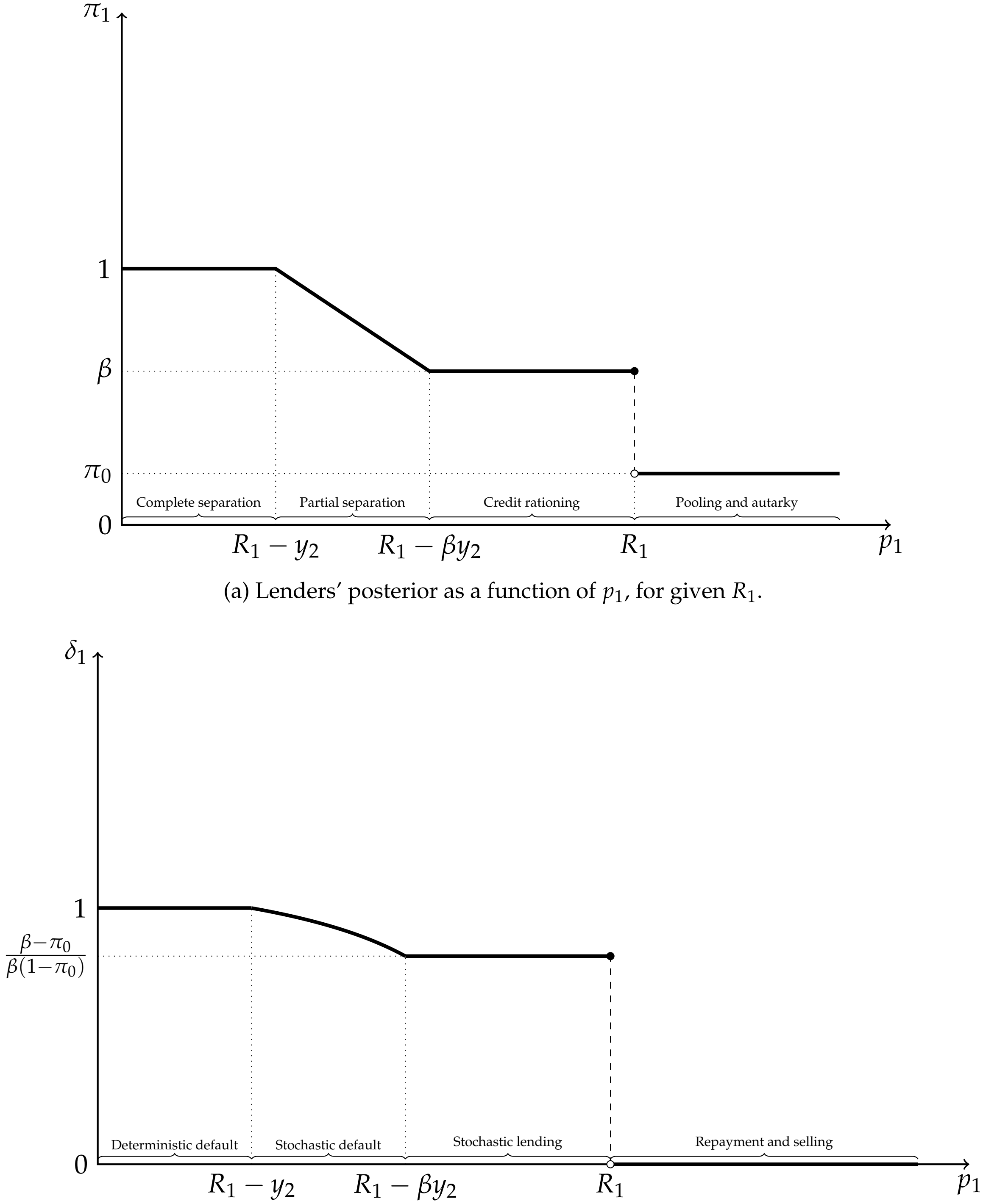
$$\underline{p}_1 = R_1 - y_2 \quad \textit{and} \quad \overline{p}_1 = R_1. \tag{4.1}$$

Proof. See Appendix A.6. □

The lenders' posterior is depicted on Figure 5(a). For $p_1 > \overline{p}_1$, debt repayment does not lead to the lenders' posterior update, so that we have $\pi_1 = \pi_0$. This happens due to the fact that for $p_1 > R_1$, the strategic borrower is better off repaying his date-0 debt and realizing his option to sell the asset. Therefore, the types pool at very high prices, and for the restriction on the lenders' prior that we focus on ($\pi_0 < \pi_0^*$), the committed type prefers to stay in autarky.

Along the same lines, for extremely low realizations of p_1, repayment indicates that the type is non-strategic and raises the lenders' posterior to $\pi_1 = 1$. In particular, whenever $p_1 + y < R_1$, the costs of loan repayment exceed the maximal possible gain for the strategic borrower. As p_1 increases from $\underline{p}_1$ to $\overline{p}_1$, the lenders' posterior decreases from one to π_0.

To gain intuition concerning the jump at $p_1 = R_1$, observe that there cannot exist an equilibrium with $\pi_1 \in (\pi_0, \beta)$: as we showed in the previous section, the committed type will not continue borrowing at date 1 unless $\pi_1 \geq \beta$. Therefore, in case if the strategic type were to repay the loan, the lenders would update their beliefs to $\pi_1 = 1$. On the other hand, the strategic type will find such a deviation profitable for p_1 close to R_1. At the same time, in order to keep the committed type indifferent between borrowing and staying in autarky, the lenders' posterior should remain at $\pi_1 = \beta$. However, since the strategic type would *strictly* prefer to apply for the loan if he could obtain credit with certainty, in order



(a) Lenders' posterior as a function of p_1, for given R_1.

(b) Strategic borrower's default probability as a function of p_1, for given R_1.

Figure 5: Default incentives and reputation as a function of p_1.

to make the strategic type indifferent between defaulting and repaying, loan provision should be stochastic: the probability that the lender accepts the contract should be less than one.

Finally, Bayes' rule for the update of π_1 can be derived from the strategic borrower's indifference condition. The lenders' posterior belief $\pi_1(p_1; R_1)$ is depicted on Figure 5(a) and the strategic type's default probability $\delta_1^*(p_1; R_1)$ is depicted on Figure 5(b).

4.2 Selling decision at date $t = 0$

Having characterized equilibrium in the continuation game after the borrower has kept the asset, we will now determine for which set of prices p_0 the committed (and thus, the strategic) type will prefer to keep the asset for borrowing.

This is stated in the next theorem.

Theorem 1. *For any $\pi_0 \in (0, \pi_0^*)$, there exists an open set of parameters (β, y_1, y_2, p_0), for which the committed type would find it profitable to keep the asset and borrow at date 0.*

When the set of parameters (β, y_1, y_2, p_0) satisfy

$$y_1 > \frac{\beta}{1-\beta} y_2, \quad \beta < \frac{3-\sqrt{5}}{2} \quad \textit{and} \quad p_0 \in \left[\frac{\beta y_2}{2(1-\beta)}, \frac{y_1}{2}\right], \tag{4.2}$$

the borrower will prefer to keep the asset in the limit as $\pi_0 \to 0$.

Proof. See Appendix A.7. □

This result states that even under the conditions when the scope for reputation building becomes vanishingly small, that is, in case if the lenders' prior π_0 tends to zero, the borrower may still want to develop reputation in order to be able to continue borrowing on the next date $t = 1$. To gain intuition for this result, it should be kept in mind that the strategic borrower's behavior is still determined by optimizing behavior of the non-strategic type: that is, some continuation games are characterized by pooling equilibria. This stands in sharp contrast with the result presented in section 3.2, in which there is no scope for mimicking, because the borrower's type is common knowledge.[15]

Theorem 1 also points out the ingredients that facilitate reputation building.

First, there should be significant gains from borrowing; in the context of this model, this implies that the discount factor has to be low. Second, the date-0 asset price must be neither too high nor too low. When p_0 is very high, for most of realizations for p_1, the strategic borrower repays the date-0 loan, which in turn does not allow complete separation of types, and thus wipes out gains from continued borrowing for the committed type. At the same time, the individual's borrowing capacity is directly affected by p_0, since this is how much (in expectation) the lenders could if the strategic type were to default with probability one. Therefore, when p_0 is very low, borrowing becomes so expensive that the committed type would rather prefer to sell the asset and remain in autarky.

[15] In particular, in the absence of reputation ($\pi_0 = 0$), for the case when $x = 0$, the borrower will sell the asset in period $t = 0$ and remain in autarky, as is evident from the characterization given in Proposition 2.

The date-0 loan contract proposed by the committed type optimally trades off the benefits from separating against the costs of the loan: R_1 is made such that in the date-1 continuation game, the strategic borrower defaults for low p_1, repays for high p_1, and randomizes for intermediate p_1.

5 Discussion and extensions

This section discusses the assumptions and implications of the model developed in section 4. Then we will give a brief account of several extensions. First, we will consider the generalized finite-horizon version with the representative borrower, in which the number of periods T tends to infinity. Second, we will take up an overlapping generations' perspective: the time horizon will be infinite, and in each period $t \geq 2$, there will exist three generations of borrowers: young, middle-aged, and old. In each generation, the proportion of committed types within each young generation will be equal to π_0. At each date t, all the generations will face the same asset price p_t. This extension will allow us to untie the impact of asset price fluctuations, on the one hand, from the borrowers' life cycle considerations.

Before we proceed to the extensions, it would be insightful to see how the results of our model would differ if the asset price did not fluctuate. Suppose we had constant asset price:

$$p_t = p \quad \text{for all } t. \tag{5.1}$$

Solving the model backwards, we first see that the individual's date-1 default and borrowing decision is independent of p_0. Hence, the properties of equilibrium constructed in the previous section will hold for the date-1 continuation game: specifically, if $p \in [0, R_1 - y]$, the types will completely separate; there will be partial separation if we had $p \in [R_1 - y, R_1]$, and there will be pooling for $p_1 > R_1$. Likewise, at date 0, the borrower would prefer to keep the asset if we had $p \in [\underline{p}_0, \overline{p}_0]$.

This implies that the complementarity between collateral and reputation depends on asset pledgeability, and not on asset price fluctuations. However, these fluctuations are important: the analysis in section 4 provides us some insights regarding comovement of asset prices and credit growth. In particular, it suggests that one would typically observe the wave of defaults when the rapid asset price increase is followed by a drastic decrease in asset prices. The analysis of the benchmark case of section 3.2 highlighted the working of this mechanism when the asset was valuable for the stream of dividends $x > 0$ for the borrowers. However, proposition 2 suggested that, if the individual does not have a chance to develop his reputation, it will either be the case that (i) he does not want to borrow in the first place (for $p_0 \geq \hat{p}_0$, he prefers to sell the asset in period 0), or (ii) he will borrow, but will not default at date 1.

5.1 Infinite horizon

Let us analyze the setup in which the time horizon may be arbitrarily long, but is finite, and consists of $T+1$ dates.[16] Take $\tau \in \{1, \ldots, T\}$ to be the last date following non-autarkic continuation game – that is, the game with $(b_{\tau-1}, R_\tau) \neq (0, 0)$.

In what follows, we consider the general process for the asset price, described by the function $F(\cdot|p)$, which satisfies the martingale property:

$$\mathbb{E}[p_{t+1}|p_t] \equiv \int p_{t+1} \mathrm{d}F(p_{t+1}|p_t) = p_t$$

First, consider the case when the asset has already been sold by date τ: $a_\tau = 0$. The continuation value of the committed type is given by

$$V_\tau(R_\tau; \pi_\tau, 0) = \max_{b_\tau, R_{\tau+1}} \Big\{ y_\tau - R_\tau + b_\tau + \beta(y_{\tau+1} - R_{\tau+1} + V_{\tau+1}) \Big\}, \tag{5.2}$$

subject to the zero-profit condition:

$$b_\tau = \pi_\tau R_{\tau+1}$$

and the feasibility constraint[17]

$$b_\tau \geq R_\tau - y_\tau.$$

In order for the committed type to be willing to stop borrowing at date τ, it must be the case that

$$\pi_\tau \leq \beta.$$

At the same time, in order to be willing to offer non-autarkic contract at date $\tau - 1$, it must be the case that

$$\pi_{\tau-1} \geq \beta.$$

However, since the sequence of the lenders' posterior is non-decreasing, we must have

$$\pi_{\tau-1} \leq \pi_\tau,$$

because each period in which the borrower repays the outstanding debt, the lenders' posterior that the type is non-strategic must increase or stay the same.

Clearly, the above three inequalities cannot hold simultaneously. Therefore, once the committed type decides to sell the asset, he must realize that the subsequent consumption path will be autarkic: from the date τ onward, we have

$$V^a_\tau = \sum_{t=0}^{T-\tau} \beta^t y_t. \tag{5.3}$$

[16]This counter includes date 0, so that T is the last date. For instance, in the canonical model developed in the previous section, we have $T = 2$.

[17]That is, the non-negativity of date-τ consumption: $c_\tau = y_\tau - R_\tau + b_\tau \geq 0$.

This is intuitive, since due to the martingale property of the price process $\{p_t\}$, together with zero dividend stream ($x = 0$), the asset is valuable only to the extent it allows the individual to increase future borrowing.

Now let us turn to the case where the individual holds the asset at date τ. The borrower will be indifferent between selling the asset and keeping it whenever

$$(\pi_\tau - \beta)y_{\tau+1} + \beta \int \max\Big\{(\pi_{\tau+1} - \beta)y_{\tau+2}, p_{\tau+1}\Big\} \mathrm{d}F(p_{\tau+1}|p_\tau) = p_\tau.$$

The right-hand side of the above expression is the net gain from selling the asset in period τ. The left-hand side is the net gain from borrowing against the asset in period τ (the term $y_{\tau+1}$ represents the maximal sum the committed borrower will be able to borrow, given the feasibility constraint), plus the option value of holding the asset until the next date.

The behavior of the strategic type pins down the evolution of the lenders' posterior beliefs $\{\pi_t\}_{t=\tau}^{T}$, jointly with the indifference condition of the strategic type (defaulting now versus repaying and defaulting on a greater debt in the future).

For each period t, the set of prices in period τ can be subdivided into two subsets: $\mathcal{P}_t^c$ and its complement, $\overline{\mathcal{P}}_t^c$, for which the committed type prefers to keep and, respectively, to sell the asset. In the specific case analyzed in the previous section, the set $\mathcal{P}_t^c$ was a closed interval at $t = 0$, and was empty for $t = 1$ and 2: the committed type *always* preferred to sell the asset at the interim stage 1. For the general structure outlined here (that is, for the horizon length $T \geq 2$ and for arbitrary distribution F), the set $\mathcal{P}_t^c$ may take a more complex structure, and may vary from one date to the other.

Similar reasoning can be applied to the two disjoint subsets $\mathcal{P}_t^s$ and $\overline{\mathcal{P}}_t^s$, for which the strategic type prefers to repay the previous-period debt versus default on it, respectively. Although it is true that, other things equal, the strategic type would prefer to repay on the loan and keep the asset as p_t increases, because the option value of selling the asset rises – at the same time, the option value of keeping the asset and defaulting on it in the future may decrease with the rise in p_t.

Therefore, although the length of the time horizon can be easily increased to length exceeding three periods, the assumption of the uniform distribution for the price process is not innocuous and cannot be dropped altogether: the interval construction of Figure 4 is sensitive to the specifics of the uniform distribution.

5.2 Overlapping generations

In order to disentangle the dynamics driven by asset price changes from the individual's life-cycle considerations, we may consider the following generalized version of the setup: the time horizon will be infinite: $t = 0, 1, 2, \ldots$, but at each date t, there will be three generations of borrowers: young (0), middle-aged (1) and old (2).

Old generation at period zero (the generation that was born at date $t = -2$) will be assumed to hold one unit of the asset, whereas the young and middle-aged individuals will be assumed to earn non-financial incomes y^0 and y^1, with $y^1 \geq y^2$. Asset can be traded among the generations, and can also be liquidated by the lenders in case of default.

In addition, each newly born generation of borrowers will contain the share π_0 of the committed borrower types, as well as $1 - \pi_0$ of the strategic types.

The date-*t* realization of the asset price leads to different consequences for the borrowers at the different stages of their life cycle (that is, for different generations). The structure of equilibrium illustrated on Figure 4 suggests that for the oldest generation, a sudden drop in the market price would lead to default of the entire sub-population of the mass $1 - \pi_0$, the strategic types. If the *level* of the asset price remains relatively high, however, only *some* of the middle-aged borrowers will choose to default, whereas if the asset price is very low, their default will be deterministic.

At the same time, the entire cohort of the middle-aged borrowers who didn't default on their previous-period loan will choose to sell the asset and either remain in autarky if the current asset price is relatively high, or choose to borrow against their reputation – in the opposite case. Interestingly, one would be able to observe *credit rationing* for the middle-aged group in case if the current asset price lies in the interval $[R_1 - \beta y_2, \overline{p}_1]$, that is, when it is quite high but not overly so.

As for the youngest generation, the model predicts massive selling of the asset when its price is either overly high or overly low, whereas moderate price level is required for the asset-backed borrowing to emerge in the first place. In particular, defaults by the middle-aged borrowers can co-exist with borrowing by the young cohort.

6 Conclusion

The paper's main focus was on the complementarity between external incentives to repay the loan (in the form of collateral), on the one hand, and internal incentives (in the form of reputation).

It was shown that the possibility to pledge the asset as collateral in the loan contract makes it more attractive for the borrower to develop reputation for being honest, that is: to borrow against the asset, repay the loan on the next date and, in case if the asset price did not increase substantially, to keep borrowing against increased reputation.

Theorem 1 shows that, subject to some parameter restrictions, this effect is present even in the limit as the prior probability that the borrower is of the committed type (π_0) tends to zero.

In the concluding comments, it would be worth mentioning the applicability of the above setup to the variety of economic situation.

The one that comes to mind is the relationship between the entrepreneur and the sequence of venture capitalists. The entrepreneur may be capable of running profitable business (with probability π_0), or else may have a 'fake' project (with probability $1 - \pi_0$).

In this context, the analogue of $\{p_t\}$ may be some publicly observable economic indicator which affects the profitability. In that respect, the result of the present paper implies that the entrepreneur will wish to borrow from the capitalist whenever the business prospects are reasonably bright, but not overly so (interval within the thresholds for p_0), and that the loan might not be feasible if it were impossible to make it contingent on these indicators (the statement of Theorem 1).

In general, any interaction between the long-lived agent and a sequence of the short-lived principals fit within the developed framework to a greater or a smaller extent.

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Appendices

A Proofs

A.1 Proof of Proposition 1

Proof. Conditional on keeping the asset at date 0, the individual will choose to sell the asset at date 1, provided that

$$p_1 \geq \underbrace{\beta(x + \mathbb{E}[p_2|p_1])}_{v_1^a(p_1)},$$

which (given the martingale property) comes down to

$$p_1 \geq \frac{\beta x}{1-\beta}.$$

The above inequality can be satisfied for at least some p_1 only if

$$p_0 \geq \frac{\beta x}{2(1-\beta)},$$

which will be assumed for now.

The net expected gain from keeping the asset until date 1 is given by

$$\begin{aligned} v_0^a(p_0) =&\beta\Big(x + \mathbb{E}\Big[\max\left\{p_1, v_1(p_1)\right\}|p_0\Big]\Big) \\ =&\beta\left[x + \frac{1}{2p_0}\left(\int_{\frac{\beta x}{1-\beta}}^{2p_0} p_1 \mathrm{d}p_1 + \frac{\beta^2 x^2}{1-\beta} + \beta\int_0^{\frac{\beta x}{1-\beta}} p_1 \mathrm{d}p_1\right)\right] \\ =&\beta x + \beta p_0 + \frac{\beta^3 x^2}{4p_0(1-\beta)}. \end{aligned}$$

The asset will be sold at date 0 provided that

$$p_0 \geq v_0^a(p),$$

or, after performing some computations, when

$$p_0 \geq \frac{\beta x}{2(1-\beta)} \left(1 + \sqrt{1+\beta}\right).$$

Therefore, we have the following two threshold prices:

$$p_0^a = \frac{\beta x}{2(1-\beta)} \left(1 + \sqrt{1+\beta}\right) \quad \text{and} \quad p_1^a = \frac{\beta x}{1-\beta}.$$

One can check that for any $\beta > 0$ (and for p_0 within the specified range), we have

$$p_0^a > p_1^a.$$

Next, it can be shown that for

$$p_0 < \frac{\beta x}{2(1-\beta)},$$

the borrower's value to holding on to the asset equals

$$v_0^a(p_0) = \beta x + \beta^2(x + p_0),$$

and the two threshold prices coincide:

$$p_0^a = p_1^a = \frac{\beta x}{1-\beta}.$$

Consequently, the asset is never sold before the end of $t = 2$, since with such a low p_0, the upper bound of the support for p_1 will fall below the threshold with probability one.

Now, for the case of full commitment, the agent will prefer to sell the asset at date 1 provided that

$$p_1 + y_2 \geq y_2 + \underbrace{x + \beta \mathbb{E}[p_2|p_1]}_{v_1^c(p_1)},$$

or

$$p_1 \geq \frac{x}{1-\beta}.$$

Discounting implies that the agent would wish to bring his consumption forward to the maximal possible extent. That is, if the asset was already sold, at date 1 the agent will borrow $b_1^c = y_2$ (and repay this sum at $t = 2$), whereas if the agent keeps the asset, he will borrow $b_1^c = y_2 + x$.[18]

[18] If the lower bound of the support for the asset price were positive: $\underline{p} > 0$, instead of zero, the agent would borrow up to $b_1^c = y_2 + x + \underline{p}$. What is essential is that the agent can repay the loan at date 2 in any circumstance, conditional on the non-negativity constraint on date-2 consumption: $c_2 \geq 0$.

What about the selling decision in period 0? Under the worst-case scenario, the realized asset price in period 1 will be zero, in which case by keeping the asset, at $t = 1$ the agent will still be able to borrow $y + x$ (to be repaid at date 2).

In case if the agent sells the asset at $t = 0$, he will be able to borrow only $y_1 + y_2$ – once again, the loan to be rolled over at $t = 1$ and repaid at date 2. If the agent keeps the asset, he will be able to borrow $y_1 + y_2 + x$ at date 0 and realize his option to sell the asset at date 1. This implies that the agent will be willing to sell the asset at date 0 provided that[19]

$$p_0 + y_1 + y_2 \geq y_1 + y_2 + \underbrace{x + \frac{\beta}{2p_0}\left(\int_{\frac{x}{1-\beta}}^{2p_0} p_1 \mathrm{d}p_1 + \frac{x^2}{1-\beta} + \beta \int_0^{\frac{x}{1-\beta}} p_1 \mathrm{d}p_1\right)}_{v_0^c(p_0)}.$$

The associated thresholds for the committed type are thus

$$p_0^c = \frac{x}{2(1-\beta)}\left(1 + \sqrt{1+\beta}\right) \quad \text{and} \quad p_1^c = \frac{x}{1-\beta}.$$

Comparing these thresholds with the ones computed for autarky, we can see that

$$p_t^c = \frac{p_t^a}{\beta} > p_t^a \quad \text{for both } t = 1, 2.$$

Finally, substituting the thresholds into the borrower's expected utility yields the expression for $U^a(p_0)$ and $U^c(p_0)$.

The borrower's autarkic utility is equal to

$$U^a(p_0) = \beta y_1 + \beta^2 y_2 + \max\left\{p_0, v_0^a(p_0)\right\},$$

where

$$v_0^a(p_0) = \begin{cases} \beta(1+\beta)x + \beta^2 p_0 & \text{if } p_0 < \dfrac{\beta x}{2(1-\beta)}, \\ \beta x + \beta p_0 + \dfrac{\beta^3 x^2}{4p_0(1-\beta)} & \text{if } p_0 \geq \dfrac{\beta x}{2(1-\beta)}. \end{cases}$$

The borrower's utility under full commitment is equal to

$$U^c(p_0) = y_1 + y_2 + \max\{p_0, v_0^c(p_0)\},$$

where

$$v_0^c(p_0) = \begin{cases} (1+\beta)x + \beta^2 p_0, & \text{if } p_0 < \dfrac{x}{2(1-\beta)}, \\ x + \beta p_0 + \dfrac{\beta x^2}{4p_0(1-\beta)} & \text{if } p_0 \geq \dfrac{x}{2(1-\beta)}. \end{cases}$$

□

[19] Similar remark applies to the case when $p_0 < \frac{x}{2(1-\beta)}$.

A.2 Proof of Proposition 2

Proof. At date 2, repayment will be made only if the gains from holding the asset, equal to the asset price plus the dividend, outweigh the gains to default, which equal R_2: that is, if the date-2 asset price exceeds the threshold:

$$p_2 \geq \underbrace{R_2 - x}_{\hat{p}_2(R_2)}.$$

In case the borrower holds the asset at the end of date 1, the competitive lender offers the contract (b_1, R_2), which solves

$$\max_{b_1, R_2} \left\{ x - R_1 + b_1 + \frac{\beta}{2p_1} \int_{R_2 - x}^{2p_1} (x + p_2 - R_2) \mathrm{d}p_2 \right\},$$

subject to

$$b_1 = \left(1 - \frac{R_2 - x}{2p_1}\right) R_2 + \frac{1}{2p_1} \int_0^{R_2 - x} p_2 \mathrm{d}p_2.$$

Expressing

$$b_1 = \left(1 - \frac{R_2}{4p_1}\right) R_2 + \frac{x^2}{4p_1}$$

from the constraint equation and maximizing with respect to R_2, we eventually get

$$\hat{b}_1 = p_1 + \frac{1 - 2\beta}{p_1} \left(\frac{x}{2(1-\beta)}\right)^2 \quad \text{and} \quad \hat{R}_2 = 2p_1 - \frac{\beta x}{1 - \beta}.$$

Observe that the loan is risky ($\hat{R}_2 > x$) only if p_1 is sufficiently large, namely if

$$p_1 \geq \frac{x}{2(1-\beta)},$$

while in the opposite case, the individual would just prefer to borrow

$$\hat{b}_1 = \hat{R}_2 = x,$$

repay the loan with certainty and realize his option to sell the asset at date 2.

So, the borrower's date-1 utility from repaying and holding the asset – to borrow against it – is equal to

$$U_1^b(p_1, R_1) = \begin{cases} x - R_1 + p_1 + \dfrac{x^2}{4p_1(1-\beta)} & \text{if } p_1 \geq \dfrac{x}{2(1-\beta)}, \\ 2x - R_1 + \beta p_1 & \text{if } p_1 < \dfrac{x}{2(1-\beta)}. \end{cases}$$

On the other hand, borrower's utility from repaying and selling the asset is equal to

$$U_1^s(p_1, R_1) = x - R_1 + p_1,$$

so that borrowing against the asset strictly dominates selling it.

Finally, the agent's utility to default is zero:

$$U_1^d = 0,$$

since he does not repay the loan issued at date 0, he loses the asset and can no longer borrow against it.

Therefore, at date 1, the loan will be repaid, if and only if

$$U_1^b(p_1, R_1) \geq 0,$$

that is, if and only if p_1 exceeds the threshold: $p_1 \geq \hat{p}_1(R_1)$, where

$$\hat{p}_1(R_1) = \begin{cases} 0 & \text{if } R_1 \leq 2x, \\ \dfrac{R_1 - 2x}{\beta} & \text{if } 2x < R_1 \leq 2x + \dfrac{\beta x}{2(1-\beta)}, \\ \dfrac{1}{2}\left[(R_1 - x) + \sqrt{(R_1 - x)^2 - \dfrac{x^2}{1-\beta}}\right] & \text{if } R_1 > 2x + \dfrac{\beta x}{2(1-\beta)}. \end{cases}$$

Now consider the agent's borrowing decision at date 0.

Along the same lines, the contract (b_0, R_1) solves

$$\max_{b_0, R_1} \left\{ b_0 + \frac{\beta}{2p_0} \int_{\hat{p}_1(R_1)}^{2p_0} U_1^b(p_1, R_1) \mathrm{d}p_1 \right\},$$

subject to

$$b_0 = \left[1 - \frac{\hat{p}_1(R_1)}{2p_0}\right] R_1 + \frac{1}{2p_0} \int_0^{\hat{p}_1(R_1)} p_1 \mathrm{d}p_1.$$

Let us rewrite the above problem in terms of the choice of the threshold $\hat{p}_1$:

$$\max_{\hat{p}_1 \in [0, 2p_0]} \left\{ \left(1 - \frac{\hat{p}_1}{2p_0}\right) R_1(\hat{p}_1) + \frac{1}{2p_0} \int_0^{\hat{p}_1} p_1 \mathrm{d}p_1 + \frac{\beta}{2p_0} \int_{\hat{p}_1}^{2p_0} U_1^b(p_1, R_1(\hat{p}_1)) \mathrm{d}p_1 \right\},$$

where

$$R_1(\hat{p}_1) = \begin{cases} 2x + \beta \hat{p}_1 & \text{if } 0 < \hat{p}_1 \leq \dfrac{x}{2(1-\beta)}, \\ x + \hat{p}_1 + \dfrac{x^2}{4\hat{p}_1(1-\beta)}, & \text{if } \hat{p}_1 \geq \dfrac{x}{2(1-\beta)}. \end{cases}$$

We perform the maximization with respect to $\hat{p}_1$ on two intervals separately.

First, take the interval

$$\hat{p}_1 \in \left[0, \frac{x}{2(1-\beta)}\right].$$

The first order condition yields:

$$-\frac{2x + \beta \hat{p}_1}{2p_0} + \beta\left(1 - \frac{\hat{p}_1}{2p_0}\right) + \frac{\hat{p}_1}{2p_0} - \frac{\beta^2}{2p_0}(2p_0 - \hat{p}_1) = 0,$$

which is a linear increasing function of $\hat{p}_1$.

Now take the interval where

$$\hat{p}_1 \in \left[\frac{x}{2(1-\beta)}, 2p_0\right].$$

The first order condition yields:

$$(2p_0 - \hat{p}_1)\left(1 - \beta - \frac{x^2}{4\overline{p}_1^2}\right) + \hat{p}_1 = 0,$$

which is also monotonically increasing in $\hat{p}_1$ and positive for all $\hat{p}_1$ within the range.

Therefore, the solution is bang-bang: we have either

$$\hat{p}_1^* = 0 \quad \text{or} \quad \hat{p}_1^* = 2p_0.$$

To determine the optimal $\hat{p}_1^*$, we compute the value of the objective function at these two candidate solutions.

When $\hat{p}_1^* = 2p_0$, we have

$$U_0^b(p_0) = p_0,$$

whereas for $\hat{p}_1^* = 0$, we have

$$U_0^b(p_0) = 2x + \beta^2 p_0.$$

So, the optimal contract is given by

$$(\hat{b}_0, \hat{R}_1) = \begin{cases} (2x, 2x) & \text{if } p_0 < \hat{p}_0, \\ \left(p_0, 2p_0 + x + \frac{x^2}{8p_0(1-\beta)}\right) & \text{if } p_0 \geq \hat{p}_0, \end{cases}$$

where

$$\hat{p}_0 = \frac{2x}{1-\beta^2}.$$

□

A.3 Proof of Lemma 1

Proof. First, consider the date-1 continuation game when there was no default, and denote the lenders' posterior that the borrower is of the 'honest' type (conditional on no default) by $\pi_1 \geq \pi_0$.

Any contract (b_1, R_2) that satisfies three conditions,

$$0 \leq R_2 \leq y_2, \quad b_1 = \pi_1 R_2 \quad \text{and} \quad b_1 + \beta(y_2 - R_2) \geq \beta y_2,$$

is consistent with equilibrium.

The first inequality is the *feasibility condition*: the borrower has to be able to repay the loan at date 2. The second is lenders' *zero-profit condition*, and the third is the *participation constraint* of the non-strategic borrower. One contract that trivially satisfies these three conditions is, of course,

$$b_1^* = R_2^* = 0,$$

which is just non-participation.

The non-trivial contract exists only it π_1 happens to be large enough, namely if

$$\pi_1 \geq \beta.$$

If the above condition is satisfied, then any contract

$$\left\{(b_1^*, R_2^*): \quad b_1^* = \pi_1 R_2^* \quad \text{and} \quad 0 \leq R_2^* \leq y_2\right\}$$

is consistent with equilibrium.

Now consider the strategic borrower's choice of whether to default at date 1.

He is *indifferent* between defaulting and repaying, provided that

$$y_1 - R_1 + b_1^* + \beta y_2 = y_1 + \beta y_2,$$

so that

$$b_1^* = R_1.$$

That is to say, the borrower has to be able to exactly roll over his debt on to the next period. If his date-1 borrowing capacity is strictly larger than the repayment on the previous-period loan ($b_1^* > R_1$), he will never default ($\delta_1 = 0$). If on the other hand he has to repay more than what he can raise ($b_1^* < R_1$), he will default with certainty.

Therefore, for a given b_1^*, the probability of default as a function of R_1, is equal to

$$\delta_1(R_1; b_1^*) \begin{cases} = 0 & \text{if } R_1 < b_1^*, \\ \in [0,1] & \text{if } R_1 = b_1^*, \\ = 1 & \text{if } R_1 > b_1^*. \end{cases}$$

Consequently, the lenders' posterior probability that the borrower is of the non-strategic type is given by

$$\pi_1(R_1; b_1^*, \pi_0) \begin{cases} = \pi_0 & \text{if } R_1 < b_1^*, \\ \in [\pi_0, 1] & \text{if } R_1 = b_1^*, \\ = 1 & \text{if } R_1 > b_1^*. \end{cases}$$

Now consider the borrowing decision at date 0. Given the strategic borrower's default probability $\delta_1(R_1; b_1^*)$ at date 1, the date-0 lenders' zero-profit condition is given by

$$b_0^* = \Big[\pi_0 + \big(1 - \pi_0\big)\big(1 - \delta_1(R_1; b_1^*)\big)\Big] R_1.$$

Given the Bayes' rule for the lenders' posterior updating,

$$\pi_1(R_1; b_1^*, \pi_0) = \frac{\pi_0}{\pi_0 + (1 - \pi_0)(1 - \delta_1(R_1; b_1^*))},$$

the zero-profit condition can be rewritten as

$$b_0(R_1; b_1^*, \pi_0) = \frac{\pi_0}{\pi_1(R_1; b_1^*, \pi_0)} R_1.$$

Now, the participation constraint of the non-strategic type is given by

$$b_0 + \beta(b_1^* - R_1) - \beta^2 R_2^* \geq 0,$$

or

$$b_0 \geq \beta R_1 - \beta(b_1^* - \beta R_2^*).$$

Jointly with the lender's date-0 zero-profit condition, this yields

$$\left(\beta - \frac{\pi_0}{\pi_1(R_1; b_1^*, \pi_0)}\right) R_1 \leq \beta(b_1^* - \beta R_2^*).$$

Since $\beta < 1$, the above condition is satisfied for all $R_1 < b_1^*$.
For $R_1 = b_1^*$, the above condition is satisfied if and only if

$$\beta^2 \leq \pi_0.$$

For $R_1 > b_1^*$, the above condition boils down to

$$(\beta - \pi_0) R_1 \leq \beta(1 - \beta) b_1^*,$$

which always holds for $\beta \leq \pi_0$, while for $\beta > \pi_0$, holds whenever

$$R_1 \leq \frac{\beta(1 - \beta)}{\beta - \pi_0} b_1^*.$$

Finally, in order for the committed borrower to be able to repay the loan at date 1, the repayment must satisfy

$$R_1 \leq y_1 + b_1^*.$$

We can now characterize an equilibrium at date 0 for a given continuation contract (b_1^*, R_2^*) and for different ranges of π_0 and β.

First, consider the case when $\pi_0 < \beta^2$.

We claim that the only possible equilibrium is autarky:

$$b_t^* = R_{t+1}^* = 0, \quad \text{for } t = 0, 1.$$

To see this, first notice that for the date-0 committed lender's zero-profit condition to be satisfied, it must be the case that

$$R_1^* < b_1^*.$$

However, we already know that the strategic type will never default on such a contract, and hence there will be no update of beliefs at date 1:

$$\pi_1 = \pi_0.$$

Next, since $\beta < 1$, we have

$$\pi_1 = \pi_0 < \beta^2 < \beta,$$

and thus the only contract available at date 2 is

$$b_1^* = R_2^* = 0.$$

Jointly with the requirement that

$$R_1^* \leq \frac{\beta(1-\beta)}{\beta - \pi_0} b_1^*,$$

this yields

$$b_0^* = R_1^* = 0.$$

Now consider the case when $\beta^2 \leq \pi_0 < \beta$.

First, we claim that we cannot have an equilibrium in which

$$R_1^* < b_1^*.$$

By contradiction, suppose this were the case. Then we would have $\pi_1 = \pi_0 < \beta$, implying that only autarkic equilibrium ($b_1^* = R_2^* = 0$) is possible at date 1. Hence, we must have $R_1^* \geq b_1^*$. When the inequality is strict, the strategic borrower defaults with probability one at date 1, and hence

$$\pi_1(R_1^*; b_1^*, \pi_0) = 1 \quad \text{and} \quad b_0^* = \pi_0 R_1^*.$$

so that the date-1 debt, conditional on the previous repayment, is riskless:

$$b_1^* = R_2^*.$$

So, any contract (b_0^*, R_1^*) with

$$b_0^* = \pi_0 R_1^* \leq \frac{\beta(1-\beta)}{\beta - \pi_0}\pi_0 b_1^* \quad \text{and} \quad R_1^* \leq y_1 + b_1^*$$

is consistent with equilibrium.

In case of equality, $R_1^* = b_1^*$, we have a class of equilibria with

$$b_0^* = \pi_0 R_2^*,$$

with the strategic borrower randomizing between repaying and defaulting, with

$$\delta_1^*(b_1^*, R_2^*; \pi_0) = 1 - \frac{\pi_0}{1-\pi_0}\left(\frac{R_2^*}{b_1^*} - 1\right).$$

Finally, take the case when $\pi_0 \geq \beta$.

Now the only binding constraint is the date 1 is feasibility. In addition to the two equilibria analyzed for the case when $\pi_0 \in [\beta^2, \beta)$, there exists another one in which the strategic borrower *never* defaults at date 1: in that case, we must have

$$b_1^* = \pi_0 R_2^*,$$

and any contract (b_0^*, R_1^*) with

$$b_0^* = R_1^* \leq y_1 + b_1^*$$

is consistent with equilibrium. □

A.4 Proof of Proposition 3

Proof. Start with the date-1 continuation game after there was a repayment on date-0 loan.

Given R_1, the lenders' zero profit condition is

$$b_1(R_2; R_1, \pi_0) = \begin{cases} R_2 & \text{if } R_2 < R_1, \\ R_1 & \text{if } R_2 \in \left[R_1, \dfrac{R_1}{\pi_0}\right]. \\ \pi_0 R_2 & \text{if } R_2 > \dfrac{R_1}{\pi_0}. \end{cases}$$

This piecewise linear function describes the solid black line on Figure 3(a).

Depending on the value of β, there are two potential candidates for the committed borrower's optimum in the date-1 continuation game.

First, the borrower may propose the contract $(b_1, R_2) = (R_1, R_1)$. This is the contract with the risk-free interest rate, and the lenders will accept such a contract only if repayment of the date-0 loan induces their posterior update to $\pi_1 = 1$. In turn, this update will be consistent with equilibrium only if the strategic type defaults with probability one. The lending contract with $b_1 = R_1$ allows the borrower who repays his date-0 debt to borrow exactly the same amount at date 1 – and the strategic type will be *indifferent* between repaying his date-0 debt R_1, borrowing $b_1 = R_1$ and defaulting on this loan at date 2, on the one hand, and defaulting on his date-0 loan, on the other. Of course, any randomization between default and repayment will also be optimal. However, for the lenders' posterior to jump up to $\pi_1 = 1$, it must be the case that the strategic type defaults with probability one on his date-0 loan. This equilibrium involves a specific tie-breaking rule in the strategic borrower's default choice.

Alternatively, the committed type may prefer to choose the pooling contract $(b_1, R_2) = (\pi_0 y_2, y_2)$ (in case if $b_1 > R_1$, the strategic type will prefer to repay his date-0 loan and pool with the committed type).

The value of β which would make the committed borrower indifferent between the two options is given by

$$\beta = 1 - \frac{(1 - \pi_0) y_2}{y_2 - R_1}.$$

Now consider the situation at date 0. The date-0 lenders anticipate that at date 1, the borrower (having repaid the date-0 debt) will offer the contract (b_1^*, R_2^*). Given the date-1 default decision of the strategic type, the date-0 lenders' zero-profit condition will be given by

$$b_0(R_1; b_1^*) \begin{cases} = R_1 & \text{if } R_1 < b_1^*, \\ \in [\pi_0 b_1^*, b_1^*] & \text{if } R_1 = b_1^*, \\ = \pi_0 R_1 & \text{if } R_1 > b_1^*. \end{cases}$$

The two potential candidates for the committed borrower's optimum are: (i) propose the contract $(b_0, R_1) = (b_1^*, b_1^*)$, which would allow to exactly roll-over the debt at date 1. In order to be willing to accept this contract at a risk-free rate, the date-0 lenders have to believe that the strategic borrower will repay the debt with probability one.

Alternatively, the committed borrower may propose the contract

$$(b_0, R_1) = (\pi_0 (b_1^* + y_1), b_1^* + y_1),$$

which would trigger default by the strategic type at date 1.

The borrower will be indifferent between the two contracts provided that

$$\beta = 1 - \frac{(1 - \pi_0) b_1^*}{y_1}.$$

Notice that the increase in R_1 expands the borrower's feasible set at date 1 and also increases (weakly) the optimally chosen b_1^*. Likewise the increase in b_1^* expands the borrower's feasible set at date 1. However, for the safe contract $(b_0, R_1) = (b_1^*, b_1^*)$ to be feasible at date 0, R_1 should not be too large, so that at date 1, the borrower offers the risky contract $(\pi_0 y_2, y_2)$: we must have

$$R_1 \leq \underbrace{\frac{(\pi_0 - \beta) y_2}{1-\beta}}_{\overline{R}_1}.$$

Otherwise, only the risky contract

$$(b_0, R_1) = (\pi_0(b_1^* + y_1), b_1^* + y_1)$$

will be feasible at date 0, and among those the best one is, of course, the one with

$$b_0^* = \pi_0(y_2 + y_1).$$

The borrower would prefer the latter provided that

$$\beta \leq 1 - \frac{(1-\pi_0)(\pi_0 - \beta)}{1-\beta} \frac{y_2}{y_1},$$

which is always satisfied for $\pi_0 \leq \beta$, whereas for the case when $\pi_0 > \beta$, it is also always satisfied provided that

$$\frac{y_2 - y_1}{y_1} \leq 3,$$

which is to say, provided that the growth in the non-financial income is not too high.

On the other hand, if the above inequality is violated, at date 0 the borrower would prefer the risky contract whenever

$$\pi_0^2 - (1+\beta)\pi_0 + \left(\beta + (1-\beta)^2 \frac{y_1}{y_2}\right) \geq 0,$$

that is, if

$$\pi_0 \geq \underbrace{\frac{1}{2}\left(1 + \beta + (1-\beta)\sqrt{1 - 4\frac{y_1}{y_2}}\right)}_{\widetilde{\pi}_0}.$$

It can be shown that, if $\pi_0 < \widetilde{\pi}_0$, the borrower would prefer the safe contract to autarky, provided that $\pi_0 > \beta$.

Finally, at date 0 the borrower will prefer the risky contract $(\pi_0 y_2, y_2)$ to autarky, provided that

$$\pi_0 \geq \underbrace{\frac{y_1}{y_1 + y_2}\beta + \frac{y_2}{y_1 + y_2}\beta^2}_{\pi_0^*},$$

which is a weighted average between β and β^2, which stands closer to β^2 the larger is the income growth between periods 1 and 2, $\frac{y_2 - y_1}{y_1}$. □

A.5 Proof of Lemma 2

Proof. Denote the lenders' posterior by

$$\pi_1 = \pi_1(p_1; R_1).$$

If the committed borrower decides to keep the asset and borrow against it, he would choose (b_1, R_2), solving

$$\max_{b_1, R_2} \left\{ y_1 - R_1 + b_1 + \beta(y_2 - R_2 + p_1) \right\},$$

subject to the lenders' zero-profit condition:

$$b_1 = \begin{cases} R_2 - (1-\pi_1)\dfrac{R_2^2}{4p_1}, & \text{if } R_2 < 2p_1, \\ \pi_1 R_2 + (1-\pi_1)p_1, & \text{if } R_2 \geq 2p_1. \end{cases}$$

As can be easily inspected, the optimal contract proposed by the committed type is given by

$$(b_1^*, R_2^*) = \begin{cases} \left(\dfrac{1-\beta^2}{1-\pi_1} p_1, \dfrac{2(1-\beta)}{1-\pi_1} p_1 \right) & \text{if } \pi_1 < \beta, \\ \left(\pi_1 y_2 + (1-\pi_1)p_1, y_2 \right) & \text{if } \pi_1 \geq \beta. \end{cases}$$

The committed type's utility from borrowing against the asset it thus given by

$$U_b^c(p_1; R_1, \pi_1) = y_1 + \beta y_2 - R_1 + \beta p_1 + \begin{cases} \dfrac{(1-\beta)^2}{1-\pi_1} p_1, & \text{if } \pi_1 < \beta, \\ (\pi_1 - \beta) y_2 + (1-\pi_1) p_1, & \text{if } \pi_1 \geq \beta. \end{cases}$$

If instead the committed type were to sell the asset and borrow against reputation, his optimal contract would be

$$(b_1^*, R_2^*) = \begin{cases} (0, 0) & \text{if } \pi_1 < \beta, \\ (\pi_1 y_2, y_2) & \text{if } \pi_1 \geq \beta. \end{cases}$$

Correspondingly, the committed type's utility from selling the asset is given by

$$U_s^c(p_1; R_1, \pi_1) = y_1 + \beta y_2 - R_1 + p_1 + \begin{cases} 0, & \text{if } \pi_1 < \beta, \\ (\pi_1 - \beta) y_2, & \text{if } \pi_1 \geq \beta. \end{cases}$$

Comparing the utility from selling the asset to the utility from keeping and borrowing against it on these two segments separately, we see that *selling the asset at date 1 and borrowing exclusively against reputation* dominates keeping the asset for the committed type.

Given R_1 and π_1, the committed type's continuation utility is equal to

$$U_1^c(p_1) = y_1 + \beta y_2 - R_1 + p_1 + \max\left\{ (\pi_1 - \beta) y_2, 0 \right\}.$$

□

A.6 Proof of Proposition 4

Proof. If the strategic type defaults on the loan, he loses both the asset and reputation, thereby consuming only his non-financial income, and his autarkic continuation utility is given by $y_1 + \beta y_2$.

If he instead mimics the committed type, repays the debt and sells the asset, his continuation utility is given by

$$U_b^s(p_1; R_1, \pi_1) = y_1 + \beta y_2 - R_1 + p_1 + \begin{cases} 0 & \text{if } \pi_1 < \beta, \\ \pi_1 y_2 & \text{if } \pi_1 \geq \beta. \end{cases}$$

As can be seen, whenever $p_1 > R_1$, the strategic type repays the debt irrespective of π_1, and thus we have

$$\pi_1 = \pi_0 < \beta.$$

Next, he is *indifferent* between defaulting and repaying whenever

$$\pi_1 = \frac{R_1 - p_1}{y_2} \quad \text{and} \quad \pi_1 \geq \beta.$$

Overall, the strategic type's randomization leads to the following lenders' posterior by the end of date 1:

$$\pi_1(p_1; R_1) = \begin{cases} 1 & \text{if } p_1 < R_1 - y_2, \\ \dfrac{R_1 - p_1}{y_2} & \text{if } R_1 - y_2 \leq p_1 < R_1 - \beta y_2, \\ \beta & \text{if } R_1 - \beta y_2 \leq p_1 \leq R_1, \\ \pi_0 & \text{if } p_1 > R_1. \end{cases}$$

It should be noted that, in order for the posterior to be kept at β for the price range

$$\pi_1 \in [R_1 - \beta y_2, R_1],$$

the *lenders have to be randomizing* as well: namely, whenever the borrower proposes the contract $(\pi_1 y_2, y_2)$, the lenders should accept this contract with probability $\alpha(p_1; R_1)$, so as to make the strategic borrower *indifferent* between defaulting and repaying:

$$\alpha(p_1; R_1) = \frac{R_1 - p_1}{\beta y_2}.$$

Given Bayes' rule for the update on behalf of lenders:

$$\pi_1(p_1; R_1) = \frac{\pi_0}{\pi_0 + (1 - \pi_0)(1 - \delta_1(p_1; R_1))},$$

we can back out the strategic borrower's equilibrium default probability:

$$\delta_1(p_1; R_1) = \begin{cases} 1 & \text{if } p_1 < R_1 - y_2, \\ 1 - \dfrac{\pi_0}{1-\pi_0}\left(\dfrac{y_2}{R_1 - p_1} - 1\right) & \text{if } R_1 - y_2 \leq p_1 < R_1 - \beta y_2, \\ \dfrac{\beta - \pi_0}{\beta(1-\pi_0)} & \text{if } R_1 - \beta y_2 \leq p_1 \leq R_1, \\ 0 & \text{if } p_1 > R_1. \end{cases}$$

The price thresholds are thus given by

$$\underline{p}_1 = R_1 - y_2 \quad \text{and} \quad \overline{p}_1 = R_1.$$

Correspondingly, the committed type's date-1 utility is given by

$$U_1^c(p_1; R_1) = y_1 + \beta y_2 - R_1 + p_1 + \begin{cases} (1-\beta)y_2, & \text{if } p_1 < R_1 - y_2, \\ R_1 - p_1 - \beta y_2, & \text{if } R_1 - y_2 \leq p_1 \leq R_1 - \beta y_2, \\ 0, & \text{if } p_1 > R_1 - \beta y_2, \end{cases}$$

which can be rewritten as

$$U_1^c(p_1; R_1) = \begin{cases} y_1 + y_2 - R_1 + p_1, & \text{if } p_1 < R_1 - y_2, \\ y_1, & \text{if } R_1 - y_2 \leq p_1 \leq R_1 - \beta y_2, \\ y_1 + \beta y_2 - R_1 + p_1, & \text{if } p_1 > R_1 - \beta y_2. \end{cases}$$

□

A.7 Proof of Theorem 1

Proof. Since $\pi_0 < \beta$, if the borrower were to sell the asset, there will be autarkic equilibrium in the corresponding continuation game, with the associated utility of

$$U_1^s = p_0 + \beta(y_1 + \beta y_2).$$

If the committed borrower were to keep the asset, he would optimally choose the contract (b_0, R_1), solving

$$\max_{b_0, R_1} \left\{ b_0 + \frac{\beta}{2p_0} \int_0^{2p_0} U_1^c(p_1; R_1) \mathrm{d}p_1 \right\}, \tag{A.1}$$

subject to the lenders' zero-profit condition:

$$b_0 = \frac{1}{2p_0} \int_0^{2p_0} \left\{ \frac{\pi_0}{\pi_1(p_1; R_1)} R_1 + \left(1 - \frac{\pi_0}{\pi_1(p_1; R_1)}\right) p_1 \right\} \mathrm{d}p_1 \tag{A.2}$$

and the feasibility constraint telling that at date 2, the borrower should be *able* to repay the loan under all circumstances:

$$R_1 \leq y_2 + p_1 + b_1^*(p_1; R_1), \quad \text{for all } p_1. \tag{A.3}$$

This constraint has to hold for all p_1, and it can be shown that the worst-case scenario is the lowest realization $p_1 = 0$, in which case the constraint translates into

$$R_1 \leq y_1 + y_2.$$

Suppose that the interval $[\beta y_2, y_1]$ is non-empty. Take the case when $\beta y_2 \leq 2p_0 \leq y_1$. In the limit as $\pi_0 \to 0$, the committed borrower's utility becomes

$$U_0^b(R_1; p_0)$$
$$= \begin{cases} -\frac{R_1^2}{4p_0} + (1-\beta)R_1 + \beta(y_1 + \beta y_2) + \beta p_0, & \text{if } 0 \leq R_1 < \beta y_2 \\ -(1-\beta)\frac{R_1^2}{4p_0} + \left[(1-\beta) - \frac{\beta^2 y_2}{2p_0}\right] R_1 - \frac{\beta^2 y_2^2}{4p_0} + \beta(y_1 + y_2) + \beta p_0, & \text{if } \beta y_2 \leq R_1 \leq 2p_0. \end{cases}$$

Taking the first order conditions, we obtain $\psi(R_1^*; p_0) = 0$, where

$$\psi(R_1; p_0) = \begin{cases} (1-\beta) - \dfrac{R_1}{2p_0} & \text{if } 0 \leq R_1 \leq \beta y_2, \\ (1-\beta) - \dfrac{\beta^2 y_2}{2p_0} - (1-\beta)\dfrac{R_1}{2p_0} & \text{if } \beta y_2 < R_1 \leq 2p_0. \end{cases}$$

When we have

$$(1-\beta) - \frac{\beta y_2}{2p_0} < 0 \quad \Longleftrightarrow \quad p_0 < \frac{\beta y_2}{2(1-\beta)},$$

the function attains its maximum on the interval $R_1 \in [0, \beta y_2)$.

The argmax is equal to

$$R_1^* = 2(1-\beta)p_0,$$

and the attained maximum is

$$\begin{aligned} U_0^b(p_0) &= -\frac{4(1-\beta)^2 p_0}{4p_0} + 2(1-\beta)^2 p_0 + \beta(y_1 + \beta y_2) + \beta p_0 \\ &= (1-\beta)^2 p_0 + \beta(y_1 + \beta y_2) + \beta p_0 \\ &= (1 - \beta + \beta^2) p_0 + \beta(y_1 + \beta y_2). \end{aligned}$$

This value is always less than the utility from selling, since $\beta < 1$.

Next, when we have

$$(1-\beta) - \frac{\beta y_2}{2p_0} \geq 0 \quad \Longleftrightarrow \quad p_0 \geq \frac{\beta y_2}{2(1-\beta)},$$

the function attains its maximum on the interval $R_1 \in [\beta y_2, 2p_0]$.

The argmax is equal to

$$R_1^* = 2p_0 - \frac{\beta^2 y_2}{1-\beta},$$

and the attained maximum is

$$\begin{aligned} U_0^b(p_0) &= \left[\frac{1-\beta}{2} - \frac{\beta^2 y_2}{4p_0}\right]\left(2p_0 - \frac{\beta^2 y_2}{1-\beta}\right) - \frac{\beta^2 y_2^2}{4p_0} + \beta(y_1+y_2) + \beta p_0 \\ &= (1-\beta)p_0 - \frac{\beta^2 y_2}{2} - \frac{\beta^2 y_2}{2} + \frac{\beta^4 y_2^2}{4(1-\beta)p_0} - \frac{\beta^2 y_2^2}{4p_0} + \beta(y_1+y_2) + \beta p_0 \\ &= \frac{[\beta^4 - \beta^2(1-\beta)]y_2^2}{4(1-\beta)p_0} + \beta(y_1+y_2) - \beta^2 y_2 + p_0 \end{aligned}$$

This value exceeds the utility from selling whenever

$$\frac{[\beta(1-\beta) - \beta^3]y_2}{4(1-\beta)p_0} \leq 1 - 2\beta,$$

so that

$$p_0 \geq \frac{\beta(1-\beta-\beta^2)y_2}{4(1-\beta)(1-2\beta)}$$

This bound is less tight than the bound $p_0 \geq \frac{\beta y_2}{2(1-\beta)}$ whenever

$$\beta \leq \frac{3-\sqrt{5}}{2}.$$

Evidently, the individual's utility from borrowing the asset is increasing in π_0, hence there will always exist the range of prices for which the borrower prefers to keep the asset. □