Capital Flows and Foreign Exchange Intervention

Paolo Cavallino†

This version: April 25, 2016

Abstract

I consider a New Keynesian model of a small open economy where international financial markets are imperfect and the exchange rate is determined by capital flows. I use this framework to study the effects of exchange rate fluctuations driven by capital flows and characterize the optimal foreign exchange intervention. Capital flow shocks cause inefficient exchange rate fluctuations that trigger boom-bust cycles in the domestic economy. The optimal policy response is to use both foreign exchange intervention and monetary policy to stabilize the economy. Foreign exchange intervention “leans against the wind” and stabilizes the path of the exchange rate, while monetary policy corrects the inefficiencies caused by price rigidity. The two tools are complements rather than substitutes. I derive the optimal foreign exchange intervention rule in closed form as a function of three implicit targets: a wedge in the Backus-Smith condition, domestic net foreign assets, and the level of foreign reserves.


Keywords: Capital Flows, Foreign Exchange Intervention, Monetary Policy

†International Monetary Fund, Research Department. Website: https://sites.google.com/site/pcavallino/home. Email: pcavallino@imf.org. Tel: +1 202 409 9064.

I would like to thank the members of my PhD committee: Xavier Gabaix, Mark Gertler, Matteo Maggiori and Thomas Philippon. My debt of gratitude to them can hardly be repaid. I would also like to thank: Jess Benhabib, Jaroslav Borovička, Pasquale Della Corte, Atish Rex Ghosh, Ricardo Lagos, Virgiliu Midrigan, Damiano Sandri, Lucio Sarno, Venky Venkateswaran, and participants at the NYU Stern Macroeconomics Lunch Seminar, NYU Job Market Presentation Group, IMF RESSI Seminar, SNB/ECB 5th Workshop on Financial Determinants of Exchange Rates, and 2016 AEA Annual Meeting. The views expressed in the paper are those of the author and do not necessarily represent those of the International Monetary Fund.
Flexible exchange rates have been praised in economic theory as a mechanism that facilitates relative price adjustments between countries and helps them achieve efficient outcomes. The countercyclical response of exchange rates to relative demand and supply shocks triggers expenditure-switching effects that smooth out output volatility and stabilizes the economy. According to this view, fluctuations in the real exchange rate are efficient responses to fundamental shocks and must therefore be welcomed. However, an alternative explanation for fluctuations in exchange rates is the presence of shocks arising in international financial markets. These fluctuations could drive the real exchange rate away from its efficient level and be the source of, rather than the cure for, economic disruption.

The evolution of financial integration, and the increased openness of both advanced and emerging economies to international capital flows, have raised new concerns among policymakers about exchange rate fluctuations driven by financial forces rather than fundamentals. As the participation of foreign investors in domestic asset markets increases, their portfolio allocation decisions become significant drivers of overall capital flows. Shifts in foreign investors portfolios can drive exchange rates away from their efficient levels and potentially affect domestic welfare through their effects on terms of trades, output and inflation. These concerns are especially strong for emerging market economies, where the relative shallowness of foreign exchange markets and the limited volume of trades strengthen the link between capital flow shocks and exchange rate volatility.

The view that exchange rates might deviate from their efficient levels as a consequence of non-fundamental shocks has a long tradition among policymakers. It was one of the key driver behind the Plaza and Louvre agreements, and it is at the heart of the well known “fear of floating” documented by Calvo and Reinhart (2002) in emerging economies. This irreconcilable dichotomy lies at the core of the debate around the exchange rate policies of emerging economies, and it is the reason why the consensus among policymakers has frequently shifted. The 1970s and 1980s were characterized by a large variety of pegs while the bipolar (float or fix) view was predominant during the 1990s. However, following the series of balance of payments crises that plagued emerging markets in the 1990s, many policymakers have settled for a floating exchange rate regime combined with inflation targeting monetary policy. Recent events seem to suggest that the consensus is on the verge of shifting once more.

The volatility of international capital flows has increased substantially since the global financial crisis of 2008-2010. To lean against the wind of capital inflows, many policymakers around the world have relied on heterodox policies such as non-conventional monetary policies, capital controls and foreign exchange interventions. According to the IMF\textsuperscript{1}, the last decade has seen an increasing number of countries actively managing their exchange rates. Brazil, Chile, Colombia, Turkey, and other emerging markets with announced inflation targeting regimes have increased both the frequency and the size of their operations in the foreign exchange market.\textsuperscript{2} Along them, also developed economies such as Israel and Switzerland have engaged in considerable manipulation of their currencies and have accumulated substantial reserves.\textsuperscript{3}

\textsuperscript{1}International Monetary Fund (2012), Annual Report on Exchange Arrangements and Exchange Restrictions, Washington, DC.
\textsuperscript{2}The Central Bank of Brazil (BC) raised foreign reserves from $54 billion at the end of 2005 to $350 billion at the end of 2011. Source: Central Bank of Brazil
\textsuperscript{3}Between March 2008 and August 2011 the Bank of Israel (BOI) accumulated $28.1 billion in foreign reserves, through daily purchases initially of $25 million and later raised to $100 million. From August 2009 to July 2011 the BOI continued to occasionally purchase foreign exchange to bring total reserves to $77.9 billion, an increases of almost 170% with respect to their level in March
Exchange rate interventions are typically attributed to precautionary or mercantilist motives. Accumulating foreign reserves might provide a cushion of liquidity if the flow of short-term debt is interrupted. Alternatively, currency interventions can be used to undervalue the domestic currency and boost the competitiveness of the export sector. Recent evidence (see for example Adler and Tovar Mora (2011) and Daude et al. (2014), among others) documents the prevalence of an alternative reason for intervening: that of stabilizing the exchange rate and reducing its volatility. The decision to intervene is increasingly driven by the goal of limiting what the policymaker perceives as unwarranted deviations of the exchange rates from their fundamental levels. Interventions correlate negatively with exchange rate pressures, i.e. they “lean against the wind”, and positively with foreign financial conditions and capital flows. Central bankers appear to be particularly worried about the negative effects of a strong currency on the competitiveness of their exports and ultimately on domestic output. Hence their attempt to fight non-fundamental appreciating pressures.4

However popular, the logic behind such interventions has not been formally tested. The goal of this paper is to characterize the optimal use of sterilized foreign exchange intervention in response to exchange rate fluctuations driven by capital flow shocks. I consider a New Keynesian model of a small open economy, as originally formulated by Gali and Monacelli (2005) and later extended to continuous time by Farhi and Werning (2012). I depart from the assumption of frictionless markets and assume that international financial markets are imperfect. Following the model developed by Gabaix and Maggiori (2015), I assume that agents in each country have limited access to international capital markets. International financiers absorb any imbalance between demand and supply of assets denominated in different currencies and clear the markets. In order to do so they require a premium in the form of expected currency appreciation/depreciation. I use this model to study the effects of shocks to foreign investors demand for domestic assets. When international asset markets are imperfect, portfolio flow shocks generate boom-bust cycles in the domestic economy. For example, an increase in the foreign demand for domestic assets appreciates the domestic currency and reduces the domestic real interest rate. Capitals flow in the domestic economy and fuel a consumption boom while output falls. The dynamics of consumption and output are reversed during the bust phase of the cycle. When foreign demand for domestic assets subsides their return increases and the external debt accumulated by domestic households must be reabsorbed. Consumption falls below its steady state level and output rises until the Home country pays off its debt.

I will show that, in response to such a shock, the optimal foreign exchange intervention “leans against the wind” and stabilizes the path of the exchange rate. Following an increase in the demand for domestic assets, the central bank increases their net supply and accumulates foreign reserves. By doing so the central bank absorbs part of the capital inflow and reduces its impact on the exchange rate. By leaning against the wind the central bank achieves two purposes. First, it reduces the initial appreciation of the domestic currency and sustains foreign demand, therefore reducing the output gap. This is the “monetary” aspect of the intervention: the central bank depreciates the nominal exchange rate in order to help domestic exporters to reduce their foreign currency prices. Second, by absorbing part of the capital inflows, it increases the

---

4Former Israel central bank governor Stanley Fisher remarked: “I have no doubt that the massive purchases [of foreign exchange] we made between July 2008 and into 2010 […] had a serious effect on the exchange rate which I think is part of the reason that we succeeded in having a relatively short recession”. Source: Levinson (2010)
domestic real interest rate and smooths out the consumption boom. Smoothing out consumption fluctuations increases welfare as it reduces the output gap caused by the wealth effect on labor supply and it allows the planner to exploit the monopsonistic/monopolistic power that the country has on its own terms of trade. However, it is never optimal for the central bank to fully stabilize domestic consumption. The reason is that, by creating a wedge between the return of domestic and foreign assets, portfolio flow shocks are akin to positive wealth shocks for the Home country. Foreign exchange interventions are costly since they reduce this wedge and its benefits. In determining the optimal amount of stabilization the central bank trades off the benefits that arise from the positive wealth shock and the inefficiencies caused by the exchange rate fluctuations. The optimal policy mix required to deal with portfolio flow shocks includes both foreign exchange intervention and monetary policy. The two tools are complements rather than substitutes. Foreign exchange intervention is the appropriate tool to smooth out consumption fluctuations as it allows the planner the choose the welfare-maximizing consumption path. However, when foreign exchange intervention is used also to reduce the output gap this leads to a stabilization bias. Monetary policy can help in reducing the output gap and reduce this burden. However, monetary policy can only move the allocation along an inefficient IS curve. Without a state-contingent labor subsidy, foreign exchange intervention is still used to shift the IS curve closer to efficiency. This conclusion is in sharp contrast with the standard view of foreign exchange interventions which emphasizes their monetary aspect.

The use of continuous-time techniques allows me to solve the planning problem in closed form and characterize the optimal foreign exchange intervention rule. I will show that the optimal policy can be implemented by a simple intervention rule that is a function of three implicit targets: a wedge in the Backus-Smith condition, domestic net foreign assets, and the level of foreign reserves.

The focus of this paper is the desirability of foreign exchange intervention in response to currency misalignments arising in financial markets. Questions regarding its effectiveness are the focus of a vast and lively literature. Most of the studies from the late 80s found that the coordinated interventions, conducted after the Plaza and Louvre agreements, were effective. Dominguez and Frankel (1993) analyze interventions using data from U.S. dollar, German mark and Swiss franc between 1982 and 1988. Their study points to the presence of a significant portfolio channel in the U.S. Dollar and German mark markets. More recently, Sarno and Taylor (2001) survey central bank interventions conducted in the 90s and conclude that foreign exchange intervention can be effective especially if it is publicly announced and provided that it is consistent with the underlying stance of monetary and fiscal policy. Menkhoff (2010) and Menkhoff (2013) survey the literature focusing on studies that use high-frequency data for both developed and emerging economies. The author finds that the evidence corroborates the hypothesis that interventions move the exchange rate level in the desired direction, especially in emerging market economies where higher reserves and shallow markets give central banks more leverage. Adler and Tovar Mora (2011) and Adler et al. (2015) provides evidence that interventions can affect the level of the exchange rate and the pace of appreciation. Although the evidence of its effectiveness is far from conclusive, it suggests that foreign exchange intervention can be a useful tool for central banks.

A recent strand of literature has focused on testing the relationship between capital flows and exchange rates. Froot and Ramadorai (2005) show that medium-term variations in expected currency returns are
associated with capital flows, while long-term variations are correlated with macroeconomic fundamentals. Hau et al. (2010), in the context of a natural experiment, provide direct evidence that exogenous capital flows affect the exchange rate. They show that countries that experienced capital inflows as a result of an increase in their weight in the MSCI World Equity Index, saw their currencies appreciate. More recently, Della Corte et al. (2016) find evidence of a global imbalance risk factor in currency excess returns. They show that the currencies of net debtor countries have higher returns, tend to be on the receiving end of carry trade flows, and depreciate when global financial conditions worsen. These results suggest that capital flows and portfolio decisions are key determinants of short to medium-term exchange rate fluctuations.

The rest of the paper is organized as follows: section 1 reviews the relevant literature; section 2 describes the model and its equilibrium; section 3 describes the log-linearized dynamics around the steady state and the planner problem; section 4 characterize the optimal foreign exchange intervention under flexible and rigid prices; section 5 studies the interaction between foreign exchange intervention and monetary policy; section 6 characterizes the optimal foreign exchange intervention rule; section 7 concludes.

1 Literature Review

This paper is related to three broad streams of literature: the international macroeconomic literature on the volatility of capital flows, the literature on international financial intermediation and the literature on optimal policy and welfare in open economies.

The volatility of capital flows and their effects on open economies are at the center of a large literature. Starting with the seminal work of Calvo (1998) most of the literature has focused on “sudden stops”, episodes in which the inflow of foreign capitals suddenly reverses. See for example Caballero and Krishnamurthy (2004), Mendoza (2010), Bianchi (2009), Bianchi and Mendoza (2010), Jeanne and Korinek (2010) and Korinek (2011). These papers emphasize the role of domestic borrowing constraints and pecuniary externalities in amplifying shocks to the external financing conditions. More recently Farhi and Werning (2014a) study the effects of foreign risk premia shocks in a small open economy with nominal rigidities. All these papers focus on capital controls as a tool to manage capital flows and stabilize the domestic economy. I contribute to this literature by studying the optimal use of an alternative tool, namely sterilized foreign exchange intervention, in response to capital flow shocks. A related strand of literature has focused on the role of capital flows in the transmission of financial conditions across countries. Rey (2015), Agrippino and Rey (2013), Shin (2014) and Blanchard et al. (2015) provide evidence of a global financial cycle driving capital flows, asset prices and credit growth across countries. The cycle might constraints and render ineffective domestic monetary policy if the capital account is not managed. Consistent with this conjecture, I find that monetary policy independence increases when foreign exchange intervention is used to deal with currency misalignments caused by capital flows.

This paper also relates to the literature on international financial intermediation and exchange rate determination in the presence of frictions. The earlier and most prominent strand of the literature on asset demand and exchange rate determination in general equilibrium has focused on complete markets. See for example Lucas Jr (1982), Backus, Kehoe and Kydland (1992), Backus and Smith (1993), Pavlova and
An other stream of literature has analyzed the role of market incompleteness in the determination of international portfolios. For recent examples see Chari, Kehoe and McGrattan (2002), Corsetti et al. (2008) and Rigobon et al. (2011). As shown by Backus and Kehoe (1989) in these classes of models sterilized intervention is not an extra policy instrument available to the central banks. When portfolio decisions are frictionless, the imperfect substitutability between assets postulated by the portfolio balance channel is not enough for sterilized intervention to have an effect on prices and quantities. In light of this result, in this paper I follow a different approach based on frictions in the intermediation process of international capital flows. Important contributions in the study of international financial frictions include Caballero and Krishnamurthy (2001), Jeanne and Rose (2002), Evans and Lyons (2002), Hau and Rey (2006) and Bruno and Shin (2015). A related stream of literature has focused on the effects of domestic financial frictions and market segmentation on the dynamics of portfolio flows and exchange rates. See for example Alvarez et al. (2009), Bacchetta and Van Wincoop (2010) and Maggiori (2013). Bruno and Shin (2014) develop a model of the international banking system where global and local funding constraints interact in the transmission of financial conditions across borders. Gabaix and Maggiori (2015) build an analytically tractable 2-period general equilibrium model where constrained international financiers intermediate capital flows across countries. They provide a novel micro-foundation to the portfolio balance channel and analyze the welfare effects of heterodox policies such as foreign intervention and capital controls. I build on their analysis and incorporate the intermediation friction in a standard New Keynesian small open economy. This allows me to provide a complete characterization of the optimal foreign exchange intervention policy and its interaction with standard monetary policy.

Finally, this paper relates to the literature on optimal policy and welfare in New Keynesian open economies. Aguiar et al. (2009), Schmitt-Grohé and Uribe (2012), Farhi and Werning (2012; 2014b; 2015) and Farhi et al. (2014) provide innovative analysis of the effects of conventional policies such as capital controls, fiscal transfers and fiscal devaluations on the welfare of a small open economy. This paper is most closely related to the literature on the desirability of stabilizing exchange rates. Much of this literature is concerned with investigating the conditions under which it is optimal to include the exchange rate in the reaction function of monetary policy. See for example Sutherland (2005), Corsetti and Pesenti (2005), Benigno (2009), De Paoli (2009) and the comprehensive review of Engel (2014). Engel (2011) characterizes the optimal response of monetary policy to currency misalignments arising from differences between home and foreign prices due to Local Currency Pricing (LCP). The focus of this paper is on currency misalignments arising in financial markets. There are few existing papers that allow for a non-trivial role for sterilized intervention. Some notable exceptions are Benes et al. (2015), Montoro and Ortiz (2013), Devereux and Yetman (2014) and Ghosh et al. (2016). These papers analyze the welfare effects of different exogenously specified intervention policies aimed at stabilizing the exchange rate. My contribution to this literature is to derive the optimal foreign exchange intervention policy in a utility maximization framework and characterize its implementation.

The theoretical underpinnings of the portfolio balance channel date back to Kouri (1976).
2 A Small Open Economy Model

Consider a continuous-time model with infinite horizon. The world economy is composed of a continuum of measure one of countries indexed by \( i \in [0, 1] \). Every country is inhabited by a continuum of measure one of identical households, and each household is composed of a measure one of family members with identical preferences over consumption paths. By assumption, there is perfect consumption insurance within the household because all members pay out their earnings to be shared equally across the entire household. In each country there is also a measure one of monopolistically competitive firms that produce a continuum of differentiated tradable goods. International financial markets are incomplete and segmented. The only assets available in the world economy are a continuum of riskless nominal bonds denominated in different currencies. Each bond pays one unit of the currency of a specific country. Households can freely trade domestic assets, i.e. bonds denominated in domestic currency, but they are constrained in their holdings of foreign assets. Hence, imbalances in financial markets might arise. The excess demand or supply of assets is absorbed, at some premium, by global financial firms. Financial intermediaries are allowed to invest in bonds denominated in different currencies and clear markets.

Uncertainty is represented by a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}(t)\}, \mathbb{P})\) on which is defined a standard Brownian motion \(Z(t)\) for \( t \in [0, \infty) \). All stochastic processes are assumed adapted to \(\{\mathcal{F}(t)\}_{t=0}^T\), the augmented filtration generated by \(Z\). In what follows I assume, without stating explicitly, all regularity conditions that ensure all processes introduced are well defined.

Since the focus of this paper is on the policy of a single economy, I will describe the model from the point of view of one country, which I call Home or domestic country, and can be though of as a particular value \( H \in [0, 1] \). To simplify the analysis I will also assume that all foreign countries are at their symmetric equilibrium at any point in time.\(^6\) I refer to them as rest of the world and denote their variables with a star superscript.

In the next sections, I describe in detail the problem facing households and firms located in the Home country. Unless otherwise noted, the problems facing foreign agents are symmetric. I will then describe the decision of financial intermediaries and the instruments available to the domestic policymaker.

**Households**

The Home country is inhabited by a continuum of measure one of identical households. Each household chooses consumption and hours of labor for all its family members. The representative household maximizes

\[
\mathbb{E} \left[ \int_0^\infty e^{-\rho t} U(C(t), L(t)) \, dt \right]
\]

(1)

where \( \rho > 0 \) is the time discount factor and \( L \) is the amount of labor supplied by each family member. The consumption index, \( C \), is a composite of tradable and non tradable goods defined by

\[
C(t) = \frac{C_T(t)^{\beta} C_{NT}(t)^{1-\beta}}{\beta^{\beta} (1-\beta)^{1-\beta}}
\]

\(^6\)This assumption allows me to keep track of only one set of international prices rather than a continuum of bilateral prices.
where $\beta \in [0, 1]$ measures the share of total expenditure allocated to non tradable goods with respect to tradable goods. The tradable goods consumption index is an aggregator of Home and imported goods, given by

$$C_T(t) \equiv \frac{C_H(t)^{1-\alpha} C_F(t)^{\alpha}}{\alpha^{\alpha} (1-\alpha)^{1-\alpha}}$$

where $\alpha \in [0, 1]$ measures the degree of Home bias in consumption and is therefore a natural index of the openness of the Home economy. As $\alpha \to 0$ the share of expenditure on foreign goods vanishes and the representative household consumes only goods produced domestically. The economy is effectively closed. As $\alpha \to 1$ the share of expenditure on domestically produced goods vanishes and the representative household consumes only goods produced abroad. Since each country has zero measure, this case describes a very open economy. The imported goods index $C_F$ is itself an aggregator of goods produced in different countries and it is defined by $C_F(t) \equiv \exp \left( \int_0^1 \ln C_i(t) \, di \right)$. Each country produces a continuum of varieties of both tradable and non tradable goods. Therefore each $C_i$, for $i \in [0, 1]$, and $C_{NT}$ are indexes of consumption of all varieties defined by

$$C_i(t) \equiv \left[ \int_0^1 C_{i,j}(t) \frac{\epsilon - 1}{\epsilon} \, dj \right]^{\epsilon - 1} \quad i \in [0, 1]$$

$$C_{NT}(t) \equiv \left[ \int_0^1 C_{NT,j}(t) \frac{\epsilon - 1}{\epsilon} \, dj \right]^{\epsilon - 1}$$

where $\epsilon > 1$ measures the elasticity of substitution across different varieties.

Let $O \equiv P_{NT}/P_T$ be the relative price of nontradables to tradables in the Home country. Define the Home terms of trade as the Home price of imported goods divided by the price of Home goods, such that an increase in the terms of trade represents a deterioration in the Home terms of trade: $S \equiv P_F/P_H$. Let $E$ be the nominal exchange rate between the Home country and the rest of the world, defined as the Home currency price of one unit of any foreign currency, such that a decrease in $E$ is an appreciation of the domestic currency. Assume that its law of motion is

$$\frac{dE(t)}{E(t)} = \mu_E(t) \, dt + \sigma_E(t) \, dZ(t)$$

where $\mu_E$ and $\sigma_E$ are the drift and volatility of $E$.

International financial markets are incomplete and segmented. The representative Home household can only invest in two assets: a Home bond paying one unit of the domestic currency, and a continuum of foreign bonds each paying one unit of a foreign currency. However, while it can freely trade domestic assets, its holding of foreign bonds is constrained. Let $D(t)$ denote the value of Home bonds held by the representative household and $F(t) \equiv \int_{i \neq H} F_i(t) \, di$ the foreign-currency value of its portfolio of foreign bonds.\footnote{Since foreign countries are identical at any point in time, the actual composition of the portfolio is indeterminate. Only its aggregate value matters.} For simplicity, I will model the demand for foreign assets as an exogenous process. The portfolio flow $F$ can be thought of as an inelastic demand coming from noise family members. It can also be motivated
as a liquidity shock or as the result of time-varying portfolio constraints. Understanding the determinants of international portfolios is a key research question in international finance, and the focus of a developing literature\(^8\), but it goes beyond the scope of this paper.

Let \( A(t) \equiv D(t) + \mathcal{E}(t) F(t) \) be the Home-currency value of the assets held by the representative household. Its dynamic budget constraint is

\[
dA(t) = A(t) i(t) \, dt + \mathcal{E}(t) F(t) \left[ (i^*(t) - i(t)) + \mu_{\mathcal{E}}(t) \right] \, dt + \sigma_{\mathcal{E}}(t) \, d\mathcal{Z}(t) \\
+ [W(t) L(t) - P(t) C(t)] \, dt + d\Sigma(t) + dT(t)
\]

where \( i \) is the domestic nominal interest rate, \( i^* \) is the nominal interest rate in the rest of the world, \( W \) is the nominal wage, \( P \) is the domestic Consumer Price Index (CPI), \( \Sigma \) represents nominal profits received from domestic firms, and \( T \) is a nominal lump-sum component of income which includes all taxes paid to and transfer received from the government.

The optimal allocation of expenditure across different goods yields the following demand functions

\[
C_T(t) = (1 - \beta) \left( \frac{P_T(t)}{P(t)} \right)^{-1} C(t) \\
C_{NT}(t) = \beta \left( \frac{P_{NT}(t)}{P(t)} \right)^{-1} C(t) \\
C_H(t) = (1 - \alpha) \left( \frac{P_H(t)}{P_T(t)} \right)^{-1} C_T(t) \\
C_F(t) = \alpha \left( \frac{P_F(t)}{P_T(t)} \right)^{-1} C_T(t)
\]

and

\[
C_{i,j}(t) = \left( \frac{P_{i,j}(t)}{P_i(t)} \right)^{-\epsilon} C_i(t) \quad \forall i \in [0,1] \text{ and } \forall j \in [0,1]
\]

Home CPI is defined as \( P \equiv P_T^{1-\beta} P_{NT}^{\beta} \). The Tradable Price Index (TPI) and Non Tradable Price Index (NTPI) are given by \( P_T \equiv P_H^{1-\alpha} P_F^{\alpha} \) and \( P_{NT} \equiv \left( \int_0^1 P_{NT,j}^{1-\epsilon} \, d\epsilon \right)^{1/\epsilon} \), respectively, while the price index for tradable goods produced at Home is \( P_H \equiv \left( \int_0^1 P_{H,j}^{1-\epsilon} \, d\epsilon \right)^{1/\epsilon} \).

The problem of the representative household is to choose consumption, labor and investment policies to maximize (1) subject to the budget constraint (2), given the exogenous demand for foreign assets \( F \). In what follows I will specialize the period utility function to take the form

\[
U(C(t), L(t)) \equiv \ln C(t) - \frac{L(t)^{1+\varphi}}{1+\varphi}
\]

Under this specification the optimal consumption/saving policy is described by the Euler equation

\[
\frac{dC(t)}{C(t)} = \left[ i(t) - \mu_P(t) - \rho + \sigma_P(t)^2 + \sigma_C(t) \sigma_P(t) + \sigma_C(t)^2 \right] \, dt + \sigma_C(t) \, d\mathcal{Z}(t)
\]

where \( \mu_P \) and \( \sigma_P \) are drift and diffusion of the CPI, such that \( dP(t) / P(t) = \mu_P(t) \, dt + \sigma_P(t) \, d\mathcal{Z}(t) \).

\(^8\)See for example Pavlova and Rigobon (2010), Rigobon et al. (2011) and Maggiori (2013), among others.
The labor supply policy is

\[ L(t) \varphi(C(t)) = \frac{W(t)}{P(t)} \]

Foreign households have symmetric preferences and solve a similar problem. With a slight abuse of notation, I denote with \( F_H^* (t) = \int_{i \neq H} F_{H,i}^* (t) dt \) the aggregate foreign-currency value of Home bonds held by foreign households. As for \( F \), I model foreign demand for domestic assets as an exogenous process. Since each country has measure zero, Home assets represents a negligible share of foreign households portfolios\(^9\). Thus, shocks to the foreign demand for Home bonds do not affect foreign prices nor aggregate quantities.

**Firms**

Each country produces tradable and nontradable goods. Each type of good is produced in a measure one of varieties by a continuum of monopolistically competitive firms, indexed by \( j \in [0, 1] \). Firms in both sectors use the same technology, described by the production function

\[ Y_{k,j}(t) = L_{k,j}(t) \tag{4} \]

for \( j \in [0, 1] \) and \( k = H, NT \). The profits generated by a generic firm \( j \) operating in the tradable goods sector are given by

\[ d\Sigma_{H,j}(t) = \left[ P_{H,j}(t) C_{H,j}(t) + E(t) P_{H,j}^*(t) C_{H,j}^*(t) - (1 - \tau_H) W(t) L_{H,j}(t) \right] dt \]

where \( P_{H,j} \) is the Home-currency price of good \( j \) when it is sold in the Home country while \( P_{H,j}^* \) is the foreign-currency price of the good when it is sold abroad. \( C_{H,j} \) represents sales of the good in the Home country while \( C_{H,j}^* \) represents its export. The Home policymaker is assumed to have only limited fiscal instruments, a constant sector-specific labor subsidy \( \tau_H \) which is chosen to maximize domestic welfare in steady state. Similarly, profits generated by a generic firm \( j \) operating in the non tradable goods sector are

\[ d\Sigma_{NT,j}(t) = \left[ P_{NT,j}(t) C_{NT,j}(t) - (1 - \tau_{NT}) W(t) L_{NT,j}(t) \right] dt \]

where \( \tau_{NT} \) is the labor subsidy for firms in the non tradable sector. There is perfect labor mobility across sectors and, therefore, a unique wage rate \( W \). Total forms profits are \( \Sigma = \int_0^1 \Sigma_{H,j} dj + \int_0^1 \Sigma_{NT,j} dj \).

In each sector firms face an identical isoelastic demand schedule for their own goods and set prices infrequently à la Calvo (Calvo (1983)). Each firm is allowed to reset its price only at stochastic dates determined by a Poisson process with intensity \( \theta \). Firms set their prices in domestic currency and the law of one price holds:

\[ P_{H,j}^*(t) = \frac{P_{H,j}(t)}{E(t)} \]

Under this assumption there is perfect exchange rate pass-through as exchange rate shocks are transmitted

\(^9\)Provided \(|F^*(t)| < \infty \) for all \( t \in [0, T] \)
one-to-one to import prices.\textsuperscript{10} The firm’s objective is to maximize the present discounted value of its stream of profits, subject to the sequence of domestic and foreign demand schedules. A tradable good firm that is allowed to reset its prices at time $t$ solves the following problem

$$\max_{P_{H,j}(t)} \mathbb{E} \left[ \int_t^\infty \theta e^{-(\rho+\theta)(k-t)} \frac{C(t) P(t)}{C(k) P(k)} \left\{ \hat{P}_{H,j}(t) Y_{H,j}(k|t) - (1 - \tau_H) W(k) L_{H,j}(k|t) \right\} dk \right]$$

subject to (4), where $Y_{H,j}(k|t) = \left( \frac{\hat{P}_{H,j}(t)}{P_H(k)} \right)^{-\epsilon} Y_H(k)$. Similar to tradable goods firms, non-tradable goods producers solve the following problem

$$\max_{P_{NT,j}(t)} \mathbb{E} \left[ \int_t^\infty \theta e^{-(\rho+\theta)(k-t)} \frac{C(t) P(t)}{C(k) P(k)} \left\{ \hat{P}_{NT,j}(t) Y_{NT,j}(k|t) - (1 - \tau_{NT}) W(k) L_{NT,j}(k|t) \right\} dk \right]$$

subject to (4), where $Y_{NT,j}(k|t) = \left( \frac{\hat{P}_{NT,j}(t)}{P_{NT}(k)} \right)^{-\epsilon} Y_{NT}(k)$.

**International Financial Intermediaries**

Households can freely trade domestic bonds, but they are constrained in their holdings of foreign assets. Hence, imbalances in financial markets might arise. In the global financial markets operate a mass one of financial firms, that can invest in assets denominated in different currencies and are therefore able to absorb any excess demand or supply. In order to do so, however, they require a premium in the form of expected currency appreciation/depreciation. Financial intermediaries are owned by foreign households and managed by their family members.\textsuperscript{11} Formally, each foreign household owns a continuum of measure one of financial intermediaries, each managed by one of its family members. Each intermediary is specialized in trading assets issued by a specific country. That is, the financial intermediary specialized in country-$i$ assets is restricted to hold a portfolio composed only of such assets and domestic bonds.\textsuperscript{12}

In order to keep the analysis tractable I assume that the management of financial firms is a short-term job. At time $t$, family members are randomly assigned to financial intermediaries. Each manager then chooses a self-financed portfolio of domestic assets and assets issued by the country in which her intermediary is specialized in. At time $t + dt$, with $dt \downarrow 0$, they collect profits and pay them back to the household, where they will be shared equally among all family members. Managers are then randomly reassigned to new intermediaries and the cycle starts over. Consider the problem facing the managers of financial intermediaries specialized in trading with the Home country living in country $i$. At time $t$, each of them chooses a portfolio of assets $\left\{ Q_{H,i}^*(t), Q_{i,i}^*(t) \right\}$ subject to the balance sheet constraint

$$\frac{Q_{H,i}^*(t)}{E(t)} + Q_{i,i}^*(t) = 0$$

\textsuperscript{10}This price-setting assumption is also known as Producer Currency Pricing (PCP) in contrast with Local Currency Pricing (LCP). Under the latter, firms choose their domestic and foreign prices independently and the law of one price does not hold.

\textsuperscript{11}This assumption is made only to simplify the exposition and it is without loss of generality.

\textsuperscript{12}This arrangement guarantees that in each market, domestic households and foreign investors have comparable masses.
where $Q_{H,i}^*$ is the Home-currency value of Home assets and $Q_{i,i}^*$ is the foreign-currency value of domestic assets held by the intermediary. At time $t + dt$, they collect returns and return to their own households. Their instantaneous net portfolio return is

$$dV^*_i(t) = \frac{Q_{H,i}^*}{E(t)} \left( i(t) - \bar{i}(t) - \mu_E(t) + \sigma_E(t)^2 \right) dt - \frac{Q_{H,i}^*}{E(t)} \sigma_E(t) dZ(t)$$

Following Gabaix and Maggiori (2015), I will assume that the borrowing process is subject to an agency friction that limits the size of the balance sheet of the intermediary and prevents perfect arbitrage between assets denominated in different currencies. At time $t$, after taking positions $\{ Q_{H,i}^*, Q_{i,i}^* \}$, managers can divert borrowed funds at rate $\left[ \frac{Q_{H,i}^*}{E(t)} \right] \sigma_E(t) v \Gamma$ per unit of time, where $\Gamma, v \geq 0$. Hence, the total amount of divertable funds is $\left( \frac{Q_{H,i}^*}{E(t)} \right)^2 \sigma_E(t)^2 \Gamma dt$. Notice that the manager, and implicitly the household owners of the intermediary, has limited commitment before returns are realized but full commitment after that. At $t + dt$ the financier is not allowed to default and any loss is absorbed by the household. Since that of the financier is a short-term job, and there is perfect consumption insurance within the household, managers will only care about the instantaneous expected return of their portfolio. Hence, they will choose not to divert funds if the following incentive compatibility constraint holds

$$\mathbb{E} [dV^*_i(t)] \geq \left( \frac{Q_{H,i}^*}{E(t)} \right)^2 \sigma_E(t)^2 \Gamma dt \quad (5)$$

Creditors correctly anticipate the incentives of the financier to divert funds and are willing to lend as long as (5) holds. Therefore, financiers solve the following problem:

$$\max_{Q_{H,i}^*} \mathbb{E} [dV^*_i(t)] = \frac{Q_{H,i}^*}{E(t)} \left( i(t) - \bar{i}(t) - \mu_E(t) + \sigma_E(t)^2 \right) dt$$

subject to $\mathbb{E} [dV^*_i(t)] \geq \left( \frac{Q_{H,i}^*}{E(t)} \right)^2 \sigma_E(t)^2 \Gamma dt$

Since the value function of the manager is linear in $Q_{H,i}^*$, while the credit constraint is convex, the constraint will always bind and the solution to the manager’s problem yields the following demand for Home bonds: $\frac{Q_{H,i}^*}{E(t)} = \frac{i(t) - \bar{i}(t) - \mu_E(t) + \sigma_E(t)^2}{\Gamma \sigma_E(t)^2}$. Since all foreign countries are identical, simple aggregation across countries yields the aggregate financial sector demand schedule for Home assets:

$$\frac{Q_H(t)}{E(t)} = \int_0^1 \frac{Q_{H,i}^*}{E(t)} \; di = \frac{i(t) - \bar{i}(t) - \mu_E(t) + \sigma_E(t)^2}{\Gamma \sigma_E(t)^2} \quad (6)$$

The financiers’ demand for Home assets is increasing in the excess return of Home bonds vis-à-vis foreign

---

13Since the financial intermediary has no capital, the foreign-currency value of the funds borrowed is always equal to $\left| \frac{Q_{H,i}^*}{E(t)} \right|$, regardless of the positions the manager is taking.

14For the constraint to make economic sense it must be the case that $\frac{Q_{H,i}^*}{E(t)} \sigma_E(t)^2 \Gamma \leq 1$. That is, the financier cannot steal more than 100% of the funds borrowed. In what follows I assume that the parameters of the model are such that this condition always holds.
bonds while it is decreasing in the volatility of the exchange rate. The parameter $\Gamma$ determines the size of the balance sheet of the financiers and is therefore an inverse measure of their risk-bearing capacity. The higher is $\Gamma$, the lower the financiers ability to sustain the currency risk of their portfolio and the higher the required compensation per unit of risk. As $\Gamma \uparrow \infty$ then $\frac{Q_i}{\epsilon} \rightarrow 0$ and financiers are unable to absorb any imbalance. Vice versa, as $\Gamma \downarrow 0$ then $i - i^* - \mu_E + \sigma_E (t)^2 \rightarrow 0$ and all bonds have the same expected rate of return, once measured in the same currency. Financiers risk-bearing capacity is so high that they arbitrage away any excess return. The parameter $\nu$ governs the sensitivity of financiers’ demand to the volatility of the exchange rate. As $\nu \downarrow 0$ this sensitivity decreases and financiers only care about excess return.

**Central Bank**

The Home central bank has two instruments to stabilize the domestic economy: monetary policy and foreign exchange intervention. Throughout the paper I specify monetary policy in terms of an interest rate rule (directly or indirectly), therefore I do not need to introduce money explicitly in the model. The second tool, currency intervention, is the main focus of this paper. The Home central bank can intervene in the asset market by buying and selling domestic and foreign bonds. Unlike households, the central bank is not constrained in his holdings of foreign assets. Let $X (t) \equiv \int_{i \neq H} X_i (t) \, di$ denote the foreign-currency value of the portfolio of foreign bonds held by the central bank, i.e. its foreign reserves. The central bank funds its holding of foreign reserves by issuing domestic bonds. Let $X_H (t)$ denote the amount of Home bonds held by the central bank. Without loss of generality, since Ricardian equivalence holds, assume that the central bank has no capital and rebates all profits and losses generated by its portfolio of assets to domestic households. Its balance sheet equation is

$$E (t) X (t) + X_H (t) = 0 \tag{7}$$

A (sterilized) foreign exchange intervention is any purchase/sale of foreign assets that alters the relative supply of Home bonds. Through foreign exchange purchases the central bank increases foreign reserves ($X (t) \uparrow$) and increases the net supply of assets denominated in Home currency ($X_H (t) \downarrow$). Vice versa, foreign exchange sales reduce the amount of foreign reserves held by the central bank ($X (t) \downarrow$) and decrease the net supply of Home-currency bonds ($X_H (t) \uparrow$). Notice that, by focusing on a cashless economy, I am ignoring the currency component of the balance sheet of the central bank. The central bank cannot increase (decrease) its holding of foreign assets by issuing (purchasing) domestic currency. Thus, foreign exchange interventions considered in this paper are sterilized interventions. Non-sterilized interventions can be modeled as a combination of sterilized intervention and interest rate policy.

The net transfer to domestic households includes central bank’s revenues/losses and taxes raised to finance the labor subsidy. It is given by

$$dT (t) = [X_H (t) (i (t) - i^* (t) - \mu_E (t)) - W (t) (\tau NT L_{NT} (t) + \tau H L_H (t))] \, dt - X_H (t) \sigma_E (t) \, dZ (t)$$

\[15\text{Money can be explicitly introduced in the model by using a cash-in-advance constraint or by augmenting the utility function to include real money balances.}

13
**Uncovered Interest Parity**

This section characterizes the link between portfolio flows and exchange rates. The time-$t$ market clearing condition for Home bonds is:

$$D(t) + F^*_H(t) + Q^*_H(t) + X_H(t) = 0$$

Using financial intermediaries aggregate demand, equation (6), I obtain the following arbitrage condition between Home and foreign bonds

$$i(t) - i^*(t) - \mu_E(t) + \sigma_E(t)^2 = -\Gamma \sigma_E(t)^2 \left[ \frac{D(t)}{E(t)} + \frac{F^*_H(t)}{E(t)} + \frac{X_H(t)}{E(t)} \right]$$

This equation provides a link between the demand for Home-currency assets and the exchange rate. The Home currency risk premium, on the left hand side of the equation, is proportional to the volatility of the exchange rate and the excess demand for Home-currency bonds that is absorbed by the financial sector. When international asset markets are imbalanced, the relative return of Home and foreign bonds must adjust in order to induce financial intermediaries to absorb the imbalance. The nominal exchange rate is the relative price between assets denominated in different currencies and provides the adjustments necessary to clear the markets. In the model, deviations from the Uncovered Interest rate Parity (UIP) are endogenous and proportional to the imbalances in demand and supply of assets denominated in different currencies. When an excess demand for Home assets arises ($D(t) + F^*_H(t) + X_H(t) > 0$) the Home currency risk premium must fall in order to induce international financiers to take short positions denominated in domestic currency. Vice versa, when the supply of Home bonds exceeds their demand ($D(t) + F^*_H(t) + X_H(t) < 0$), the risk premium of the Home currency rises.

By using the balance sheet equation of Home households, the excess demand for Home bonds can be further decomposed as follows

$$i(t) - i^*(t) - \mu_E(t) + \sigma_E(t)^2 = -\Gamma \sigma_E(t)^2 \left( \frac{A(t)}{E(t)} + \frac{F^*_H(t)}{E(t)} - F(t) + \frac{X_H(t)}{E(t)} \right)$$

This decomposition highlights three sources of demand for Home assets. The first component is associated with capital account, or net foreign assets, flows. These flows are generated by the desire of the Home households to reallocate their consumption intertemporally and they are accompanied by symmetric trade flows. The second component of the demand for Home assets is associated with portfolio flows. Portfolio flows are induced by the desire of domestic or foreign households to alter the composition of their portfolio of assets and therefore alter the country’s gross external position. Without any form of asset markets segmentation, portfolio flows don’t have any impact on the exchange rate as households in other countries would be willing to take the other side of the portfolio as long as uncovered interest parity holds. However, when asset markets are segmented the relative demand for domestic and foreign assets matter, and changes in countries’ gross external position affect the path of the exchange rates. Shocks to portfolio flows will be the main focus.
of this paper. Without loss of generality, I will assume $F = 0$ and consider shocks to the foreign demand for domestic assets, $F^*_H$. Although I model these shocks as exogenous, shifts in foreign investors demand for Home bonds capture a broad range of situations: they might be triggered by shocks to the perceived risk of investing in the home country, or by shocks to the overall riskiness of the investor’s portfolio which induce yield-searching or safe-heaven type of flows, or they may represent investor’s preferences for a particular country’s bonds. Importantly, these shocks induce exchange rate fluctuations that would not arise in standard models with frictionless asset markets. In this sense, they capture exchange rate movements that are not driven by standard macroeconomic fundamentals. The last component of the demand for Home assets is given by the central bank intervention. By expanding and contracting its balance sheet, the Home central bank can alter the relative supply of assets and therefore the size of the imbalance that financiers must absorb. As we have seen above, this affects the Home currency risk premium and therefore the path of the exchange rate. Formally, foreign exchange intervention is central bank intermediation of international flows that coexists and complements intermediation provided by financial intermediaries. Foreign exchange intervention affects the exchange rate to the extent there exists limits to arbitrage in private intermediation, that is $\Gamma > 0$.\footnote{In closed economy, similar forms of intervention have been studied in the context of large scale asset purchases and unconventional monetary policy. See for example Gertler and Karadi (2011), Gertler et al. (2012) and Gertler et al. (2013)}

**Equilibrium**

Following the literature on open economies with incomplete markets, define the consumption wedge as the marginal utility of a unit of Home currency for foreign households relative to Home households. Formally

$$\Lambda (t) \equiv \frac{1}{Q(t)} \frac{C(t)}{C^*}$$

where $Q \equiv E P^*/P$ is the real exchange rate, defined as the relative price of one unit of foreign consumption in terms of domestic consumption such that a decrease in $Q$ is an increase in the purchasing power of the Home currency. If international asset markets were frictionless ($\Gamma = 0$), marginal utilities in the Home country and in the rest of the world would grow at the same rate once converted in the same units. Thus, $\Lambda$ would be constant for all $t$ and the real exchange rate would be proportional to their ratio: $Q(t) \propto \frac{C(t)}{C^*}$. This is the well-known Backus-Smith condition (Backus and Smith (1993)) when there is perfect risk sharing across countries. This condition fails in the present model since households have to trade with constrained financiers. Fluctuations in $\Lambda$ introduce a time-varying wedge between the marginal rate of substitution between foreign and domestic consumption and its marginal rate of transformation, the real exchange rate. Using the Euler equations of domestic and foreign agents, we can derive the law of motion of the consumption wedge:

$$\frac{d\Lambda (t)}{\Lambda (t)} = \left( i(t) - i^* - \mu \varepsilon (t) + \sigma \varepsilon (t)^2 + \sigma \Lambda (t) \sigma \varepsilon (t) + \sigma^2 \right) dt + \sigma \Lambda (t) dZ (t)$$

\footnote{In closed economy, similar forms of intervention have been studied in the context of large scale asset purchases and unconventional monetary policy. See for example Gertler and Karadi (2011), Gertler et al. (2012) and Gertler et al. (2013).}
The nominal wage level is determined by the households labor supply schedule:

\[ \Delta \]

where the consumption wedge, since it tends to reduce its drift \( \left( E \left[ \frac{dA(t)}{A(t)} \right] \right) \). Either domestic consumption increases relative to foreign consumption or the real exchange rate appreciates, or both. The domestic currency is overvalued as the real exchange rate falls below its efficient level, defined as the value that would arise under complete and frictionless markets given the current relative consumption. Vice versa, an increase in the Home currency risk premium reduces the consumption wedge and the domestic currency is undervalued/overvalued and thus differ from the global optimum.

Let \( Y_k \equiv \int_0^1 Y_{k,j} (t) \, d\tilde{j} \) for \( k = H, NT \) be aggregate domestic output in each sector. Market clearing in the goods markets require

\[
Y_H (t) = C_H (t) + C_H^* (t) \\
Y_{NT} (t) = C_{NT} (t)
\]

The labor market clearing condition is

\[
L (t) = \int_0^1 L_{H,j} (t) \, dj + \int_0^1 L_{NT,j} (t) \, dj = \Delta_H (t) Y_H (t) + \Delta_{NT} (t) Y_{NT} (t)
\]

where \( \Delta_k (t) \equiv \int_0^1 \left( \frac{P_k(t)}{P_k(t)} \right)^{-\epsilon} \, dj \), for \( k = H, NT \), are indexes of price dispersion in each sector. The nominal wage level is determined by the households labor supply schedule: \( W(t) = L(t)^{\frac{\epsilon}{1-\epsilon}} C(t) P(t) \). The law of motion of Home net foreign assets, evaluated in foreign currency, is

\[
dA^S (t) = \left[ A^S (t) i^* (t) + (A^S (t) + X_H^S (t)) \left( i (t) - i^* - \mu_E (t) + \sigma_E (t)^2 \right) \right. \\
\left. + \left( Y_H (t) \frac{S(t)}{S(t)} + \frac{O(t)}{S(t)^{1-\alpha}} Y_{NT} (t) - \frac{O(t)^{\alpha}}{S(t)^{1-\alpha}} C(t) \right) P_F^* \\
- \left( A^S (t) + X_H^S (t) \right) \sigma_E (t) dZ (t) \right] dt
\]

where a dollar superscript denotes the foreign-currency value of the variable. This is the Home country aggregate budget constraint. The real exchange rate is given by \( Q(t) = \frac{S(t)^{1-\alpha} P_F^*}{O(t)^{\alpha} P_F} \). Finally, market clearing in the Home bonds market yields the UIP equation described in the previous section

\[
i(t) - i^* - \mu_E (t) + \sigma_E (t)^2 = -\Gamma \sigma_E (t)^{\nu} \left( A^S (t) + F_H^S (t) + X_H^S (t) \right)
\]

The model is closed by the firms pricing policies which determine the laws of motion of prices and their dispersions. I will consider three different price-setting assumptions. First I will consider the case of flexible prices. When firms are allowed to reset their prices at every \( t \) then the optimal pricing setting decisions
are \( P_{k,j}(t) = \frac{\epsilon}{\tau_k}(1 - \tau_k)W(t) \), for \( k = NT, H \). Second I will consider the extreme case of rigid prices. Under this assumption firms are not able to change their prices at any point in time. Therefore \( P_{k,j}(t) = P_{k,j}(0) \), for \( k = NT, H \) and \( \forall t \in [0, \infty) \). Finally, I will consider the intermediate case of staggered pricing. When firms are allowed to reset their prices only infrequently, PPI inflation in sector \( k \) is given by \( \pi_k(t) = \theta - \epsilon \left( \Theta_k(t) \Phi_k(t) \right)^{1-\epsilon} - 1 \), where \( \Theta_k \) and \( \Phi_k \) are expected present discounted values of future revenues and costs in sector \( k \), respectively. Their equations, together with the law of motion of price dispersion, are reported in the appendix.

The next lemma characterizes the symmetric steady state of the economy and derives the optimal static labor subsidies.

**Lemma 1.** Suppose prices are flexible, and there are no portfolio flows: \( F_H^S = 0 \). Then the symmetric steady state of the economy has \( \Lambda = 1, \ A^S = X^S_H = 0 \) and the optimal labor subsidies are \( \tau_H = 1 - \frac{1}{\lambda(1-\alpha)} \) and \( \tau_{NT} = 1 - \frac{1}{\lambda_T} \).

At a symmetric steady state, the central bank of the Home country does not manipulate the exchange rate. This should not be surprising since at a symmetric steady state there are no capital flows between the Home country and the rest of the world. Therefore financial market imperfections are mute in equilibrium. This result, however, highlights the fact that the use of foreign exchange intervention in this model is not driven by mercantilist motives. The policymaker does not have any incentive to sustain a permanently undervalued currency with the goal of increasing foreign demand for domestic goods.

### 3 The Log-Linearized Model

In order to study the effects of portfolio flow shocks on the small open economy and characterize the optimal policy response I use a log-linearized version of the equilibrium conditions described in the previous section. Before doing so, however, in order to preserve the financial frictions that is at the heart of the model I take the limit for \( \nu \downarrow 0 \). I characterize optimal policies by considering a second order approximation of the welfare function around the deterministic steady state. The optimal policies obtained from the solution of the linear-quadratic problem are linear and in many cases an analytical solution can be derived. In what follows lowercase letters denote log-deviations from steady state, if the variable has a non-zero steady state value. Otherwise, they denote simple deviations from steady state.

The equations that approximate the local dynamics of the equilibrium around its steady state are the log-linearized analogues of the stochastic differential equations described in the previous section. The consumption wedge and net foreign assets evolve as follows:

\[
\begin{align*}
\frac{d\lambda(t)}{dt} &= -\Gamma \left( a^S(t) + f_H^S(t) - x(t) \right) \\
\frac{da^S(t)}{dt} &= \left( \rho a^S(t) - \eta \lambda(t) \right)
\end{align*}
\]

where \( \eta = \alpha (1 - \beta) P^*C^* \), subject to the initial condition \( a^S(0) = 0 \) and the terminal condition...
\[
\lim_{t \to \infty} e^{-\rho t} q^S(t) = 0. \]

The laws of motion of domestic outputs and inflations depend on the price setting assumption. If prices are sticky then domestic tradable and non-tradable output follows

\[
dy_H(t) = (i(t) - \rho - \pi_H(t)) \, dt - \alpha d\lambda(t)
\]

\[
dy_{NT}(t) = (i(t) - \rho - \pi_{NT}(t)) \, dt
\]

These are the log-linearized IS curves of the model. PPI inflation in each sector evolves according to

\[
d\pi_H(t) = [\rho \pi_H(t) - k(1+\omega) y_H(t) - k \psi y_{NT}(t) - \alpha k \lambda(t)] \, dt
\]

\[
d\pi_{NT}(t) = [\rho \pi_{NT}(t) - k \omega y_H(t) - k (1+\psi) y_{NT}(t)] \, dt
\]

where \( \kappa \equiv \theta (\rho + \theta), \omega \equiv \varphi \frac{(1-\alpha)(1-\beta)}{1-\alpha + \alpha \beta} \) and \( \psi \equiv \varphi \frac{\beta}{1-\alpha + \alpha \beta} \). These are the log-linearized New Keynesian Phillips curves of the model. Notice that both the IS curve and the NKPC for tradable goods include an additional term that captures the effect of the consumption wedge on tradable output and inflation. When prices are completely flexible domestic output is proportional to the consumption wedge and it is given by \( y_H(t) = -\alpha \frac{1+\psi}{1+\rho} \lambda(t) \) and \( y_{NT}(t) = \alpha \frac{\omega}{1+\rho} \lambda(t) \). Finally, when prices are completely rigid output is determined by \( y_H(t) = e(t) + (1-\alpha) \lambda(t) \) and \( y_{NT}(t) = \lambda(t) + e(t) \).

The loss function for the Home policymaker is derived from a second order approximation to the domestic households utility, given in equation (1), and a second order approximation to the Home country budget constraint, given by equation (14). Details of the derivation are provided in the appendix. The policymaker wishes to minimize

\[
\mathbb{L} = \frac{1}{2} \int_{0}^{\infty} e^{-\rho t} \left\{ t.i.p. + x(t)^2 + \phi_\lambda \lambda(t)^2 + \phi_{\pi_H} \pi_H(t)^2 + \phi_{\pi_{NT}} \pi_{NT}(t)^2 \right. + \phi_y [\omega y_H(t)^2 + \psi y_{NT}(t)^2 + (\omega y_H(t) + \psi y_{NT}(t))^2] \left. \right\} \, dt
\]  

\[
\phi_\lambda = \frac{\eta (1-\alpha)^2}{2 \Gamma(2-\alpha)} \quad \phi_y = \frac{\eta (1-\alpha + \alpha \beta)}{2 \varphi \alpha \Gamma(2-\alpha)(1-\beta)} \quad \phi_{\pi_H} = \frac{\eta (1-\alpha + \alpha \beta)}{2 \varphi \alpha \Gamma(2-\alpha)(1-\beta)} \\
\phi_{\pi_{NT}} = \frac{\eta \beta}{\kappa 2 \varphi \alpha \Gamma(2-\alpha)(1-\beta)}
\]

There are four sources of welfare loss. The last three terms in the loss function are common in the New Keynesian literature. The last term captures the welfare cost of deviating from the efficient level of output, while the third and fourth terms capture the welfare losses induced by inflation through its effect on prices dispersion and domestic total factor productivity. The first two terms are new and arise as a consequence of the imperfection in the international asset market. The second term captures the welfare cost caused by an inefficient intertemporal allocation of consumption relative to the path of the real exchange rate.\(^{17}\) The log-linearized consumption wedge is \( \lambda(t) = e(t) - q(t) \). A positive consumption wedge is associated with an appreciation of the real exchange rate relative to domestic consumption, while a negative consumption

\(^{17}\)An additional term related to imperfect international risk-sharing appears in the quadratic loss function of many models with incomplete markets. See for example De Paoli (2009) for a small open economy model and Corsetti et al. (2010) for two-country model. In all these models, however, this additional term disappears when the elasticity of intertemporal substitution and the elasticity of substitution between domestic and foreign goods are unitary.
wedge is associated with a depreciation of the real exchange rate relative to domestic consumption. By iterating the budget constraint forward and using initial and terminal conditions I obtain \( \int_0^\infty e^{-\rho t} \lambda(t) \, dt = 0 \). Thus, periods with positive consumption wedge must be followed or preceded by periods with negative consumption wedge. These fluctuations are inefficient since they reduce the present discounted value of the domestic consumption stream. Domestic consumption is too high when the domestic consumption bundle is relatively more expensive (appreciated real exchange rate) and too low when it is relatively cheaper. A reallocation of consumption from periods with positive consumption wedge to periods with negative consumption wedge would raise its present discounted value. Finally, the first term in the loss function comes from a second order approximation of the Home budget constraint, given by equation (14), and captures the monetary cost of using foreign exchange interventions. The monetary cost arises from the fact that by intervening in the foreign exchange market the central bank moves the return of domestic bonds against its own position and, in equilibrium, against the position taken by domestic households. For example, a foreign exchange purchase increases foreign reserves and reduces the central bank holdings of Home bonds. By increasing the supply of Home-currency bonds, however, the central bank depreciates the exchange rate and increases their return relative to foreign bonds, therefore increasing the cost of funding reserves. Vice versa, a foreign exchange sale increases the central bank holding of Home-currency assets and appreciates the exchange rate, reducing their return. Since financiers must take the opposite position in order for markets to clear, returns adjust in their favor and against the central bank.

The loss function is derived without specific assumptions about how prices are set (indeed whether prices are sticky or not). This highlights the fact that the loss in welfare arises not specifically from price stickiness but from prices that do not deliver the efficient allocation. In what follows I study the effects of and characterize the optimal policy response to a shock to the foreign demand for Home assets, \( f^* \). The next proposition defines the log-linearized planning problem.

**Problem 1.** The planning problem is to choose an interest rate policy \( i(t) \) and a foreign exchange intervention policy \( x(t) \) that minimize (21) subject to (15), (16), (17), (18), (19), 20 and

\[
\frac{d f^*_H(t)}{dt} = -\varrho f^*_H(t) \tag{22}
\]

given initial conditions \( f^*_H(0) = \varepsilon, a^S(0) = 0 \) and the terminal condition \( \lim_{t \to \infty} e^{-\rho t} a^S(t) = 0 \).

4 Optimal Foreign Exchange Intervention

In this section I characterize the optimal use of foreign exchange intervention alone. That is, I assume that the central bank commits to maintaining the nominal interest rate at its steady state level \( i(t) = \rho \) and uses only foreign exchange interventions. I will study the planner problem under flexible and rigid prices as these two extreme cases provide similar analytical solutions which deliver sharp intuition for the underlying mechanism of the model. The intermediate case of sticky prices will be analyzed in the next section when I will allow the central bank to use both foreign exchange intervention and monetary policy.
4.1 Optimal Foreign Exchange Intervention with Flexible Prices

In this section I characterize the optimal foreign exchange intervention policy under flexible prices. When firms are allowed to reset their prices instantaneously output gaps in the tradable and non-tradable sectors are

\[
y_H(t) = -\alpha(1+\psi)\lambda(t) \quad \text{and} \quad y_{NT}(t) = \alpha\omega(1+\psi)\lambda(t),
\]

respectively. The planner problem simplifies to

\[
\min_x \frac{1}{2} \int_0^{\infty} e^{-\rho t} \left( x(t)^2 + \phi \lambda(t)^2 \right) dt
\]

subject to 15, 16 and 22, given initial conditions \( f^*_H(0) = \epsilon, \quad a^S(0) = 0 \) and the terminal condition \( \lim_{t \to \infty} e^{-\rho t} a^S(t) = 0 \). The relative weight on the consumption wedge is given by

\[
\phi^F = \phi + \phi_\alpha\omega(1+\psi)
\]

and therefore includes both the welfare cost induced by the output gaps and that induced by an inefficient intertemporal allocation of consumption. I will consider the case where the central bank can credibly commit at time \( t = 0 \) to the entire future path of foreign reserves. The next proposition characterizes the allocation implemented by the optimal foreign exchange intervention policy.

**Proposition 1.** The solution to the problem

\[
\min_x \frac{1}{2} \int_0^{\infty} e^{-\rho t} \left( x(t)^2 + \phi \lambda(t)^2 \right) dt
\]

subject to (15), (16), and (22), with \( a^S(0) = 0 \) and \( f^*_H(0) = \epsilon \) is

\[
\lambda(t) = \Gamma \begin{bmatrix} \frac{1}{\rho + \xi + \zeta} \\ \frac{1}{\xi + \frac{1-\Xi}{\rho + \theta + \xi}} \\ \frac{1-\Xi}{\rho + \theta + \xi} \end{bmatrix}^T \begin{bmatrix} x(t) \\ a^S(t) \\ f^*_H(t) \end{bmatrix}
\]

where \( \xi, \zeta, \text{and } \Xi \) are defined in the appendix. The states \( \begin{bmatrix} x & a^S & f^*_H \end{bmatrix} \) evolve as

\[
\begin{bmatrix}
dx(t) \\
da^S(t) \\
df^*_H(t)
\end{bmatrix} =
\begin{bmatrix}
-\Xi - (1-\Xi)\xi & -\phi_\lambda^2 \rho + \xi + \zeta & \phi \lambda^2 \frac{\rho + \theta + \xi}{(\rho + \theta + \xi)(\rho + \theta + \xi)} \\
-\eta \Gamma \rho + \xi + \zeta & -\Xi - (1-\Xi)\zeta & -\eta \Gamma \frac{\Delta}{\rho + \theta + \xi} + \frac{1-\Delta}{\rho + \theta + \xi} \\
0 & 0 & -\theta
\end{bmatrix}
\begin{bmatrix}
x(t) \\ a^S(t) \\ f^*_H(t)
\end{bmatrix}
\]

with \( \begin{bmatrix} x(0) & a^S(0) & f^*_H(0) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \epsilon \end{bmatrix} \).

In order to describe the effects of portfolio flow shocks and the optimal policy response I will focus on the case of an increase in the foreign demand for domestic assets, that is \( \epsilon > 0 \). Since the solution is linear, both the effects and the optimal policy response to a decrease in the demand for domestic assets (\( \epsilon < 0 \)) are symmetric. Figure 1 plots the allocation implemented by the optimal foreign exchange intervention under flexible prices, in response to a time zero positive shock to foreign investors demand for domestic assets, and compares it with the allocation that arises without intervention. The analytical solution for the allocation without intervention is reported in the appendix. The numerical simulation is meant to describe
the qualitative properties of the allocation and should not be thought of as a serious calibration exercise, for which the model is clearly too stylized.

At $t = 0$ foreign households want to increase their holdings of Home bonds. The excess demand for Home-currency assets generates an imbalance in the asset market that financiers will have to absorb. The Home currency appreciates on the spot ($e(0) < 0$) and depreciates in expectation ($de(0) > 0$) such that the price of domestic assets increases and their foreign-currency return falls. The excess return of foreign assets vis-à-vis domestic assets induces financiers to short Home bonds until the imbalance is absorbed. The appreciation of the Home currency induces domestic producers to reduce their prices, while the expected depreciation increases import prices expected inflation. Both effects increase CPI inflation and reduce the real borrowing cost for Home agents therefore inducing domestic households to accumulate debt. Foreign capital flows into the Home economy ($da^S(0) < 0$) and finances a consumption boom.

The effect of the shock on domestic output and employment is the result of two forces. On one hand, the appreciation of the exchange rate makes foreign goods relatively cheaper. Thus, domestic and foreign demand for Home goods decreases. Producers react to the loss in competitiveness by reducing their prices. On the other hand, the consumption boom triggered by the fall in the real interest rate faced by Home households increases domestic aggregate demand. In equilibrium domestic consumption of both tradable and non-tradable goods increases while foreign demand falls. As a consequence, output in the non-tradable sector rises while tradable output decreases. The net effect is negative as total output, proxied by employment, falls below its efficient level. The reason behind this result lies in the wealth effect on labor supply. The increase in domestic consumption pushes up real wages which, in turn, prevent prices from falling enough to restore full employment.\footnote{Notice that the sign of the response of total output is independent on the relative sizes of the two sectors, since}

$$l(0) = -\frac{\alpha}{1+\varphi} \frac{(1-\alpha)(1-\beta)}{1-\alpha+\alpha\beta} \lambda(0)$$

This is further evidence that the underlying reason behind the recessionary impact of the capital inflow is the increase in the real wage which hits both sector equally.

\footnote{Notice that the sign of the response of total output is independent on the relative sizes of the two sectors, since}

$\lambda(0) = -\frac{\alpha}{1+\varphi} \frac{(1-\alpha)(1-\beta)}{1-\alpha+\alpha\beta} \lambda(0)$
bank absorbs part of the imbalances arising in the international asset market and reduces the impact of the portfolio flow shock on the exchange rate. The intervention reduces the initial appreciation of the domestic currency and moderates its expected depreciation. This, in turn, increases the domestic real interest rate and smooths out the consumption boom. Smoothing out consumption fluctuations increases domestic welfare for two reasons: it stabilizes domestic output and employment, and it maximizes the present discounted value of consumption.

The optimal level of employment is chosen by the planner by maximizing (3) subject to the technological constraints $Y_H(t) = L_H(t)$ and $Y_{NT}(t) = L_{NT}(t)$, and the market clearing conditions (11), (12), and (13) given the consumption/output possibility set implied by the consumption wedge $\Lambda$. In the special case where the elasticities of intertemporal and intratemporal substitutions are equal to one, the optimal allocation implies a constant employment $L(t) = (1 - \alpha + \alpha\beta)^{\frac{1}{1+\phi}}$. The possibility of altering the domestic terms of trade through foreign exchange intervention does not alter the well known result, derived by Gali and Monacelli (2005) in a model with complete markets, that under a Cole-Obstfeld parametrization a constant level of employment is optimal. The presence of imperfections in international financial markets, however, affects the ability of the planner to implement the optimal allocation. The labor subsidies that implement the efficient level of employment are

$$\tau_{OPT}^H(t) = 1 - \frac{\alpha + (1 - \alpha) \Lambda(t)}{\mathcal{M} (1 - \alpha) \Lambda(t)}$$

$$\tau_{OPT}^{NT}(t) = \frac{\alpha + (1 - \alpha) \Lambda(t) \tau_H(t)}{\alpha + (1 - \alpha) \Lambda(t)}$$

If markets are frictionless then $\Lambda$ is constant and constant labor subsidies are enough to render the flexible price equilibrium level of employment efficient. When domestic consumption increases, a depreciation of the exchange rate reduces the relative price of both tradable and non-tradable domestic goods. The increase in demand offsets the increase in real wages triggered by the domestic consumption boom. Hence employment remains constant. When international financial markets are imperfect and the exchange rate is determined by financial forces, this expenditure-switching mechanism breaks down. A positive portfolio flow shock simultaneously appreciates the exchange rate and causes a consumption boom in the Home economy. The wealth effect on labor supply pushes up real wages while the appreciation of the exchange rate reduces foreign demand. As a result employment falls. The planner would like to raise labor subsidies in order to reduce real wages and restore the efficient level of employment. If fiscal policy is not flexible enough to correct these distortions through appropriate time-varying subsidies, fluctuations in consumption driven by portfolio flow shocks result in inefficient levels of employment. By stabilizing these fluctuations, the central bank reduces wage inflation/deflation and mitigate the welfare losses caused by the associated output gaps.

The second reason for stabilizing consumption fluctuations has to do with dynamic terms of trade management. An increase in Home consumption appreciates the terms of trade via two channels: the wealth effect on labor supply described above and the Home bias effect on domestic demand. Since Home consumption is biased toward domestic goods, and the law of one price holds, an increase in domestic con-
sumption raises domestic demand for Home goods and appreciates the terms of trade. Therefore, domestic households consume more exactly when the cost of their consumption bundle is high. The domestic planner would like to exploit its monopsonistic power by reducing consumption and depreciating the terms of trade. Vice versa, a decrease in domestic consumption below its steady state level depreciates the terms of trade and reduce export revenues. The domestic planner would like to exploit its monopolistic power by increasing consumption and appreciating the terms of trade. This logic is similar to Costinot et al. (2014), where the planner wishes to manipulate the dynamic path of the terms of trade in order to maximize the present discounted value of a time-varying endowment.

Despite the inefficiencies induced by fluctuations in domestic consumption, it is never optimal for the central bank to fully stabilize the path of the exchange rate. The reason is that, by creating a wedge between the return of domestic and foreign assets, portfolio flow shocks are akin to positive wealth shocks for the Home country. Although domestic households cannot take advantage of the arbitrage opportunity provided by deviations from UIP, they can still exploit the lower real cost of borrowing (higher real return on saving) induced by a positive (negative) portfolio flow shock. By leaning against the wind the central bank mitigates the effects of the shock on the path of the exchange rate and on the real interest rate available to domestic households. In determining the optimal amount of stabilization the central bank trades off the benefits that arise from a cheaper cost of borrowing (or higher return on savings) and the inefficiencies described above.

### 4.2 Optimal Foreign Exchange Intervention with Rigid Prices

When prices are rigid the domestic terms of trade are given by \( s(t) = e(t) \). If the central bank keeps the nominal interest rate at its steady state level \( i(t) = \rho \), then \( e(t) = -\lambda(t) \) and output gaps in the tradable and non-tradable sectors are \( y_H(t) = -\alpha\lambda(t) \) and \( y_{NT}(t) = 0 \), respectively. Similarly to the flexible price case, the planner problem can be simplified to

\[
\min_{x} \frac{1}{2} \int_{0}^{\infty} e^{-\rho t} \left( x(t) + \phi^R \lambda(t)^2 \right) dt
\]

subject to 15, 16 and 22, given initial conditions \( f^S_H(0) = \varepsilon \), \( a^S(0) = 0 \) and the terminal condition \( \lim_{t \to \infty} e^{-\rho t} a^S(t) = 0 \). The relative weight on the consumption wedge is given by \( \phi^R \equiv \phi\lambda + \phi_y\alpha^2\omega(1 + \omega) \).

The planner problem is similar to the one solved under flexible prices. Its solution has the structure described in proposition (1). The only difference between the two problems is the relative weight that the planner attaches to the stabilization of the consumption wedge. When prices are rigid domestic producers are unable to reduce them in order to regain the competitiveness lost due to an overvalued currency. Domestic demand shift away from domestic goods and toward foreign goods, reducing domestic output. The consumption boom, however, raises domestic demand for both tradable and non-tradable goods. When the elasticities of substitution across goods and across periods are identical and equal to one, the two effects have the same magnitude and perfectly offset each other. Thus, domestic demand for both tradable and non-tradable goods remain constant. Foreign demand, however, falls sharply as the foreign price of domestic goods increases one-for-one with the nominal exchange rate. Hence, while non-tradable output remain
at its efficient level, tradable output falls. The effects of the portfolio flow shock on aggregate output and employment are more severe than in the flexible prices case. Now domestic prices remain too high not only because of the wealth effect on labor supply triggered by the consumption boom, but also because of the nominal friction that prevents domestic producers from adjusting them downwards. Notice that the path of the consumption wedge is unaltered by the presence of nominal rigidities.

With rigid prices the central bank responds to an increase in the foreign demand for Home-currency assets by intervening more aggressively. In fact, it’s easy to check that $\phi^R > \phi^F$. When prices are flexible the main goal of the intervention is to mitigate the depreciation of the exchange rate that follows the portfolio flow shock. By doing so it stabilizes domestic consumption and alleviates the inefficiencies described in the previous section. Since producers are allowed to adjust their prices freely, the nominal appreciation of the exchange rate per se does not generate any welfare loss. When prices are rigid, however, the nominal appreciation reduces the competitiveness of domestic goods and causes a bigger fall in aggregate output. Now the central bank has an additional reason for intervening as it wishes to stabilize not only the dynamic of the exchange rate, but also its level. By fighting the appreciation, the intervention reduces the foreign price of domestic goods and sustains foreign demand. This is the “monetary” aspect of foreign exchange intervention: the central bank “leans against the wind” in order to mitigate the effect of the portfolio flow shock on the competitiveness of domestic exports.

5 Optimal Monetary Policy and Foreign Exchange Intervention

In this section I consider the case when prices adjust slowly and study the interaction between foreign exchange and monetary policies. In order to simplify the equations I consider a model in which there are only tradable goods ($\beta = 0$). The presence of non-tradable goods, as we have seen before, does not affect the main conclusions of the analysis.

When the central bank uses both tools available, its problem is the one described in Problem 1. Unfortunately the problem does not admit a closed form solution and can be solved only numerically. The details of the solution are reported in the appendix. The blue line in Figure (3) depicts the allocation implemented by the jointly optimal, time-zero, foreign exchange and interest rate policies. When dealing with a capital inflow, the central bank increases the supply of domestic bonds, and accumulates reserves, while simultaneously reducing the nominal interest rate. In order to understand the role played by each instrument and their interaction, it is instructive to focus on them separately. The red line in Figure (3) depicts the allocation implemented by using only foreign exchange intervention (with $i(t) = \rho$) while the green line depicts the allocation implemented by using monetary policy only ($x(t) = 0$). The details of the solution to these problems, and their analytical expression when available, are reported in the appendix.

Foreign exchange intervention, as we have seen in section 4.1, is the appropriate tool to smooth out consumption fluctuations. It allows the planner to choose the welfare-maximizing consumption path by trading off the benefits and costs of the real interest rate shocks. However, when it is used to deal also with the inefficiency generated by sticky prices, excessive exchange rate stabilization arises. The central bank intervenes more aggressively in order to stabilize output and reduce deflation. Monetary policy, if
available, can help in reducing the output gap and mitigate this burden. This is in sharp contrast with the standard view of foreign exchange intervention which emphasizes their role in managing the trade balance. The monetary aspect of foreign exchange intervention arises only when monetary policy is not available, for example because the zero lower bound is binding.

Monetary policy, on the other hand, cannot deal with the inefficiencies induced by fluctuations in the consumption path. In the previous sections we have seen that a positive portfolio flow shock appreciates the exchange rate and reduces the domestic real interest rate. The latter effect triggers a consumption boom which reduces output and the present discounted value of the consumption stream. The appreciation of the domestic currency, on the other hand, increases the foreign price of domestic goods which, if prices are sticky, depresses domestic output even further. One might naively guess that, in analogy with the use of foreign exchange intervention, the central bank should tighten monetary policy in order to mitigate the consumption boom and its negative effects on domestic welfare. This logic however does not take into account the effects of such a policy on the price of domestic assets and their returns. An increase in the nominal interest rate would make domestic assets even more attractive for foreign investors. In order for markets to clear, the exchange rate would have to appreciate even more, depressing foreign demand further. Therefore, by simultaneously reducing domestic consumption and domestic demand, a tighter monetary policy would cause a bigger fall in output. A quick inspection of the equations reveals that the consumption wedge \( \lambda \) and the foreign-currency value of the Home net foreign asset position \( a^S \) are independent of the degree of price stickiness and of the output gap. Hence monetary policy is unable to affect them. By using the domestic and foreign demand schedules we can rewrite the consumption wedge as the gap between domestic and foreign demand:

\[
\lambda(t) = c_H(t) - c^*_H(t)
\]

Monetary policy moves domestic and foreign demand in the same direction, therefore it has limited power in reducing the tension between them generated by the portfolio flow shock. In the special case considered in this paper where all elasticities of substitution are equal to one \( (\gamma = \eta = \varsigma = 1) \), the effects of monetary policy on domestic and foreign demand have exactly the same magnitude. Thus, their ratio is independent of the path of the nominal interest rate chosen by the central bank. This result has two striking implications. First of all, monetary policy cannot be used to maximize the present discounted value of the path of domestic consumption as it cannot simultaneously reduce consumption and depreciate the terms of trade. Second, its ability to reduce the output gap is limited by wages inflation triggered by the consumption boom. In response to a positive portfolio flow shock, the central bank reduces the nominal interest rate and reduces the attractiveness of domestic assets. This depreciates the exchange rate and helps domestic producers to reduce the foreign price of domestic goods. The expansionary monetary policy also increases domestic consumption and therefore domestic demand. This reduces the output gap even further but exacerbates wages inflation and eventually leads to positive PPI inflation. Portfolio flow shocks shift the IS and worsen the output inflation trade-off for monetary policy. Foreign exchange intervention can be used to shift the IS curve closer to its efficient position. The effectiveness of monetary policy is independent of foreign exchange intervention only in two extreme cases. When \( \theta \to 0 \) and prices become fully rigid, the central bank can independently use monetary policy to fully stabilize output, since there is no inflation cost. This is
true also when an appropriate state-contingent labor subsidy is in place. In this case the divine coincidence holds and monetary policy alone can simultaneously achieve zero inflation and zero output gap.

In dealing with the effects of portfolio flow shocks the central bank optimally uses both instruments. Foreign exchange intervention and monetary policy are complementary tools rather than substitutes. Unfortunately, no closed form solution exists for Problem 1 and the optimal policies cannot be derived analytically. In order to characterize the optimal foreign exchange intervention rule I will study a sequential planner’s problem in which the central bank first chooses the optimal path for $\lambda$ and then, taking it as given, chooses the optimal paths for $y_H$ and $\pi_H$. The sequential problem admits a closed form solution and its solution approximates quite well the solution to the joint problem, Problem 1. In the next section I will use this solution to derive the optimal foreign exchange intervention policy.

**Problem 2.** The planner first chooses the path of $x$ that minimizes

$$
\mathbb{L}_1 = \frac{1}{2} \int_0^\infty e^{-\rho t} \left( x(t)^2 + \phi \lambda(t)^2 \right) dt
$$

subject to 15, 16 and 22, given initial conditions $f_H^s(0) = \varepsilon$, $a^s(0) = 0$ and the terminal condition $\lim_{t \to \infty} e^{-\rho t} a^s(t) = 0$. Taking the path of $\lambda$ as given, the planner then chooses the path of $\tau$ that minimizes

$$
\mathbb{L}_2 = \frac{1}{2} \int_0^\infty e^{-\rho t} \left[ \phi \pi_H(t)^2 + \phi y(t) (1 + \omega) y_H(t)^2 \right] dt
$$

subject to (17) and (19).

**Proposition 2.** The solution to Problem (2) is

$$
\begin{bmatrix}
\pi_H(t) \\
\lambda(t)
\end{bmatrix} =
\begin{bmatrix}
\frac{\kappa (1 + \omega)}{\rho + \eta}, \\
\frac{\hat{\gamma}(\xi)}{\rho + \eta} + \frac{\hat{\gamma}^2}{\rho + \eta} \hat{\gamma}(\rho + \eta)
\end{bmatrix}^\top
\begin{bmatrix}
y_H(t) \\
x(t) \\
a^s(t) \\
f_H^s(t)
\end{bmatrix}
$$

while the states $s = \begin{bmatrix} y_H \ x \ a^s \ f_H^s \end{bmatrix}^\top$ evolve as $ds = M \, dt$ where

$$
M =
\begin{bmatrix}
\frac{-l}{(1 - \varepsilon) \hat{\gamma}^2} & \frac{\hat{\gamma}^2}{\rho + \xi + \varepsilon} & 0 & 0 & 0 \\
\frac{-\hat{\gamma}^2}{\rho + \xi + \varepsilon} & \varepsilon \hat{\gamma} - (1 - \Xi) \hat{\gamma} & \frac{-\eta \hat{\gamma}}{\rho + \xi + \varepsilon} & 0 & 0 \\
\frac{\hat{\gamma}^2}{\rho + \xi + \varepsilon} & \frac{\phi \hat{\gamma}^2}{\rho + \xi + \varepsilon} & 0 & 0 & 0 \\
\frac{\hat{\gamma}^2}{\rho + \xi + \varepsilon} & \frac{\phi \hat{\gamma}^2}{\rho + \xi + \varepsilon} & 0 & 0 & 0 \\
\end{bmatrix}
$$

26
with \( y_H(0) = x(0) = a^S(0) \) and \( f^{*S}_H(0) = \varepsilon \). The functions \( \Upsilon(\cdot) \), \( \hat{\Upsilon}(\cdot) \), and the parameter \( \iota \) are defined in the appendix.

Figure (4) plots the allocations implemented by the solution to Problem (1), blue line, and Problem (2), red line, where the parameter \( \phi \) is chosen to minimize (21). The two allocations are almost indistinguishable from each other. In light of the previous discussion on the relationship between the two tools this is not a surprising result. The planner wants to stabilize the path of \( \lambda \) for two reasons. First, because it enters directly into the loss function with weight \( \phi \lambda \). Second, because it affects the paths of the output gap and domestic inflation. The sequential problem does not ignore the second channel, but rather it collapses both channels into the choice of a single parameter, \( \phi \). In fact, the optimal \( \phi \) that minimize (21) in Problem (2) is strictly greater than \( \phi_\lambda \). The remaining difference between the two solutions is only due to the relative dynamics of \( \lambda, y_H, \) and \( \pi_H \), and is therefore negligible.

6 Optimal Foreign Exchange Intervention Rule

In this section I exploit the analytical solutions derived in Proposition 1 and 2 to characterize the optimal intervention rule in terms of endogenous variables.

Lemma 2. Let

\[
\{ \hat{x}(t) \}_{t \geq 0} = \arg \min_{x} \frac{1}{2} \int_{0}^{\infty} e^{-\rho t} \left( x(t)^2 + \phi \lambda(t)^2 \right) dt
\]

subject to (15), (16), and (22), with \( a^S(0) = 0 \) and \( f^{*S}_H(0) = \varepsilon \). Then \( \dot{x} \) satisfies the following differential equation

\[
dx(t) = \Psi_\lambda \lambda(t) + \Psi_a a^S(t) + \Psi_x x(t)
\]

where

\[
\Psi_\lambda = \phi \Gamma \frac{\rho + \varrho + \frac{\xi}{\rho + \xi + \bar{\xi}}}{\rho + \varrho + \Xi \xi + (1 - \Xi) \bar{\xi}} > 0
\]

\[
\Psi_a = -\phi \Gamma \left( \rho + \varrho + \frac{\xi}{\rho + \xi + \bar{\xi}} \right) \left( \rho + \frac{\Xi \xi}{\rho + \xi + \bar{\xi}} + \Xi \xi + (1 - \Xi) \bar{\xi} \right) < 0
\]

\[
\Psi_x = -\frac{\Xi \Xi \left( \rho + \varrho + \frac{\xi}{\rho + \xi + \bar{\xi}} \right) + \Xi \left( 1 - \Xi \right) \left( \rho + \varrho + \frac{\xi}{\rho + \xi + \bar{\xi}} \right)}{\rho + \varrho + \Xi \xi + (1 - \Xi) \bar{\xi}} < 0
\]

Furthermore

\[
\frac{\partial |\Psi_\lambda|}{\partial \varrho} > 0 \quad \frac{\partial |\Psi_a|}{\partial \varrho} > 0 \quad \frac{\partial |\Psi_x|}{\partial \varrho} < 0
\]

\[
\frac{\partial |\Psi_\lambda|}{\partial \Gamma} > 0 \quad \frac{\partial |\Psi_a|}{\partial \Gamma} > 0 \quad \frac{\partial |\Psi_x|}{\partial \Gamma} < 0
\]
\[
\frac{\partial |\Psi_\lambda|}{\partial \phi} > 0 \quad \frac{\partial |\Psi_a|}{\partial \phi} > 0 \quad \frac{\partial |\Psi_x|}{\partial \phi} = \begin{cases} > 0 & \varrho \downarrow 0 \\ < 0 & \varrho \uparrow \infty \end{cases}
\]

Lemma 2 shows how the central bank can implement the optimal foreign exchange intervention using three implicit targets: the consumption wedge \(\lambda\), Home net foreign assets \(a^S\), and foreign reserves \(x^S\).

The first target of the intervention, the consumption wedge, captures the goal of macroeconomic stabilization pursued by the central bank. Since \(\Psi_\lambda > 0\), the central bank commits to accumulate (decumulate) reserves and increase (decrease) the net supply of domestic bonds when the consumption wedge is positive (negative). The stabilizing effect of such policy can be understood by looking at equation (15). By iterating it forward we obtain

\[
\lambda(t) = \Gamma \int_t^\infty \left( a^S(u) - x(u) + f_H^S(u) \right) du
\]

The second target of the optimal foreign exchange intervention, net foreign assets, captures the interaction between public and private flows. Since \(\Psi_a < 0\), the central bank commits to accumulate (decumulate) reserves and increase (decrease) the net supply of domestic assets when the country is a net debtor (creditor). By doing so the central bank moves the return of domestic assets against the position held by domestic agents. Ceteris paribus, an increase in the net supply of domestic assets increases their return making borrowing more expensive for Home households. Vice versa, a decrease in the net supply reduces their return and the incentive to save. By committing to increase the cost (reduce the profitability) of their positions, the central bank discourages domestic households from accumulating assets. Once the portfolio flow subsides, these assets will have to be absorbed by financial intermediaries causing the same types of inefficiencies generated by the initial phase of the cycle. Thus, by discouraging a build up in imbalances the central bank mitigates the severity of the bust phase of the cycle.

Finally the third target of the intervention, the level of foreign reserves, captures the monetary cost of

\[19\text{A similar argument holds for the real exchange rate.}\]
the intervention. Since $\Psi_x < 0$, the path of foreign reserves is mean reverting. That is, the central bank commits to eventually reverse the intervention and bring the level of reserves back to their steady state level. The monetary cost of the intervention arises from the fact that, when the central bank intervenes in the asset market it moves returns against its own positions. Notice also that, since holding reserves is costly, the central bank optimally smooths out the intervention rather than intervening aggressively at the time of the shock. At $t = 0$, following a positive portfolio flow shock, the central bank keeps reserves at their steady state level, $x(0) = 0$, and increases the supply of domestic bonds only gradually over time.

The comparative statics results reported in Lemma 2 shed some light on the determinants of the optimal foreign exchange intervention strategy. First, the central bank intervenes more aggressively against a transitory shock while it is more accommodative when the shock is persistent. The weights on the consumption wedge and net foreign assets are increasing in $\varrho$, meaning that, keeping constant the effects of the shock on $\lambda$ and $a^S$, the size of the intervention is increasing in the speed at which the shock subsides. Not only the intervention is larger in size, but it also relatively more persistent. In fact, the speed at which reserves revert to their steady state level is a decreasing function of $\varrho$. Vice versa, when the shock is very persistent the optimal intervention is smaller in size and it has a relatively shorter duration. Second, as we have seen in the previous section the size of the optimal intervention is increasing in the desire for stabilization, measured by $\phi$. Its effect on the duration of the intervention, however, depends on the persistence of the shock. When the portfolio flow shock is transitory (persistent) an increase in the desire for stabilization increases (decreases) the relative duration of the intervention. Finally, the central bank intervenes more aggressively when international financial markets are distressed, i.e., when the risk-bearing capacity of the financial sector is low ($\Gamma \uparrow$). The intervention is larger in size but has a relatively shorter duration.

7 Conclusions

I considered a New Keynesian model of a small open economy where international financial markets are imperfect and the exchange rate is determined by capital flows. I used this framework to study the effects of exchange rate fluctuations driven by capital flows and characterize the optimal foreign exchange intervention policy. When there are frictions in financial intermediation across countries, capital flows shocks cause inefficient exchange rate fluctuations that trigger boom-bust cycles in the domestic economy. The optimal policy response is to partially stabilize these fluctuations using both foreign exchange intervention and monetary policy. The optimal foreign exchange intervention “leans against the wind” and stabilizes the path of the exchange rate: following an increase (decrease) in the foreign demand for domestic assets, the central bank increases (decreases) their net supply and accumulates (decumulates) foreign reserves. By doing so the central bank stabilizes the path of the exchange rate and smooths out fluctuations in domestic consumption. Simultaneously, the central bank reduces (increases) the nominal interest rate in order to reduce the relative price of domestic goods and mitigate the output gap. I showed that foreign exchange intervention is not a mere substitute for monetary policy. Rather, the two tools complement each other. Finally, I showed how the foreign exchange intervention policy can be implemented by a simple rule which is a function of three endogenous targets: the consumption wedge, the domestic net foreign asset position, and the level of
accumulated foreign reserves.

These results shed light on the surge in foreign exchange interventions observed since the onset of the global financial crisis, in emerging and advanced economies alike. Foreign exchange intervention is an additional tool for central banks which does not substitute for monetary policy, but rather complements it when shocks in international financial markets spill over and threatens domestic stability.
References


_ , Peter Karadi et al., “Qe 1 vs. 2 vs. 3...: A framework for analyzing large-scale asset purchases as a monetary policy tool,” international Journal of central Banking, 2013, 9 (1), 5–53.


Figure 1: FX Intervention Under Flexible Prices

Impulse responses for the model with flexible prices to a positive shock to foreign demand for Home assets. The blue line represents the allocation implemented by the optimal FX intervention, while the red line represents the allocation without intervention. Parameter values: $\rho = 0.01$, $\alpha = 0.4$, $\beta = 0.1$, $\varphi = 3$, $\Gamma = 1$, $\varepsilon = 0.0473$, $\varrho = 0.2$, $\varphi_x = 0.05$. 
Figure 2: FX Intervention Under Rigid Prices

Impulse responses for the model with rigid prices to a positive shock to foreign demand for Home assets. The blue line represents the allocation implemented by the optimal FX intervention, while the red line represents the allocation without intervention. In both cases monetary policy is idle ($i = \rho$). Parameter values: $\rho = 0.01$, $\alpha = 0.4$, $\beta = 0.1$, $\varphi = 3$, $\Gamma = 1$, $\varepsilon = 0.0473$, $\varrho = 0.2$, $\phi_x = 0.05$. 
Impulse responses for the model with sticky prices to a positive shock to foreign demand for Home assets. The blue line represents the allocation implemented by the optimal joint use of FX intervention and monetary policy, the red line represents the allocation implemented by the optimal FX intervention alone ($i = \rho$), and the green line represents the allocation implemented by the optimal monetary policy ($x(t) = 0$) alone. Parameter values: $\rho = 0.01$, $\alpha = 0.4$, $\varphi = 3$, $\theta = 0.7$, $\epsilon = 6$, $\beta = (1 - \alpha)^{1/\varphi}$, $\Gamma = 1$, $\varepsilon = 0.0473$, $\rho = 0.2$, $\phi_x = 0.05$. 

38
Figure 4: Sequential Problem vs Joint Problem

Impulse responses for the solution to the joint problem (blue line) and to the sequential problem (red line). Parameter values: $\rho = 0.01$, $\alpha = 0.4$, $\varphi = 3$, $\theta = 0.7$, $\epsilon = 6$, $\beta = (1 - \alpha)^{-1}$, $\Gamma = 1$, $\varepsilon = 0.0473$, $\varrho = 0.2$, $\phi_x = 0.03$, $\phi = 3.324$
Appendix

A  The Household’s Problem

The household’s problem is:

\[
V(A, S) = \max_{C,L} \mathbb{E} \left[ \int_0^T e^{-\rho t} \left( \ln C - \frac{L^{1+\varphi}}{1+\varphi} \right) dt \right]
\]

subject to

\[
dA = [Ai - \mathcal{E}F (i - i^* - \mu\varepsilon) + WL - PC + \Sigma + T] dt + \sigma_A dZ
\]

\[
dS = S\mu dt + S\sigma dZ
\]

where \( S \) is a generic vector of states and \( \sigma_A \) is a function of the states only. The HJB for this problem is

\[
\rho V = \sup_{C,L} \left[ \ln C - \frac{L^{1+\varphi}}{1+\varphi} + A(N) \right]
\]

where

\[
A(V) = V_A [Ai - \mathcal{E}F (i - i^* - \mu\varepsilon) + WL - PC + \Pi + T]
\]

\[
+ \sum_s V_s S_{\mu s} + \frac{1}{2} \left( \sigma^2 A_{AA} - 2 \sum_s S_{\sigma A} S_{V A S} + \sum_s S^2 S_{V S S} \right)
\]

is the infinitesimal generator operator. The first order conditions with respect to \( C \) and \( L \) are:

\[
CP = \frac{1}{V_A}
\]

\[
L^\varphi = V_A W
\]

Derive both sides of the HJB with respect to \( A \) to obtain the law of motion of \( V_A \):

\[
\frac{dV_A}{V_A} = (\rho - i) dt + \sigma_A dZ
\]

where \( \sigma_{V_A} = -\frac{V_A}{V_A^2} \mathcal{E} F \sigma_{\varepsilon} + \sum_s \frac{V_A S_{\sigma S}}{V_A} S_{\sigma S} \). Finally we can apply Ito’s Lemma to the first order condition to derive the Euler equation

\[
\frac{dC}{C} = (i - \mu_P - \rho + \sigma_P^2 + \sigma_C \sigma_P + \sigma_C^2) dt + \sigma_C dZ
\]

where \( \sigma_C = -\sigma_P - \sigma_{V_A} \).
B Calvo Pricing in Continuous Time

B.1 Producer Price Dynamics

Domestic producer price indexes are defined by

\[ P_k(t) \equiv \left[ \int_0^1 P_{k,j}(t)^{1-\epsilon} \, dj \right]^{1/\epsilon} \]

In order to derive its law of motion, let’s write it down in discrete time first and then take the limit as the length of the time interval goes to zero. The length of a period is \([t, t + dt)\). During a period a fraction \(1 - e^{-\theta dt}\) of firms receive the Calvo signal that will allow them to set a new price at time \(t + dt\). The remaining fraction \(e^{-\theta dt}\) will not be able to change price and will be stuck with the price posted at \(t\). Let \(S(t)\) be the set of firms not re-optimizing their posted price at time \(t\). Using the fact that all firms resetting prices will choose an identical price \(\bar{P}_k(t)\) we obtain

\[ P_k(t + dt) = \left[ \int_0^1 P_{k,j}(t + dt)^{1-\epsilon} \, dj \right]^{1/\epsilon} = \left[ \int_{S(t)} P_{k,j}(t)^{1-\epsilon} \, dj + \left( 1 - e^{-\theta dt} \right) \bar{P}_k(t + dt)^{1-\epsilon} \right]^{1/\epsilon} \]

where the last equality follows from the fact that the distribution of prices among firms not adjusting at time \(t + dt\) corresponds to the distribution of posted prices at time \(t\), though with a total mass reduced to \(e^{-\theta dt}\).

The equation above can be rewritten as

\[ P_k(t + dt)^{1-\epsilon} - P_k(t)^{1-\epsilon} = - \left( 1 - e^{-\theta dt} \right) P_k(t)^{1-\epsilon} + \left( 1 - e^{-\theta dt} \right) \bar{P}_k(t + dt)^{1-\epsilon} \]

Now, we cannot take the the limit as \(dt \to 0\) since the last term is not time \(t\) measurable. Therefore add and subtract \(\left( 1 - e^{-\theta \Delta t} \right) \bar{P}_k(t)^{1-\epsilon}\) to obtain

\[ P_k(t + dt)^{1-\epsilon} - P_k(t)^{1-\epsilon} = \left( 1 - e^{-\theta \Delta t} \right) \left( \bar{P}_k(t)^{1-\epsilon} - P_k(t)^{1-\epsilon} \right) + \left( 1 - e^{-\theta \Delta t} \right) \Delta \bar{P}_k(t)^{1-\epsilon} \]

Now Taylor expand \(e^{-\theta \Delta t}\) around \(dt = 0\) and take the limit for \(dt \to 0\) to obtain

\[ \frac{dP_k(t)}{P_k(t)} = \frac{\theta}{1-\epsilon} \left[ \left( \frac{P_k(t)}{\bar{P}_k(t)} \right)^{1-\epsilon} - 1 \right] \, dt \]

B.2 Price Dispersion

The aggregate loss of efficiency induced by price dispersion among firms is \(\Delta_k(t) \equiv \int_0^1 \left[ \frac{P_{k,j}(t)}{P_k(t)} \right]^{\epsilon} \, dj\).

Its law of motion can be derived as in the previous section. Let’s write the discrete time analogue and
appropriately shrink the length of the period

\[ \Delta_k (t + dt) = \int_0^1 \left[ \frac{P_{k,j} (t + dt)}{P_k (t + dt)} \right]^{-\epsilon} \, dj \]

\[ = P_k (t + dt)^\epsilon \int_{S(t)} P_{k,j} (t)^{-\epsilon} \, dj + \left( 1 - e^{-\theta dt} \right) P_k (t + dt)^{-\epsilon} \]

\[ = \left( \frac{P_k (t + dt)}{P_k (t)} \right)^\epsilon \int_{S(t)} P_{k,j} (t)^{-\epsilon} \, dj + \left( 1 - e^{-\theta dt} \right) P_k (t + dt)^{-\epsilon} \]

As before we add and subtract the last term lagged, Taylor expand the exponential terms and take the limit as \( dt \to 0 \) to obtain

\[ d\Delta_k (t) = \left[ \theta \left( \frac{P_k (t)}{P_k (t)} \right)^{-\epsilon} + \Delta_k (t) (\epsilon \pi_k (t) - \theta) \right] \, dt \]

or

\[ d\Delta_k (t) = \left[ \theta \left( 1 - \frac{\epsilon - 1}{\theta} \pi_k (t) \right) \right]^{-1} \, + \Delta_k (t) (\epsilon \pi_k (t) - \theta) \] \, dt

**B.3 Optimal Price Setting**

A measure one of monopolistic firms (indexed by \( j \in [0, 1] \)) engage in infrequent price setting a la Calvo. Each firm re-optimizes its price \( P_{k,j} (t) \) only at discrete dates determined by a Poisson process with intensity \( \theta \). The time \( \delta \) between two re-optimizations is distributed according to the exponential density: \( \theta e^{-\theta \delta} \). A firm that is allowed to re-optimize its price at time \( t \) maximizes the present discounted value of future profits\(^{20}\):

\[ \max_{P_{k,j} (t)} \mathbb{E} \left[ \int_t^\infty \frac{C (t)}{C (u)} \frac{P (t)}{P (u)} e^{-(\rho + \theta) (u - t)} \left\{ \hat{P}_{k,j} (t) Y_{k,j} (u | t) - C_k (Y_{k,j} (u | t)) \right\} \, du \right] \]

subject to the demand schedule

\[ Y_{k,j} (u | t) = \left[ \frac{\hat{P}_{k,j} (t)}{P_k (u)} \right]^{-\epsilon} Y_k (u) \]

where \( C (\cdot) \) is the firms nominal cost function. The first-order condition associated with the problem is

\[ \mathbb{E} \left[ \int_t^\infty \frac{C (t)}{C (u)} \frac{P (t)}{P (u)} e^{-(\rho + \theta) (u - t)} Y_{k,j} (u | t) \left\{ \hat{P}_{k,j} (t) - \mathcal{M} M C_k (Y_{k,j} (u | t)) \right\} \, du \right] = 0 \]

\[ \hat{P}_{k,j} (t) = P_k (t) \frac{\mathbb{E} \left[ \int_t^\infty \frac{C (t) P (t)}{C (u) P (u)} e^{-(\rho + \theta) (u - t)} \left[ \frac{P^*_k (t)}{P^*_k (u)} \right]^{-\epsilon} Y^*_k (u) \mathcal{M} M C_k (u) \, du \right]}{\mathbb{E} \left[ \int_t^\infty \frac{C (t) P (t)}{C (u) P (u)} e^{-(\rho + \theta) (u - t)} \left[ \frac{P^*_k (t)}{P^*_k (u)} \right]^{-1 - \epsilon} P^*_k (u) \mathcal{E} (u) Y^*_k (u) \, du \right]} \]

\(^{20}\)We assume that firms commit to supply whatever quantity demanded at the posted price, even if that implies negative profits.
where $MC_k$ is the nominal marginal cost function and $\mathcal{M} \equiv \frac{\epsilon}{1-\epsilon}$. Note that in the limiting case of no price rigidities ($\theta \to \infty$), this condition collapses to the familiar optimal price-setting condition under flexible prices $P_{k,j} (t) = \mathcal{M}MC_k (Y_{k,j} (t))$.

The firm’s cost function is $C_k (Y_{k,j} (u|t)) = Y_{k,j} (u|t) (1 - \tau_k) W (u)$, therefore the nominal marginal cost is

$$ MC_k (Y_{k,j} (u|t)) = (1 - \tau_k) W (u) \equiv MC_k (u) $$

The FOC can be rewritten as

$$ \hat{P}_{k,j} (t) = P_k (t) \frac{\Theta_k (t)}{\Phi_k (t)} $$

where $\Theta_k$ and $\Upsilon_k$ are defined by

$$ \Theta_k (t) \equiv \mathbb{E} \left[ \int_t^\infty \left\{ e^{-(\rho + \theta)(u-t)} \frac{Y_k (u)}{C (u)} P (u) \left( \frac{P_k (t)}{P_k (u)} \right)^{-\epsilon} \mathcal{M}MC_k (u) \right\} du \right] \tag{B.1} $$

$$ \Phi_k (t) \equiv \mathbb{E} \left[ \int_t^\infty \left\{ e^{-(\rho + \theta)(u-t)} \frac{P_k (u) Y_k (u)}{C (u)} P (u) \left( \frac{P_k (t)}{P_k (u)} \right)^{1-\epsilon} \right\} du \right] \tag{B.2} $$

Using the result that the dynamics of the price level is locally deterministic we can rewrite the formulas above as

$$ \Theta_k (t) = \mathbb{E} \left[ \int_t^\infty \left\{ e^{-\int_t^u [\rho + \theta - \epsilon \pi_k (s)] ds} \frac{\mathcal{M}MC_k (u) Y_k (u)}{C (u) P (u)} \right\} du \right] $$

$$ \Phi_k (t) = \mathbb{E} \left[ \int_t^\infty \left\{ e^{-\int_t^u [\rho + \theta - (\epsilon - 1) \pi_k (s)] ds} \frac{P_k (u) Y_k (u)}{C (u) P (u)} \right\} du \right] $$

The Feynman-Kac representation formula establishes that $\Theta_k$ and $\Upsilon_k$ are the unique solutions to the partial differential equations

$$ (\rho + \theta - \epsilon \pi_k (t)) \Theta_k (t) = \mathcal{A} (\Theta_k) + \frac{\mathcal{M}MC_k (t) Y_k (t)}{C (t) P (t)} $$

$$ [\rho + \theta - (\epsilon - 1) \pi_k (t)] \Phi_k (t) = \mathcal{A} \Upsilon_k (t) + \frac{P_k (t) Y_k (t)}{C (t) P (t)} $$

where the operator $\mathcal{A}$ is the infinitesimal generator of the stochastic process, defined as $\mathcal{A} f = \mu \nabla_x f (x) + \frac{1}{2} \text{tr} \left[ \sigma^2 H (f) (\sigma^x)^T \right]$. Hence, their laws of motion are

$$ \frac{d\Theta_k (t)}{\Theta_k (t)} = \left[ \rho + \theta - \epsilon \pi_k (t) - \frac{\mathcal{M}MC_k (t) Y_k (t)}{C (t) P (t) \Theta_k (t)} \right] dt + \sigma_{\Theta_k} (t) dZ (t) $$

$$ \frac{d\Phi_k (t)}{\Phi_k (t)} = \left[ \rho + \theta - (\epsilon - 1) \pi_k (t) - \frac{Y_k (t) P_k (t)}{C (t) P (t) \Upsilon_k (t)} \right] dt + \sigma_{\Upsilon_k} (t) dZ (t) $$

Finally, we can rewrite inflation as $\pi_k (t) = \frac{\theta}{1-\epsilon} \left[ \left( \frac{\Theta_k (t)}{\Phi_k (t)} \right)^{1-\epsilon} - 1 \right]$.  

43
C Loss Function

The welfare function is

\[ W = \int_{0}^{\infty} e^{-\rho t} \left[ \ln C - \frac{L^{1+\varphi}}{1+\varphi} \right] dt \]

where

\[ C = \Lambda S^{1-\alpha} \frac{P^*}{P_F^*} C^* \]
\[ L = \Delta_H Y_H + \Delta_{NT} Y_{NT} \]

and

\[ S = \frac{Y_H}{(1-\beta) [\alpha + (1-\alpha) \Lambda]} \frac{P_F}{P_F^*} C^* \]
\[ \mathcal{O} = \beta \Lambda S^{1-\alpha} \frac{P^*}{Y_{NT} P_F^*} C^* \]

Log-linearize both \( C \) and \( L \) to obtain

\[ \ln C = t.i.p. + \ln \Lambda + (1-\alpha) \ln S - \ln \mathcal{O} \]
\[ = t.i.p. + (1-\beta) \ln \Lambda - (1-\alpha) (1-\beta) \ln [\alpha + (1-\alpha) \Lambda] - (1-\alpha) (1-\beta) \ln Y_H + \beta \ln Y_{NT} \]
\[ \simeq t.i.p. + \alpha (1-\beta) (2-\alpha) \lambda - \alpha (1-\alpha)^2 (1-\beta) \frac{1}{2} \lambda^2 + (1-\alpha) (1-\beta) y_H + \beta y_{NT} \]

\[ \frac{L^{1+\varphi}}{1+\varphi} \simeq t.i.p. + (1-\alpha) (1-\beta) \delta_H + \beta \delta_{NT} + (1-\alpha) (1-\beta) y_H \]
\[ + (1+\omega) \frac{(1-\alpha) (1-\beta)}{2} y_H^2 + \beta y_{NT} + (1+\psi) \frac{\beta}{2} y_{NT}^2 \]

Thus

\[ \mathbb{W} = \int_{0}^{\infty} e^{-\rho t} \left[ t.i.p. + \alpha (2-\alpha) \lambda - \frac{(1-\alpha)^2}{2} \lambda^2 \right] dt \]

Now use the following second order approximation to the budget constraint

\[ \int_{0}^{\infty} e^{-\rho t} \left( \lambda + \frac{\Gamma}{\eta} x^2 \right) dt = 0 \]

to replace the linear term in \( \mathbb{W} \) and obtain

\[ \mathbb{W} = \int_{0}^{\infty} e^{-\rho t} \left[ \frac{t.i.p. - \alpha (2-\alpha) \Gamma}{\eta} z^2 - \frac{(1-\alpha)^2}{2} \lambda^2 - (1-\alpha) \frac{\xi}{2 \beta} \pi_H^2 - (1-\alpha) \frac{1+\omega}{2} y_H^2 - \frac{\beta}{1-\beta} \frac{1+\psi}{2} y_{NT}^2 \right] dt \]

Finally, the loss function is

\[ \mathbb{L} = \frac{1}{2} \int_{0}^{\infty} e^{-\rho t} \left\{ x^2 + \phi \lambda^2 + \phi_{\pi_H} \pi_H^2 + \phi_{\pi_{NT}} \pi_{NT}^2 + \phi_y \left[ \omega y_H^2 + \psi y_{NT}^2 + (\omega y_H + \psi y_{NT})^2 \right] \right\} dt \]
where

\[
\begin{align*}
\phi_\lambda &= \frac{\eta(1-\alpha)^2}{2\Gamma(2-\alpha)} \\
\phi_{\pi H} &= \frac{\eta(1-\alpha)}{2\alpha\Gamma(2-\alpha)} \\
\phi_{\pi NT} &= \frac{\eta\beta}{2\alpha\Gamma(2-\alpha)(1-\beta)} \\
\phi_y &= \frac{\eta(1-\alpha+\alpha\beta)}{2\beta\alpha\Gamma(2-\alpha)(1-\beta)}
\end{align*}
\]

**D Proofs**

**Proof of Lemma 1** Under flexible prices, the deterministic planner problem is

\[
\begin{align*}
\max_{Y_H, Y_{NT}, X} \int_0^\infty e^{-\rho t} & \left\{ (1-\beta) \ln \Lambda - (1-\alpha) (1-\beta) \ln [\alpha + (1-\alpha) \Lambda] \\
& + (1-\alpha) (1-\beta) \ln Y_H + \beta \ln Y_{NT} - \frac{(Y_H+Y_{NT})^{1+\phi}}{1+\phi} \right\} \, dt
\end{align*}
\]

subject to

\[
\begin{align*}
d\Lambda &= -\Lambda \Gamma \left( A^S + F^S_H - X \right) \, dt \\
dA^S &= \left[ A^S \rho - \Gamma \left( A^S + F^S_H - X \right) (A^S - X) + \alpha (1-\beta) (1-\Lambda) P^* C^* \right] \, dt
\end{align*}
\]

The Lagrangian is

\[
\mathcal{L} = (1-\beta) \ln \Lambda - (1-\alpha) (1-\beta) \ln [\alpha + (1-\alpha) \Lambda] \\
+ (1-\alpha) (1-\beta) \ln Y_H + \beta \ln Y_{NT} - \frac{(Y_H+Y_{NT})^{1+\phi}}{1+\phi} \\
- \lambda_\Lambda \Lambda \Gamma \left( A^S + F^S_H + X^S_H \right) \\
+ \lambda_A \left[ A^S \rho - \Lambda \Gamma \left( A^S + F^S_H + X^S_H \right) (A^S + X^S_H) + \alpha (1-\beta) (1-\Lambda) P^* C^* \right]
\]

The FOCs are

\[
(1-\alpha) (1-\beta) \frac{1}{Y_H} = (Y_H + Y_{NT})^\phi \\
\beta \frac{1}{Y_{NT}} = (Y_H + Y_{NT})^\phi
\]

\[
\lambda_\Lambda = -\lambda_A \left( 2A^S + F^S_H - 2X \right)
\]

and

\[
\begin{align*}
d\lambda_\Lambda &= \left\{ \rho \lambda_\Lambda - (1-\beta) \frac{1}{\lambda} + \frac{(1-\alpha)^2(1-\beta)}{\alpha(1-\alpha)\lambda} + \lambda_A \Gamma \left( A^S + F^S_X - X \right) \right\} \, dt \\
&+ \lambda_A \Gamma \left( A^S + F^S_H - X \right) (A^S - X) + \alpha (1-\beta) \lambda_A P^* C^* \, dt
\end{align*}
\]

Therefore

\[
Y_H = \frac{(1-\alpha)(1-\beta)}{(1-\alpha+\alpha\beta)^{1+\phi}} \\
Y_{NT} = \frac{\beta}{(1-\alpha+\alpha\beta)^{1+\phi}}
\]
and \( L = (1 - \alpha + \alpha \beta)^{1/\varphi} \). When prices are flexible prices are set as a constant markup over wages, therefore

\[
P_H = \mathcal{M} (1 - \tau_H) W = \mathcal{M} (1 - \tau_H) L^\varphi \Lambda P_H S P^*_F C^*
\]

\[
P_{NT} = \mathcal{M} (1 - \tau_{NT}) W = \mathcal{M} (1 - \tau_{NT}) L^\varphi \Lambda S^{1-\alpha} P^*_F C^* \]

Therefore the optimal labor subsidies are

\[
\tau_{\text{OPT}}^H (t) = 1 - \frac{\alpha + (1 - \alpha) \Lambda (t)}{\mathcal{M} (1 - \alpha) \Lambda (t)}
\]

\[
\tau_{\text{OPT}}^{NT} (t) = \frac{\alpha + (1 - \alpha) \Lambda (t) \tau_H (t)}{\alpha + (1 - \alpha) \Lambda (t)}
\]

Using the first order conditions it’s easy to check that, in a symmetric steady state with \( A^S = 0 \) and \( F^S_H = 0 \) we have \( X^S_H = 0 \) and \( \Lambda = 1 \).

**Natural Allocation**  When prices are flexible and the central bank does not intervene in the asset market, the allocation solves

\[
d\lambda = -\Gamma \left( a^S + f^S_H \right) dt
\]

\[
da^S (t) = \left( \rho a^S - \eta \lambda \right) dt
\]

\[
df^S_H = -\rho f^S_H
\]

given the initial conditions \( f^S_H (0) = \varepsilon, a^S (0) = 0 \) and the terminal condition \( \lim_{t \to \infty} e^{-\rho t} a^S (t) = 0 \).

Let \( z^T = [\lambda \ a^S \ f^S_H] \). The natural allocation solves the system of differential equations \( dz = Ax dt \), where

\[
A = \begin{bmatrix}
0 & -\Gamma & -\Gamma \\
-\eta & \rho & 0 \\
0 & 0 & -\varrho
\end{bmatrix}
\]

The eigenvalues of \( A \) are \( [\varrho \ -\xi \ \rho + \xi] \), where \( \xi = \frac{-\rho + \sqrt{\rho^2 + 4 \eta \Gamma}}{2} > 0 \). The first two eigenvectors of \( A \) are

\[
v_1 = \begin{bmatrix}
-\frac{\Gamma (\rho + \varrho)}{\eta \Gamma - \varrho (\rho + \varrho)} \\
-\frac{\eta \Gamma - \varrho (\rho + \varrho)}{\eta \Gamma - \varrho (\rho + \varrho)} \\
1
\end{bmatrix}
v_2 = \begin{bmatrix}
\frac{\rho + \xi}{\eta} \\
1 \\
0
\end{bmatrix}
\]

The solution is

\[
\begin{bmatrix}
\lambda \\
a^S \\
f^S_H
\end{bmatrix} = \zeta_1 \begin{bmatrix}
-\frac{\Gamma (\rho + \varrho)}{\eta \Gamma - \varrho (\rho + \varrho)} \\
-\frac{\eta \Gamma - \varrho (\rho + \varrho)}{\eta \Gamma - \varrho (\rho + \varrho)} \\
1
\end{bmatrix} e^{-\varrho t} + \zeta_2 \begin{bmatrix}
\frac{\rho + \xi}{\eta} \\
1 \\
0
\end{bmatrix} e^{-\xi t}
\]
where the parameters $\zeta_1$ and $\zeta_2$ are determined by the initial conditions $a^S(0) = 0$ and $f^{sS}_H(0) = \varepsilon$. Thus we obtain

$$ \lambda = \Gamma \left[ \frac{1}{\frac{1}{\rho + \varepsilon + \xi}} \right] \left[ \begin{array}{c} a^S \\ f^{sS}_H \end{array} \right] $$

where the states evolve as follows

$$ \left[ \begin{array}{c} d a^S \\ d f^{sS}_H \end{array} \right] = \left[ \begin{array}{cc} -\xi & -\frac{\eta \Gamma}{\rho + \varepsilon + \xi} \\ 0 & -\rho \end{array} \right] \left[ \begin{array}{c} a^S \\ f^* \end{array} \right] $$

**Proof of Propositions 1**

The planner’s problem is

$$ \min_x \frac{1}{2} \int_0^\infty e^{-\rho t} \left( x^2 + \phi \lambda^2 \right) dt $$

subject to

$$ d\lambda = -\Gamma \left( a^S + f^{sS}_H - x \right) dt $$

$$ da^S = \left( \rho a^S - \eta \lambda \right) dt $$

$$ d f^{sS}_H = -\phi f^{sS}_H $$

given the initial conditions $f^{sS}_H(0) = \varepsilon$, $a^S(0) = 0$ and the terminal condition $\lim_{t \to \infty} e^{-\rho t} a^S(t) = 0$. The Hamiltonian associated with this problem is

$$ H = \frac{\phi}{2} \lambda^2 + \frac{1}{2} x^2 + \mu^\lambda \left( -\Gamma a^S + \Gamma x - \Gamma f^{sS}_H \right) + \mu^a(t) \left( \rho a^S - \eta \lambda \right) $$

The FOC is

$$ x = -\Gamma \mu^\lambda $$

while the laws of motion of the costates are

$$ d\mu^\lambda = \rho \mu^\lambda - \phi \lambda + \eta \mu^a $$

$$ d\mu^a = \Gamma \mu^\lambda $$

subject to the initial condition $\mu^\lambda(0) = 0$. We can use the first order conditions to replace $\mu^\lambda$ and $d\mu^a$ to obtain a second order differential equation that the optimal foreign exchange intervention must satisfy

$$ d dx = \rho dx + \left( \eta \Gamma + \phi \Gamma^2 \right) x - \phi \Gamma^2 a^S - \phi \Gamma^2 f^{sS}_H $$
with \( x(0) = 0 \). Let \( z^T = \begin{bmatrix} dx & x & \lambda & a^g & f_H^{ss} \end{bmatrix} \), then the system of differential equations that must be solved is \( dz = Az dt \), where

\[
A = \begin{bmatrix}
\rho & \eta \Gamma + \phi \Gamma^2 & 0 & -\phi \Gamma^2 & -\phi \Gamma^2 \\
1 & 0 & 0 & 0 & 0 \\
0 & \Gamma & 0 & -\Gamma & -\Gamma \\
0 & 0 & -\eta & \rho & 0 \\
0 & 0 & 0 & 0 & -\rho
\end{bmatrix}
\]

The eigenvalues of \( A \) are \(-\rho - \xi - \eta - \rho + \xi - \rho + \eta\), where

\[
\xi = -\rho + \sqrt{\rho^2 + 4\Xi^2} \\
\Xi = \frac{2\eta \Gamma + \phi \Gamma^2 - \sqrt{\phi \Gamma^2 (4\eta \Gamma + \phi \Gamma^2)}}{2}
\]

\[
\xi = -\rho + \sqrt{\rho^2 + 4\Xi^2} \\
\Xi = \frac{2\eta \Gamma + \phi \Gamma^2 + \sqrt{\phi \Gamma^2 (4\eta \Gamma + \phi \Gamma^2)}}{2}
\]

such that \( \eta \Gamma = \Xi \Xi \) and \( \phi \Gamma^2 = (\Xi - \Xi)^2 \). The solution is

\[
\begin{bmatrix}
dx \\
x \\
\lambda \\
a^g \\
f_H^{ss}
\end{bmatrix} = \zeta_1 \begin{bmatrix}
\rho \Omega^2 (\Xi - \Xi)^2 \\
-\Omega^2 (\Xi - \Xi)^2 \\
\Xi (\Omega^2 - \Xi \Xi) \\
(\Xi^2 - \Omega^2) (\Xi^2 - \Omega^2)
\end{bmatrix} e^{-\xi t} + \zeta_2 \begin{bmatrix}
\frac{\Xi - \Xi}{\eta} \\
\frac{\Xi - \Xi}{\rho + \eta + \xi} \\
0 \\
0
\end{bmatrix} e^{-\eta t} + \zeta_3 \begin{bmatrix}
\frac{\Xi - \Xi}{\eta} \\
\frac{\Xi - \Xi}{\rho + \eta + \xi} \\
0 \\
0
\end{bmatrix} e^{-\xi t}
\]

where \( \Omega^2 = \rho (\rho + \eta) \) and the coefficients \( \zeta_1 \ z_2 \ z_3 \) are determined using the initial conditions \( a^g(0) = 0, x_H^{ss}(0) = \varepsilon \) and \( f_H^{ss}(0) = \varepsilon \). Thus we obtain

\[
\lambda(t) = \Gamma \begin{bmatrix}
\frac{1}{\rho + \xi + \eta} \\
\frac{\Xi - \Xi}{\rho + \xi + \eta} \\
\frac{\Xi - \Xi}{\rho + \xi + \eta} \\
\frac{\Xi - \Xi}{\rho + \xi + \eta}
\end{bmatrix}^T \begin{bmatrix}
x(t) \\
a^g(t) \\
f_H^{ss}(t)
\end{bmatrix}
\]

where \( \Xi = \frac{\Xi}{\Xi + \Xi} \) are defined in the appendix. The states \( [x \ a^g \ f_H^{ss}] \) evolve as

\[
\begin{bmatrix}
dx(t) \\
da^g(t) \\
f_H^{ss}(t)
\end{bmatrix} = \begin{bmatrix}
-\Xi \xi - (1 - \Xi) \xi \\
-\eta \Gamma \frac{\Xi - \Xi}{\rho + \xi + \eta} \\
\frac{\phi \Gamma^2 \xi}{\rho + \xi + \eta} \\
\frac{\phi \Gamma^2 (\rho + \xi + \eta)}{\rho + \xi + \eta}
\end{bmatrix} \begin{bmatrix}
x(t) \\
a^g(t) \\
f_H^{ss}(t)
\end{bmatrix}
\]
with \[ \begin{bmatrix} x(0) & a^s(0) & f^s_H(0) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \varepsilon \end{bmatrix}. \]

**Optimal Joint FX and Monetary Policies**  The planner’s problem is

\[
\min_{x,i} \frac{1}{2} \int_0^\infty e^{-\rho t} \left[ x^2 + \phi_\lambda x^2 + \phi_\pi \pi_H^2 + \omega (1 + \omega) \phi_y y_H^2 \right] dt
\]

subject to

\[
d\lambda = -\Gamma \left( a^s + f^s_H - x \right) dt \tag{D.1}
\]

\[
da^s = \left( \rho a^s - \eta \lambda \right) dt \tag{D.2}
\]

\[
dy_H = \left[ i - \rho - \pi_H + \alpha \Gamma \left( a^s + f^s_H - x \right) \right] dt \tag{D.3}
\]

\[
d\pi_H = \left[ \rho \pi_H - \kappa (1 + \omega) y_H - \alpha \kappa \lambda \right] dt \tag{D.4}
\]

\[
df^s_H = -\phi f^s_H \tag{D.5}
\]

given the initial conditions \( f^s_H(0) = \varepsilon \), \( a^s(0) = 0 \) and the terminal condition \( \lim_{t \to \infty} e^{-\rho t} a^s(t) = 0 \). The Hamiltonian associated with this problem is

\[
H = \frac{1}{2} x^2 + \phi_\lambda x^2 + \frac{\phi_\pi}{2} \pi_H^2 + \omega (1 + \omega) \frac{\phi_y}{2} y_H^2 - \mu^\lambda \Gamma \left( a^s + f^s_H - x \right) + \mu^a \left( \rho a^s - \eta \lambda \right)
\]

\[
+ \mu^{y_H} \left[ i - \rho - \pi_H + \alpha \Gamma \left( a^s + f^s_H - x \right) \right] + \mu^{\pi_H} \left[ \rho \pi_H - \kappa (1 + \omega) y_H - \alpha \kappa \lambda \right]
\]

The FOCs are

\[
x = \alpha \Gamma \mu^{y_H} - \Gamma \mu^\lambda
\]

\[
\mu^{y_H} = 0
\]

while the laws of motion of the costates are

\[
d\mu^\lambda = \rho \mu^\lambda - \phi_\lambda \lambda + \eta \mu^a + \alpha \kappa \mu^{\pi_H}
\]

\[
da^\alpha = \Gamma \mu^\lambda - \alpha \Gamma \mu^{y_H}
\]

\[
d\mu^{y_H} = \rho \mu^{y_H} - \omega (1 + \omega) \phi_y y_H + \kappa (1 + \omega) \mu^{\pi_H}
\]

\[
d\mu^{\pi_H} = -\phi_\pi \pi_H + \mu^{y_H}
\]

The optimal monetary policy is

\[
i - \rho = \left( 1 - \kappa \frac{\phi_\pi}{\omega \phi_y} \right) \pi_H \, dt + \alpha d\lambda
\]
while the optimal foreign intervention satisfies

$$ddx = \rho dx + (\eta \Gamma + \phi \lambda \Gamma^2) x - \phi \lambda \Gamma^2 \left( a^S + f^S_H \right) + \kappa \alpha \Gamma \phi_{xH} \tau_H$$

Let $z^T = \begin{bmatrix} y_H & \pi_H & dx & x & \lambda & a^S & f^S_H \end{bmatrix}$, then the system of differential equations that must be solved is $dz = Az\, dt$ where

$$A = \begin{bmatrix} 0 & -\frac{\kappa}{\omega} & \eta \Gamma + \phi \lambda \Gamma^2 & 0 & 0 & 0 & 0 \\ -\kappa (1 + \omega) & \rho & 0 & 0 & -\alpha \kappa & 0 & 0 \\ 0 & \kappa \alpha \Gamma \phi_{xH} & \rho & \eta \Gamma + \phi \lambda \Gamma^2 & 0 & -\phi \lambda \Gamma^2 & -\phi \lambda \Gamma^2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma & 0 & -\Gamma & -\Gamma \\ 0 & 0 & 0 & 0 & -\eta & \rho & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\varrho \end{bmatrix}$$

The solution has the following form

$$z = \zeta_1 v_1 e^{\nu_1 t} + \zeta_2 v_2 e^{\nu_2 t} + \zeta_3 v_3 e^{\nu_3 t} + \zeta_4 v_4 e^{\nu_4 t} + \zeta_5 v_5 e^{\nu_5 t} + \zeta_6 v_6 e^{\nu_6 t} + \zeta_7 v_7 e^{\nu_7 t}$$

where $\nu_j$ and $v_j$ are the eigenvalues and associated eigenvectors of $A$. Unfortunately, the eigenvalues of $A$ cannot be derived analytically, therefore the system can only be solved numerically.

**Optimal FX Intervention** The problem is similar to the one solved in the previous section, but this time $i = \rho$. Then the optimal foreign exchange intervention satisfies

$$ddx = \rho dx + [\eta \Gamma + \alpha^2 \Gamma^2 \omega (1 + \omega) \phi_y + \phi \lambda \Gamma^2] x - [\alpha^2 \Gamma^2 \omega (1 + \omega) \phi_y + \phi \lambda \Gamma^2] \left( a^S + f^S_H \right) + \alpha \Gamma \omega \left[ (1 + \omega) \phi_y - \kappa \phi_{xH} \right] \tau_H + \alpha \Gamma \kappa \omega H^{yH}$$
Let $z^\top = \begin{bmatrix} y_H & \pi_H & \mu^{y_H} & \mu^{\pi_H} & dx & x & \lambda & a^S & f_H^s \end{bmatrix}$, then the system of differential equations that must be solved is $dz = Az\,dt$ where

$$A^\top = \begin{bmatrix}
0 & -\kappa (1 + \omega) & -\omega (1 + \omega) \phi_y & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & \rho & 0 & -\phi_{\pi_H} & \alpha \Gamma (1 + \omega) \phi_y & 0 & 0 & 0 & 0 \\
-\omega (1 + \omega) \phi_y & 0 & \rho & 1 & \alpha \Gamma \omega & 0 & 0 & 0 & 0 \\
0 & 0 & \kappa (1 + \omega) & 0 & 0 & 0 & \rho & 1 & 0 \\
0 & 0 & 0 & 0 & \alpha^2 \Gamma^2 (1 + \omega) \phi_y & 0 & \Gamma & 0 & 0 \\
0 & -\alpha \kappa & 0 & 0 & 0 & 0 & \eta \Gamma & \phi_\lambda \Gamma^2 & 0 \\
0 & 0 & 0 & 0 & -\alpha^2 \Gamma^2 (1 + \omega) \phi_y & 0 & -\Gamma & \rho & 0 \\
0 & 0 & 0 & 0 & -\phi_\lambda \Gamma^2 & 0 & -\phi_\lambda \Gamma^2 & 0 & -\rho
\end{bmatrix}$$

given the initial conditions $f_H^s(0) = \varepsilon$, $a^S(0) = \mu^{y_H}(0) = \mu^{\pi_H}(0) = 0$ and the terminal condition $\lim_{t \to \infty} e^{-\rho t} a^S(t) = 0$. The solution has the following form

$$z = \zeta_1 v_1 e^{\nu_1 t} + \zeta_2 v_2 e^{\nu_2 t} + \zeta_3 v_3 e^{\nu_3 t} + \zeta_4 v_4 e^{\nu_4 t} + \zeta_5 v_5 e^{\nu_5 t} + \zeta_6 v_6 e^{\nu_6 t} + \zeta_7 v_7 e^{\nu_7 t} + \zeta_8 v_8 e^{\nu_8 t} + \zeta_9 v_9 e^{\nu_9 t}$$

where $\nu_j$ and $v_j$ are the eigenvalues and associated eigenvectors of $A$. As before, the eigenvalues of $A$ cannot be derived analytically, therefore the system can only be solved numerically.

**Optimal Monetary Policy** The problem is similar to the one solved before, but this time $x = 0$. Let $z^\top = \begin{bmatrix} y_H & \pi_H & \lambda & a^S & f_H^s \end{bmatrix}$, then the system of differential equations that must be solved is $dz = Az\,dt$ where

$$A = \begin{bmatrix}
0 & -\frac{\kappa}{\omega} \phi_{\pi_H} & 0 & 0 & 0 \\
-\kappa (1 + \omega) & \rho & -\alpha \kappa & 0 & 0 \\
0 & 0 & \rho & -\Gamma & -\Gamma \\
0 & 0 & -\eta & \rho & 0 \\
0 & 0 & 0 & 0 & -\rho
\end{bmatrix}$$

given the initial conditions $f_H^s(0) = \varepsilon$, $a^S(0) = y_H(0) = 0$ and the terminal condition $\lim_{t \to \infty} e^{-\rho t} a^S(t) = 0$. The eigenvalues of $A$ are $[-\rho \ -\xi \ -\nu \ \rho + \xi \ \rho + \nu]$ where $\nu =$
\[-\rho+\sqrt{\frac{\rho^2+4\kappa^2(1+\omega)\frac{\hat{\phi}_y}{\hat{\phi}_y}}{2}}.\] The solution is

\[
\begin{bmatrix}
 y_H \\
 \pi_H \\
 \lambda \\
 a^S \\
 f^{S^*}_{H}
\end{bmatrix} = \zeta_1 \begin{bmatrix}
 -\Upsilon(h) \\
 -\hat{\phi}_y (\rho + \theta) \\
 -\Gamma(\rho + \theta) \\
 -\Gamma(\rho + \theta) \\
 \Gamma_\eta - \varrho(\rho + \theta)
\end{bmatrix} e^{-\varrho t} + \zeta_2 \begin{bmatrix}
 \Upsilon(h) \\
 \frac{\hat{\phi}_y (1+\omega)}{\eta_1} \\
 \frac{\hat{\phi}_y}{\eta_1} \\
 \frac{\hat{\phi}_y}{\eta_1} \\
 1
\end{bmatrix} e^{\eta t} + \zeta_3 \begin{bmatrix}
 1 \\
 0 \\
 0 \\
 0
\end{bmatrix} e^{\eta t}
\]

where \(\Upsilon(h) = \frac{\hat{\phi}_y}{\hat{\phi}_y} \frac{\kappa \hat{\Gamma}(\rho+h)}{h(\rho+h)-\kappa^2 \frac{1+\omega}{\omega} \frac{\hat{\phi}_y}{\hat{\phi}_y}}\), and the parameters \(\zeta_1\), \(\zeta_2\), and \(\zeta_3\) are determined by the initial conditions \(a^S(0) = y_H(0) = 0\) and \(f^{S^*}_{H}(0) = \varepsilon\). Thus we obtain

\[
\begin{bmatrix}
 \pi_H(t) \\
 \lambda(t)
\end{bmatrix} = \begin{bmatrix}
 \frac{\kappa(1+\omega)}{\rho+t} \\
 \frac{\hat{\Upsilon}(\xi)}{\Gamma} \\
 \frac{\hat{\Upsilon}(\xi)-\Upsilon(\varrho)}{\xi-\varrho(\rho+\varrho+\xi)} \\
 \frac{\hat{\phi}_y}{\eta_1} \\
 \frac{\hat{\phi}_y}{\eta_1}
\end{bmatrix} \begin{bmatrix}
 y_H(t) \\
 \eta a^S(t) \\
 f^{S^*}_{H}(t)
\end{bmatrix}
\]

where \(\hat{\Upsilon}(h) = \left[h - \frac{1}{\rho+t} \kappa^2 \frac{1+\omega}{\omega} \frac{\hat{\phi}_y}{\hat{\phi}_y} \right] \frac{\kappa \hat{\Gamma}(\rho+h)}{h(\rho+h)-\kappa^2 \frac{1+\omega}{\omega} \frac{\hat{\phi}_y}{\hat{\phi}_y}} \Upsilon(h)\), while the states evolve as

\[
\begin{bmatrix}
 \frac{dy_H(t)}{dt} \\
 \frac{da^S(t)}{dt} \\
 \frac{df^{S^*}_{H}(t)}{dt}
\end{bmatrix} = \begin{bmatrix}
 -\varrho \\
 \frac{(1-\xi)\Upsilon(\xi)}{\eta_1} -\xi \\
 \frac{(1-\xi)\Upsilon(\xi)-(1-\xi)\Upsilon(\varrho)}{\xi-\varrho(\rho+\varrho+\xi)} - \varrho
\end{bmatrix} \begin{bmatrix}
 y_H(t) \\
 a^S(t) \\
 f^{S^*}_{H}(t)
\end{bmatrix}
\]

with \(y_H(0) = a^S(0)\) and \(f^{S^*}_{H}(0) = \varepsilon\).

**Proof of Proposition 2** The planner solves

\[
\min_i \frac{1}{2} \int_0^\infty e^{-\rho t} \left[\phi_{\pi,\pi} \pi_H^2 + \omega (1+\omega) \phi_y y_H^2\right] dt
\]

subject to (D.3) and (D.4). The optimal monetary policy is still

\[
i = \rho + \left(1 - \frac{\phi_{\pi,\pi}}{\phi_{\pi,\pi}}\right) \pi_H dt + \alpha d\lambda
\]

while the foreign exchange intervention satisfies

\[
ddx = \rho dx + (\eta \Gamma + \phi \Gamma^2) x - \phi \Gamma^2 \left(a^S + f^{S^*}_{H}\right)
\]
Let \( z^\top = \begin{bmatrix} y_H & \tau_H & dx & x & \lambda & a^S & f_H^{sS} \end{bmatrix} \), then the system of differential equations that must be solved is \( dz = A z dt \) where

\[
A = \begin{bmatrix}
0 & -\kappa \phi_x / \phi_y & 0 & 0 & 0 & 0 & 0 \\
-\kappa (1 + \omega) & \rho & 0 & 0 & -\alpha \kappa & 0 & 0 \\
0 & 0 & \rho & \eta \Gamma + \phi \Gamma^2 & 0 & -\phi \Gamma^2 & -\phi \Gamma^2 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Gamma & 0 & -\Gamma & -\Gamma \\
0 & 0 & 0 & 0 & -\eta & \rho & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\rho
\end{bmatrix}
\]

given the initial conditions \( f_H^{sS}(0) = \varepsilon \), \( a^S(0) = x(0) = y_H(0) = 0 \) and the terminal condition \( \lim_{t \to \infty} e^{-\rho t} a^S(t) = 0 \). The eigenvalues of \( A \) are 

\[
[ -\rho, -\xi, -\bar{\xi}, -\eta, \rho + \xi, \rho + \bar{\xi}, \rho + \iota \]
\]

where 

\[
\iota = -\rho + \sqrt{\rho^2 + 4 \kappa^2 \frac{\lambda \phi_H \gamma_H}{\phi_y}} \frac{1}{2}
\]

The solution is

\[
\begin{bmatrix}
y_H \\
\tau_H \\
dx \\
x \\
a^S \\
f_H^{sS}
\end{bmatrix} = \begin{bmatrix}
\zeta_1 \\
\zeta_2 \\
\zeta_3
\end{bmatrix} e^{-\rho t} + \begin{bmatrix}
\zeta_2 \\
\zeta_3 \\
\zeta_2
\end{bmatrix} e^{-\xi t}
\]

\[
= \begin{bmatrix}
\frac{\kappa \phi_x}{\omega \phi_y} Y(\phi) (\Omega^2 - \Xi^2) \\
\rho Y(\phi) (\Omega^2 - \Xi^2) \\
\rho \Omega^2 (\Xi - \Xi) \\
-\Omega^2 (\Xi - \Xi) \\
\frac{\phi + \rho}{\eta} \Xi (\Omega^2 - \Xi^2) \\
\Xi (\Omega^2 - \Xi^2) \\
(\Xi^2 - \Omega^2) (\Xi^2 - \Omega^2)
\end{bmatrix} + \begin{bmatrix}
\frac{\kappa \phi_x}{\omega \phi_y} Y(\xi) \\
\xi Y(\xi) \\
\Xi Y(\xi) \\
\Xi (\Omega^2 - \Xi^2) \\
\frac{\phi + \xi}{\eta} \\
\Xi (\Omega^2 - \Xi^2) \\
(\Xi^2 - \Omega^2) (\Xi^2 - \Omega^2)
\end{bmatrix} e^{-\xi t}
\]
where the parameters $\zeta_1$, $\zeta_2$, and $\zeta_3$ are determined by the initial conditions $a^s(0) = y_H(0) = 0$ and $f^{sH}(0) = \varepsilon$. Therefore we obtain

\[
\begin{bmatrix}
\pi_H(t) \\
\lambda(t)
\end{bmatrix} = \begin{bmatrix}
\frac{\kappa^{1+\omega}}{\rho+\xi}\frac{\hat{Y}(\xi) - \hat{Y}(\xi)}{[\rho+\nu+\nu](\rho+\nu+\nu)} & 0 \\
\frac{\hat{\epsilon}}{\eta} \frac{\hat{Y}(\xi) + \frac{1-\Xi}{\eta} \hat{\Gamma}(\xi)}{[\rho+\xi+\xi](\rho+\xi+\xi)} & \frac{\phi}{\rho+\xi+\xi} + \frac{(1-\Xi)\hat{\Gamma}}{\rho+\xi+\xi}
\end{bmatrix}^T \begin{bmatrix}
y_H(t) \\
x(t) \\
a^s(t) \\
f^{sH}(t)
\end{bmatrix}
\]

while the states $s = \begin{bmatrix} y_H & x & a^s & f^{sH} \end{bmatrix}^T$ evolve as $ds = Msdt$ where

\[
M = \begin{bmatrix}
\frac{\phi\Gamma}{\rho+\xi+\xi} & \frac{\Xi(\rho+\xi) + (1-\Xi)(\rho+\xi)}{\rho+\xi+\xi} & 0 & 0 \\
\frac{\Xi(\rho+\xi) - (1-\Xi)(\rho+\xi)}{\rho+\xi+\xi} & \frac{\Xi(\rho+\xi) - (1-\Xi)(\rho+\xi)}{\rho+\xi+\xi} & \frac{\phi\Gamma^2}{\rho+\xi+\xi} & 0 \\
\frac{\Xi(\rho+\xi) - (1-\Xi)(\rho+\xi)}{\rho+\xi+\xi} & \frac{\Xi(\rho+\xi) - (1-\Xi)(\rho+\xi)}{\rho+\xi+\xi} & \frac{\Xi(\rho+\xi) - (1-\Xi)(\rho+\xi)}{\rho+\xi+\xi} & \frac{\phi\Gamma^2}{\rho+\xi+\xi} - \eta\Gamma^2 \left(\frac{\Xi(\rho+\xi) + (1-\Xi)(\rho+\xi)}{\rho+\xi+\xi} \right)
\end{bmatrix}
\]

with $y_H(0) = x(0) = a^s(0)$ and $f^{sH}(0) = \varepsilon$.

**Proof or Lemma 2** The optimal intervention rule can be derived by manipulating the closed form solutions of Proposition (1) and (2). The explicit expressions of the derivatives with respect to $\varrho$ are

\[
\begin{align*}
\frac{\partial |\Psi_\lambda|}{\partial \varrho} &= \frac{\phi\Gamma}{\rho+\xi+\xi} \frac{\Xi(\rho+\xi) + (1-\Xi)(\rho+\xi)}{\rho+\varrho + \Xi(\rho+\xi) + (1-\Xi)(\rho+\xi)} \geq 0 \\
\frac{\partial |\Psi_a|}{\partial \varrho} &= \frac{\phi\Gamma}{\eta} \frac{\Xi(\rho+\xi) + (1-\Xi)(\rho+\xi)}{\rho+\varrho + \Xi(\rho+\xi) + (1-\Xi)(\rho+\xi)} \geq 0 \\
\frac{\partial |\Psi_x|}{\partial \varrho} &= -\frac{(1-\Xi)(\Xi(\rho+\xi))}{\rho+\varrho + \Xi(\rho+\xi) + (1-\Xi)(\rho+\xi)} < 0
\end{align*}
\]

The explicit expressions of the derivatives with respect to $\phi$ and $\Gamma$ have been derived using the symbolic toolkit available in Wolfram Mathematica. Their expressions are too big to be reported here and are available upon request.