Mechanism Design with Aftermarkets.
Part I: Cutoff Mechanisms.

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Abstract

I study a model of mechanism design in which the mechanism is followed by an aftermarket, that is, a post-mechanism game played between the agent and other market participants. The designer has preferences that depend on the final outcome but she cannot directly redesign the aftermarket. However, she can influence its information structure by disclosing information elicited by the mechanism.

I identify a class of cutoff mechanisms which elicit and release information in a way that is always incentive compatible, regardless of the form of the aftermarket and underlying distribution of types. Under a richness condition, only cutoff mechanisms have this property. Optimization in the class of cutoff mechanisms is tractable and yields results with applications to the design of auctions followed by bargaining or resale markets, and optimal level of post-transaction transparency in over-the-counter markets.

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1 Introduction

“The game is always bigger than you think” – this phrase succinctly captures a prevalent feature of practical mechanism design problems – they can rarely be fully understood without the wider market context. When a seller designs an auction, she should not ignore future resale or bargaining opportunities which might influence bidders’ endogenous valuations for the object. A dealer in a financial over-the-counter market understands that a counterparty in a transaction may not be the final holder of the asset. Yet, most theoretical models analyze the design problem in a vacuum.\footnote{Some notable exceptions are discussed in the literature review.}

In this paper, I revisit the canonical mechanism design problem of allocating an object to one of several agents. Unlike in the standard model, the mechanism is followed by an aftermarket, modeled as a post-mechanism game played between the agent who acquired the object and other market participants (third parties). The aftermarket is beyond the control of the mechanism designer but she may have preferences over equilibrium outcomes of the post-mechanism game.

Although the designer is unable to directly redesign the aftermarket, she can influence its information structure by releasing information elicited by the mechanism. As a result, the design problem is augmented with an additional choice variable – a disclosure rule. For example, if a bidder who wins an object engages in bargaining over acquisition of complementary goods after the auction, the designer has to decide how much information about bids to reveal after the auction. The choice of a disclosure rule influence the bargaining position of the bidder in the aftermarket.

The resulting structure of the problem can be described as a composition of mechanism and information design. In the first step, the mechanism has to elicit information from the agents in an incentive compatible way. In the second step, the mechanism discloses this information to other market participants in order to induce the optimal information structure in the aftermarket. The two parts of the problem interact in a non-trivial way: the amount of information elicited by the mechanism determines the amount of information available for disclosure; disclosure influences the incentives of agents to reveal their private information to the mechanism.

The paper focuses on two questions: (1) how much information \textbf{can} be elicited and revealed (implementability), and (2) how much information the designer \textbf{should} elicit and reveal given her objective function (optimal mechanisms). These questions are intricately connected to the decision of how to allocate the object.
Suppose that a designer considers a particular allocation and disclosure rule. Does there exist a mechanism with transfers which implements this rule? In the presence of an aftermarket, the answer depends on the details of the model. For example, the prior distribution of agents’ types influences beliefs, and hence equilibrium and expected payoffs in the aftermarket. Indirectly, the prior distribution interacts with incentive compatibility constraints in the mechanism. This implies that the set of feasible (implementable) mechanisms varies with the distribution. Implementability is also sensitive to fine details of the aftermarket. As a result, the optimal design problem is intractable in most cases. In part II of the paper, I impose more structure and develop methods that allow to solve special instances of the problem. In part I, I take an alternative route and restrict the class of mechanisms that can be used.

The class I study has the property that each mechanism in the class is always implementable – regardless of the prior distribution of types and details of the aftermarket. Such property is possessed by cutoff mechanisms. Informally, in a cutoff mechanism, the signal sent by the designer cannot directly depend on the report of the agent who participates in the aftermarket. It can depend on reports of agents who do not participate in the aftermarket, and on the realization of stochastic elements of the mechanism. The informational content of such signal is determined by the allocation rule.

To understand how a cutoff mechanism works, consider allocating an object to one of \( N \geq 1 \) agents. For any allocation rule, in order to receive the object, an agent has to outbid some random threshold, which I call a cutoff – be it a bid of another agent or a (possibly random and personalized) reserve price set by the seller. A cutoff mechanism releases information about the realization of the cutoff. For example, an efficient auction corresponds to a cutoff mechanism if the message sent by the auctioneer after the auction depends only on the second bid. The second-highest bid is a cutoff from the perspective of the winner. When the second bid is perfectly disclosed, the posterior distribution of the winner’s type is a truncation of the prior. If no information is revealed, the posterior belief is that of a first-order statistic. Thus, by changing the disclosure policy of the cutoff, the auctioneer influences the information structure of the aftermarket. The cutoff is defined endogenously by the allocation rule. If the designer introduces a random reserve price in the auction, the relevant cutoff becomes the maximum of the realized reserve price and the second bid.

The surprising property of cutoff mechanisms is not that they are always implementable – it is that they are the only mechanisms with this property. There exist non-cutoff mechanisms which are implementable but they are implementable only if
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some restrictions are imposed on the set of possible prior distributions and aftermarkets. Thus, in many settings, non-cutoff mechanisms may be less attractive from a practical perspective.

Cutoff mechanisms turn out to be relatively tractable. Their qualitative properties depend on the number of agents in the mechanism.

If one agent participates in the mechanism, a cutoff mechanism reveals information about realization of stochastic elements of the mechanism, for example, a random reserve price. If the allocation rule is fixed at a suboptimal level, it may be beneficial to disclose such information. However, if the allocation and disclosure rule are designed jointly, there always exists an optimal mechanism which reveals no information. This conclusion holds irrespective of the objective function and form of the aftermarket.

Using this insight, I apply the one-agent model to study optimal level of transparency in financial over-the-counter markets. A dealer (mechanism designer) chooses a mechanism to sell an asset to another dealer (agent) who resells to a customer (third party) in the aftermarket. Under the assumption that the initial seller has no private information, both the profit-maximizing and the welfare-maximizing mechanisms release no information. Moreover, there are cases in which the two mechanisms are identical. Thus, the model points out circumstances under which market regulation, such as requiring more information to be disclosed, is ineffective or even undesirable.

In Section 5, I extend the model to allow for exogenous private information of the designer. In the case of an OTC market, the initial seller faces a random cost of supplying the asset. I show that the welfare-maximizing mechanism reveals information only about the random cost. The result provides theoretical support for the use of financial benchmarks, such as LIBOR, which are designed to disclose dealers’ costs.

If multiple players participate in the mechanism, a cutoff mechanism can use reports of agents who do not acquire the good to construct the signal. How much information about the winner’s type can be revealed depends on the allocation rule. Unlike in the one-agent case, information revelation can be strictly optimal even in the joint design problem. The optimal disclosure rule is influenced by the alignment of preferences of the designer and the third party, and the structure of the aftermarket.

I consider several applications of the multiple-agent model. In the first application, the aftermarket features a third-party regulator who makes a decision influencing the ex-post payoff of the winner of the object. For example, a firm enters an industry by acquiring another company, or buying a license. The industry is regulated by a government agency. When the seller of the object and the regulator have aligned
preferences, it is optimal to run an efficient auction and reveal the second-highest value. If the designer and the regulator play a zero-sum game, no information is disclosed. This is consistent with the empirical observation that auctions in the public sector tend to reveal more information than auctions in the private sector.

In the second application, I assume that the auctioneer is constrained to allocate the object to the highest-value bidder but chooses how much information to reveal. The winner of the object bargains after the auction with a third party. For example, an agent who wins a contract in a procurement auction negotiates with a subcontractor. For negotiations to succeed, the third party has to incur a random cost. The gains from negotiation depend on the type of the winner which is unknown to the third party. I show that optimal disclosure depends on the convexity/concavity of the distribution of the third party’s cost. Due to properties of cutoff mechanisms, the optimal mechanism can be implemented robustly, i.e., even if the designer does not know the distribution of types or details of the bargaining protocol.

In the third application, I consider a model of resale to a third-party buyer. Both the allocation and the disclosure rule are chosen. Under conditions, in the welfare-maximizing mechanism, the designer allocates the object randomly if types of agents are low, and runs an efficient auction if at least two bidders express a high enough value for the object. Then, the designer announces whether the auction took place or not. The announcement serves as a signal of high competition in the case when the auction took place, leading to a higher resale price in the aftermarket.

For a fixed aftermarket and distribution of types, restricting attention to cutoff mechanisms is not without loss of generality. Part II of this paper provides a partial answer to the question of how much value the designer is losing by doing so. I show that in a tractable class of problems in which the third party takes a binary action, cutoff mechanism are in fact optimal. The conclusion requires that the preferences of the agent and the third party are sufficiently misaligned.

The rest of the paper is organized as follows. In Section 2, I introduce a special case of the model, with one agent and a resale game in the aftermarket. This allows me to highlight main ideas in a simple setting. In Section 3, the model and the results are extended to allow for multiple agents and a general aftermarket. Section 4 presents applications of the general model. In Section 5, I discuss extensions of the model, and Section 6 concludes. Most proofs are collected in the appendix.
1.1 Literature review

From a conceptual and methodological perspective, this paper combines mechanism design with information design. In a seminal paper, Myerson (1981) solves the problem of allocating a single asset in a mechanism design framework. The designer is allowed to choose an arbitrary mechanism, subject to incentive-compatibility and participation constraints. In contrast, as characterized by a survey paper of Bergemann and Morris (2016b), information design takes the mechanism (or game) as given and considers optimization over information structures. In my model, the principal designs the mechanism and the information structure jointly.

My analysis makes use of the concavification argument introduced by Aumann and Maschler (1995), and adopted to the Bayesian persuasion model by Kamenica and Gentzkow (2011). The argument establishes that in a large class of information design problems, optimization can be performed in the space of distributions over posterior beliefs, and thus the optimal payoff is a concave closure of the function that maps each posterior belief into the corresponding expected payoff for the designer. One of the main technical contributions of my paper is to find a connection between the mechanism design problem and the concavification result via the introduction of cutoffs. Because the model allows for an arbitrary aftermarket, the information design problem embedded in my problem is more general than Bayesian persuasion. Thus, the analysis is connected to multiple-player generalizations of Bayesian persuasion, e.g. Bayes Correlated Equilibrium of (see Bergemann and Morris, 2016a).

With regard to the structure of the problem, the literature most closely related to my model is a series of papers by Giacomo Calzolari and Alessandro Pavan on sequential agency. In a sequential agency problem, the agent contracts with multiple principals, and upstream principals decide how much information to reveal to downstream principals. Thus, the allocation rule and the disclosure rule are designed jointly. The literature can be divided into two strands.

The first strand deals with the methodological problem of finding the correct revelation principle for the model of sequential agency. If downstream principals do not observe the decisions made by upstream principals, the agent’s message space may have to be enriched to include reports other than the type. This problem is addressed in Calzolari and Pavan (2008) and Calzolari and Pavan (2009), and in papers referenced

\footnote{Because signals are public in my model, the mechanism designer can only access a subset of BCE of the post-mechanism game.}
therein. These issues are orthogonal to my theoretical analysis in Section 3 which models the aftermarket as a “black-box” game (a mapping from types and beliefs into final expected payoffs). However, because an agency problem is a special case of a game, the above considerations become relevant in solving examples of my model.

The second strand of the literature attempts to solve applications of sequential agency problems. Calzolari and Pavan (2006a) consider a model of a revenue-maximizing monopolist selling an object to an agent who can later resell to a third party. They consider a simple version with binary types which allows them to derive a closed-form solution. They show that it is sometimes optimal to distort the allocation and send explicit signals to influence the outcome of the second-stage game. Introduction of cutoffs as a way to talk about allocation and revelation in a mechanism provides a structural insight into the trade-offs present in the model of Calzolari and Pavan (2006a). Out of four mechanism that can be optimal in their setting (depending on parameters), three are cutoff mechanisms. My model is more general in that it allows (i) an arbitrary objective function, (ii) multiple agents, (iii) arbitrary second-stage game, and (iv) a general type space.

Calzolari and Pavan (2006b) show in a fairly general setup of two-stage sequential agency (with one agent) that, under conditions, it is optimal to reveal no information in the upstream mechanism. This conclusion is similar to my result about optimality of no-revelation in one-agent problems. However, the results are not related otherwise. None of the three economic assumptions of the main theorem of Calzolari and Pavan (2006b) are assumed in my analysis. For example, the upstream principal in Calzolari and Pavan (2006b) has no direct preferences over the outcome of the second stage. Although this is allowed by my model, I focus on exactly opposite cases when the principal cares about the final allocation (e.g. because she maximizes total surplus). Moreover, in my model, the preferences of the agent and the third party are typically not separable in the outcomes of the two stages.

A large literature analyzes the consequences of resale after auctions (e.g. Gupta and Lebrun, 1999, Zheng, 2002, Haile, 2003, Hafalir and Krishna, 2008, Hafalir and Krishna, 2009). The structure of the problem is similar to my model, except that the second stage is a game between the bidders, rather than between the winning bidder and a third party. This makes the analysis of the problem qualitatively different. In the literature on auctions with resale, the revelation rule is either (i) made redundant

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3 I comment on this further in Part II of the paper (TBA) which establishes conditions under which cutoff mechanisms are optimal.
by assuming an information structure in the resale stage (e.g. types are revealed, as in Gupta and Lebrun, 1999), (ii) fixed for the purpose of the analysis (as in Haile, 2003 who assumes that all bids are revealed), or (iii) only relevant to the extent that it permits implementing the optimal allocation in an equilibrium of the auction (as in Zheng, 2002, where the optimal allocation and payoff are known ex-ante, and no revelation rule can increase the payoff of the mechanism designer). In contrast, the revelation rule is a choice variable in my model, plays an active role in the design problem, and in particular interacts non-trivially with the optimal allocation rule.

A number of papers analyze the consequences of post-auction interaction between bidders. Katzman and Rhodes-Kropf (2008) and Zhong (2002) examine the effect of different bid announcement policies on revenue in standard auctions followed by Bertrand and Cournot competition. Lauermann and Virág (2012) consider a model where bidders exercise a common outside option after the auction. Dworczak (2015) analyzes the consequences of different bid disclosure rules in a setting where bidders trade other units of the object after the mechanism. All of these papers (with the exception of one section in Dworczak, 2015) compare a small number of fixed auction formats (e.g. first-price, second-price) and announcement rules (full revelation of bids, revelation of the winning bid etc.). In contrast, this paper proposes a mechanism design approach in which the designer can choose an arbitrary allocation and revelation rule. I show in examples that a simple auction design, e.g. a second-price auction with revelation of the price paid by the winner, may be optimal even if the full set of mechanisms is available.

A different model combining mechanism design with persuasion is considered by Kolotilin, Li, Mylovanov and Zapechelnyuk (2015). In their setting, the agent reports private information to the designer who then communicates her private information back to the agent. In my model, the designer communicates to the third party, and also controls allocation of a physical object. As far as I can tell, these differences make the two problems unrelated in terms of structural insights.

The literature on information disclosure in auctions is very rich but has mainly focused on ex-ante revelation, i.e. disclosure of information before or during the auction (seminal examples include Milgrom and Weber, 1982a, Milgrom and Weber, 1982b, and Eső and Szentes, 2007). My analysis focuses on ex-post revelation of the information elicited by the mechanism.

My paper is conceptually related to the idea that public intervention in the market may change the information structure, hence effecting the final market outcome. A
series of papers by Philippon and Skreta (2012), Tirole (2012), Fuchs and Skrzypacz (2015) analyze the problem of overcoming adverse selection in a market by an intervention that alleviates the lemons problem. I consider a similar effect in Subsection 2.6, where the form of the optimal mechanism is driven by the adverse selection problem in the aftermarket.

2 Simple Model of Resale

In this section, I consider a simplified version of the model. There is one agent, and the aftermarket is a resale game. The third party has a higher value than the agent, and the bargaining protocol is take-it-or-leave-it offers. This simple setting allows me to highlight main ideas without obscuring the picture with additional details associated with the general case. In Section 3, I extend the analysis to multiple-agent mechanisms, continuous type spaces, and a general aftermarket.

2.1 Model

A seller (mechanism designer) owns an indivisible object that she can allocate to an agent. The agent has value $\theta \in \Theta$ for holding the object, where $\Theta$ is an arbitrary finite subset of non-negative real numbers. Agent’s type is distributed according to a prior full-support probability mass function $f$. If the agent acquires the object, she can resell it to a third party with value $v(\theta)$, where $v: \Theta \to \mathbb{R}$ is some non-decreasing function. I assume that $v(\theta) > \theta$, for all $\theta \in \Theta$.

The market game consists of two stages: (1) implementation of the mechanism, and (2) the aftermarket. In the first stage, the seller chooses and publicly announces a direct mechanism $(x, \pi, t)$, where $x: \Theta \to [0, 1]$ is an allocation function, $\pi: \Theta \to \Delta(S)$ is a signal function with some finite signal set $S$, and $t: \Theta \to \mathbb{R}$ is a transfer function. If the agent reports $\hat{\theta}$, she receives the good with probability $x(\hat{\theta})$ and pays $t(\hat{\theta})$. Conditional on selling the good, the designer draws and publicly announces a signal $s \in S$ according to distribution $\pi(\cdot | \hat{\theta})$.

In the second stage, the third party observes the signal realization $s$, and Bayes-updates her beliefs. The signal is the only source of information about the outcome of

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4 Focusing on direct mechanisms is without loss of generality, by the Revelation Principle. However, restricting attention to direct mechanisms will be consequential for some practical properties discussed in Subsection 2.7.
the mechanism for the third party.\footnote{It is irrelevant whether the third party can observe that the agent acquired the good in the mechanism because she is going to condition on this event when making or accepting an offer in the aftermarket.} I let $f^s$ denote the updated belief over the agent’s type. With probability $\eta \in [0, 1]$, the third party makes a take-it-or-leave-it offer to the agent, and with probability $1 - \eta$, the agent makes a take-it-or-leave-it offer to the third party. I adopt the tie-breaking rule that the offer is accepted if and only if the conditional expected value of the offeree is at least as high as the proposed price.

Both the agent and the third party are expected-utility maximizers with quasi-linear utility. I denote by $\text{BNE}(\bar{f})$ the set of mixed-strategy Bayesian Nash equilibria of the aftermarket, given posterior belief $\bar{f}$ held by the third party. Fixing an equilibrium selection $\sigma^f \in \text{BNE}(\bar{f})$, for any $\bar{f}$, let $\sigma^f_a$ and $\sigma^f_{tp}$ denote the equilibrium distributions of prices offered by the the agent and the third party, respectively, conditional on being the proposer. Finally, let

$$u(\theta; \bar{f}) \equiv \int_{\mathbb{R}} \max\{\theta, p\} \left( \eta d\sigma^f_{tp}(p) + (1 - \eta)1_{\{p \leq v(\theta)\}} d\sigma^f_a(p) \right)$$

denote the expected continuation payoff of an agent with type $\theta$ who acquired the object in the first-stage mechanism, conditional on equilibrium $\sigma^f$ under posterior belief $\bar{f}$.

I will distinguish two cases depending on the severity of the adverse selection problem in the aftermarket. The lemons condition is \textit{locally slack} if $\max_{\hat{\theta} < \theta} v(\hat{\theta}) \geq \theta$, for all $\theta \in \Theta$. The lemons condition \textit{binds locally} if the opposite strict inequality holds for all $\theta \in \Theta$. The distinction is relevant for the analysis of local incentive-compatibility constraints. Suppose that the posterior belief of the third party is concentrated on two adjacent types. In the first case, the adverse selection problem disappears; in the second case, the lemons problem is still present.

### 2.2 Implementability

In order to find optimal mechanisms, I first characterize mechanisms which are feasible. I call $(x, \pi)$ a \textit{mechanism frame}.

\textbf{Definition 1.} A mechanism frame $(x, \pi)$ is \textit{implementable} if there exist transfers $t$ such that the agent participates and reports truthfully in the first-stage mechanism,
taking into account the continuation payoff from the aftermarket:

\[
\sum_{s \in S} u(\theta; f^s)\pi(s|\theta)x(\theta) - t(\theta) \geq 0, \tag{IR}
\]

\[
\theta \in \argmax_{\theta} \sum_{s \in S} u(\theta; f^s)\pi(s|\hat{\theta})x(\hat{\theta}) - t(\hat{\theta}), \tag{IC}
\]

for all \( \theta \in \Theta \).

Equations (IR) and (IC) are the standard participation and incentive-compatibility constraints. However, the problem of implementing a mechanism frame \((x, \pi)\) in the presence of the aftermarket differs from implementing an allocation rule \(x\) in a one-stage design problem. Myerson (1981) proves that in a one-stage allocation problem, \(x\) is implementable if and only if \(x\) is non-decreasing. The set of feasible mechanisms is independent of the distribution of types \(f\). With the aftermarket, whether \((x, \pi)\) is implementable or not depends on the prior distribution \(f\). The distribution \(f\) affects the equilibrium \(\sigma^{f^s}\) by changing the posterior belief \(f^s\) of the third party for any signal \(s\).

In the next subsection, I identify a class of mechanism that generalize the Myerson monotonicity property to the two-stage setting. These mechanisms will be implementable regardless of the underlying distribution of types \(f\).

### 2.3 Cutoff mechanisms

In order to define a cutoff mechanism, I introduce the notion of a random-cutoff representation of an allocation rule. To simplify exposition, I assume throughout that \(x(\max(\Theta)) = 1\), i.e. the highest type always receives the good (the assumption is relaxed in the general model).

#### 2.3.1 Random-cutoff representation of an allocation rule

Fix a non-decreasing allocation rule \(x(\theta)\) on \(\Theta\). Using the fact that \(x(\max(\Theta)) = 1\), let \(c_x\) be a random variable on \(\Theta\) with cumulative distribution function \(x(\theta)\).\(^6\) By definition, \(x(\theta) = \mathbb{P}(\theta \geq c_x)\). Thus, the allocation rule \(x(\theta)\) can be implemented by drawing a cutoff from the distribution of \(c_x\), and giving the good to the agent if and only

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\(^6\) I abuse notation slightly because cdfs are defined on the entire real line while \(x\) is defined on \(\Theta\). This is a harmless abuse because \(x\) can be trivially extended to a step function on the real line.
if the reported type $\theta$ is greater than the realized cutoff. I will call $c_x$ a *random-cutoff representation* of $x$. Let $dx$ denote the probability mass function of $c_x$.

Conversely, fix a random variable $c$ distributed according to a cdf $G$ with support on $\Theta$. Then, $G(\theta)$ is a non-decreasing allocation rule on $\Theta$, and $c = c_G$. That is, $c$ is a random-cutoff representation of allocation $G$.

Therefore, there is a one-to-one correspondence between a subset of allocation rules and random cutoffs: non-decreasing allocation rules are cdfs of cutoff.

For a fixed allocation rule $x$, the support of random cutoff $c_x$ is the set of points in $\Theta$ at which $x$ increases strictly. In particular, degenerate (deterministic) cutoffs correspond to allocation rules that give the object with probability one to all types above some threshold. Although types and cutoffs are formally elements of the same space $\Theta$, I use $C \equiv \Theta$ to denote the space of cutoffs.

### 2.3.2 Definition of a cutoff mechanism

In a cutoff mechanism, the signal distribution depends only on the realization of the random cutoff representing the allocation rule, rather than on the reported type directly.

**Definition 2.** A mechanism frame $(x, \pi)$ is a *cutoff mechanism* if $x$ is non-decreasing, and the signal $\pi$ can be represented as

$$\pi(s|\theta)x(\theta) = \sum_{c \leq \theta} \gamma(s|c)dx(c),$$

for each $\theta \in \Theta$ and $s \in S$, for some signal function $\gamma : C \rightarrow \Delta(S)$.

In a cutoff mechanism $(x, \pi)$, the agent reports $\theta$, and the seller draws a cutoff $c$ from distribution $dx$. If $\theta \geq c$, the agent acquires the good in exchange for a transfer, and the designer draws a signal to be announced from the distribution $\gamma(\cdot|c)$. If $\theta < c$, the agent does not acquire the good (it is irrelevant whether and which signal is sent in this case). The signal is informative about the type of the agent because the third party conditions on the event that the agent acquired the good, i.e. $\theta \geq c$.

Although $\gamma$ is defined on the entire set $C \equiv \Theta$ for notational convenience, the properties of $\pi$ depend only on how $\gamma$ is defined on the support of $dx$. Signals sent conditional on realizations $c$ with $dx(c) = 0$ are irrelevant because they occur with probability zero.
2.3.3 Characterization of cutoff mechanisms

Proposition 1 establishes a key property of cutoff mechanisms.

Proposition 1. A cutoff mechanism is implementable for any prior distribution of types $f$.

I do not provide a proof of Proposition 1 because it follows from a more general Theorem 1 stated and proven in Section 3. However, the intuitive argument is simple. In a cutoff mechanism the report of the agent does not directly influence the signal. The agent can change the outcome only by manipulating the probability with which she acquires the good. Higher types value each ex-post outcome weakly more than low types – the continuation payoff $u(\theta; \bar{f})$ is non-decreasing in $\theta$. This implies a single-crossing property and hence existence of transfers that rule out profitable misreporting aimed at changing the probability of acquiring the good in the mechanism.

Implementability of cutoff mechanisms regardless of the distribution $f$ is reminiscent of why pivot mechanisms are dominant-strategy implementable. In a pivot mechanism, the report of an agent doesn’t influence the transfer the agent is paying, except when it changes the allocation. In a cutoff mechanism, the report doesn’t influence the signal, except when it changes the allocation.

I conclude the subsection with a simple example illustrating the analysis.

Example 1. Suppose that $\Theta = \{l, h\}$, $v(\theta) \equiv v$, $f = (f_l, f_h)$, and $\eta = 1$ (the third party makes an offer). It is without loss of generality to assume that the signal space is binary, $S = \{0, 1\}$, and that the third party quotes a price $l$ or $h$. For any allocation rule $x = (\beta, 1)$, $\beta \in [0, 1]$, I characterize the set of implementable signal structures. The random cutoff representation of $x$ is a binary variable with realizations $\{l, h\}$ and pmf $dx = (\beta, 1 - \beta)$.

Let $\pi_l = \pi(1|l)$ and $\pi_h = \pi(1|h)$ be the probabilities of sending signal 1 conditional on allocating the good to the low type and the high type, respectively. The pair $(\pi_l, \pi_h)$ fully determines the signal structure. Without loss of generality, I assume that 1 is the high signal, i.e. $\pi_l \leq \pi_h$.

Cutoff mechanisms. By inspection of Definition 2, a necessary and sufficient condition for $(x, \pi)$ be a cutoff mechanism under $x$ is that

$$\pi_l \beta \leq \pi_h, \quad (2.2)$$

$$1 - \pi_l \beta \leq \pi_h, \quad (2.3)$$
where condition (2.2) is vacuously satisfied given \( \pi_l \leq \pi_h \). The above conditions say that each of the two signals has a higher ex-ante probability of being sent when the type of the agent is high. The resulting set of signal structures implementable in a cutoff mechanism is depicted in Figure 2.1. Point \((0, 1)\) in that figure corresponds to a fully informative signal, and the point \((0, 1 - \beta)\) to full disclosure of the realized cutoffs. At the diagonal \( \pi_l = \pi_h \) the signal is uninformative.

**Fig. 2.1:** Signals implementable in cutoff mechanisms

**Fig. 2.2:** Signals implementable under distribution \( f = (1/2, 1/2) \)

**Fig. 2.3:** Signals implementable under distribution \( f = (3/4, 1/4) \)

**Fig. 2.4:** Signals implementable under distribution \( f = (1/4, 3/4) \)

*All implementable mechanisms.* Whether a mechanism frame \((x, \pi)\) is implementable depends on the reaction of second-stage prices to the information revealed by the mechanism. If the same price is quoted by the third party under the two signals, the mechanism frame is trivially implementable. Otherwise, the high price is quoted when the
signal realization is 1, and the low price when it is 0. Given \((x, \pi)\), Bayes’ rule determines posterior beliefs for every signal, and hence prices. By direct calculation, \((x, \pi)\) is implementable in this case if and only if

\[
\beta \pi_l h + \beta (1 - \pi_l) l - (\pi_h h + (1 - \pi_h) l) \geq \beta h - h.
\]

This condition is equivalent to (2.3). Therefore, a mechanism is not implementable if and only if (i) prices under two signals are different and (ii) the mechanism is not a cutoff mechanism. Figures 2.2 - 2.4 depict the regions where condition (i) holds, for three different priors (gray area).

If the prior is uniform, the region coincides with the set of non-cutoff mechanisms. Thus, only cutoff mechanisms are implementable. In two other cases, implementable mechanisms are a strict superset of cutoff mechanisms.

Implementability puts an upper bound on the informativeness of signal structures. The low type has a relatively higher preference for signals that lead to a high resale price but a high resale price can only occur under signals that are sent sufficiently often conditional on a high type. If signals are too informative, this difference in willingness to pay for signals cannot be undone with transfers. Cutoff mechanisms are implementable because they use the allocation function as a counter-balancing leverage. When the allocation rule is constant (\(\beta = 1\)), cutoff mechanisms reveal no information. The steeper the allocation function (lower \(\beta\)), the larger the set of \(\pi\) that can be implemented using a cutoff mechanism. Full separation of types is only possible when \(\beta = 0\), i.e. when the high type is the only holder of the good.

In Example 1, although non-cutoff mechanisms are sometimes implementable (for some prior distributions \(f\)), only cutoff mechanisms are implementable always. If the third party makes an offer with positive probability (\(\eta > 0\)), this is a general result.

**Proposition 2.** Suppose that \(\eta > 0\) and the lemons condition is locally slack. If \((x, \pi)\) is implementable for every distribution of types \(f\), then \((x, \pi)\) is a cutoff mechanism.

I first prove a lemma which provides an alternative characterization of cutoff mechanisms.

---

7 The two signals lead to different prices if

\[
(v - h)(\pi_l \beta f_l + \pi_h f_h) > (v - l)\pi_l \beta f_l, \tag{2.4}
\]

\[
(v - h)((1 - \pi_l)\beta f_l + (1 - \pi_h) f_h) < (v - l)(1 - \pi_l) \beta f_l. \tag{2.5}
\]
Lemma 1. A mechanism frame \((x, \pi)\) is implementable for every distribution of types \(f\) if and only if
\[
\pi(s|\theta)x(\theta) \text{ is non-decreasing in } \theta, \tag{M}
\]
for every signal \(s \in S\).

Condition (M) is a direct analog of the Myerson monotonicity condition from one-stage allocation problems. In the two-stage problem, implementability for every distribution requires that for any signal \(s\) the probability of allocating the good and sending signal \(s\) is non-decreasing in the type. Condition (M) implies Myerson monotonicity (it is enough to add up \(\pi(s|\theta)x(\theta)\) across all \(s\)). Note that Lemma 1 implies Proposition 1 because, by definition, \(\pi(s|\theta)x(\theta)\) is non-decreasing in \(\theta\) in cutoff mechanisms.

I sketch the proof of Lemma 1 below. The remaining details and the proof of Proposition 2 can be found in Appendix A.

Proof of Lemma 1. I first prove that condition (M) is necessary. It is enough to show that \(\pi(s|\theta)x(\theta) \geq \pi(s|\hat{\theta})x(\hat{\theta})\) for any two adjacent types \(\theta > \hat{\theta}\).

In Appendix A.1, I prove existence of a distribution \(f\) with the following properties: (i) when the third party makes an offer, she offers price \(\theta\) after seeing signal \(s\) if and only if \(\pi(s|\theta)x(\theta) \geq \pi(s|\hat{\theta})x(\hat{\theta})\); otherwise, she offers price \(\hat{\theta}\), (ii) when the agent makes an offer, the unique equilibrium conditional on \(s\) involves a constant price \(p^s \in [\theta, v(\hat{\theta})]\), and hence trade with probability one. The distribution \(f\) that achieves these two properties puts all mass on \(\{\hat{\theta}, \theta\}\), and is such that in the absence of additional information, the third party is indifferent between offering price \(\theta\) and \(\hat{\theta}\). The second property holds under \(f\) because \(v(\hat{\theta}) \geq \theta\), by assumption that the lemons condition is slack locally, and thus conditional expected gains from trade are positive regardless of the exact posterior beliefs of the third party.

Given a distribution \(f\) with properties (i) and (ii), we can observe that \(u(\theta; f^s) = u(\hat{\theta}; f^s)\) exactly when \(\pi(s|\theta)x(\theta) \geq \pi(s|\hat{\theta})x(\hat{\theta})\). In the opposite case \(\pi(s|\theta)x(\theta) <\)

\(^8\) Formally, \(\hat{\theta}\) is the largest type strictly smaller than \(\theta\) (which exists due to finiteness of \(\Theta\)).
\( \pi(s|\hat{\theta})x(\hat{\theta}) \), we have \( u(\theta; f^*) > u(\hat{\theta}; f^*) \) (the last observation relies on the assumption \( \eta > 0 \)). Inequality (2.6) becomes

\[
\sum_{\{s \in S : \pi(s|\theta)x(\theta) < \pi(s|\hat{\theta})x(\hat{\theta})\}} \alpha_s \left[ \pi(s|\theta)x(\theta) - \pi(s|\hat{\theta})x(\hat{\theta}) \right] \geq 0, \tag{2.7}
\]

where \( \alpha_s \equiv u(\theta; f^*) - u(\hat{\theta}; f^*) \) is strictly positive for each \( s \) in the summation. We have obtained that a sum of strictly negative terms is non-negative. This is only possible when the set of indices in the sum is empty: \( \{s \in S : \pi(s|\theta)x(\theta) < \pi(s|\hat{\theta})x(\hat{\theta})\} = \emptyset \). Thus, condition (M) holds for every signal \( s \).

To prove that condition (M) implies implementability, I use a condition for checking implementability in arbitrary type and allocation spaces from Dworczak and Zhang (2015).\(^9\) Given a set of types and their final allocations, the assignment is implementable if and only if the matching between types and final allocations is efficient (see Dworczak and Zhang, 2015, for details and formal definitions). Because \( u(\theta; f^*) \) is non-decreasing in \( \theta \) for any \( f^* \), matching efficiency is implied by pairwise stability: total surplus cannot be increased by swapping the allocations of some pair of types. Condition (2.6) is sufficient (and necessary) to ensure that \( \theta \) and \( \hat{\theta} \) cannot profitably swap their final allocations. Thus, it is enough to prove that inequality (2.6) holds for all \( \theta > \hat{\theta} \). The fact that \( u(\theta; f^*) \) is non-decreasing in \( \theta \) implies that the first square bracket is non-negative in each term of the sum in (2.6), and condition (M) implies the same about the second square bracket. Thus, inequality (2.6) always holds, regardless of the underlying prior distribution \( f \).

The assumption \( \eta > 0 \) implies that beliefs about the type of the agent influence the outcome of the aftermarket. In the opposite case, if the agent always makes the offer, it is possible (for example when \( v(\theta) \) is constant) that beliefs held by the third party are irrelevant, and hence each \( \pi \) is trivially implementable. When \( \eta = 0 \), Proposition 2 remains true if the lemons problem is severe enough so that beliefs about the type of the agent are always relevant.

**Proposition 2′.** Suppose that \( \eta = 0 \), the lemons condition binds locally, and a pure strategy is selected for the aftermarket. If \((x, \pi)\) is implementable for every distribution of types \( f \), then \((x, \pi)\) is a cutoff mechanism.

\(^9\) See Rochet (1987) for the classical formulation of the implementability condition in arbitrary type spaces.
The proof is fully analogous to the proof of Proposition 2 and thus omitted.

Lemma 1 can be used to extend the conclusion of Proposition 1. The proof of sufficiency of condition (M) only used the fact that \( u(\theta; f^*) \) is non-decreasing in \( \theta \). This implies that cutoff mechanism are implementable not only for every distribution of types but also for a large class of bargaining protocols in the aftermarket. For example, let \( \mathcal{P} : \Delta(\Theta) \to \mathbb{R} \) be an arbitrary mapping from posterior beliefs of the third party into a deterministic price quoted in the second stage. Then, \( u(\theta; f^*) \equiv 1_{\{\mathcal{P}(f^*) \geq v(\theta)\}} \max(\theta, \mathcal{P}(f^*)) + 1_{\{\mathcal{P}(f^*) < v(\theta)\}} \theta \) is non-decreasing in \( \theta \).

**Corollary 1.** A cutoff mechanism is implementable for any protocol \( \mathcal{P} \).

Propositions 2, 2′ and Corollary 1 imply the following conclusion.

**Corollary 2.** A mechanism frame \( (x, \pi) \) is implementable for all distributions \( f \), all non-decreasing valuation functions \( v \), and all \( \eta > 0 \) if and only if \( (x, \pi) \) is a cutoff mechanism.

### 2.4 Optimal cutoff mechanisms

In this subsection, I consider optimization in the class of cutoff mechanisms. The objective function of the mechanism designer is given by

\[
\sum_{\theta \in \Theta} \sum_{s \in \mathcal{S}} V(\theta; f^*) \pi(s | \theta) x(\theta) f(\theta),
\]

where

\[
V(\theta; f^*) = \mathbb{E}_{\sigma^f, \nu}(\theta, p) = \int_{\mathbb{R}} \nu(\theta, p) d\sigma^f(p),
\]

and \( \nu \) is an arbitrary upper semi-continuous function of the type of the agent and the price quoted in the second stage. For example, to maximize total surplus, set 

\[
\nu(\theta, p) = 1_{\{\theta \leq p \leq v(\theta)\}} v(\theta) + (1 - 1_{\{\theta \leq p \leq v(\theta)\}}) \theta.
\]

This specification allows for an arbitrary objective of the designer, as long as she is an expected-utility maximizer. In particular, it encompasses expected revenue maximization because the final allocation determines transfers.\(^{10}\) Throughout, I fix a selection \( \sigma^f \) from the set of Bayes Nash equilibria of the post-mechanism game, for any posterior belief \( \bar{f} \). If there are multiple equilibria for

\(^{10}\) With a continuous type space, this follows from the payoff equivalence theorem, see for example Milgrom (2001). I work with a finite type space in this section so payoff equivalence does not hold. However, Dworczak and Zhang (2015) show that for a fixed \( (x, \pi) \), the set of implementing transfers is a complete lattice with a unique highest element.
some $\tilde{f}$, the selection could be made by the mechanism designer (without influencing subsequent results).

### 2.4.1 Optimization over disclosure policies

In the first step of the derivation of optimal mechanisms, I treat the allocation function $x$ as given, and optimize over disclosure rules $\pi$ subject to $(x, \pi)$ being a cutoff mechanism.

An allocation function $x$ can be represented by a random cutoff $c_x$. In a cutoff mechanism, the signal only depends on the realization of the random variable $c_x$. By Proposition 1, each cutoff mechanism is implementable. Thus, because the objective function does not depend explicitly on transfers, we can ignore constraints (IC) and (IR) in the optimization problem. The mechanism design problem becomes a pure communication problem in which the designer chooses a disclosure policy of the random cutoff in order to induce the optimal distribution over posterior beliefs. This problem is formally equivalent to the Bayesian persuasion problem formulated by Kamenica and Gentzkow (2011) and first analyzed by Aumann and Maschler (1995).

For any posterior belief $G$ of the cutoff held by the third party (after observing some signal), let

$$f^G(\theta) \equiv \Pr_{c \sim G}(\theta \mid \theta \geq c) = \frac{G(\theta) f(\theta)}{\sum_\tau G(\tau) f(\tau)}, \quad (2.9)$$

be the corresponding posterior belief over the type of the agent (conditional on the agent acquiring the good). Equivalently, $f^G$ is the belief over the type of the agent held by the third party who believes that the mechanism designer implemented the allocation rule $G(\theta)$.

Next, let

$$\mathcal{V}(G) = \sum_{\theta \in \Theta} V(\theta; f^G) G(\theta) f(\theta) \quad (2.10)$$

be the conditional expected payoff to the mechanism designer conditional on inducing a posterior belief $G$ over the cutoff. Equivalently, $\mathcal{V}(G)$ is the expected payoff to the mechanism designer that would arise if the allocation function were $G$ (instead of the actual $x$) and the mechanism revealed no additional information to the third party. I assume that the selection $\sigma$ is such that $\mathcal{V}$ is upper-semi continuous.\(^{11}\)

---

\(^{11}\) Such a selection always exists because $V$ is upper semi-continuous, and the BNE correspondence is upper hemi-continuous. If the the selection is picking the designer-preferred equilibrium for every posterior belief, then $\mathcal{V}$ is upper semi-continuous.
Proposition 3. For every allocation function \( x \), the problem of maximizing (2.8) over \( \pi \) subject to \((x, \pi)\) being a cutoff mechanism is equivalent to

\[
\max_{\tau \in \Delta(\Delta(C))} \mathbb{E}_{G \sim \tau} V(G) \tag{2.11}
\]

subject to

\[
\mathbb{E}_{G \sim \tau} G(\theta) = x(\theta), \quad \forall \theta \in \Theta. \tag{2.12}
\]

Proposition 3 follows directly from the main result of Kamenica and Gentzkow (2011) but I nevertheless provide some discussion of the proof in Appendix A.3.

Equation (2.11) says that the mechanism designer seeks to maximize her expected payoff over distributions over posterior beliefs of the third party. Condition (2.12) is the so-called Bayes-plausibility constraint which states that the induced posterior beliefs over cutoffs have to average out to the prior belief (beliefs are represented by cdfs). Using the random-cutoff representation, the prior belief over cutoffs is simply the allocation function \( x \).

Formulation (2.11) - (2.12), together with the random-cutoff representation, yield an alternative interpretation of one-agent cutoff mechanisms. Any cutoff mechanism \((x, \pi)\) can be represented as a probability distribution \( \tau \) over implementable mechanism frames \((G, \emptyset)\) which reveal no information. Here, \( G \) is treated as an allocation rule, and \( \emptyset \) represents no announcement. The designer draws a mechanism \((G, \emptyset)\) from the distribution \( \tau \) and announces which mechanism is used but the mechanism itself reveals no information. Due to (2.12), the allocation rule \( x \) is implemented in expectation.\(^{12}\)

Proposition 3 implies that the concavification result known from Bayesian persuasion can be applied in my setting. Let \( \mathcal{X} \) be the set of all non-decreasing allocation functions on \( \Theta \).

Corollary 3. The optimal expected payoff to the mechanism designer in the problem (2.11)-(2.12) is equal to the concave closure of \( V \),

\[\text{co} V(x) \equiv \sup \{ z : (x, z) \in CH(\text{graph}(V)) \}, \]

where \( CH \) denotes the convex hull, and \( \text{graph}(V) \equiv \{ (\tilde{x}, \tilde{z}) \in \mathcal{X} \times \mathbb{R} : \tilde{z} = V(\tilde{x}) \} \).

I illustrate the above results with a simple numerical example.\(^{13}\)

\(^{12}\) Under this interpretation, Proposition 1 becomes trivial. Because each \( G \) is non-decreasing, the mechanism \((G, \emptyset)\) is trivially implementable. The mechanism can send signals before the agent reports!

\(^{13}\) The example uses a continuous type space in order to simplify calculations. Section 3 formally
Example 2. Suppose that $\eta = 1$, $v \equiv 1$, and $\theta$ is uniformly distributed on $[0, 1]$. Let $x(\theta) = \beta$ for $\theta < 2/3$, and $x(\theta) = 1$ for $\theta \geq 2/3$. I consider a binary allocation rule because it leads to a binary distribution of the cutoff allowing a graphical analysis. The designer maximizes total surplus over revelation policies using a cutoff mechanism.

The random-cutoff representation $c_x$ of the allocation function $x$ is a binary variable with realizations $\{0, 2/3\}$ and distribution $dx = (\beta, 1-\beta)$. Therefore, feasible posterior beliefs over cutoffs form a one-dimensional family indexed by the posterior probability that the cutoff is equal to 0 which I denote by $\alpha$. For each $\alpha$, let $x^\alpha(\theta) = \alpha$ for $\theta < 2/3$, and $x^\alpha(\theta) = 1$ for $\theta \geq 2/3$, be the corresponding allocation rule. The resulting posterior belief $f^{x^\alpha}$ over the type is given by (2.9) with $G \equiv x^\alpha$. Since the third party always makes the offer, the second-stage equilibrium is summarized by $p^\star(x^\alpha)$ – the optimal price quoted by the third party given belief $x^\alpha$ over cutoffs.\footnote{By direct calculation,}

\[
V(x^\alpha) = \int_0^1 \left(1_{\{p^\star(x^\alpha) \geq \theta\}} + 1_{\{p^\star(x^\alpha) < \theta\}} \theta\right) x^\alpha(\theta) f(\theta) d\theta.
\]

The function $V$ is strictly concave and increasing in $\alpha$ on $[0, 1/4]$, drops discontinuously at $\alpha = 1/4$,\footnote{This is because the price jumps at $\alpha = 1/4$, see footnote 14.} and is linear increasing on $[1/4, 1]$ (see Figure 2.5). The concave closure coincides with $V$ on $[0, 1/4]$, and is linear on $[1/4, 1]$.

If $\beta \leq 1/4$, $\text{co}V(x) = V(x)$, and thus the optimal mechanism reveals no information. In the opposite case, $\text{co}V(x) > V(x)$, the unique optimal cutoff mechanism releases information in the form of a binary signal which induces a posterior belief $\alpha = 1/4$, or $\alpha = 1$, with appropriate probabilities. This mechanism frame can be implemented as follows. A cutoff $c$ is drawn from the binary distribution $(\beta, 1-\beta)$ on $\{0, 2/3\}$. If $c = 0$, all types receive the good, and the mechanism discloses that $c = 0$ with conditional probability $\lambda$ that solves $1/4 = (1-\lambda)\beta/[(1-\lambda)\beta + (1-\beta)]$. With the remaining probability $1-\lambda$, no message is sent. If $c = 3/4$, the good is awarded to all types above $3/4$, and no message is sent. The probability $\lambda$ was chosen so that conditional on no message, the posterior probability of $c = 0$ is exactly $1/4$.\footnote{This shows that the results of this subsection hold with a continuous type space.}
2.4.2 Joint optimization

In the second step, I consider joint optimization over \((x, \pi)\) in the class of cutoff mechanisms. I say that a mechanism \((x, \pi)\) reveals no information if every signal realization \(s\) is uninformative about the type of the agent: \(\pi(s|\theta) = \pi(s|\hat{\theta})\) for all \(\theta, \hat{\theta} \in \Theta, s \in S\).

Proposition 4. The problem of maximizing (2.8) over \((x, \pi)\) subject to \((x, \pi)\) being a cutoff mechanism has an optimal solution that reveals no information.

Note that the conclusion of Proposition 4 holds regardless of the objective function.

Proof. By Corollary 3, the value to the designer at an optimal solution is \(\sup_{x \in \mathcal{X}} \text{co}V(x)\). By definition of the concave closure, \(\sup_{x \in \mathcal{X}} \text{co}V(x) = \sup_{x \in \mathcal{X}} V(x)\). An optimal solution exists because \(\text{co}V\) is an upper semi-continuous function on a compact set.\(^{16}\) By definition, \(V(x)\) is the expected payoff to the mechanism designer when \(x\) is the allocation function and the mechanism reveals no information.

To gain intuition for Proposition 4, recall that any cutoff mechanism can be interpreted as randomization over no-information-revealing, implementable mechanisms \((G, \emptyset)\), with the public message disclosing which of the mechanisms \((G, \emptyset)\) was used. Each \((G, \emptyset)\) induces a certain posterior belief in the aftermarket, and a conditional expected payoff to the mechanism designer. One of these mechanisms, denoted \((G^*, \emptyset)\),

\(^{16}\) \(\mathcal{X}\) is compact because it is a closed subset of \([0, 1]\) which is compact by Tychonoff’s theorem.
must yield the highest conditional expected payoff. The designer can weakly increase her ex-ante expected payoff by choosing \((x, \pi) = (G^*, \emptyset)\).

In other words, the mechanism design problem is a Bayesian persuasion problem in which the prior distribution (over cutoffs) can be chosen – hence, there is no need to induce a distribution over posteriors. I illustrate this point by revisiting Example 2.

**Example 3.** Consider the setting of Example 2. Suppose first that we can additionally optimize over \(\beta\), the probability that types below \(2/3\) receive the good. This corresponds to choosing the optimal binary prior distribution over cutoffs \(\{0, 2/3\}\), and means that we can choose an arbitrary point on the concave closure of \(V\) in Figure 2.5. The expected payoff to the mechanism designer is maximized at \(\beta = 1\). At \(\beta = 1\), the function \(V\) coincides with its concave closure, so no-revelation is optimal.

Now suppose that we can choose an arbitrary allocation function \(x\). By Proposition 4, we know that no revelation is optimal. It is enough to solve an unconstrained problem

\[
\max_{x \in X} V(x) = \int_0^1 \left( 1_{\{p^*(x) \geq \theta\}} + 1_{\{p^*(x) < \theta\} \theta}\right) x(\theta) f(\theta) d\theta,
\]

(2.13)

where

\[
p^*(x) \in \arg\max_p (1 - p) \int_0^p x(\theta) f(\theta) d\theta.
\]

The solution is \(x \equiv 1\), i.e. it is optimal to give the good to all types and reveal no information. The proof is non-trivial because (2.13) is an infinite-dimensional non-linear program due to the impact of \(x\) on the price \(p^*\). I do not provide the proof here. In Part II of the paper (TBA), I show that the above problem has the same solution even when we optimize in the class of all implementable mechanisms, and allow more general prior distribution over types. Because cutoff mechanisms are a subclass of implementable mechanisms, the above conclusion follows from that result.

Proposition 4 does not imply that the designer ignores the effect on the aftermarket when choosing the optimal mechanism. The choice of the allocation function, even in the absence of explicit signals, has a non-trivial impact on the information structure in the resale game because the third party conditions on the event that the agent acquired the good. In Example 3, it is optimal to give the good to all types which leads to a resale price of \(1/2\) (recall that \(f\) is uniform). However, if the designer allocated only to types above a threshold \(r\), the resale price would be \((1 + r)/2\). By not allocating to types \([0, r]\), the designer could change the information structure in the aftermarket,
and induce beneficial trade between the third party and types \([1/2, (1 + r)/2]\).

The conclusion of Proposition 4 fails in two cases that I will study later. First, when the mechanism designer has private information, she may decide to disclose it, even if the information is not directly relevant for the third party (see Subsection 5.1). Second, the result fails when there are multiple agents in the mechanism (see Subsection 3.3.2).

### 2.5 Information structures induced by cutoff mechanisms

Using a cutoff mechanism, the designer can induce an arbitrary distribution over posterior beliefs of the cutoff, as long as posterior beliefs average out to the prior belief according to the the Bayes-plausibility constraint (2.12). In this section, I characterize feasible distributions over posterior beliefs of the type of the agent induced by cutoff mechanisms. The characterization is useful in applications because it allows me to work directly with beliefs over the type which are more natural than the artificially derived cutoff.

I start with some auxiliary definitions. For a fixed allocation rule \(x\), I call \(f^x(\theta)\), defined by (2.9), the no-communication posterior. The no-communication posterior is the belief over the type of the agent, conditional on the agent acquiring the good, held by the third party when the allocation function is \(x\), and the mechanism reveals no information. Distribution \(f_1\) likelihood-ratio dominates distribution \(f_2\) (denoted \(f_1 \gtrsim_{\text{MLR}} f_2\)) if \(f_1(\theta)/f_2(\theta)\) is non-decreasing whenever it is defined.

**Proposition 5.** A finite-support distribution of beliefs \(\rho \in \Delta(\Delta(\Theta))\) is a conditional distribution of the posterior beliefs over the agent’s type (conditional on the agent acquiring the good) induced by a cutoff mechanism with allocation \(x\) if and only if

\[
\bar{f} \gtrsim_{\text{MLR}} f, \quad \forall \bar{f} \in \text{supp}(\rho).
\]

and

\[
\mathbb{E}_{\bar{f} \sim \rho} \bar{f}(\theta) \equiv \sum_{\bar{f} \in \text{supp}(\rho)} \bar{f}(\theta) \rho(\bar{f}) = f^x(\theta),
\]

The proof is in the appendix. Condition (2.15) is the standard Bayes-plausibility constraint, except that the posterior beliefs have to average out to the no-communication posterior, instead of to the prior. This is because the distribution over beliefs is taken to be conditional on allocating the good. Condition (2.14) is an additional constraint on posterior belief – each posterior has to likelihood-ratio dominate the prior.
Proposition 5 provides a new way of characterizing the optimal payoffs achievable to the mechanism designer. Let
\[
W(\tilde{f}) = \sum_{\theta \in \Theta} V(\theta; \tilde{f}) \tilde{f}(\theta)
\]
be the expected payoff to the mechanism designer that would arise if the prior distribution over types were \(\tilde{f}\), and the mechanism allocated to all types and revealed no information. Let \(M_f \equiv \{\tilde{f} \in \Delta(\Theta) : \tilde{f} \succ^{MLR} f\}\) be the set of distributions that likelihood-ratio dominate the prior \(f\). Note that \(M_f = \{f^x : x \in \mathcal{X}\}\).

**Proposition 6.** The optimal expected payoff to the mechanism designer in the problem (2.11)-(2.12) (for a fixed allocation \(x\)) is equal to
\[
\left(\sum_{\theta \in \Theta} x(\theta) f(\theta)\right) \text{co}^{M_f} W(f^x)
\]
(2.16)
where
\[
\text{co}^{M_f} W(f^x) \equiv \sup\{z : (f^x, z) \in \text{CH}(\text{graph}(W)|_{M_f})\},
\]
and \(\text{graph}(W)|_{M_f}\) is the graph of \(W\) restricted to domain \(M_f\).

The above characterization of the optimal payoff is analogous to the one from Corollary 3 but differs in that the concave closure is taken in the space of conditional distributions over beliefs over the type of the agent. I emphasize that the distribution over beliefs is conditional on allocating the good – in Corollary 3, I consider ex-ante distributions over beliefs (over the cutoff). In Proposition 6, I consider ex-post (conditional on allocating the good) distributions over beliefs (over the type). Hence, to obtain the expected payoff in Proposition 6, the conditional expected payoff \(\text{co}^{M_f} W(f^x)\) is multiplied by the unconditional probability of allocating the good, \(\sum_{\theta \in \Theta} x(\theta) f(\theta)\).

The above characterization is useful for several reasons. In Section 3, I use it to provide sufficient conditions for optimality of no and full revelation of the cutoff in the multiple-agent model. In Section 4, the result is used to solve an application to post-auction bargaining. In Appendix A.6, I show that Proposition 6 implies a clean characterization of the cost of requiring incentive compatibility in the design problem (the optimal payoff of the mechanism designer is compared to the hypothetical payoff she would receive if she had access to agent’s private information).
2.6 Application - Regulating an OTC market

I illustrate the theoretical results of this section by an application to regulating an over-the-counter market. The main goal is to show how the theory can be useful in guiding choices in a real-life problem. Transparency of financial OTC markets is an important topic in recent policy debate and a growing body of theoretical and empirical literature (see for example Bessembinder and Maxwell, 2008, Asquith, Covert and Pathak, 2013, Duffie, Dworczak and Zhu, 2015 and Asriyan, Fuchs and Green, 2015).

The seller and the agent are dealers in an OTC market. If the agent acquires the asset, with some probability \( \lambda \in (0, 1] \) she has a chance to resell the asset to a third party (who is, for example, an individual investor or a firm).\(^{17}\) The third party has a value for the asset that depends on the value of the agent but she does not observe the agent’s private information. However, she can observe the signal revealed ex-post in the transaction between the dealers – the disclosure rule corresponds to the informational transparency of the market.

The seller has a cost \( k \in [0, 1) \) of allocating the asset.\(^{18}\) I assume that \( \theta \) has a continuous distribution \( F \) with density \( f \) on \( [0, 1] \) which is regular.\(^{19}\) The value of the third party \( v(\theta) \) is strictly increasing with \( v(0) < k \) and \( v(1) > 1 \) (these assumptions rule out uninteresting boundary solutions). In the OTC market, it is more realistic to assume that the agent has full bargaining power, \( \eta = 0 \). However, the qualitative conclusions continue to hold when the third party has bargaining power. I restrict attention to pure-strategy equilibria in the post-mechanism game. Finally, I assume that when the seller allocates to all types above her cost, a price equal to 1 is not an equilibrium of the aftermarket:

\[
\int_k^1 (v(\theta) - 1)f(\theta)d\theta < 0. \tag{2.17}
\]

The assumption implies that the second-stage lemons problem will guide choices of information structures in the first-stage mechanism.

An alternative interpretation of the model, useful for the OTC application, is that probability \( x \) is a quantity of a perfectly divisible asset. The initial seller has quantity

---

\(^{17}\) The baseline model of Subsection 2.1 formally only allows \( \lambda = 1 \). The general model from Section 3 covers the case \( \lambda \leq 1 \).

\(^{18}\) This cost could be a value of the seller (outside option), the cost of intermediation, or a cost of acquiring the asset in some other market.

\(^{19}\) That is, the virtual surplus function \( J(\theta) \equiv \theta - (1 - F(\theta))/f(\theta) \) is increasing. See Section 3 for a formal extension of the model to a continuous type space.
normalized to 1, and all players’ values are linear in quantity.

### 2.6.1 Regulator’s preferred mechanism

In this subsection, I assume that a regulator can choose a mechanism that the seller is obliged to implement. This is a strong assumption but constitutes a useful benchmark. In Appendix A.11, I analyze a problem where the regulator can only impose a disclosure rule, and show that similar conclusions hold. The regulator’s objective is to maximize efficiency.

If the allocation rule is \( x \), and the mechanism reveals no information, the price in the aftermarket is given by

\[
p(x) = \max \{ p \in [0, 1] : \int_0^p (v(\theta) - p) x(\theta) f(\theta) d\theta \geq 0 \}. \tag{2.18}
\]

If the seller sells her full quantity to all types above her cost, the price is less than 1, and hence some gains from trade are lost due to the lemons problem. Suppose that the seller asks the agent to report the type in the mechanism, and then publicly announces the report. Such mechanism, if truthful, solves the lemons problem, and leads to full efficiency. However, this is not a cutoff mechanism – for some distribution of types, the agent would find it profitable to misreport, regardless of transfers.\(^{20}\) In fact, if the allocation rule is a threshold rule (allocates full quantity to all types above a threshold), no information can be revealed by a cutoff mechanism, because the random cutoff is degenerate (deterministic).

Information can be revealed if the mechanism screens types by offering different quantities for sale. In this case, \( x \) is non-constant, the distribution over cutoffs is no longer degenerate, and the mechanism sends messages of the form “quantity sold was at least \( x \)”. Asquith et al. (2013) analyze consequences of introducing informational requirements (TRACE) in the corporate bond market which forced dealers to reveal prices and exact quantity (up to a cap) after each transaction. The analysis of cutoff mechanisms implies that is is not always possible to reveal so much information. At least in some cases, the dealers would be forced to use a mixed-strategy (if the IC constraint is violated), or to leave the market (if the IR constraint is violated) to protect their private information. This is consistent with the conclusions of Asquith et al. (2013) who show that the volume of trade went down in many segments of the market. They

\(^{20}\) In Part II of the paper (TBA), I show that in fact this conclusion holds for all distributions, i.e. such mechanism is never implementable.
also provide anecdotal evidence that dealers found trading more difficult after TRACE was introduced.

The designer faces a complicated trade-off between costly screening (reducing quantity sold to lower types) and lower information asymmetry in the aftermarket. The trade-off would be difficult to resolve if not for Proposition 4 (and its extension to continuous type spaces found in Subsection 3.3.2) which guarantees existence of an optimal mechanism which reveals no information. Since no explicit announcements are made in the optimal mechanism, it is enough to solve the unconstrained problem

\[
\max_{x \in X} V_{\text{eff}}(x) \equiv \max_{x \in X} \int_0^{p(x)} [\lambda v(\theta) + (1 - \lambda)\theta - k] x(\theta) f(\theta) d\theta + \int_{p(x)}^1 (\theta - k)x(\theta) f(\theta) d\theta,
\]

where \( p(x) \) is given by (2.18). The problem is still non-trivial because the objective function is non-linear in \( x \), due to its impact on the price in the aftermarket.

**Proposition 7.** The problem of maximizing \( V_{\text{eff}}(x) \) over \( x \in X \) admits a solution of the form \( x(\theta) = 1_{\{\theta \geq r^*_{\text{eff}}\}} \) for some \( r^*_{\text{eff}} \in [0, 1] \).

The proof and the definition of \( r^*_{\text{eff}} \) can be found in Appendix A.7. The optimal scheme is a posted-price mechanism with all quantity offered for sale.\(^{21}\) The price is chosen by optimally trading off the losses from not allocating the asset to low types against higher realized gains from trade in the aftermarket.

### 2.6.2 Seller’s preferred mechanism

I now analyze how the market outcome differs in the absence of regulation, i.e. when the seller chooses a profit-maximizing cutoff mechanism. The problem of maximizing revenue admits representation (2.8), because \((x, \pi)\) pins down agent’s information rents, and hence transfers (see Appendix A.8 for a detailed derivation of the objective function). Thus, I can apply Proposition 4 again, and derive the following result.

**Proposition 8.** The profit-maximizing mechanism reveals no information and allocates the good to all types above a threshold: \( x(\theta) = 1_{\{\theta \geq r^*_{\text{rev}}\}} \) for some \( r^*_{\text{rev}} \in [0, 1] \).

Moreover, \( r^*_{\text{eff}} \leq r^*_{\text{rev}} \) with equality if and only if \( p(1_{\{\theta \geq r^*_{\text{eff}}\}}) = 1 \) and \( \lambda \geq \lambda^* \) for some \( \lambda^* < 1 \). That is, the welfare- and profit-maximizing mechanisms coincide when trade

\(^{21}\) I emphasize that the boundary solution for quantity in the optimal mechanism does not follow solely from the assumption that utilities are linear in quantity – equation (2.19) is not linear in quantity. Rather it follows from the monotone structure of the problem (see the proof of Proposition 7).
happens in the aftermarket with probability one conditional on the third party being present, and the third party is present with sufficiently high probability.

The proof and the definitions of $r_{\text{rev}}^*$ and $\lambda^*$ can be found in Appendix A.9.

### 2.6.3 Conclusions

Both the efficient and the profit-maximizing schemes are posted-price mechanisms which reveal no information. The threshold (price) can be either (i) strictly lower for the efficient mechanism, or (ii) the same in both mechanisms.

In the less surprising case (i), the profit-maximizing seller excludes types $\theta \in [r_{\text{eff}}^*, r_{\text{rev}}^*]$ in order to reduce information rents of the agent. This lowers total surplus. In case (ii), the mechanism chosen by the seller also achieves maximal total surplus. There is no trade-off between efficiency and revenue. Such outcome is possible when the lemons problem is not too severe, so that the probability of trade in the aftermarket is high (which is why a high $\lambda$ is needed). In this case, the agent acts primarily as an intermediary, and thus the seller can extract rents without excluding low types. In Appendix A.10, I illustrate the above discussion with a numerical example.

The above analysis implies that increased transparency is not necessarily desirable from an efficiency viewpoint. However, the results should be properly interpreted. The model only considers revealing the information that the seller does not initially have, i.e. the private information of the agent elicited by the mechanism. Duffie et al. (2015), in a different model, analyze disclosure of information initially controlled by the seller and show that transparency typically improves welfare. In Section 5.1, I extend the model to allow for private information of the seller, and show that information disclosure may be optimal in that case.

### 2.7 Flexibility and Robustness

In this subsection, I discuss some practical issues related to cutoff mechanisms and their properties.

**Definition 3.** A mechanism frame $(x, \pi, t)$ is flexible if $(x, \pi)$ is implementable for every distribution of types.

Flexibility of a mechanism means that for every distribution of types, there exist transfers (which may depend on the distribution and other details) which implement
the mechanism frame. Under conditions, flexibility is the defining property of cutoff mechanisms.

Three examples demonstrate the usefulness of this property in real-life mechanism design problems. First, unlike in the traditional theoretical approach, many practical situations require one mechanism to handle multiple instances of the problem. Consider designing informational requirements for a financial over-the-counter (OTC) market. The regulator cannot condition the design on the distribution of types and other details which might vary across different dealer-customer interactions. In a particular instance of the problem, a dealer in the OTC market (seller) might have a good estimate of the distribution of values of a visiting buyer (e.g. observes whether the buyer is an individual customer or a large hedge fund) but the regulator does not have access to that information. Flexibility means that the dealer can find prices that implement the recommended policy in every instance of the problem. Many auction houses use the same design across thousands of auctions for diverse items, despite the fact that the distribution of values clearly depends on the characteristics of a particular item.

Second, even if the mechanism is intended for a particular one-time problem, practical considerations often force the designer to design the mechanism in steps. In the design of big spectrum auctions (e.g. the Incentive Auction in the US), major parts of the mechanism have to be determined and fixed long before the implementation to give time to regulators to approve it, and participants to understand it and voice concerns. Closer to implementation, as more information about the problem may arrive, minor adjustments are possible but the designer is committed to the major part of the design. In this context, flexibility can be seen as a modeling approach in which the mechanism frame is the major part, and transfers can be adjusted.

Third, it is a desirable property of mechanisms that truthful equilibria not only exist but are easily seen as being optimal (see for example Li, 2015). With this respect, failure of flexibility means that proving that a mechanism is implementable necessarily requires some knowledge of the underlying distribution of types. To the extent that the mechanism designer cannot be certain of some restrictions on possible distributions, mechanisms that are not in the class of cutoff mechanisms may be of limited usefulness in practice.
Relation to robustness

Flexible implementation allows transfers to be a function of the distribution of types. For the sake of discussion in this paper, I informally define an even stronger notion of robust implementation.

**Definition 4.** A mechanism frame \((x, \pi)\) is robustly implementable if there exists an indirect mechanism (whose description does not depend on the distribution of types) which implements \((x, \pi)\) for every distribution of types.

Under robust implementation, the designer does not need to know the distribution of types to implement \((x, \pi)\). For a given distribution, by the Revelation Principle, every indirect mechanism is equivalent to some direct mechanism. If an indirect mechanism implements \((x, \pi)\) for every distribution, it has to be that there always exist transfers that implement \((x, \pi)\) in a direct mechanism. Thus, flexibility is as a necessary condition for robust implementation.

Robustness is a desirable property for practical purposes but may be cumbersome to work with in optimization. Flexibility is a much easier condition to work with, and it often turns out that the optimal cutoff mechanisms can be implemented in a robust way (see Subsection 4.2 for a practical example, and Subsection 5.3 for a general discussion). In any case, robust mechanisms are a subclass of flexible (cutoff) mechanisms, so a designer interested in robustness (e.g. not knowing the distribution) has no reason to look beyond the class of cutoff mechanisms.

Direct versus indirect mechanisms

In discussing flexibility and robustness, I have focused on the case when the mechanism designer wants to implement the same mechanism frame \((x, \pi)\) regardless of the distribution of types. For a fixed distribution \(f\), focusing on mechanism frames as a primitive description of mechanisms is without loss of generality (by the Revelation Principle). However, when the mechanism is fixed but the distribution \(f\) varies, it might be natural to look at other representations of mechanisms. For example, in the case when the post-mechanism interaction is a resale game, the designer might want to implement the same final (ex-post) allocation regardless of the distribution of types. This approach

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22 The notion of robustness defined in this paper is still weaker than some other notions of robustness from the literature which might additionally require that the equilibrium is independent of the higher-order beliefs of agents, see for example Bergemann and Morris (2005).
could lead to a different theory of flexible mechanisms. The advantage of considering mechanism frames is that their description is independent of the description of the aftermarket. In Section 3, I allow an arbitrary game in the aftermarket and the final allocation of the good may not be enough to describe the final outcome. Mechanism frames remain a valid description of the first-stage mechanism regardless of the form of the aftermarket.

Alternatively, a designer might want to fix an indirect mechanism, allowing the allocation and disclosure rule to be endogenously determined along with the varying distribution $f$. Optimization in the class of all indirect mechanisms is intractable. Traction can be gained by restricting the set of feasible indirect mechanisms to a small class. See Dworczak (2015) for an example of such analysis.

### 3 General Model

In this section, I extend the simple model of Section 2 by allowing (i) a general aftermarket, (ii) multiple agents in the mechanism, (iii) continuous distributions of types, and (iv) a more general definition of flexibility. The key assumptions maintained in the general model are that (i) the designer allocates a single object, and (ii) only the agent who acquired the good participates in the aftermarket.\(^{23}\)

The mechanism designer owns an indivisible object that she can allocate to one of $N$ agents. $N$ also denotes the set of agents. If agent $i$ acquires the object, she participates in the post-mechanism game described below. Agent $i \in N$ has a type $\theta_i \in [0, 1]$. Types are distributed according to a prior joint distribution with density $f$ on $[0, 1]^N$, with marginals $f_i$. Let $\Theta_i = \text{supp}(f_i)$, and $\Theta \equiv \times_{i \in N} \Theta_i$. Throughout, bold symbols denote vectors, in particular $\theta \equiv (\theta_1, \theta_2, ..., \theta_N)$ and $\theta_{-i} \equiv (\theta_1, ..., \hat{\theta}_i, ..., \theta_N)$. I consider two cases.\(^{24}\)

1. Continuous distribution. Each $\Theta_i$ is an interval, and $f$ is a density with respect to the Lebesgue measure on the Borel $\sigma$–field of $\Theta$. I denote the set of all such distributions by $\mathcal{F}_c$.

2. Discrete distribution. Each $\Theta_i$ is finite, and $f$ is a density with respect to discrete uniform measure on $\Theta$. The set of all such distributions is $\mathcal{F}_d$.

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\(^{23}\) Assumption (ii) is relaxed in Subsection 5.4.

\(^{24}\) The theory to be developed works for more general distributions but the two cases seem to be sufficient for practical purposes and allow me to simplify exposition.
I adopt the convention that for a set $X$, function $g$ on $X$, and discrete density $f$ on $X$,

$$\int_X g(x) f(x) dx \equiv \sum_{x \in \text{supp}(f)} g(x) f(x).$$

With this notation, I will not distinguish between the case of continuous and discrete distributions, and unless explicitly stated, all claims pertain to both cases.

A direct mechanism is a tuple $(x, \pi, t)$, where $x : \Theta \to [0, 1]^N$ is an allocation function with $\sum_{i \in N} x_i(\theta) \leq 1$, for all $\theta$, $\pi : \Theta \to \Delta(S)^N$ is a signal function with signal space $S$ (endowed with a respective $\sigma$-field) and $t : \Theta \to \mathbb{R}^N$ is a transfer function. If agent $i$ reports $\hat{\theta}_i$, and other agents report truthfully, she receives the good with probability $x_i(\hat{\theta}_i, \theta_{-i})$ and pays $t_i(\hat{\theta}_i, \theta_{-i})$. Conditional on allocating the good to agent $i$, the designer draws and publicly announces a signal $s \in S$ according to distribution $\pi_i(\cdot | \hat{\theta}_i, \theta_{-i})$. No other signal is sent. To make sure that integrals are well-defined in the continuous case, I assume that $x_i(\theta)$ and $\pi_i(\cdot | \theta)$ are measurable functions of $\theta$, for any measurable set $S \subseteq S$, for all $i$.

If the distribution $f$ is continuous, it is convenient to equate mechanisms that differ on a measure-zero set of type profiles. I will not distinguish between mechanisms $(x, \pi, t)$ and $(x', \pi', t')$ if $x(\theta) = x'(\theta)$, $\pi(\cdot | \theta) \equiv \pi'(\cdot | \theta)$, and $t(\theta) = t'(\theta)$, for almost all $\theta$ with respect to Lebesgue measure. Such mechanisms are identical from an ex-ante perspective for a Bayesian agent. Consequently, “for all types” should be interpreted as “for almost all types” when the distribution $f$ is continuous. Another consequence is that profitable deviations are allowed for a measure-zero set of types of any agent.

I call a mechanism simple if the signal space $S$ is finite. Looking at simple mechanisms is with loss of generality when $\Theta$ is infinite but it greatly simplifies exposition and intuition. I consider mechanisms with infinite signal spaces in Subsection 3.1.2. For simple mechanisms, $\pi_i(s | \theta)$ is well-defined as the probability of sending signal $s$ conditional on agent $i$ winning and report profile $\theta$.

For sake of generality, I do not explicitly assume that there is a third party in the aftermarket. Instead, the post-mechanism game is described in reduced form by the conditional expected payoffs it generates for the agent given the information revealed by the mechanism. Formally, the aftermarket $A$ is a collection of payoff functions

$$A \equiv \{ u_i(\theta; \bar{f}, e) : \theta \in \Theta_i, \bar{f} \in \Delta(\Theta_i), e \in E \}_{i \in N},$$

where $u_i(\theta; \bar{f}, e)$ is the conditional expected payoff to agent $i$ with type $\theta \in \Theta_i$ condi-
tional on holding the good. The payoff function \( u_i \) is indexed by two variables: \( \bar{f} \) which denotes a belief over agent \( i \)'s type, and \( e \in \mathcal{E} \). I call \( e \) an “environment”; \( \mathcal{E} \) is the set of environments. An environment may include any payoff-relevant features of the post-mechanism game such as parameters governing the bargaining protocol, characteristics of the third-party players, etc. For each \( i \) and \( e \), \( u_i(\theta; \bar{f}, e) \) is assumed to be bounded, measurable in \( \theta \), and upper semi-continuous in \( \bar{f} \) (in the weak* topology on the space of distributions).

The “black-box” approach to modeling the aftermarket implicitly entails the following assumptions. A game is played after the mechanism between agent \( i \) who acquired the good (whose identity becomes known), and some number of third-party players. Third-party players have the same prior belief \( f \) of the agents’ types, and observe the public signal \( s \) sent by the mechanism which leads to a posterior belief over agent \( i \)'s type denoted \( \bar{f} = f_i^s \). Given belief \( \bar{f} \) and environment \( e \), the game has an equilibrium which yields an expected payoff \( u_i(\theta; \bar{f}, e) \) to type \( \theta \) of agent \( i \). If there are multiple equilibria, the function \( u_i \) encodes equilibrium selection (in applications, equilibrium selection can be included as part of the design problem). By assumption, the signal sent by the mechanism influences the aftermarket only through the posterior belief \( \bar{f} \). Other roles of the signal (for example, as a coordination device) can be incorporated into the model by considering an appropriate equilibrium concept (e.g. a version of correlated equilibrium, see Bergemann and Morris, 2016a).

### 3.1 Implementability and cutoff mechanisms

To define implementability in the multiple-agent model, I first consider dominant-strategy implementation. Bayesian implementation is discussed in Subsection 3.2.

Let \( f_i^s \) be the belief over the type of the agent \( i \) who acquired the good, conditional on signal \( s \), given prior joint belief \( f \), under a mechanism frame \((x, \pi)\), assuming truthful reporting. For simple mechanisms,

\[
f_i^s(\tau) = \frac{\int_{\Theta_{-i}} \pi_i(s | \tau, \theta_{-i}) x_i(\tau, \theta_{-i}) f_{-i}(\theta_{-i} | \tau) d\theta_{-i}}{\int_{\Theta} \pi_i(s | \theta) x_i(\theta) \bar{f}(\theta) d\theta}, \quad \forall \tau \in \Theta_i, \tag{3.1}
\]

where \( f_{-i}(\theta_{-i} | \tau) \) is the conditional density of profile \( \theta_{-i} \) conditional on \( \theta_i = \tau \).

**Definition 5.** A mechanism frame \((x, \pi)\) is dominant-strategy implementable if there exist transfers \( t \) such that agents participate and report truthfully in the first-stage
mechanism, taking into account the continuation payoff from the aftermarket:

\[
\int_S u_i(\theta_i; f_i^s, c) d\pi_i(s|\theta_i, \theta_{-i}) x_i(\theta_i, \theta_{-i}) - t_i(\theta_i, \theta_{-i}) \geq 0, \quad (IR)
\]

\[
\theta_i \in \arg\max_{\hat{\theta}_i \in \Theta_i} \int_S u_i(\theta_i; f_i^s, c) d\pi_i(s|\hat{\theta}_i, \theta_{-i}) x_i(\hat{\theta}_i, \theta_{-i}) - t_i(\hat{\theta}_i, \theta_{-i}), \quad (IC)
\]

for all \(i \in N\), \(\theta_i \in \Theta_i\), and \(\theta_{-i} \in \Theta_{-i}\).

To define cutoff mechanisms for the general model, I let \(C_i \equiv \Theta_i \cup \{\hat{\theta}_i\}\) be the space of cutoffs for agent \(i\). Element \(\hat{\theta}_i\) is an artificial type larger than any \(\theta_i \in \Theta_i\). It is included to allow the possibility that the highest type does not receive the good with probability one. Suppose that the interim allocation rule \(x_i(\theta_i, \theta_{-i})\) is non-decreasing in \(\theta_i\) for any \(\theta_{-i}\), a property that is necessary and sufficient for implementability in the absence of the aftermarket (Myerson, 1981). A non-decreasing function is continuous almost everywhere, and thus there exists a non-decreasing, right-continuous \(x'_i(\theta_i, \theta_{-i})\) which differs from \(x_i(\theta_i, \theta_{-i})\) on a measure-zero set of types \(\theta_i\). Because I equate mechanisms that differ on measure-zero set of types, I can without loss of generality assume that \(x_i(\theta_i, \theta_{-i})\) is right-continuous. Thus, \(x_i(\theta_i, \theta_{-i})\) can be extended to a cumulative distribution function on \(C_i\) by defining \(x_i(\hat{\theta}_i, \theta_{-i}) = 1\). The random variable defined by this cdf is the random-cutoff representation of \(x_i(\theta_i, \theta_{-i})\) (see Subsection 2.3.1). I will denote the distribution of the random cutoff by \(dx_i(\cdot; \theta_{-i})\).

For any measurable function \(g\) on \(C_i\), \(\int g(c) dx_i(c, \theta_{-i})\) denotes the Lebesgue integral of \(g\) with respect to the distribution of cutoffs induced by the interim allocation rule \(x_i(\theta_i, \theta_{-i})\) on \(C_i\). Because the allocation for agent \(i\) depends on the reports of other agents, the distribution of cutoffs depends on \(\theta_{-i}\).

**Definition 6.** A mechanism frame \((x, \pi)\) is a *simple cutoff mechanism* if \(S\) is finite, \(x_i(\theta_i, \theta_{-i})\) is non-decreasing in \(\theta_i\) for all \(\theta_{-i}\), and the signal function \(\pi_i\) can be represented as

\[
\pi_i(s|\theta_i, \theta_{-i}) x_i(\theta_i, \theta_{-i}) = \int_0^{\theta_i} \gamma_i(s|c, \theta_{-i}) dx_i(c, \theta_{-i}), \quad (3.2)
\]

for each \(\theta_i \in \Theta_i\), \(\theta_{-i} \in \Theta_{-i}\), and \(s \in S\), for some measurable signal function \(\gamma_i : C_i \times \Theta_{-i} \rightarrow \Delta(S)\), for all \(i \in N\).

In the multiple-agent setting, a cutoff mechanism reveals information about the random cutoff for every fixed report profile of other agents. The signal sent when agent

\[\int_S u_i(\theta_i; f_i^s, c) d\pi_i(s|\theta_i, \theta_{-i}) = \sum_{s \in S} u_i(\theta_i; f_i^s, c) \pi_i(s|\theta_i, \theta_{-i}).\]

\(25\) For simple mechanisms, \(\int_S u_i(\theta_i; f_i^s, c) d\pi_i(s|\theta_i, \theta_{-i}) = \sum_{s \in S} u_i(\theta_i; f_i^s, c) \pi_i(s|\theta_i, \theta_{-i}).\)
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\( i \) is the winner can also depend on the reports of all other agents. If the random cutoff is degenerate conditional on \( \theta_{-i} \), for example \( x_i(\theta_i; \theta_{-i}) = 1_{\{\theta_i \geq z(\theta_{-i})\}} \) for some function \( z \), then \( \pi_i(s|\theta_i; \theta_{-i}) = \gamma_i(s|z(\theta_{-i}), \theta_{-i}) \), for all \( \theta_i \geq z(\theta_{-i}) \), i.e. the signal depends only on the profile of reports, except the report of agent \( i \).

The next definition generalizes the flexibility property of cutoff mechanisms (see Subsection 2.7). I use \( \mathcal{F} \) to denote a generic subset of distributions.

**Definition 7.** A mechanism frame \((x, \pi)\) is flexible with respect to \((\mathcal{F}, \mathcal{E})\), if \((x, \pi)\) is implementable for any prior distribution \( f \in \mathcal{F} \) and any environment \( e \in \mathcal{E} \).

In Section 2, flexibility was defined with respect to \( \mathcal{F} = \Delta(\Theta) \). Corollary 1 states that in the resale model of Section 2, cutoff mechanisms are also flexible with respect to \( \mathcal{E} = \mathbb{R}^{\Delta(\Theta)} \), where each environment \( e \in \mathcal{E} \) corresponds to a market-protocol \( \mathcal{P} \).

The following definition generalizes the property of the aftermarket necessary to obtain flexibility of cutoff mechanisms.

**Definition 8 (Monotonicity).** The aftermarket \( \mathcal{A} \) is monotone (under \( \mathcal{E} \)), if for any agent \( i \in N \), any belief \( \bar{f} \in \Delta(\Theta_i) \), and any environment \( e \in \mathcal{E} \), the expected utility function \( u_i(\theta; \bar{f}, e) \) is non-decreasing in \( \theta \).

**Theorem 1.** For any set of environments \( \mathcal{E} \), if \( \mathcal{A} \) is monotone under \( \mathcal{E} \), a simple cutoff mechanism is implementable for any prior distribution \( f \) and any environment \( e \in \mathcal{E} \).

To state the converse part, I impose a richness condition which is a joint condition on the aftermarket and the set of prior distributions and environments.

**Definition 9 (Richness).** The tuple \((\mathcal{F}, \mathcal{E}, \mathcal{A})\) satisfies richness if for any simple mechanism frame \((x, \pi)\), all \( i \in N \), all types \( \theta_i > \hat{\theta}_i \) and \( \theta_{-i} \), there exists a prior distribution \( f \in \mathcal{F} \) and an environment \( e \in \mathcal{E} \) such that

\[
\pi_i(s|\theta_i, \theta_{-i})x_i(\theta_i, \theta_{-i}) < \pi_i(s|\hat{\theta}_i, \theta_{-i})x_i(\hat{\theta}_i, \theta_{-i}) \implies u_i(\theta_i; f_i^*, e) > u_i(\hat{\theta}_i; f_i^*, e), \tag{3.3}
\]

\[
\pi_i(s|\theta_i, \theta_{-i})x_i(\theta_i, \theta_{-i}) > \pi_i(s|\hat{\theta}_i, \theta_{-i})x_i(\hat{\theta}_i, \theta_{-i}) \implies u_i(\theta_i; f_i^*, e) \leq u_i(\hat{\theta}_i; f_i^*, e). \tag{3.4}
\]

The richness condition requires that continuation payoffs are sensitive to the information revealed by the mechanism. The premise in condition (3.3) can be interpreted as “bad news” about the agent’s type – after observing a signal \( s \) that satisfies the left-hand side inequality (assuming that \( \theta_{-i} \) is known), the posterior probability of the lower type \( \hat{\theta}_i \) increases. Under some prior distribution \( f \) and environment \( e \), Richness
requires that the expected payoff of the higher type $\theta_i$ strictly exceeds the expected payoff of the lower type $\hat{\theta}_i$ when the mechanism sends “bad news”. The reverse inequality holds when the mechanism sends “good news” (condition 3.4).

For further intuition, consider the resale game of Section 2 in the one-agent model, and assume that the third party makes the offer. The resale game satisfies the richness condition (with $\mathcal{F} = \Delta(\Theta)$ and $\mathcal{E} = \emptyset$) because for any $\theta > \hat{\theta}$, we can find a prior distribution $f$ such that “bad news” induces a second-stage price $\hat{\theta}$, and “good news” induces a second-stage price $\theta$ (see the proof of Lemma 1 in Appendix A.1). The payoff of type $\theta$ strictly exceeds the payoff of type $\hat{\theta}$ conditional on signal $s$ if and only if type $\theta$ does not resell the good which happens exactly when the price is $\hat{\theta}$.

Remark 1. The richness condition can be relaxed by allowing $f$ to belong to the closure of $\mathcal{F}$, as long as $u_i$ is continuous along some sequence of distributions $f_n \in \mathcal{F}$ converging to $f$. This often simplifies demonstrating richness for continuous distributions.

Under the richness condition, I can prove the converse to Theorem 1.

**Theorem 2.** Suppose that a simple mechanism frame $(x, \pi)$ is flexible with respect to $(\mathcal{F}, \mathcal{E})$ which, together with the aftermarket $\mathcal{A}$, satisfy the richness condition. Then, $(x, \pi)$ is a (simple) cutoff mechanism.

**Corollary 4.** If $(\mathcal{F}, \mathcal{E}, \mathcal{A})$ satisfy Monotonicity and Richness, flexibility is a defining property of cutoff mechanisms in the set of simple mechanisms.

To prove Theorem 1 and Theorem 2, I use a lemma which provides an alternative characterization of flexibility via a monotonicity condition.

**Lemma 2.** Suppose that $\mathcal{A}$ is monotone under $\mathcal{E}$, and $(\mathcal{F}, \mathcal{E}, \mathcal{A})$ satisfy Richness. A simple mechanism frame $(x, \pi)$ is flexible with respect to $(\mathcal{F}, \mathcal{E})$ if and only if

$$\pi_i(s|\theta_i, \theta_{-i})x_i(\theta_i, \theta_{-i}) \text{ is non-decreasing in } \theta_i,$$

or all $s \in \mathcal{S}$, $\theta_{-i} \in \Theta_{-i}$.

The proofs of the above results are in Appendix B.

### 3.1.1 Examples

I present two examples of settings that satisfy the assumptions of the theory. The first example shows that the resale game of Section 2 can be extended to the multiple-
agent setting, and that it satisfies the richness condition with $\mathcal{F}$ being the class of all independent distributions.

**Example 4** (Resale). Suppose that agent $i$ who acquires the good in the mechanism plays a resale game with a third party, under the assumptions of Section 2 and $\eta > 0$. Assume (for now) that the value of the third party $v$ is constant, and larger than $\max_i \max(\Theta_i)$. Let $\mathcal{F}$ be the set of all joint distributions $f \in \Delta(\Theta)$ with independent marginals. Then, for any equilibrium selection for the resale game giving rise to expected payoff functions $u_i$, $\mathcal{A}$ is monotone, and $(\mathcal{F}, \mathcal{E} = \emptyset, \mathcal{A})$ satisfies richness. See Appendix B.4 for the proof.

If $(\mathcal{F}, \mathcal{E}, \mathcal{A})$ satisfy Richness, and $\mathcal{F} \subseteq \mathcal{F}'$, $\mathcal{E} \subseteq \mathcal{E}'$, then $(\mathcal{F}', \mathcal{E}', \mathcal{A})$ also satisfy Richness. Define $\mathcal{E}' = V \times [0, 1] \times [0, 1]$, with a typical element $e = (v, \eta, \lambda)$. The function $v : \Theta \to \mathbb{R}$ outputs the value of the third party as a function of the profile of types $\theta$, and $V$ is the set of functions which are non-decreasing in each variable. Parameter $\eta$ is the probability that the third party makes the offer, and $\lambda$ is the probability that the third party is present. Then, $(\Delta(\Theta), \mathcal{E}', \mathcal{A})$ satisfy Monotonicity and Richness. By Corollary 4, cutoff mechanisms are implementable for any distribution $f$ and any environment $e \in \mathcal{E}$, and no other mechanism has this property.26

As long as $\mathcal{F}$ includes all independent distributions, and $\mathcal{E}$ includes the game described in the first paragraph, the richness condition holds. Informally, if the designer thinks it is possible that the winner will resell her object to a high-value third-party and wants the mechanism to be flexible, she must use a cutoff mechanism.

**Example 5** (Buying a complementary good). The mechanism designer sells an item to one of $N$ bidders. For simplicity, assume that types of all agents come from the same type space $\Theta$. The winner buys a second (complementary) good in the after-market (examples include buying infrastructure after winning a spectrum license, or subcontracting in order to complete a project after winning a procurement auction). A third-party seller quotes a monopoly price which the agent can accept or reject. If agent $i$ with type $\theta_i$ acquires both goods, she obtains her full value $\theta_i$. If she doesn’t acquire the second good, she enjoys a reservation value $r(\theta_i)$, for some function $r : \Theta \to \mathbb{R}$ which is non-decreasing and satisfies $r(\theta) < \theta$ for all $\theta \in \Theta$. Let $\mathcal{R}$ be the set of all such functions $r$.26

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26 Functions $u_i$ generated by this game do not satisfy the assumption that only the belief over the type of the winner matters for the payoff. This is because the value of the third party depends on the entire type profile. Nevertheless, all previous results continue to hold in that case.
When $\mathcal{F}$ is the set of all joint product distributions on $\Theta^N$, and $\mathcal{E} = \mathcal{R}$ is the set of environments, the aftermarket is monotone (because for any price $p$ quoted by the third party, $\max\{r(\theta), \theta - p\}$ is non-decreasing in $\theta$), and $(\mathcal{F}, \mathcal{E}, \mathcal{A})$ satisfy richness. The proof, similar to the proof of the analogous claim in Example 4, is omitted. ■

3.1.2 Infinite Signal Spaces

In this subsection, I define and characterize cutoff mechanisms with infinite signal spaces. This case adds technical complications while offering few or no economic insights but is needed to formally consider full revelation in models with continuous type spaces. The details are relegated to Appendix B.5.

Because the signal matters only to the extent that it influences posterior beliefs, it is without loss of generality to assume $S \equiv \Theta$. Accordingly, $S$ is endowed with the same Borel $\sigma$-field as the type space.

Definition 10. A mechanism frame $(x, \pi)$ is a cutoff mechanism if $x_i(\theta_i, \theta_{-i})$ is non-decreasing in $\theta_i$ for all $\theta_{-i}$, and the signal function $\pi_i$ can be represented as

$$\pi_i(S|\theta_i, \theta_{-i})x_i(\theta_i, \theta_{-i}) = \int_0^{\theta_i} \gamma_i(S|c, \theta_{-i})dx_i(c, \theta_{-i}),$$

(3.5)

for each $\theta_i$, $\theta_{-i}$, and measurable $S \subseteq S$, for some signal function $\gamma_i : C_i \times \Theta_{-i} \rightarrow \Delta(S)$.

The only difference relative to a simple cutoff mechanism is that the signal space $S$ is allowed to be infinite, and condition (3.5) is required to hold for all measurable subsets $S$ of $\Theta$, instead of only for all singletons $\{s\}$.

In Appendix B.5, I show that a cutoff mechanism is implementable for all monotone aftermarkets and all prior distributions of types. Under a suitable richness condition, the converse conclusion holds. I also prove an approximation result: a mechanism frame is a cutoff mechanism if and only if it is the limit of simple cutoff mechanisms with the same allocation rule. The result provides a formal tool to connect the analysis of simple cutoff mechanisms to the general case.

3.2 The symmetric model

Having analyzed implementability in a very general setting, I specialize to a symmetric model to present results on Bayesian implementation, reduced-form mechanisms, and
optimal mechanisms.\(^{27}\)

I assume that \(\Theta_i = \Theta\), for all \(i\), and the prior distribution \(\mathbf{f}\) is a product distribution with identical marginals \(f\), with cdf \(F\). The aftermarket payoffs are independent of the identity of the winner, \(u_i \equiv u\), for all \(i\). Because agents are ex-ante identical, it is without loss of generality to focus on symmetric mechanisms. A symmetric mechanism \((x, \pi, t)\) consists of mappings \(x : \Theta^N \rightarrow [0, 1]\), \(\pi : \Theta^N \rightarrow \Delta(S)\), and \(t : \Theta^N \rightarrow \mathbb{R}\).

For each \(i\), \(x(\theta_i, \theta_{-i})\) is the probability that agent \(i\) with type \(\theta_i\) receives the good, \(^{28}\) \(d\pi(\cdot | \theta_i, \theta_{-i})\) is the probability distribution over signals conditional on agent \(i\) winning the object, and \(t(\theta_i, \theta_{-i})\) is the transfer paid by agent \(i\), given reports \(\theta_{-i}\) of other players. It is without loss of generality to take \(S \equiv \Theta\).

Given a symmetric mechanism frame \((x, \pi)\), its reduced form under distribution \(f\) is defined by

\[
x_f(\theta) = \int_{\Theta_{-i}} x(\theta, \theta_{-i}) f_{-i}(\theta_{-i}) d\theta_{-i},
\]

and

\[
\pi_f(S | \theta) x_f(\theta) = \int_{\Theta_{-i}} \pi(S | \theta, \theta_{-i}) x(\theta, \theta_{-i}) f_{-i}(\theta_{-i}) d\theta_{-i},
\]

for all measurable \(S \subseteq S\) and \(\theta \in \Theta\).\(^ {29}\)

**Definition 11.** A symmetric mechanism frame \((x, \pi)\) is Bayesian implementable under distribution \(f\) if there exists a transfer function \(t : \Theta \rightarrow \mathbb{R}\) such that its reduced form \((x_f, \pi_f)\) satisfies

\[
\int_S u(\theta; f^s, e) d\pi_f(s | \theta) x_f(\theta) - t(\theta) \geq 0, \quad \text{(BIR)}
\]

\[
\theta \in \text{argmax}_{\hat{\theta} \in \Theta} \int_S u(\theta; f^s, e) d\pi_f(s | \hat{\theta}) x_f(\hat{\theta}) - t(\hat{\theta}), \quad \text{(BIC)}
\]

for all \(\theta \in \Theta\), where \(f^s\) is the posterior belief over the winner’s type conditional on signal \(s\).

Under Bayesian implementation, only interim expected allocation and signal functions matter to agents. The second-stage posterior belief \(f^s\) depends only on the interim expected functions. If the designer’s preferences (to be specified) do not depend on the posterior beliefs over types of agents who didn’t acquire the good, it is without loss

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\(^{27}\) Most results extend directly to an asymmetric model but exposition becomes more cumbersome.

\(^{28}\) Since \(x\) is an allocation function, we have \(\sum_{i \in N} x(\theta_i, \theta_{-i}) \leq 1\), for all \(\theta \in \Theta\).

\(^{29}\) It is irrelevant how we define \(\pi_f(S | \theta)\) for \(\theta\) such that \(x_f(\theta) = 0\).
of generality to represent mechanisms by their reduced forms. This is captured by the following definition of (Bayesian) equivalence of mechanism frames.

**Definition 12.** Two symmetric mechanism frames \((x, \pi)\) and \((x', \pi')\) are (Bayesian) equivalent under \(f\) if they induce the same reduced form (up to relabeling of signals).

Instead of starting from a mechanism frame and deriving its reduced form, it is often convenient to work directly with reduced forms.

**Definition 13.** A pair \((\bar{x}, \bar{\pi})\) is a reduced-form mechanism frame under prior \(f\) if \(\bar{x} : \Theta \to [0, 1], \bar{\pi} : \Theta \to \Delta(S)\) for some signal space \(S\), and

\[
\bar{x}(\theta) = \int_{\Theta_{-i}} x(\theta, \theta_{-i}) f(\theta_{-i}) d\theta_{-i}, \tag{3.6}
\]

for some joint allocation rule \(x\).

A reduced-form mechanism frame \((\bar{x}, \bar{\pi})\) is only meaningful when we fix a prior distribution \(f\). Condition (3.6) ensures that the interim expected allocation rule is feasible under prior \(f\), i.e. induced by some joint allocation function \(x\).

**Definition 14.** A reduced-form mechanism frame \((\bar{x}, \bar{\pi})\) is a reduced-form cutoff mechanism if \(\bar{x}(\theta)\) is non-decreasing in \(\theta\), and the signal function \(\bar{\pi}\) can be represented as

\[
\bar{\pi}(S|\theta)\bar{x}(\theta) = \int_0^\theta \gamma(S|c) d\bar{x}(c), \tag{3.7}
\]

for each \(\theta\), and measurable \(S \subseteq S\), for some signal function \(\gamma : C \to \Delta(S)\).

The definition of a reduced-form cutoff mechanism is very similar to the definition of one-agent cutoff mechanisms from Section 2. Not surprisingly, every cutoff mechanisms induces a reduced-form cutoff mechanisms.

**Proposition 9.** For any symmetric cutoff mechanism \((x, \pi)\), and any prior distribution \(f\), \((x_f, \pi_f)\) is a reduced-form cutoff mechanism.

Proposition 9 becomes useful in conjunction with the next result which establishes that the converse statement is also true.

**Theorem 3.** For any reduced-form cutoff mechanism \((\bar{x}, \bar{\pi})\) under prior \(f\), there exists a symmetric cutoff mechanism \((x, \pi)\) such that \((x_f, \pi_f) = (\bar{x}, \bar{\pi})\).
Theorem 3 has the flavor of a BIC-DIC equivalence result which states that in some settings Bayesian and dominant-strategy implementation are equivalent if mechanisms are identified by their interim expected allocations.\(^\text{30}\) In my setting, the result says that if there exists a mechanism which induces a reduced-form cutoff mechanism (and hence is Bayesian implementable), then there exists an equivalent cutoff mechanism (which is hence dominant-strategy implementable). Compared to a standard BIC-DIC equivalence result, Theorem 3 strengthens the conclusion at the cost of strengthening the premise.

Proposition 9 and Theorem 3 imply that a designer interested in dominant-strategy implementation can restrict attention to reduced-form cutoff mechanisms which are much easier to work with. Using the famous characterization of reduced-form auctions developed by Matthews (1984) and Border (1991), I obtain the following corollary.

**Corollary 5.** A pair \((\bar{x}, \bar{\pi})\) is a reduced form of a cutoff mechanism under distribution \(f\) if and only if \(\bar{x}(\theta)\) is non-decreasing in \(\theta\), the signal function \(\bar{\pi}\) can be represented as in (3.7), and

\[
\int_{\tau}^{1} \bar{x}(\theta)f(\theta)d\theta \leq \frac{1 - F_{N}(\tau)}{N}, \quad \forall \tau \in \Theta.
\]

(M-B)

Equation (M-B) is the so-called Matthews-Border condition which ensures that the interim expected allocation \(\bar{x}\) can be induced by some joint allocation \(x\) under \(f\).

A major benefit of working with reduced forms is that the signal function becomes one-dimensional. In Subsection 3.1, I showed that a cutoff mechanism may reveal information about the entire vector \(\theta_{-i}\). The analysis of reduced-form mechanisms indicates that some of this information is redundant. As an example, let the allocation rule be \(x(\theta_{i}, \theta_{-i}) = 1_{\{\theta_{i} > \theta^{(1)}_{-i}\}}\), where \(\theta^{(1)}_{-i}\) denotes the first-order statistic of \(\theta_{-i}\). Consider the reduced form of \(x\) under prior \(f\), \(x_{f}(\theta) = F_{N}^{-1}(\theta)\). Under \(x_{f}\), the distribution of the cutoff in a reduced-form cutoff mechanism is simply the distribution of the first-order statistic of \(\theta_{-i}\), conditional on agent \(i\) acquiring the object.

**Corollary 6.** If the allocation rule is given by \(x(\theta_{i}, \theta_{-i}) = 1_{\{\theta_{i} > \theta^{(1)}_{-i}\}}\), then, up to equivalence, any cutoff mechanism implementing \(x\) only reveals information about the second highest reported type.

\(^{30}\) The classical reference is Manelli and Vincent (2010) but in the proof I use a more general approach introduced by Gershkov, Goeree, Kushnir, Moldovanu and Shi (2013). The main theorem of Gershkov et al. (2013) is not directly applicable because in my setting payoffs are not necessarily linear in types. However, their proof technique can be used to prove Theorem 3.
Corollary 6 could be proven directly by noting that $\theta_{-i}^{(1)}$ is a sufficient statistic for the posterior belief over the type $\theta_i$ of the winner. Corollary 5 implies that such a one-dimensional sufficient statistic can be found for any non-decreasing allocation rule $x(\theta)$. Up to equivalence, a cutoff mechanism only reveals information about the reduced-form cutoff with distribution determined by the interim expected allocation $x_f(\theta)$.

### 3.3 Optimal cutoff mechanisms

In this subsection, I consider optimization in the symmetric model in the class of cutoff mechanisms. The objective of the designer is given by

$$\sum_{i \in N} \int_\Theta \int_\mathcal{S} V(\theta_i; f_s^*) d\pi(s|\theta_i, \theta_{-i}) x(\theta_i, \theta_{-i}) f(\theta) d\theta,$$

where $V : \Theta \times \Delta(\Theta) \to \mathbb{R}$ is upper-semi continuous in the second argument (in the weak* topology on $\Delta(\Theta)$). The payoff of the mechanism designer depends only on the type of the agent who acquires the good, and on the posterior belief over that type.\(^{31}\)

Under the symmetry assumption, the objective function becomes

$$N \int_\Theta \int_\mathcal{S} V(\theta; f^*) d\pi_f(s|\theta) x_f(\theta) f(\theta) d\theta,$$

where $(x_f, \pi_f)$ is the reduced form induced by $(x, \pi)$.

#### 3.3.1 Optimization over disclosure policies

Thanks to the reduced-form representation, the results from Section 2 generalize immediately to the multiple-agent setting. The function $x_f$, treated as a cdf, defines a prior distribution over cutoffs. Given posterior belief $G$ of the cutoff (which can also be treated as an interim expected allocation rule), the belief over the type of the agent who acquired the good is given by the density

$$f^G(\theta) = \frac{G(\theta)f(\theta)}{\int_\Theta G(\tau)f(\tau)d\tau}. \quad (3.10)$$

\(^{31}\) There are interesting situations that do not satisfy this assumption. As noted in footnote 26, the characterization of implementable mechanisms extends to these cases. However, finding an optimal mechanism requires solving a multi-dimensional information disclosure problem which is typically intractable.
Next, let
\[ V(G) = N \int_{\Theta} V(\theta; f^G) G(\theta)f(\theta) d\theta \] (3.11)
be the expected payoff to the mechanism designer conditional on inducing a posterior belief \( G \) over the cutoff.

By Proposition 9 and Theorem 3, optimization over cutoff mechanisms can be performed directly in the space of reduced-form cutoff mechanisms. For a fixed allocation \( x \) and distribution \( f \), a reduced-form cutoff mechanism is formally identical to a one-agent cutoff mechanism from Section 2. Thus, we can use the proof of Proposition 3 to establish the following result (because the proof is fully analogous, it is omitted).\(^{32}\)

**Proposition 10.** For every allocation rule \( x \), non-decreasing in each variable, the problem of maximizing (3.9) over \( \pi \) subject to \((x, \pi)\) being a cutoff mechanism is equivalent to solving
\[
\max_{\tau \in \Delta(\Delta(C))} \mathbb{E}_{G \sim \tau} V(G)
\] (3.12)
subject to
\[
\mathbb{E}_{G \sim \tau} G(\theta) = x_f(\theta), \forall \theta \in \Theta.
\] (3.13)

Applying the main result of Kamenica and Gentzkow (2011), I obtain the concave-closure characterization of the optimal payoff. Recall that \( \mathcal{X} \) denotes the set of one-dimensional non-decreasing allocation rules on \( \Theta \).

**Corollary 7.** The optimal expected payoff to the mechanism designer in the problem (3.12)-(3.13) is equal to
\[
\text{co}V(x_f) \equiv \sup\{ z : (x_f, z) \in CH(\text{graph}(V)) \},
\]
where \( \text{graph}(V) \equiv \{(\bar{x}, \bar{z}) \in \mathcal{X} \times \mathbb{R} : \bar{z} = V(\bar{x})\} \).

### 3.3.2 Joint optimization

I turn attention to joint optimization over \((x, \pi)\) in the class of cutoff mechanisms.

**Corollary 8.** The problem of maximizing (3.9) over the set of cutoff mechanisms is equivalent to solving
\[
\max_{\bar{x} \in \mathcal{X}} \text{co}V(\bar{x})
\] (3.14)

\(^{32}\) The only difference in the proof is that the space of cutoffs \( C \) is potentially infinite. The Online Appendix of Kamenica and Gentzkow (2011) extends their results to a continuous state space, so this technical complication does not cause any problems.
subject to
\[ \int_r^1 \bar{x}(\theta) f(\theta) d\theta \leq \frac{1 - F_N(\tau)}{N}, \quad \forall \tau \in \Theta. \] (3.15)

Corollary 8 follows directly from Corollary 5, Proposition 10, and Corollary 7. Optimization over cutoff mechanisms can be performed in the space of reduced-form cutoff mechanisms which means that the interim expected allocation rule \( \bar{x} \) must satisfy the Matthews-Border condition (3.15).

In the case \( N = 1 \), constraint (3.15) is trivially satisfied. Thus, Corollary 8 implies that Proposition 4 can be extended – the same proof can be used to show that no disclosure is always optimal, regardless of the post-mechanism games and also when the type space is continuous.

**Corollary 9.** If \( N = 1 \), the problem of maximizing (3.9) over the set of cutoff mechanisms always has an optimal solution that reveals no information.

When \( N \geq 2 \), constraint (3.15) alters the conclusion from the one-agent setting. It may be optimal to disclose information. This is because the concave closure of \( V \) is taken in the space of all non-decreasing interim allocation rules (equivalently: all posterior beliefs over cutoffs), while the actual rule \( \bar{x} \) must be chosen from a subset of rules that satisfy the Matthews-Border condition (3.15). In other words, it might be optimal to induce posterior beliefs that do not correspond to an interim expected allocation that satisfies (3.15). For example, if the second bid is disclosed in an efficient auction, the posterior belief over the type of the winner is a truncation. Such belief does not correspond to any interim allocation rule that satisfies the Matthews-Border condition, i.e. there does not exist a symmetric no-information-revealing mechanism that always induces a posterior belief in the form of a truncation – the belief can only be accessed by making an explicit announcement. A simple example illustrating this discussion is provided in Section 4.1.

### 3.3.3 Optimization in the space of beliefs over types

The reduced-form representation can be used to apply the results of Subsection 2.5 to the multiple-player model. Recall that \( W(\bar{f}) = \int_\Theta V(\theta; \bar{f}) \bar{f}(\theta) d\theta \) denotes the conditional expected payoff to the mechanism designer given posterior belief \( \bar{f} \) over the type of the winner, and \( M^f \) denotes the set of beliefs over \( \Theta \) that likelihood-ratio dominate the prior \( f \). Let \( \mathcal{X}_{MB} \) denote the set of non-decreasing interim expected allocation
functions that satisfy the Matthews-Border condition (M-B). Combining Corollary 5 with Proposition 6, I obtain an alternative characterization of the optimal payoff.

**Corollary 10.** The problem of maximizing (3.9) over the set of cutoff mechanisms is equivalent to solving

$$
\max_{\bar{x} \in \Lambda_{MB}} \left( N \int_{\Theta} \bar{x}(\theta) f(\theta) d\theta \right) co^{M_f} \mathcal{W}(f^x).
$$

In particular, if \( \mathcal{W} \) is convex on \( M_f \), it is optimal to disclose the cutoff and types of losers; if \( \mathcal{W} \) is concave on \( M_f \), it is optimal to reveal no information.

Corollary 10 is analogous to Corollary 8 except that the concave closure is taken in the space of conditional posterior distributions over the type of the winner, instead of in the space of interim expected allocations. It is often more natural to work with the former space because posterior beliefs over types directly influence the aftermarket payoffs. In Subsection 4.2, I present an application where convexity/ concavity of \( \mathcal{W} \) follows from assumptions about the primitive parameters of the post-mechanism game.

## 4 Applications

In this section, I illustrate theoretical results with three examples. The first one shows how the optimal level of information disclosure depends on the alignment of preferences of the designer and the third party. In the second example, I consider optimization over disclosure rules in an efficient auction followed by negotiations. In the third, I analyze joint optimization in a model where the aftermarket is a resale game. Throughout, I assume that the designer uses a cutoff mechanism.

### 4.1 Communication between regulators

Suppose that the aftermarket is an interaction between the winner of the object (with type \( \theta \)) and a third-party regulator who chooses a decision \( a \in A \) to maximize the expectation of some social welfare function \( W(a, \theta) \). I assume that \( A \) is finite, and that \( W(a, \theta) \) is non-decreasing in \( \theta \) for all \( a \). Let \( a^*(\bar{f}) \in \arg\max A \mathbb{E}_f W(a, \theta) \) be the optimal decision given belief \( \bar{f} \) of the type of the winner held by the regulator.

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33 In Section 2.5, I worked with a finite type space but all results generalize easily to a continuous type space.
The payoff of the agent (which will not play a role in the analysis) is assumed to be non-decreasing in her type for any decision \( a \in A \) (so that all cutoff mechanisms are implementable, by Theorem 1). I further assume that the mechanism designer wants to maximize a social welfare function \( V_1(\theta) + W(a, \theta) \) which includes the surplus from the first-stage mechanism and from the second-stage interaction between the agent and the third-party regulator. The surplus from the mechanism \( V_1(\theta) \) is assumed to be non-decreasing in the type of the winner \( \theta \).

In this setting, the function \( V \) is given by

\[
V(G) = N \max_{a \in A} \int \left[ V_1(\theta) + W(a, \theta) \right] G(\theta)f(\theta)d\theta.
\] (4.1)

\( V \) is convex in \( G \) because it is a point-wise maximum of linear functionals. The concave closure of \( V \) corresponds to the payoff from full disclosure of the cutoff. By Corollary 7, full disclosure of the cutoff is optimal. If the realization of the cutoff \( c \) is announced, the posterior belief over the type of the winner is a truncation of \( f \) at \( c \). Given an interim expected allocation function \( \bar{x} \), the expected payoff from full disclosure is

\[
\text{co}V(\bar{x}) = \int_C V(1_{\geq c})d\bar{x}(c),
\]

where \( 1_{\geq c} \) denotes the interim allocation rule (and cdf of the cutoff) \( x(\theta) = 1_{\{\theta \geq c\}} \).

Using integration by parts,

\[
\int_C V(1_{\geq c})d\bar{x}(c) = \int_C N \left[ V_1(c) + W(a^*(f^1_{\geq c}), c) \right] f(c) \int_c^c d\bar{x}(z)dc.
\]

By Corollary 8, we have to solve

\[
\max_{\bar{x} \in X} \int_\Theta w(\theta)\bar{x}(\theta)f(\theta)d\theta
\]

subject to the Matthews-Border condition (3.15). Because \( V_1(\theta) \) and \( W(a, \theta) \) are non-decreasing in \( \theta \), \( w(\theta) \) is also non-decreasing. Condition (3.15) has to hold with equality for every \( \tau \). This uniquely pins down \( \bar{x} \) as the cdf of the first-order statistic of \( N - 1 \) draws from \( f \). Thus, \( \bar{x} \) corresponds to the efficient allocation rule.

Summarizing, the optimal mechanism allocates the object to the highest type, and reveals all information about the cutoff which corresponds to the second highest bid.
Under conditions (see Subsection 5.3 for details), the mechanism can be robustly implemented as a second-price auction in which the price paid by the winner is revealed. Note that this does not contradict Corollary 9. When $N = 1$, the auction boils down to allocating the good to all types of the agent at a constant price.

Full disclosure of the cutoff is optimal because the preferences of the mechanism designer and the third party are perfectly aligned. In the opposite case of perfectly conflicting interests, it is optimal to withhold all information elicited by the mechanism.

Suppose that the third-party regulator wants to minimize the function $W$, instead of maximizing it, i.e. the regulators play a zero-sum game. With this modification, the function $V$ becomes

$$V(G) = N \int_{\Theta} V_1(\theta)G(\theta)f(\theta)d\theta + N \min_{a \in A} \int_{\Theta} W(a, \theta)G(\theta)f(\theta)d\theta. \quad (4.2)$$

The first term is affine in $G$, and the second term is concave. Therefore, $V$ is a concave functional, and it is equal everywhere to its concave closure. By Corollary 7, no-communication is optimal, regardless of the choice of the allocation function $\bar{x}$.

Both of the above cases are extreme. In a typical situation, the (mis)alignment of preferences between the designer and the third party is imperfect, and the optimal disclosure policy is determined by the details of the post-mechanism game.

### 4.2 Post-auction negotiations

A mechanism designer allocates an object to one of $N \geq 2$ identical bidders. Types are distributed independently according to a full-support density $f$ with cdf $F$. I impose a constraint that in the first stage the object has to be allocated to the highest bidder. After the object is allocated, the winner negotiates with a third party in the aftermarket. Examples include subcontracting after winning a government contract in a procurement auction, bargaining over roaming agreements after spectrum auctions, and resale.

I adopt a tractable reduced-form approach to the bargaining game. Having observed the signal released by the mechanism, the third party has to pay a cost $k$ in order to start negotiations. If the cost is paid, negotiations lead to a total surplus of $\Delta + \delta \theta$, where $\theta$ is the type of the winner. The third party captures a fraction $\eta \in (0, 1)$ of the additional surplus generated by the negotiations ($\eta$ is the Nash bargaining parameter).

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34 This may be seen as a consequence of institutional constraints, or the inability of the designer to prevent resale between bidders after the auction but before the aftermarket.
In the opposite case, negotiations do not take place, and total surplus is equal to the value of the winner $\theta$. One interpretation of the game is that the third party incurs a due diligence cost – by paying the cost $k$ she learns the value of the winner and the subsequent negotiation stage is a full-information Nash bargaining game. I assume that $\Delta + \delta \theta \geq \theta$, for all $\theta, \delta > 0$, and that the cost $k$ is random from the point of view of the first stage, distributed according to cdf $H$ on $[k, \bar{k}]$.

If $\mathcal{F}$ is the set of independent distributions, and $\mathcal{E}$ includes all possible values of the parameters ($\Delta$, $\delta$, $\eta$, and $H$), then the setting satisfies Monotonicity and Richness. By Corollary 4, cutoff mechanisms are characterized by flexibility.

The auctioneer maximizes the total expected value of allocating the good over cutoff mechanisms that allocate to the highest type in the first stage.

**Proposition 11.** If $H(x)x$ is convex, it is optimal to disclose the second highest value. If $H(x)x$ is concave, it is optimal to disclose no information.

Using a carefully constructed auction, the designer can implement the optimal mechanism robustly, in the sense defined in Section 2.7. Details of this construction and a discussion is provided in Subsection 5.3. Transfers are pinned down by the equilibrium bidding strategies of agents. The designer does not need to know anything about the distribution of types (or details of the aftermarket) to guarantee that the mechanism is implementable. A guarantee of optimality can be obtained if the designer knows enough about the distribution $H$ – for example that $H(x)x$ is convex.

The optimal mechanism can be implemented as a second price auction in which either the price paid by the winner is revealed, or there is no announcement. Moreover, the optimal mechanism is robust.

To gain intuition for the result, note that posterior beliefs matter in the aftermarket only via the expectation of the type of the winner. The problem of the designer is to induce an optimal distribution of the posterior expected type of the winner by disclosing information about the cutoff. By Corollary 6, the cutoff corresponds to the second bid. The third party is willing to pay the cost $k$ if and only if it does not exceed the expected conditional gain from negotiating $\eta(\Delta + (\delta - 1)E[\theta|s])$, where $E[\theta|s]$ is the expected type of the winner conditional on signal $s$. The designer wants to increase the probability that negotiations take place. If $H$ is convex, probability increases when beliefs are more

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35 I exclude the private cost of the third party from the objective function of the designer. If the designer cares about the cost incurred by the third party, the analysis changes a bit, and in particular the optimal mechanism can depend on the bargaining parameter $\eta$. 

4 Applications

If $H$ is concave, the highest probability of negotiating is obtained by pooling all information. However, the objective is to maximize the value. High probability of negotiations is most valuable when the social gains from negotiating are highest. This additional effect favors information revelation because disclosure introduces positive correlation between the gains from negotiating and the probability that negotiations take place. For optimality of full disclosure of the second bid, it is enough that $H$ is not “too concave”, so that $H(x)x$ is convex. If $H$ remains concave after multiplying by $x$, the first effect dominates and no revelation is optimal.

If $H(x)x$ is neither convex nor concave, the problem can still be solved if $H$ is sufficiently regular by applying the duality approach of Kolotilin (2016) or Dworczak and Martini (2016). They provide a general solution method for a class of Bayesian persuasion problems in which the preferences of the Sender only depend on the posterior mean. Corollary 10 allows to formulate the above problem as a Bayesian persuasion problem with exactly that feature, with the caveat that posterior distributions have to likelihood-ratio dominate the prior.\footnote{This can be incorporated into the analysis of Dworczak and Martini (2016) by requiring that the distribution of posterior means puts all mass on means above the prior mean. One can solve this relaxed problem, and check ex-post that the optimal distribution over posterior means can be induced by a distribution over posterior beliefs that likelihood-ratio dominate the prior belief.}

4.3 An auction with resale

A mechanism designer allocates an object to one of $N \geq 2$ identical bidders. Types are distributed independently according to a full-support density $f$ with cdf $F$. In the first-stage, agents are imperfectly informed about their ex-post value of holding the good. Type $\theta_i$ of agent $i$ is the probability with which her value is high ($h$) if she acquires the object. With probability $1 - \theta_i$, the value is low ($l$). The value of the third party is $v$, with $v > h > l$. The third party has full bargaining power, and makes a take-it-or-leave-it offer to the agent who either accepts or rejects. The designer wants to maximize total expected surplus.

**Proposition 12.** If $E_f[\theta_N^{(1)}] \geq (h - l)/(v - l)$, where $\theta_N^{(1)}$ denotes the first-order statistic, the optimal mechanism is an efficient auction with no information revelation.

In the opposite case $E_f[\theta_N^{(1)}] < (h - l)/(v - l)$, suppose that the distribution $F$ satisfies a regularity condition defined in Appendix C.2. Then, the optimal mechanism takes the following form. The designer names a price $p$, and agents simultaneously accept
or reject. If all agents reject, the object is allocated uniformly at random. If exactly one agent accepts, she gets the object at price p. If more than one agent accepts, the designer runs an tie-breaking auction with reserve price p among agents who accepted. The designer only reveals whether the auction took place or not.

If the regularity condition fails, the optimal mechanism is either the one described above, or an auction with a strictly positive reserve price and no information revelation. The regularity condition is satisfied by all distributions \( F(\theta) = \theta^\kappa \) for \( \kappa > 0 \).

The designer wants resale to happen, so she tries to induce a high price in the aftermarket. The condition \( \mathbb{E}_f[\theta^{(1)}_{N}] \geq (h-l)/(v-l) \) implies that the third party quotes the high price \( h \) in the absence of additional information when the good is allocated to the highest type in the mechanism. In this case, an efficient auction with no information revelation achieves the upper bound on surplus, and is hence optimal.

In the more interesting case, an efficient auction with no revelation leads to a low price \( l \) in the resale stage. In order to induce a high price with positive probability, the designer runs a two-step procedure, and announces whether the second step (the tie-breaking auction) was reached. The auction is a signal of high value of the winner of the object. The price \( p \) is set in such a way that conditional on announcing that the auction took place, the third party is indifferent between the high and low price (and quotes the high price). Note that this is a cutoff mechanism because whether the auction takes place or not depends on the second highest type.

Suppose that no agent accepts the initial price \( p \). Then, the designer knows that the low price \( l \) will be offered in the second stage, so to maximize the probability of resale, she should allocate to the lowest type. However, incentive-compatibility constraints make it impossible to allocate to low types more often than to high types. Therefore, the mechanism allocates the object by a uniform lottery.

Government licenses are often allocated by administrative decisions, or through auctions. The above mechanism indicates that revealing which method was used constitutes an important message for the market – an auction is a signal that there was high competition for the license, and thus the winner is likely to have a high value.

The case when \( F \) fails the regularity condition is discussed in Appendix C.2.
5 Extensions

5.1 Private information of the seller in one-agent problems

If only one agent participates in the first-stage mechanism, there always exists an optimal mechanism which reveals no information (Corollary 9). This conclusion seems to be strong, especially in the context of optimal transparency of over-the-counter markets in Subsection 2.6. In the models of Asriyan et al. (2015) and Duffie et al. (2015), information revelation is optimal under conditions. To reconcile my findings with the findings of that literature, I extend the model to allow for private information of the seller. I assume that the mechanism designer (seller) privately observes a realization of some random variable $k \in \mathcal{K}$ which influences her payoff from allocating the asset. I provide a general characterization, and then apply it to the model of Subsection 2.6.

5.1.1 Extended model and characterization of optimal payoffs

Let $\alpha_0$ denote the prior probability mass function of the random variable $k \in \mathcal{K}$ observed by the seller. I assume that $\mathcal{K}$ is finite. In the general model of Section 3 with $N = 1$, a mechanism frame is a collection $\{(x_k, \pi_k)\}_{k \in \mathcal{K}}$, indexed by realizations of the random variable $k$. The mechanism frame $\{(x_k, \pi_k)\}_{k \in \mathcal{K}}$ is a cutoff mechanism if $(x_k, \pi_k)$ is a cutoff mechanism for each $k$. I assume that $k$ does not enter into agent’s utility, and that the designer’s payoff depends only on the posterior beliefs over the type of the agent – it does not depend directly on the posterior beliefs over $k$. This assumption is not essential and could be relaxed at the cost of complicating notation. In Appendix D.1, I briefly discuss implementability of cutoff mechanisms in the extended model. Here, I focus on optimization. The objective function of the designer is

$$\sum_{k \in \mathcal{K}} \alpha_0(k) \int_{\Theta} \int_{S} V(\theta; f^s, k) d\pi_k(s|\theta) x_k(\theta) f(\theta) d\theta,$$

where $V(\theta; f^s, k)$ is indexed by $k$.

In the above framework, the designer decides about disclosure of two types of information: her exogenous information about $k$ and the endogenous information about cutoffs $c$. The prior distribution of $k$ is given, while the prior distribution of $c$ is a choice variable, determined by the chosen allocation rule. The distribution of the cutoff may depend on the realization of $k$. The problem can be formulated as inducing an optimal distribution over posterior joint distributions $H$ over $(k, c)$ subject to a condition that
the marginal posterior distributions over $k$ average out to the prior $\alpha_0$. Given a joint distribution $H$, let $dH_k(\cdot)$ denote the marginal distribution over $k$ in the form of a pmf, and let $H_{c|k}(\cdot | k)$ denote the conditional distribution over $c$ given $k$ in the form of a cdf. For any distribution $\alpha$ over $\mathcal{K}$, let

\[ U(\alpha) \equiv \max_{H \in \Delta(\mathcal{K} \times \mathcal{C}) : \text{marg}_k(H) = \alpha} \mathcal{V}(H), \tag{5.2} \]

where

\[ \mathcal{V}(H) = \sum_{k \in \mathcal{K}} dH_k(k) \int_\Theta V(\theta; f^H, k) H_{c|k}(\theta | k) f(\theta) d\theta, \]

and $dH_k(k) = \alpha(k)$ if marg$_k(H) = \alpha$. The function $\mathcal{V}(H)$ is the expected payoff of the mechanism designer when the joint distribution over $(k, c)$ is $H$, and the mechanism reveals no information ($f^H$ is the corresponding posterior belief held by the third party). Accordingly, $U(\alpha)$ is the maximum expected payoff achievable to the mechanism designer for a fixed distribution $\alpha$ over $\mathcal{K}$, assuming that the designer chooses the optimal allocation rule for each $k$ but no information is revealed.

I state the following result without a proof (the proof uses similar arguments to the ones used multiple times in this paper).

**Proposition 13.** The maximal value of the objective function (5.1) attained over the set of cutoff mechanisms $\{(x_k, \pi_k)\}_{k \in \mathcal{K}}$ is given by $\text{co} U(\alpha_0)$ – the value of the concave closure of the function $U$ at the prior $\alpha_0$.

Proposition 13 implies that the problem can be solved in two steps. In the first step, for every distribution $\alpha$ over $\mathcal{K}$, we compute the optimal mechanism assuming no communication (this yields the function $U$). In the second step, the function $U$ is concavified which corresponds to finding the optimal revelation policy. When the prior $\alpha_0$ is degenerate (the seller observes no information), Proposition 13 boils down to a one-agent version of Corollary 8. More generally, the following corollary holds.

**Corollary 11.** The optimal mechanism in the extended model with one agent reveals information only about the random variable $k$ (i.e. the realization of $k$ is a sufficient statistic for the distribution of signals released by the mechanism). Unless the function $U$ is concave, there exist prior distributions $\alpha_0$ for which some information revelation is strictly optimal.

Corollary 11 follows directly from Proposition 13. The optimal mechanism concavifies the function $U$ which corresponds to inducing a distribution over posterior beliefs
over $k$. Conditional on inducing a posterior belief over $k$, no further information is revealed, by definition of $V$. If the function $U$ is not concave everywhere, then at some prior $\alpha_0$, $\co U(\alpha_0) > U(\alpha_0)$ which means that some information is revealed (see Kamenica and Gentzkow, 2011).

The optimal mechanism reveals information about $k$ although $k$ does not directly influence the payoffs of the agent and the third party. However, disclosing information about $k$ indirectly reveals information about the cutoff. Since the the realization of $k$ may influence the allocation chosen by the seller (for example, when $k$ is the cost of the seller), the distribution of cutoffs depends on the realization of $k$ which is unknown to the third party. By learning about $k$, the third party learns more about the allocation rule used by the seller, and that is informative about the type of the agent.

### 5.1.2 Implications for the OTC market analysis

In Subsection 2.6, both the welfare- and profit-maximizing mechanism can be implemented as a deterministic posted-price mechanism. In practice, prices in financial markets constantly change. Dealers’ costs vary with the market conditions and other variables. I extend the model from Subsection 2.6 by assuming that the cost $k$ is random, distributed according to $\alpha_0$ on $\mathcal{K} \subset [0, 1]$. The realization is observed by the seller and the agent but not the third party.\footnote{This assumption is not essential for the analysis, as long as a cutoff mechanism is used. It can be interpreted as saying that dealers know the cost of borrowing in the inter-dealer market but individual customers do not.}

**Proposition 14.** The welfare-maximizing mechanism in the extended model is a posted-price mechanism with a price that varies with $k$, and a signal distribution that only depends on the realization of $k$.

The optimal signal is sensitive to parameters of the model. Suppose that $f$ is uniform, $k$ takes on two values, and $v(\theta) = \delta \theta$, $\delta > 1$. Depending on the numerical value of $\delta$, it is possible to construct examples in which either full revelation or no revelation of $k$ is optimal (see Appendix D.2). This is because the optimal mechanism trades off several effects whose magnitudes depend on details of the model.

To say more about the optimal mechanism, I consider a robust approach to the disclosure problem. I have so far assumed that the third party only receives signals from the mechanism. In practice, the third party may observe other signals, or acquire
more information, from sources not controlled by the mechanism designer.\footnote{For example, in financial OTC markets, customers often search for the best quote before transacting, and thus they can acquire more information about \( k \) by seeing quotes of other dealers, assuming that costs of dealers are correlated.}

To model this in a tractable way, I assume that the mechanism designer does not know the distribution of the exogenous signal about \( k \) that the third party observes. Mechanisms are evaluated according to their worst-case performance. Formally, (1) the designer chooses the mechanism frame \( \{(x_k, \pi_k)\}_{k \in \mathcal{K}} \), (2) Nature chooses a distribution of additional exogenous signals about \( k \), (3) the mechanism is implemented, (4) the third party observes the signal \( s \) released by the mechanism and the additional signal drawn from the distribution chosen by Nature, and (5) the aftermarket game is played. Nature tries to minimize the expected payoff of the mechanism designer.

**Proposition 15.** *The welfare-maximizing mechanism under the worst-case criterion is a posted-price mechanism with full disclosure of the cost \( k \).*

The mechanism described in Proposition 15 can be interpreted as a financial market benchmark such as LIBOR (LIBOR discloses information about the borrowing cost of dealers). Duffie et al. (2015), in their model, provide sufficient conditions for optimality of announcing a benchmark. In my framework, although it may not be optimal when the exact distribution of exogenous signals is known, announcing a benchmark gives the highest welfare guarantee to an ambiguity averse regulator.

### 5.2 A canonical indirect representation of cutoff mechanisms

Cutoff mechanisms have been defined and analyzed as direct mechanisms. Although direct mechanisms are convenient for theoretical analysis, they are rarely used in practice. In this section, I define a canonical class of indirect mechanisms – called Generalized Clock Auctions (GCAs) – and show that cutoff mechanism frames correspond to monotone equilibria of these mechanisms. The goal is not to advocate the use of GCAs in practical problems but to provide better intuition for how much information can be revealed by a cutoff mechanism. As it turns out, cutoff mechanisms can be associated with (possibly noisy or partial) disclosure of the history of bidding in a clock auction which ends when at most one bidder remains active.
5.2.1 Generalized Clock Auction

I consider the symmetric setting of Section 3.2, and assume that the type space $\Theta$ is finite for ease of exposition. A Generalized Clock Auction (GCA) is characterized by a sequence of prices and a disclosure rule. Let $T = \{0, 1, 2, ..., T\}$ be the set of rounds. In round 0, agents simultaneously decide whether they want to participate or not.\(^{39}\) In every subsequent round $t \in T$, the timing is as follows (the term “announce” pertains to communication between the auctioneer and the bidders; these announcements are not observed by the third party):

1. A price $p^t$ is announced;
2. Bidders simultaneously (and covertly) decide to stay in the auction or to exit;
3. The auctioneer observes bidders’ decisions and implements the relevant outcome (to be specified);
4. The auctioneer announces to all bidders the outcome of the round (whether the auction continues, the set of active bidders, the winner if the auction ends).

The outcome of a round is determined in the following way. If at least two bidders decide to stay (and $t < T$), the auction continues. Otherwise, a “tentative winner” is chosen uniformly at random among active bidders. If all bidders decided to exit or the tentative winner is an agent who decided to stay, this constitutes the final outcome and the the auctioneer announces the winner and that all bidders became inactive. If there exists an agent who decided to stay but she is not the tentative winner, then that agent becomes the final winner and the auctioneer announces that all bidders but the winner became inactive. At $t = T$, all bidders must exit. After the object is allocated, a signal $s$ is released publicly according to the disclosure rule. The distinction between the winner being active or inactive at the end of the auction is irrelevant for the final outcome but will be used in the definition of the disclosure rule.

Let $H^t$ be the public history of the game up to and including round $t$, in the procedure described above, and let $H^t$ be the set of all public histories.\(^{40}\) Public history in this context can be identified with the sequence of announcements made by the auctioneer to the bidders during the auction. A Generalized Clock Auction (GCA) is a sequence of functions $\{(Y^t, P^t)\}^T_{t=1}$, where $Y^t : H^t \rightarrow \Delta(S)$, for some (finite) signal

\(^{39}\) This preliminary round is necessary to allow for the possibility that the good is not allocated.

\(^{40}\) Formally, a public history is the largest information set contained in information sets of all bidders.
space $S$, and $P^t : \mathcal{H}^{t-1} \rightarrow \Delta(\mathbb{R})$. In each round $t$, given a history $H^{t-1}$, a price $p^t$ is drawn from the distribution $P^t(H^{t-1})$. If the auction ends in round $t$, the signal is drawn and announced according to distribution $Y^t(H^t)$. Hence, the signal $s$ is an arbitrary garbling of the entire public history of the auction.

Prices do not have to change monotonically in a GCA. Because the signal distribution changes during the auction, it is as if a different good were offered for sale in every round. Hence, prices may have to decrease when the signal distribution gets less attractive for bidders.

Because technical details associated with defining strategies and equilibria of GCAs are not relevant for the message of this subsection, I relegate them to Appendix D.4. Informally, in a monotone equilibrium, lower types exit before higher types.

**Proposition 16.** If $(x, \pi)$ is a mechanism frame implemented by a monotone equilibrium of a GCA, then $(x, \pi)$ is a cutoff mechanism.

The proof of Proposition 16, as well a discussion of its assumptions, can be found in Appendix D.5.

To formulate the converse result, I restrict attention to a subclass of allocation rules. The subclass includes all practically relevant cases and allows me to use a simple GCA.

Define a hierarchical allocation rule $x^{\kappa_1...\kappa_k}(\theta)$ for any sequence $\kappa_1 < ... < \kappa_k$, with $\kappa_m \in \Theta$ for all $m$, by

$$x^{\kappa_1...\kappa_k}(\theta_i, \theta_{-i}) = \begin{cases} \frac{1}{|\{j \in \mathbb{N} : \kappa_m \leq \theta_j < \kappa_{m+1}\}|} & \text{if } \kappa_m \leq \theta_i < \kappa_{m+1} \text{ and } \forall j, \theta_j < \kappa_{m+1}, \\
0 & \text{otherwise}, \end{cases}$$

where by convention $\kappa_{k+1} = \infty$. For example, if $\Theta = \{\theta_1, ..., \theta_n\}$, then $x^{\theta_1...\theta_n}(\theta)$ is the efficient allocation rule, $x^{\theta_m...\theta_n}(\theta)$ additionally excludes types $\theta_1, ..., \theta_{m-1}$, and $x^{\theta_1}(\theta)$ corresponds to a uniform lottery. A symmetric allocation rule $x$ is called *decomposable* if it is a convex combination of hierarchical allocation rules.

**Proposition 17.** Suppose that $x$ is decomposable. Any symmetric cutoff mechanism frame $(x, \pi)$ can be implemented (up to equivalence) in a pure-strategy equilibrium of a Markov GCA in which randomization over prices may only happen in round 0 (subsequent prices are deterministic functions of the time and the number of active bidders).

Decomposability of $x$ implies that it is enough to keep track of the number of active bidders in any round because the allocation does not depend on the types of bidders
who exited in previous round.\footnote{The main difficulty in the proof is to show that decomposability of $x$ implies existence of a signal \distribution $\pi'$, equivalent to $\pi$, which inherits this property. The original \distribution $\pi$ may depend on types of bidders who exited in previous rounds, and thus it cannot be implemented with a deterministic price path.} In order to implement an arbitrary cutoff mechanism frame, I would have to allow for stochastic non-Markov prices, and the construction of the GCA would be significantly more complicated.

### 5.3 Robust Implementation by Standard Auctions

In this section, I analyze conditions under which a cutoff mechanism frame can be implemented in a robust way, i.e. without knowing the distributions of types and details of the post-mechanism game (see Section 2.7). It is assumed throughout that agents know the distribution and the post-mechanism game.\footnote{It is enough to assume that agents have the same Bayesian beliefs about these variables, and the same higher-order belief about the beliefs of the the third party.} I assume $N \geq 2$ and work with a continuous type space $\Theta = [0, 1]$ for convenience.

It is known (see for example Bergemann and Morris, 2013) that using direct mechanisms is not without loss of generality when agents have more information about the distribution than the designer. If arbitrary indirect mechanisms are allowed, the designer can elicit information about the distribution by asking agents to report it, and punishing if reports disagree. Applicability of such schemes in practice might raise serious doubts. I focus on simple indirect mechanisms with a one-dimensional message space, such as standard auctions.

In general, robust implementation of a cutoff mechanism in an auction with a one-dimensional bid space may be impossible. If the equilibrium bidding function is an injection on the set of types, there exists a mapping from bids into signals that induces the direct disclosure rule $\pi$. However, this mapping varies with $f$, so if the designer does not know $f$, she may be unable to invert the bidding function to recover it.

There are important cases in which the inversion need not be done. Suppose that $x$ allocates the object to the highest-value agent. Corollary 10 provides sufficient conditions under which full revelation of the cutoff, or no revelation, are optimal. These two mechanisms are solutions to two practical problems considered in Section 4. Can these mechanism frames be implemented robustly by standard auctions?

By Corollary 6, it is enough to consider signals that only depend on the second-highest type. Define a function $v^\pi : \Theta^2 \rightarrow \mathbb{R}$ by $v^\pi(\theta, \hat{\theta}) = u(\theta; f^\pi_{\hat{\theta}}, e)$, where $f^\pi_{\hat{\theta}}$ is the
posterior belief over the type of the winner when the second-highest value is \( \hat{\theta} \) and the signal function is given by \( \pi \). The function \( v^\pi(\theta, \hat{\theta}) \) is the expected continuation value of the winner with type \( \theta \) conditional on winning against second-highest type \( \hat{\theta} \).\footnote{When \( \pi \) is the no-revelation rule, \( f^\pi_{\hat{\theta}} \) is the distribution of the first-order statistic of \( N \) draws from \( f \) (and does not depend on \( \hat{\theta} \)). If \( \pi \) is the full-disclosure rule, \( f^\pi_{\hat{\theta}} \) is the truncation of \( f \) at \( \hat{\theta} \).} To simplify proofs, I assume that \( v^\pi(\theta, \theta) \) is strictly positive and differentiable in \( \theta \).

The function \( v^\pi \) is similar to an object studied by Milgrom and Weber (1982a) in the context of auctions with affiliated values. In the setting of Milgrom and Weber (1982a), the value of the winner depends on the value of the second-highest bidder \( \hat{\theta} \) due to statistical correlation of types and the assumption of non-private values. In my setting, the value of the winner depends on \( \hat{\theta} \) because the bid of the second-highest bidder influences the signal sent by the mechanism (and hence the continuation payoff of the winner). In Milgrom and Weber (1982a), affiliation of types implies that their analog of \( v^\pi(\theta, \theta) \) is non-decreasing in \( \theta \), a property necessary for existence of a monotone equilibrium in standard auctions. In my model, \( v(\theta, \hat{\theta}) \) is non-decreasing in \( \theta \) under the assumption of a monotone aftermarket. In general, there is no reason to expect monotonicity in \( \hat{\theta} \), and hence \( v^\pi(\theta, \theta) \) may fail to be increasing in \( \theta \). A sufficient condition for monotonicity of \( v^\pi(\theta, \theta) \) is that a “higher” posterior belief of the third party leads to a higher payoff for the agent in the aftermarket. An example is provided at the end of the subsection.

**Proposition 18.** Suppose that \( x(\theta_i, \theta_{-i}) = 1_{\{\theta_i \geq \theta_{-i}\}} \), for all \( \theta \in \Theta \). If \( \pi \) is the full-disclosure rule, \((x, \pi)\) can be implemented robustly by

- a second-price auction where the price paid by the winner is disclosed, or
- a first-price auction where the second-highest bid is disclosed,

if and only if \( v^\pi(\theta, \theta) \) is strictly increasing in \( \theta \). Moreover, \((x, \pi)\) can always be robustly implemented by an all-pay auction where the second-highest bid is disclosed.

If \( \pi \) is the no-revelation rule, \((x, \pi)\) can be implemented robustly by any of the above auction formats (with no revelation).

A candidate equilibrium bidding function is determined by the local (first-order) optimality condition of each agent. The bidding function has to be strictly increasing to guarantee that full disclosure of the bid (or price) leads to full disclosure of the second-highest type (and is also needed for equilibrium existence). In a SPA,
candidate bidding function in the full-disclosure case is \( v^\pi(\theta, \theta) \), and so \( v^\pi \) must be strictly increasing in \( \theta \). In a FPA, it is enough that \( (\int_0^\theta v^\pi(\tau, \tau) dF^{N-1}(\tau))/(F^{N-1}(\theta)) \) is strictly increasing in \( \theta \). This condition holds for all \( F \) if and only if \( v^\pi(\theta, \theta) \) is strictly increasing in \( \theta \). If robustness is required only for a subset of prior distributions (e.g. because the designer has some information about it), a weaker condition on \( v^\pi \) may suffice. Finally, in an all-pay auction, an agent with type \( \theta \) bids \( \int_0^\theta v^\pi(\tau, \tau) dF^{N-1}(\tau) \).

The bidding function is always increasing, because, unlike in a first- or second-price auction, it is not obtained by conditioning on winning.

The above results can be applied to the solution of the post-auction bargaining model of Subsection 4.2. In the full-disclosure case,

\[
v^\pi(\theta, \hat{\theta}) = \theta + (1 - \eta) H \left( \eta(\Delta + (\delta - 1) \mathbb{E}[\theta_N^{(1)} | \theta_N^{(2)} = \hat{\theta}]) \right) (\Delta + (\delta - 1) \theta),
\]

where \( \mathbb{E}[\theta_N^{(1)} | \theta_N^{(2)} = \hat{\theta}] \) is the expected value of the highest type when the second-highest type is \( \hat{\theta} \). If \( F \) and \( H \) have full support, \( v^\pi(\theta, \theta) \) is strictly increasing when \( \delta > 1 \). By Proposition 18, a SPA with revelation of the price robustly implements the optimal mechanism. Otherwise, an all-pay auction may be used.

### 5.4 What if the loser also interacts in the aftermarket?

In the preceding sections, I assumed that only the agent who acquires the object interacts in the aftermarket. In many cases, other agents may also engage in post-mechanism interactions. For example, a loser may try to purchase a similar object in the aftermarket, or negotiate to gain access to an object owned by another market participant.

To allow for this possibility, I study the one-agent model from Section 2 but with an arbitrary post-mechanism game. The aftermarket is formally a pair \( (\mathcal{A}_l, \mathcal{A}_w) \) with

\[
\mathcal{A}_j \equiv \{u_j(\theta; \bar{f}) : \theta \in \Theta, \bar{f} \in \Delta(\Theta)\},
\]

for \( j \in \{l, w\} \), where subscript \( l \) denotes the aftermarket for a “loser” (agent does not acquire the good), and \( w \) – the aftermarket for a “winner”. I assume that continuation payoffs are non-decreasing in the type, and, for ease of exposition, that \( \Theta \) is finite. A mechanism frame in the extended setting is \( (x, \pi_l, \pi_w) \), where \( \pi_l \) is the signal distribution conditional on not allocating the object, and \( \pi_w \) is the signal distribution conditional on allocating the object. Because there is only one agent, only one of the signal structures is used ex-post. Third-party players in the aftermarket know whether
the agent acquired the good or not.

**Definition 15.** A mechanism frame \((x, \pi_l, \pi_w)\) is a *cutoff mechanism* if \(x\) is non-decreasing, and the signals \(\pi_l\) and \(\pi_w\) can be represented as

\[
\pi_l(s|\theta)(1-x(\theta)) = \sum_{c>\theta} \gamma_l(s|c)dx(c), \tag{5.3}
\]

\[
\pi_w(s|\theta)x(\theta) = \sum_{c\leq\theta} \gamma_w(s|c)dx(c), \tag{5.4}
\]

for each \(\theta \in \Theta\) and \(s \in S\), for some signal functions \(\gamma_l : C \to \Delta(S)\) and \(\gamma_w : C \to \Delta(S)\).

Condition (5.4) is analogous to condition (2.1) in the definition of cutoff mechanisms from Section 2. Condition (5.3) is a mirror image of condition (5.4), with the sum indexed by all cutoff levels exceeding the type of the agent.

Instead of formulating a richness condition similar to the one defined in Section 3, I simplify the analysis by requiring the mechanism frames to be implementable for all distributions \(f\) and all monotone aftermarkets \((A_l, A_w)\).

**Proposition 19.** A mechanism frame \((x, \pi_l, \pi_w)\) is implementable for all distributions \(f\) and all monotone aftermarkets \((A_l, A_w)\) if and only if \(x\) is a constant allocation rule (in which case no information can be revealed).

The conclusion continues to hold when only the loser interacts, and the winner enjoys the utility of holding the object: \(u_w(\theta; \bar{f}) \equiv \theta\), for all \(\bar{f} \in \Delta(\Theta)\).

In the absence of any conditions on the aftermarket, no information about the agent’s type can be elicited in a cutoff mechanisms. Proposition 19 should not come as a surprise. Suppose that in the loser’s aftermarket, the same object is allocated but with utility doubled for every type. In this case, higher types have a relative preference for *not* acquiring the object in the first stage which reverts the direction of single-crossing. To avoid the negative result, it is necessary to assume that the game played when the object is acquired is in some sense preferred to playing the game when the object is not allocated. Preference for winning the object should be expressed in relative terms (consistent with single-crossing) rather than absolute terms (absolute differences in utility can always be undone with transfers).

**Definition 16.** The winner’s aftermarket \(A_w\) is single-crossing-separated from the
loser’s aftermarket $A_l$ if for any $\theta > \hat{\theta}$, there exists $d(\theta, \hat{\theta}) > 0$, such that for all $\bar{f}$,

$$u_w(\theta; \bar{f}) - u_w(\hat{\theta}; \bar{f}) \geq d(\theta, \hat{\theta}) \geq u_l(\theta; \bar{f}) - u_l(\hat{\theta}; \bar{f}).$$

Single-crossing separation requires that the difference in utilities between any two types in $A_w$ can be separated from the difference in utilities between these two types in $A_l$, uniformly in posterior beliefs. For example, the condition is satisfied with $d(\theta, \hat{\theta}) = \theta - \hat{\theta}$ when $A_l$ is buying an identical object from a different seller, and there is no winner’s aftermarket, i.e. $u_w(\theta; \bar{f}) \equiv \theta$. Alternatively, the winner’s aftermarket yields an additive payoff $u_w(\theta; \bar{f}) = \theta + v_w(\theta; \bar{f})$, for some $v_w(\theta; \bar{f})$, non-decreasing in $\theta$.

**Proposition 20.** A mechanism frame $(x, \pi_l, \pi_w)$ is implementable for all distributions $f$ and all monotone aftermarkets $(A_l, A_w)$ such that $A_w$ is single-crossing-separated from $A_l$ if and only if $(x, \pi_l, \pi_w)$ is a cutoff mechanism.

Using Proposition 20 and arguments used in preceding sections, one can show that in the extended setting (1) for a fixed allocation function $x$, the problem of finding an optimal cutoff mechanism is a Bayesian persuasion problem, and (2) there always exists an optimal cutoff mechanism which reveals no information.

The extension to multiple players is straightforward in the case when only the loser interacts (i.e. $u_w(\theta; \bar{f}) \equiv \theta$). In this case, it is the bid of the winner, not the losers, that can be revealed without compromising incentives. If losing and winning players both interact and all signals are public, the problem becomes significantly more complicated, because third-party players observe two signals, both of which could contain non-trivial information about the same agent.

### 6 Conclusions

In this paper, I studied mechanism design in a setting where the mechanism is followed by an aftermarket, i.e. a post-mechanism game played between the agent who acquired the object and third-party market participants. Existence of an exogenous aftermarket creates a new tool in the design problem – the disclosure rule. By disclosing information elicited by the mechanism, the designer can influence the information structure of the post-mechanism game.

I introduced a tractable class of cutoff mechanisms which are characterized by being always implementable – regardless of the aftermarket and for all prior distributions of
types. I applied the theory to study optimal transparency of financial over-the-counter markets, communication between regulators, post-auction bargaining, and mechanisms with resale.

It is useful to distinguish three sources of information that the mechanism can attempt to disclose: (1) private information of agents who participate in the aftermarket, (2) private information of agents who do not participate in the aftermarket, and (3) private information of the designer, including outcomes of endogenous randomization in the mechanism. Although final payoffs are determined by posterior beliefs about the first type of information, the analysis of cutoff mechanisms reveals that only the last two sources can be used robustly, i.e. irrespective of the fine details of the model. I suspect that this conclusion holds more generally. Extending the results of this paper to an arbitrary first-stage social choice problem is an interesting direction for future research.

References


A Proofs and Supplementary Materials for Section 2

A.1 Proof of Lemma 1

In this subsection, I complete the proof of Lemma 1. Recall that I have to prove existence of distribution \( f \) such that (i) when the third party makes an offer, she offers price \( \theta \) after seeing signal \( s \) if and only if \( \pi(s|\theta)x(\theta) \geq \pi(s|\hat{\theta})x(\hat{\theta}) \); otherwise, she offers price \( \hat{\theta} \), (ii) when the agent makes an offer, the unique equilibrium conditional on \( s \) involves a constant price \( p^* \in [\theta, v(\hat{\theta})] \) (and hence trade with probability one).
First, let’s define
\[ p^*_\langle x, \pi, f\rangle(s) \in \arg\max_p \sum_{\theta \leq p} (v(\theta) - p)\pi(\theta|x(\theta)f(\theta)), \]
as the optimal price quoted by the third party when she makes an offer, given mechanism frame \((x, \pi)\), distribution \(f\), and conditional on signal \(s\). If distribution \(f\) is supported on the set \(\{\hat{\theta}, \theta\}\), the optimal price can be either \(\hat{\theta}\) or \(\theta\). Price \(\hat{\theta}\) is uniquely optimal if
\[
(\theta - \hat{\theta})\pi(s|\hat{\theta})x(\hat{\theta})f(\hat{\theta}) > (v(\theta) - \theta)\pi(s|\theta)x(\theta)f(\theta).
\]
Price \(\theta\) is uniquely optimal if the opposite strict inequality holds. Define \(f\) as the unique distribution supported on \(\{\hat{\theta}, \theta\}\) such that \(f(\hat{\theta})/f(\theta) = (v(\theta) - \theta)/(\theta - \hat{\theta})\). Then, \(f\) achieves property (i).

To show that property (ii) holds under \(f\) as well, I use the assumption that \(v(\hat{\theta}) = \max_{\tau < \theta} v(\tau) \geq \theta\), where the first equality follows from the fact that \(\hat{\theta}\) is the largest type smaller than \(\theta\). The inequality \(v(\hat{\theta}) \geq \theta\) implies that conditional expected gains from trade are strictly positive for any non-degenerate belief of the third party. Every price quoted in equilibrium must be accepted with positive probability. Given the tie-breaking assumption about acceptance decisions, each price is accepted with probability one. If there were two prices offered by the agent in equilibrium, each type of the agent would deviate to quoting the higher price. Thus, there must be a constant price \(p^s \in [\theta, v(\hat{\theta})]\) in the second-stage equilibrium conditional on every signal \(s\) sent in the mechanism.

### A.2 Proof of Proposition 2

To simplify notation, I will denote \(\underline{\theta} = \min\{\Theta\}, \bar{\theta} = \max\{\Theta\}\), and for any \(\theta\), let \(\theta^+\) be the smallest type larger than \(\theta\), and let \(\theta^-\) be the largest type smaller than \(\theta\), whenever they exist.

Fix a mechanism frame \((x, \pi)\) that is implementable for every \(f\). By summing up over signals, for each \(\theta\),
\[
\sum_{s \in S} \left[ \pi(s|\theta^+)x(\theta^+) - \pi(s|\theta)x(\theta) \right] = x(\theta^+) - x(\theta).
\]
By Lemma 1, we know that \((x, \pi)\) satisfies condition (M), hence each term in the sum
on the left hand side is non-negative. It follows that \( x(\theta^+) \geq x(\theta) \), and

\[
\pi(s|\theta^+)x(\theta^+) - \pi(s|\theta)x(\theta) \leq x(\theta^+) - x(\theta), \forall s \in S, \forall \theta. \tag{A.1}
\]

Given the above properties, I will prove that if a mechanism frame \((x, \pi)\) satisfies condition (M), then we can inductively construct a signal function \(\gamma: C \rightarrow \Delta(S)\) such that

\[
\pi(s|\theta)x(\theta) = \sum_{c \leq \theta} \gamma(s|c)dx(c), \forall s \in S; \tag{A.2}
\]

where recall that \(dx\) denotes the probability mass function over cutoffs induced by the allocation function \(x\).

The induction is over \(\Theta\) (types are totally ordered because \(\Theta\) is a subset of the real line). For type \(\theta\), define \(\gamma(s|\theta) := \pi(s|\theta)\), for any \(s \in S\). Then, we have

\[
\pi(s|\theta)x(\theta) = \pi(s|\theta)dx(\theta) = \sum_{c \leq \theta} \gamma(s|c)dx(c).
\]

Inductive hypothesis: Suppose we have constructed \(\gamma(s|c)\) for \(s \in S\) and \(c < \theta\) such that equation (A.2) holds for all \(\hat{\theta} < \theta\).

Inductive step: I will construct \(\gamma(s|\theta)\) for \(s \in S\), and show that condition (A.2) holds for \(\theta\).

There are two cases. If \(x(\theta) = x(\theta^-)\), then \(dx(\theta) = 0\), so the claim holds trivially (it does not matter how we define \(\gamma\) for this \(\theta\)). Otherwise, let us define, for each \(s \in S\),

\[
\gamma(s|\theta) = \frac{\pi(s|\theta)x(\theta) - \pi(s|\theta^-)x(\theta^-)}{x(\theta) - x(\theta^-)}.
\]

This is a well defined probability for each \(s \in S\) due to condition (M) and inequality (A.1). Moreover, we have

\[
\pi(s|\theta)x(\theta) = \gamma(s|\theta)\left(x(\theta) - x(\theta^-)\right) + \pi(s|\theta^-)x(\theta^-).
\]

Using the inductive hypothesis, and the fact that \(x(\theta) - x(\theta^-) = dx(\theta)\), we obtain

\[
\pi(s|\theta)x(\theta) = \gamma(s|\theta)dx(\theta) + \sum_{c \leq \theta^-} \gamma(s|c)dx(c) = \sum_{c \leq \theta} \gamma(s|c)dx(c)
\]

which ends the induction and the proof.
A.3 Proof of Proposition 3

We start from the problem of maximizing

\[ \sum_{\theta \in \Theta} \sum_{s \in S} V(\theta; f^s) \pi(s | \theta) x(\theta) f(\theta) \]

over \( \pi \) subject to \((x, \pi)\) being a cutoff mechanism. For any cutoff mechanism, by definition, there exists a function \( \gamma \) such that \( \pi(s | \theta) x(\theta) = \sum_{c \leq \theta} \gamma(s | c) dx(c) \). Thus, the problem becomes

\[ \max_{\gamma} \sum_{\theta \in \Theta} \sum_{s \in S} V(\theta; f^s) \sum_{c \leq \theta} \gamma(s | c) dx(c) f(\theta). \]

Equivalently,

\[ \max_{\gamma} \sum_{s \in S} \left( \sum_{c} \gamma(s | c) dx(c) \right) \sum_{\theta \in \Theta} V(\theta; f^s) \left( \frac{\sum_{c \leq \theta} \gamma(s | c) dx(c)}{\sum_{c} \gamma(s | c) dx(c)} \right) f(\theta). \]

In the above expression, \( \varsigma_s \) is the unconditional probability of sending signal \( s \), and the remaining expression is equal to \( V(G_s) \), as defined in (2.10), where \( G_s \) is the posterior cumulative distribution function of the cutoff conditional on signal \( s \). Thus, the objective function can be written as

\[ \mathbb{E}_{s \sim \varsigma_s} V(G_s). \]

(A.3)

To confirm that \( V \) depends solely on the posterior belief over the cutoff, note that

\[ V(G_s) = \mathbb{E}_{c \sim G_s} \sum_{\theta \in \Theta} V(\theta; f^s) 1_{\{\theta \geq c\}} f(\theta). \]

Thus, the problem is formally equivalent to the Bayesian persuasion problem of Kamenica and Gentzkow (2011). Their main insight is that instead of choosing a distribution \( \varsigma \) over signals, we can be choosing directly a distribution over posterior beliefs \( \tau \in \Delta(\Delta(C)) \), subject to a Bayes-plausibility constraint. This gives us equations (2.11) and (2.12) (equation (2.12) is the Bayes-plausibility constraint on the beliefs about the cutoff \( c \) phrased in terms of its cdf).
A.4 Proof of Proposition 5

I first show that every distribution over beliefs of the cutoff $\tau \in \Delta(\Delta(C))$ which is feasible under allocation $x$ defines a distribution over beliefs of the type $\rho \in \Delta(\Delta(\Theta))$ which satisfies conditions (2.15)-(2.14). For every $G \in \text{supp}(\tau)$, let $f^G$, defined by (2.9), be the corresponding posterior belief over the type. Clearly, each such $f^G$ satisfies condition (2.14) because $G$ is a non-decreasing function. To show condition (2.15), define

$$\rho(f^G) = \frac{\sum_{\theta \in \Theta} G(\theta)f(\theta)\tau(G)}{\sum_{\tilde{G} \in \text{supp}(\tau)} \sum_{\theta \in \Theta} \tilde{G}(\theta)f(\theta)\tau(\tilde{G})}. \quad \text{(A.4)}$$

The expression $\rho(f^G)$ is the probability of inducing belief $f^G$ conditional on allocating the good (the probability distribution over cutoffs $\tau$ is unconditional, hence the need to transform the probabilities by conditioning on the event that the good was allocated). Because $\tau$ is a feasible distribution, i.e. it satisfies condition (2.12),

$$\sum_{\tilde{G} \in \text{supp}(\tau)} \sum_{\theta \in \Theta} \tilde{G}(\theta)f(\theta)\tau(\tilde{G}) = \sum_{\theta \in \Theta} x(\theta)f(\theta).$$

Then, we have

$$\sum_{G \in \text{supp}(\tau)} f^G(\theta)\rho(f^G) = \sum_{G \in \text{supp}(\tau)} \frac{G(\theta)f(\theta)}{\sum_{\tilde{G} \in \text{supp}(\tau)} \sum_{\theta \in \Theta} \tilde{G}(\theta)f(\theta)\tau(\tilde{G})} \frac{\sum_{\tilde{\theta} \in \Theta} G(\tilde{\theta})f(\tilde{\theta})}{\sum_{\tilde{\theta} \in \Theta} x(\tilde{\theta})f(\tilde{\theta})} \tau(G)$$

$$= \left(\sum_{G \in \text{supp}(\tau)} G(\theta)f(\theta)\right) f(\theta) = \frac{x(\theta)f(\theta)}{\sum_{\tilde{\theta} \in \Theta} x(\tilde{\theta})f(\tilde{\theta})} = f^x(\theta),$$

which is condition (2.15).

To show the opposite direction, start with a distribution over beliefs of the agent’s type conditional on allocating the good, $\rho \in \Delta(\Delta(\Theta))$, satisfying conditions (2.15) and (2.14) for allocation $x$. For each $\tilde{f} \in \text{supp}(\rho)$, define

$$G_{\tilde{f}}(\theta) := \left(\frac{x(\tilde{\theta})f(\tilde{\theta})}{\tilde{f}(\theta)}\right) \frac{\tilde{f}(\theta)}{\tilde{f}(\theta)}, \forall \theta \in \Theta,$$

where $\tilde{\theta} = \max\{\Theta\}$. Because $\tilde{f}$ likelihood-ratio dominates $f$, the function $G_{\tilde{f}}(\theta)$ is non-decreasing and bounded above by 1. Thus, it defines an allocation function, and hence also a distribution over cutoffs. Define a distribution over distributions over the
cutoff \( \tau \in \Delta(\Delta(C)) \) by

\[
\tau(G_f) = \rho(\bar{f}) \frac{\sum_{\theta \in \Theta} x(\theta) f(\theta)}{x(\bar{\theta}) f(\bar{\theta})}, \quad \forall \bar{f} \in \text{supp}(\rho).
\]

This is a well defined distribution because, by condition (2.15),

\[
\sum_{\bar{f} \in \text{supp}(\rho)} \tau(G_f) = \left( \sum_{\bar{f} \in \text{supp}(\rho)} \rho(\bar{f}) \right) \frac{\sum_{\theta \in \Theta} x(\theta) f(\theta)}{x(\bar{\theta}) f(\bar{\theta})} = f(\bar{\theta}) \frac{\sum_{\theta \in \Theta} x(\theta) f(\theta)}{x(\bar{\theta}) f(\bar{\theta})} = 1.
\]

We can now check that condition (2.12) is satisfied:

\[
\sum_{f} \tau(G_f) G_f(\theta) = \sum_{f} \rho(\bar{f}) \frac{\sum_{\theta \in \Theta} x(\theta) f(\theta)}{f(\theta)} = f(\bar{\theta}) \frac{\sum_{\theta \in \Theta} x(\theta) f(\theta)}{f(\theta)} = x(\theta).
\]

Therefore, \( \tau \) is a feasible distribution over beliefs of the cutoff given allocation function \( x \). It remains to show that the unconditional distribution \( \tau \) over beliefs of the cutoff gives rise to the conditional distribution \( \rho \) over beliefs of the type (conditional on allocating the good). By direct calculation, equation (A.4) holds for \( \rho \) and \( \tau \) defined as above.

### A.5 Proof of Proposition 6

The proof follows almost directly from Proposition 5. Starting from the objective function (2.11) and a distribution \( \tau \) over beliefs over cutoffs, we have

\[
\mathbb{E}_{G \sim \tau} V(G) = \sum_{G \in \text{supp}(\tau)} \sum_{\theta \in \Theta} V(\theta; f^G) G(\theta) f(\theta) \tau(G)
\]

\[
= \left( \sum_{\theta \in \Theta} x(\theta) f(\theta) \right) \sum_{G \in \text{supp}(\tau)} \sum_{\theta \in \Theta} V(\theta; f^G) f^G(\theta) \frac{\sum_{\theta \in \Theta} G(\theta) f(\theta)}{\sum_{\theta \in \Theta} x(\theta) f(\theta)} \tau(G)
\]

\[
= \left( \sum_{\theta \in \Theta} x(\theta) f(\theta) \right) \mathbb{E}_{\bar{f} \sim \rho} \mathcal{W}(\bar{f}),
\]

where the last equality follows from the proof of Proposition 5 – the distribution \( \rho \) is the conditional distribution over beliefs of the type (conditional on allocating the good) corresponding to the unconditional distribution of beliefs over the cutoff \( \tau \).

Given the above representation of the objective function and Proposition 5, the
concave closure characterization follows from the usual argument.

### A.6 Additional content for Section 2.5 - Cost of Incentives

The goal of this subsection is to characterize the utility cost of requiring incentive compatibility in the design problem. I compare the design problem of Section 2 with an auxiliary problem in which the designer observes the type of the agent. I maintain the assumptions and notation of Section 2 throughout.

Consider the problem of maximizing (2.8) over $\pi$ for a fixed $x$ over all mechanisms, ignoring the incentive constraints, i.e. as if the designer could observe the type of the agent. By Kamenica and Gentzkow (2011), the value of this auxiliary problem (conditional on allocating the good) is equal to

$$\text{co } W(f^x) \equiv \sup \{ z : (f^x, z) \in \text{CH}(\text{graph}(W)) \},$$

where the domain of $W$ is unrestricted, i.e. equal to $\Delta(\Theta)$. By applying Proposition 6 from Section 2.5, we can conclude that the expression

$$\Delta W(f^x) \equiv \text{co } W(f^x) - \text{co } M^f W(f^x)$$

measures the utility cost for the mechanism designer of not observing the agent’s type, for a fixed allocation function. Incentive compatibility restricts the set of posterior beliefs that the mechanism can access. In the relaxed problem, the mechanism can explore a larger space over which the prior is decomposed into posteriors – leading to a potentially higher value of the concave closure. I illustrate the above characterization with a numerical example.

**Example 6.** Let $\Theta = \{1/2, 3/4\}$, $f = (3/4, 1/4)$, $\eta = 1$, and $v \equiv 1$. Fix $x = (1/2, 1)$ and suppose we want to maximize social surplus. The no-communication posterior is $f^x = (3/5, 2/5)$. The set of posterior beliefs can be indexed by $\alpha$ – the probability of high type. The third party quotes the high price if and only if $\alpha \geq 1/2$. Therefore, the expected payoff to the designer when the posterior belief is $\alpha$ (conditional on allocating the good) is

$$W(\alpha) = \begin{cases} (1 - \alpha) + \alpha^3 & \text{if } \alpha < 1/2 \\ 1 & \text{if } \alpha \geq 1/2. \end{cases}$$

A posterior belief $(1 - \alpha, \alpha)$ likelihood-ratio dominates the prior $f$ if and only if $\alpha \geq 1/4$. 


Figure A.1 illustrates the conditional expected payoff to the mechanism designer under the optimal cutoff mechanism, and under the optimal mechanism when the designer observes the type of the agent.

Fig. A.1: Function $\mathcal{W}$ for Example 6 (solid line), $\text{co}^M \mathcal{W}$ (blue chain line), $\text{co} \mathcal{W}$ (red dotted line).

If the designer optimizes over allocation rules $x \in \mathcal{X}$ as well, the optimal ex-ante expected payoff under a cutoff mechanism is given by

$$\max_{x \in \mathcal{X}} \left( \sum_{\theta \in \Theta} x(\theta) f(\theta) \right) \text{co}^M \mathcal{W}(f^x) = \max_{x \in \mathcal{X}} \left( \sum_{\theta \in \Theta} x(\theta) f(\theta) \right) \mathcal{W}(f^x),$$

where the equality follows from Proposition 4 (and could also be proven directly using a similar argument). If the designer observes the agent’s type, the optimal expected payoff is given by

$$\max_{x \in \mathcal{X}} \left( \sum_{\theta \in \Theta} x(\theta) f(\theta) \right) \text{co} \mathcal{W}(f^x) \geq \max_{x \in \mathcal{X}} \left( \sum_{\theta \in \Theta} x(\theta) f(\theta) \right) \mathcal{W}(f^x).$$

The above inequality can be strict because the domain of $\mathcal{W}$, $\Delta(\Theta)$, is a strict superset of $M_f = \{f^x : x \in \mathcal{X}\}$, and thus the concave closure may lie strictly above the global maximum of the function $\mathcal{W}$ on $M_f$. The conclusion of Proposition 4 fails when the designer can observe the type of the agent.
A.7 Proof of Proposition 7

The problem (2.19) of the regulator can be equivalently stated as

\[
\max_{x \in X, p \in [0, 1]} \int_0^p [\lambda v(\theta) + (1 - \lambda)\theta - k] x(\theta)f(\theta)d\theta + \int_p^1 (\theta - k)x(\theta)f(\theta)d\theta
\]

subject to

\[
\int_0^p (v(\theta) - p)x(\theta)f(\theta)d\theta \geq 0.
\]

We can solve the problem in two steps, by first optimizing over \(x\), and then over \(p\). For a fixed \(p\), the problem is linear, and we can apply optimal control techniques. I will show that for any \(p\), the optimal \(x\) is a threshold rule.

I will first prove that \(x(\theta) = 1\) for \(\theta \geq p\) at the optimal solution. In the case \(p \geq k\), this is obvious. Suppose that \(p < k\). Then, \(x(\theta) = 1\) for \(\theta \geq k\), and \(x(\theta) = x(p)\) for \(\theta \in (p, k)\). The last conclusion follows from the fact that the objective function is maximized by minimizing \(x\) point-wise in the interval \((p, k)\) and \(x\) has to be non-decreasing. Because the objective function is linear in \(x\) on \([0, k]\), and the constraint is preserved when \(x\) is multiplied by a positive scalar, we must have either \(x(p) = 0\) or \(x(p) = 1\) (boundary solution). In the first case, we conclude that \(x(\theta) = 1\{\theta \geq k\}\), and thus it is impossible that \(p < k\). In the second case, we obtain the desired conclusion.

By the above paragraph, we can ignore the term \(\int_p^1 (\theta - k)x(\theta)f(\theta)d\theta\) in the optimization. To deal with the constraint (A.11), I introduce an auxiliary state variable \(\Gamma\) with \(\Gamma'(\theta) = (v(\theta) - p)x(\theta)f(\theta)\), \(\Gamma(0) = 0\) and \(\Gamma(p) \geq 0\). By Mangasarian Sufficiency Theorem (see for example Seierstad and Sydsaeter, 1987), to prove optimality of a feasible candidate solution \(x^*\), it is enough to find a continuous and piece-wise continuously differentiable function \(q(\theta)\) such that, for all \(\theta \in [0, 1]\),

\[
x^*(\theta) \in \arg \max_x H(x, \theta, q) \equiv \arg \max_x (\lambda v(\theta) + (1 - \lambda)\theta - k + q(\theta)(v(\theta) - p)) x(\theta)f(\theta),
\]

\[
q'(\theta) = 0, \quad q(p) \geq 0 \quad (= 0 \text{ if } \Gamma(p) > 0)
\]

\(H(x, \theta, q(\theta))\) is concave in \(x\).

Define \(q(\theta) \equiv q_0 \geq 0\). Then, \((\lambda v(\theta) + (1 - \lambda)\theta - k + q(\theta)(v(\theta) - p))\) is strictly increasing, so the optimal \(x^*\) is equal to \(1_{\{\theta \geq r\}}\) for some \(r \in [0, 1]\). Because \(H\) is linear in \(x\), and there exists an \(r \in [0, 1]\) such that (A.11) holds, an optimal solution exists that takes this form.
Because an optimal \( x \) is a threshold rule for every \( p \), it is without loss of generality to restrict attention to threshold rules when looking for the solution to problem (A.10).

Abusing notation slightly, let \( p(r) = p(1_{\theta \geq r}) \), where \( p(x) \) for \( x \in \mathcal{X} \) is defined in (2.18). Then, the optimal allocation function is given by \( x^*(\theta) = 1_{\{\theta \geq r^*_{\text{eff}}\}} \), where

\[
r^*_{\text{eff}} = \arg\max_r \int_r^{p(r)} [\lambda v(\theta) + (1 - \lambda)\theta - k] f(\theta) d\theta + \int_1^{p(r)} (\theta - k) f(\theta) d\theta.
\]

This finishes the proof of Proposition 7. I will prove some properties of \( r^*_{\text{eff}} \) that will be useful in further proofs.

Let’s define \( \bar{r} \) as

\[
\int_r^1 (v(\theta) - 1) f(\theta) d\theta = 0. \tag{A.7}
\]

Because \( v(1) > 1 \), \( \bar{r} \) is well defined, and by the assumption \( \int_1^k (v(\theta) - 1) f(\theta) < 0 \), we have \( \bar{r} > k \). It follows that if \( p(r^*_{\text{eff}}) = 1 \), then \( r^*_{\text{eff}} = \bar{r} \). On the other hand, \( r^*_{\text{eff}} \) cannot be lower than \( \underline{r} \) defined by

\[
\lambda v(\underline{r}) + (1 - \lambda)\underline{r} = k. \tag{A.8}
\]

Suppose that \( r < \bar{r} \). For \( r \in [\underline{r}, \bar{r}] \), we have

\[
\int_r^{p(r)} (v(\theta) - p(r)) f(\theta) d\theta = 0. \tag{A.9}
\]

Using the implicit function theorem

\[
p'(r) = \frac{(v(r) - p(r)) f(r)}{(v(p(r)) - p(r)) f(p(r)) - F(p(r)) + F(r)}. \]

The denominator must be negative because, by definition of \( p(r) \), the expression \( \int_r^p (v(\theta) - p) f(\theta) d\theta \) changes sign from positive to negative as a function of \( p \) at \( p = p(r) \). Moreover, \( v(r) < p(r) \) because \( v \) is strictly increasing. Thus, \( p'(r) > 0 \).

In the opposite case \( r \geq \bar{r} \), we have \( r^*_{\text{eff}} = \underline{r} \). Indeed, it is never optimal to choose an \( r \) below \( \underline{r} \), and because in this case \( p(r) = 1 \) for all \( r \geq \underline{r} \), it is also suboptimal to choose \( r > \underline{r} \).
A.8 Derivation of the objective function for subsection 2.6.2

Given \((x, \pi)\), the final (expected) allocation \(y\) after the second stage is given by

\[
y(\theta) = \lambda \int_S x(\theta) 1_{\{\theta \geq p(f^*)\}} d\pi(s \mid \theta) + (1 - \lambda)x(\theta),
\]

for all \(\theta\), where \(d\pi(\cdot \mid \theta)\) is the distribution over signals conditional on \(\theta\), and \(p(f^*)\) is the equilibrium price given posterior belief \(f^*\) (note that \(f^*\) depends on \(x\)). Using the envelope formula, we can calculate expected utility \(U(\theta)\) of type \(\theta\) as

\[
U(\theta) = U(0) + \int_0^\theta y(z) dz = U(0) + \lambda \int_0^\theta \left( \int_{s \in S} x(z) 1_{\{z \geq p(f^*)\}} d\pi(s \mid z) \right) dz + (1 - \lambda) \int_0^\theta x(z) dz.
\]

In a profit-maximizing mechanism, \(U(0) = 0\), thus expected transfers are given by

\[
t(\theta) = \lambda \left( \int_S \max(\theta, p(f^*)) d\pi(s \mid \theta) x(\theta) - \int_0^\theta \left( \int_S x(z) 1_{\{z \geq p(f^*)\}} d\pi(s \mid z) \right) dz \right) + (1 - \lambda) \left( \theta x(\theta) - \int_0^\theta x(z) dz \right).
\]

Using integration by parts, seller’s expected profit can be expressed as

\[
\int_0^1 \int_S \left[ \lambda (p(f^*)) 1_{\{\theta \leq p(f^*)\}} + 1_{\{\theta > p(f^*)\}} J(\theta) \right] d\pi(s \mid \theta) x(\theta) f(\theta) d\theta,
\]

where \(J(\theta) \equiv \theta - (1 - F(\theta))/f(\theta)\) is the virtual surplus function. The objective function takes the form (2.8) (in its continuous version, formally introduced in Section 3), so we can apply Proposition 4 (extended to a continuous type space in Section 3) to conclude that the optimal mechanism reveals no information. Thus, the problem is to maximize

\[
\int_0^{p(x)} (\lambda p(x) + (1 - \lambda)J(\theta) - k) x(\theta) f(\theta) d\theta + \int_1^{p(x)} (J(\theta) - k) x(\theta) f(\theta) d\theta.
\]

\footnote{See Section 3 for a formal derivation of the continuous type space model.}
A.9 Proof of Proposition 8

The problem of the seller can be equivalently stated as

$$\max_{x \in X, p \in [0, 1]} \int_0^p [\lambda p + (1 - \lambda)J(\theta) - k] x(\theta) f(\theta) d\theta + \int_0^1 (J(\theta) - k) x(\theta) f(\theta) d\theta \quad (A.10)$$

subject to

$$\int_0^p (v(\theta) - p)x(\theta) f(\theta) d\theta \geq 0. \quad (A.11)$$

Because I assumed that $J(\theta)$ is non-decreasing, by the same argument as in the proof of Proposition 7, the optimal $x^*$ is without loss of generality a threshold rule: $x^*(\theta) = 1_{\{\theta \geq r\}}$ for some $r \in [0, 1]$.

Then the optimal threshold level $r^*_{rev}$ can be defined as

$$r^*_{rev} = \arg\max_{r} \int_r^{p(r)} [\lambda p(r) + (1 - \lambda)J(\theta) - k] f(\theta) d\theta + \int_{p(r)}^1 (J(\theta) - k) f(\theta) d\theta.$$

To prove the second part of the Proposition, assume first that we are in the case $r < \bar{r}$, where $\bar{r}$ and $\underline{r}$ are defined by equation (A.8) and (A.7). Because $J(\theta) \leq \theta$, we can show (just like for $r^*_{eff}$ in the proof of Proposition 7) that $r^*_{rev} \geq \underline{r}$. Next, we have

$$V_{rev}(r) = V_{eff}(r) - \int_r^{p(r)} \left( \lambda(v(\theta) - p(r)) + (1 - \lambda) \frac{1 - F(\theta)}{f(\theta)} \right) f(\theta) d\theta - \int_{p(r)}^1 (1 - F(\theta)) d\theta.$$

For $r \in (\underline{r}, \bar{r})$, using equation (A.9), we obtain,

$$V^\prime_{rev}(r) = V^\prime_{eff}(r) + (1 - \lambda)(1 - F(r)) + \lambda(1 - F(p(r))p'(r) > V^\prime_{eff}(r).$$

Because of the assumption $v(0) < k$, $r^*_{eff}$ is never equal to 0, and due to $v(1) > 1$, it is never equal to 1. Thus, the first-order condition must hold: $V^\prime_{eff}(r^*_{eff}) = 0$. It follows that if $r^*_{eff} < \bar{r}$, we must have $r^*_{rev} > r^*_{eff}$. By definition of $\bar{r}$, $p(r^*_{eff}) < 1$.

In the opposite case $r^*_{eff} = \bar{r}$, we have $p(r^*_{eff}) = 1$, and I need to show that $r^*_{rev} \geq \bar{r}$ with equality for sufficiently large $\lambda$. By the above analysis, we know that $r^*_{rev}$ cannot be strictly lower than $\bar{r}$, so I only need to prove that there exists $\lambda^* < 1$ such that $r^*_{rev} = \bar{r}$ for all $\lambda \geq \lambda^*$. 
For \( r \geq \bar{r} \), because \( p(r) = 1 \), we have

\[
V_{\text{rev}}(r) = \int_{r}^{1} (\lambda + (1 - \lambda)J(\theta) - k) f(\theta) d\theta.
\]

Because \( J \) is increasing, \( r = \bar{r} \) is a solution if and only if

\[
\lambda + (1 - \lambda)J(\bar{r}) - k \geq 0.
\]

This allows us to define

\[
\lambda^* = \begin{cases} 
\frac{k - J(\bar{r})}{1 - J(\bar{r})} & \text{if } J(\bar{r}) \leq k \\
0 & \text{if } J(\bar{r}) > k.
\end{cases}
\]

Clearly, \( \lambda^* < 1 \), and due to \( v(1) > 1 \), \( \bar{r} < 1 \), so \( \lambda^* \) is well defined.

Finally, I consider the second case \( r \geq \bar{r} \) (when \( r^*_{\text{eff}} = r \)). By definition of \( \bar{r} \), \( p(r^*_{\text{eff}}) = 1 \), so I have to prove that \( r^*_{\text{rev}} \geq \bar{r} \) with equality if and only if \( \lambda \) is sufficiently high. This follows from the same reasoning as above, where in the derivation of \( \lambda^* \), \( \bar{r} \) is replaced by \( r \). By definition (A.8) and assumption \( v(1) > 1 \), we have \( \bar{r} < 1 \), so \( \lambda^* < 1 \) is well defined.

### A.10 Example for Subsection 2.6.2

Suppose that \( \lambda = 1 \), \( v(\theta) = \delta \theta + \Delta \) for \( \delta \in (0, 2) \), \( \Delta \in [0, k] \), \( \delta + \Delta > 1 \), and \( F \) is the uniform distribution on \([0, 1]\). If the allocation function gives the good to all types above threshold \( r \), the price in the aftermarket is

\[
p_r = \max \left\{ \frac{\delta r + 2\Delta}{2 - \delta}, 1 \right\}.
\]

Define \( r = (k - \Delta)/\delta \), \( \bar{r} = (2 - \delta - 2\Delta)/\delta > k \) (by assumption 2.17), and assume that \( \bar{r} \leq \bar{r} \).\(^{45}\) Then, by direct calculation,

\[
r^*_{\text{eff}} = \begin{cases} 
\frac{4(\delta - 1)A + (2 - \delta)^2k}{\delta(4 - 3\delta)} & \text{if } \delta \leq \frac{4 - 2\Delta - 2k}{3 - k} \\
\bar{r} & \text{if } \delta > \frac{4 - 2\Delta - 2k}{3 - k}
\end{cases}
\]

\(^{45}\) In the opposite case \( \bar{r} > \bar{r} \), we would have \( r^*_{\text{eff}} = r^*_{\text{eff}} = \bar{r} \).
and

\[ r_{\text{rev}}^* = \begin{cases} \frac{\delta - 2\Delta + (2 - \delta)k}{2\delta} & \text{if } \delta \leq \frac{4 - 2\Delta - 2k}{3 - k} \\ \bar{r} & \text{if } \delta > \frac{4 - 2\Delta - 2k}{3 - k} \end{cases}. \]

For example, when \( \delta = 1 \), we have \( r_{\text{eff}}^* = k \), and \( r_{\text{rev}}^* = 1/2 - \Delta + k/2 > k \). The efficient and profit-maximizing allocations are closer when the cost is higher, and when the lemons problem becomes less severe (\( \Delta \) rises).

### A.11 Regulation - Information Disclosure Requirements

In Subsection 2.6.1, I assumed that the regulator can impose the whole mechanism frame. In Subsection 2.6.2, the mechanism was chosen by the seller. In this appendix, I study an intermediate setting in which the seller decides about the allocation rule, and the regulator imposes a disclosure policy (requirement). I focus on the case \( r_{\text{eff}}^* < r_{\text{rev}}^* \).

I consider two versions of the problem. In the first, the seller chooses a mechanism \((x, \pi)\), and then the regulator can impose a (potentially different) revelation policy \( \pi' \) treating \( x \) as given. In the second, the regulator chooses a revelation policy first, and then the seller best-responds with a mechanism that implements this revelation policy.

To formulate the first version of the problem, let \( V_{\text{rev}}(x, \pi) \) and \( V_{\text{eff}}(x, \pi) \) be the expected revenue to the seller and expected social surplus, respectively, when the mechanism is \((x, \pi)\). Let

\[ \pi^*(x) \in \arg\max_{\pi: (x, \pi) \text{ is a cut. mech.}} V_{\text{eff}}(x, \pi) \]

be a selection from the set of optimal revelation policies for any given allocation rule \( x \), and consider the problem

\[ \max_{x \in \mathcal{X}} V_{\text{rev}}(x, \pi^*(x)). \quad (A.12) \]

A solution to problem \( (A.12) \) is given by \( x^*(\theta) = 1_{\{\theta \geq r_{\text{rev}}^*\}} \). Indeed, if the seller chooses \( x^* \), the corresponding distribution over cutoffs is degenerate (deterministic). In any cutoff mechanism, by definition, \( \pi(s|\theta)x^*(\theta) \) does not depend on \( \theta \) for all \( \theta \geq r_{\text{rev}}^* \), thus any feasible signal is uninformative of the type of the agent who acquired the asset.\(^{46}\) It follows that \((x^*, \pi^*(x^*))\) reveals no information, and hence is equivalent to the seller’s preferred mechanism from Subsection 2.6.2. Because \( x^* \) achieves the upper bound on the seller’s profit, it is an optimal choice for \( (A.12) \). The regulation is ineffective.

\(^{46}\) This conclusion is also true when we look at all implementable mechanisms, not just cutoff mechanisms, see Part II of this paper (TBA).
The above formulation gives the seller an ability to commit to an allocation rule. I now analyze a model in which the regulator has the first-mover advantage. Because only information about the cutoffs can be revealed in a cutoff mechanism, suppose that the regulator can choose an arbitrary signal function \( \gamma : C \rightarrow S \) which specifies disclosure of cutoffs in a mechanism chosen by the seller. Given \( \gamma \), the seller chooses \( x \in X \) to maximize \( V_{rev}(x, \pi) \), where \( \pi \) is determined by \( x \) and \( \gamma \) according to (2.1).

Again, the seller can achieve her optimal profit from subsection 2.6.2 by choosing \( x^\star(\theta) = 1_{\{\theta \geq \hat{r}_{rev}\}} \), regardless of the choice of the regulator. When \( x^\star \) is chosen by the seller, every \( \gamma \) leads to the same uninformative signal because the distribution of the cutoff is degenerate. The regulator cannot increase total surplus by requiring information to be revealed.

B Proofs and Supplementary Materials for Section 3

B.1 Proof of Lemma 2

The proof is similar to the proof of Lemma 1, so I will skip some analogous details.

I first prove that condition \((M)\) is necessary. Fix a mechanism frame \((x, \pi, \theta_i > \hat{\theta_i} \text{ and } \theta_{-i})\). Since \((x, \pi)\) is assumed implementable, condition \((IC)\) has to hold for \(\theta_i\) and \(\hat{\theta_i}\). In particular, type \(\theta_i\) cannot find it profitable to report \(\hat{\theta_i}\), and vice versa. Summing up the two resulting inequalities, we get rid of transfers and obtain (using the fact that the mechanism is simple)

\[
\sum_{s \in S} \left[ u_i(\theta_i; f_i^s, e) - u_i(\hat{\theta}_i; f_i^s, e) \right] \left[ \pi_i(s|\theta_i, \theta_{-i})x_i(\theta_i, \theta_{-i}) - \pi_i(s|\hat{\theta}_i, \theta_{-i})x_i(\hat{\theta}_i, \theta_{-i}) \right] \geq 0.
\]

(B.1)

For the sake of simplifying the expressions, let \(\alpha_s(\tau) \equiv u_i(\tau; f_i^s, e)\) and \(\beta_s(\tau) \equiv \pi_i(s|\tau, \theta_{-i})x_i(\tau, \theta_{-i})\). By the richness assumption, there exist \(f \in F\), and \(e \in E\) such that conditions (3.3) and (3.4) hold. Under these \(f\) and \(e\), inequality (B.1) becomes

\[
\sum_{\{s \in S: \beta_s(\theta_i) < \beta_s(\hat{\theta}_i)\}} \left[ \alpha_s(\theta_i) - \alpha_s(\hat{\theta}_i) \right] \left[ \beta_s(\theta_i) - \beta_s(\hat{\theta}_i) \right] \geq 0,
\]

with \(\alpha_s(\theta_i) > \alpha_s(\hat{\theta}_i)\) for each signal \(s\) in the summation, by condition (3.3). We have thus obtained that a sum of strictly negative terms is non-negative. This is only possible when the set of indices in the sum is empty: \(\{s \in S: \beta_s(\theta_i) < \beta_s(\hat{\theta}_i)\} = \emptyset\). Because
\( \theta_i > \hat{\theta}_i \) were arbitrary, condition \((M)\) holds for every signal \( s \). And because \( i \) and \( \theta_{-i} \) were arbitrary, the first part of Lemma 2 is proven.

To prove that condition \((M)\) implies implementability for any \( f \in F \) and \( e \in E \), I will use a condition for checking implementability in arbitrary type and allocation spaces from Dworczak and Zhang (2015). Fixing an arbitrary \( \theta_{-i} \), it is enough to check that \((IR)\) and \((IC)\) hold for agent \( i \). When the type space is continuous, I use the result from the appendix in Dworczak and Zhang (2015) which states that it is enough to check the condition for all finite subsets of the type space.

Because \( \mathcal{A} \) is monotone, for any \( i, f \in F, \) and \( e \in E, u_i(\theta; f^s_i, e) \) is non-decreasing in \( \theta \). Similarly as in the proof of Lemma 1, it is enough to prove that condition \((B.1)\) holds for any \( \theta_i, \hat{\theta}_i, \) and \( \theta_{-i} \). The fact that \( u_i(\theta; f^s_i, e) \) is non-decreasing in \( \theta \) implies that the first square bracket is non-negative in each term of the sum, and condition \((M)\) implies the same about the second square bracket. Thus, inequality \((B.1)\) always holds, because the left hand side is a sum of non-negative terms.

### B.2 Proof of Theorem 1

Theorem 1 becomes an easy corollary of Lemma 2. I only used the assumption that \( \mathcal{A} \) is monotone in the part of the proof of Lemma 2 which established that condition \((M)\) is sufficient for flexibility with respect to \((F, E)\). If \((x, \pi)\) is a cutoff mechanism, condition \((M)\) holds trivially (by definition of cutoff mechanisms).

### B.3 Proof of Theorem 2

In the proof of Lemma 2, only the richness assumption was used to establish that condition \((M)\) is necessary for flexibility with respect to \((F, E)\). Therefore, it is enough to prove that condition \((M)\) implies that a simple mechanism frame \((x, \pi)\) is a cutoff mechanism.

Fix \( f \in F, e \in E, i \in N \) and \( \theta_{-i} \in \Theta_{-i} \). Let \( \beta_s(\tau) \equiv \pi_i(s|\tau, \theta_{-i})x_i(\tau, \theta_{-i}) \). By Lemma 2, \( \beta_s(\tau) \) is a non-decreasing function, for any \( s \). Summing over \( s \in S \), we get that \( x_i(\theta_i, \theta_{-i}) \) is non-decreasing in \( \theta_i \).

I will now construct the signal function \( \gamma \) that satisfies equation \((3.2)\). Each \( \beta_s(\tau) \) is a measurable function of \( \tau \) because both \( x_i(\tau, \theta_{-i}) \) and \( \pi_i(s|\tau, \theta_{-i}) \) are measurable in \( \tau \). Because \( \beta_s(\tau) \) is non-decreasing, it has one-sided limits everywhere and is continuous almost everywhere. According to the convention that I identify mechanisms that differ
on a measure-zero set of types, it is without loss of generality to assume that $\beta_s(\tau)$ is right-continuous in $\tau$. It follows that $\beta_s$ induces a positive measure $\mu_s$ on $C_i$ defined by

$$
\mu_s((a, b] \cap C_i) = \beta_s\left(\max_{\nu \in [a, b] \cap C_i} b'\right) - \beta_s\left(\min_{\nu' \in [a, b] \cap C_i} b'\right),
$$

for any interval $(a, b]$ in $[0, 1]$ (a $\sigma-$additive measure on the Borel $\sigma-$field is uniquely defined by the values it takes on intervals).

I will show that the measure $\mu_s$ is absolutely continuous with respect to the distribution of the random cutoff $dx_i(\cdot, \theta_{-i})$. For any $a, b \in C_i$, $a < b$, we have

$$
\beta_s(b) - \beta_s(a) \leq \sum_{s \in S} [\beta_s(b) - \beta_s(a)] = x_i(b, \theta_{-i}) - x_i(a, \theta_{-i}).
$$

It follows that if $x_i(b, \theta_{-i}) = x_i(a, \theta_{-i})$, then $\beta_s(b) - \beta_s(a) = 0$. Thus, $\mu_s$ is absolutely continuous with respect to $dx_i(\cdot, \theta_{-i})$.

By the Radon-Nikodym Theorem, there exists a measurable positive function $g_s$ on $C_i$ which is a density of $\mu_s$ with respect to $dx_i(\cdot, \theta_{-i})$. In particular,

$$
\pi_i(s | \theta_i, \theta_{-i}) x_i(\theta_i, \theta_{-i}) \equiv \mu_s([0, \theta_i] \cap C_i) = \int_0^{\theta_i} g_s(c) dx_i(c, \theta_{-i}),
$$

for all $\theta_i$ and each $s \in S$. Moreover, we have

$$
\sum_{s \in S} \pi_i(s | \theta_i, \theta_{-i}) x_i(\theta_i, \theta_{-i}) = x_i(\theta_i, \theta_{-i}) = \int_0^{\theta_i} \sum_{s \in S} g_s(c) dx_i(c, \theta_{-i}),
$$

thus, for all $\theta_i$,

$$
\int_0^{\theta_i} \left(\sum_{s \in S} g_s(c) - 1\right) dx_i(c, \theta_{-i}) = 0.
$$

Because the above equality holds for all integrals over an interval of the form $[0, \theta_i]$, by a standard argument, it holds for all integrals over Borel subsets of $[0, 1]$. It follows that $\sum_s g_s(c) = 1$, $dx_i-$almost everywhere. Therefore, we can define

$$
\gamma_i(s | c, \theta_{-i}) = g_s(c),
$$

for $dx_i-$almost all $c \in C_i$ (and in an arbitrary way on the remaining set of $c$ of $dx_i-$measure 0). Then, $\gamma_i$ is a well defined signal function, and that equation (3.2) from
the definition of cutoff mechanisms holds for all \( s \), and all \( \theta_i \). Because \( i \) and \( \theta_{-i} \) were arbitrary, \((x, \pi)\) is a cutoff mechanism.

### B.4 Proof of the claim in Example 4

The proof that the resale game satisfies the richness condition is very similar to the part of the proof of Lemma 1 contained in Appendix A.1, and I thus skip some details.

Fix a mechanism frame \((x, \pi), i \in N, \theta_i > \hat{\theta}_i, \) and \( \theta_{-i} \). I have to find a prior joint distribution \( f \) such that conditions (3.3) and (3.4) hold. Suppose first that we consider discrete distributions.

Let \( f = \times_{j \in N} f_j \) be a product pmf with marginals \( f_j \), for \( j \in N \). For any \( j \neq i \), let \( f_j(\theta_j) = 1 \) (degenerate distribution), and let \( \text{supp}(f_i) = \{\theta_i, \hat{\theta}_i\} \). Define

\[
p^{*}\pi_{(x, \pi)}(s; f, \theta_{-i}) \in \arg\max_{p} \sum_{\theta \leq p} (v - p)\pi_i(s|\theta, \theta_{-i})x(\theta, \theta_{-i})f_i(\theta),
\]

as the optimal price quoted by the third party when she makes an offer, conditional on signal \( s \). Price \( \hat{\theta}_i \) is uniquely optimal if

\[
(\theta_i - \hat{\theta}_i)\pi_i(s|\hat{\theta}_i, \theta_{-i})x_i(\hat{\theta}_i, \theta_{-i})f_i(\hat{\theta}_i) > (v - \theta_i)\pi_i(s|\theta_i, \theta_{-i})x_i(\theta_i, \theta_{-i})f_i(\theta_i).
\]

Price \( \theta_i \) is uniquely optimal if the opposite strict inequality holds. Define \( f_i \) as the unique distribution such that \( f_i(\hat{\theta}_i)/f_i(\theta_i) = (v - \theta_i)/\theta_i \). The choice of \( f \) implies that in the absence of additional information, the third party is indifferent between prices \( \theta_i \) and \( \hat{\theta}_i \).

If the agent makes an offer, because the value of the third party is constant, she quotes a price \( v \) and extracts the whole surplus.

Given the aftermarket, recalling that \( \eta > 0 \) is the probability that the third party makes the offer, we have

\[
u_i(\theta; f^*_i, e) = \eta \max\{\theta, p(f^*_i)\} + (1 - \eta)v,
\]

where \( p(f^*_i) \in \arg\max_{p} \sum_{\theta \leq p} (v - p)f^*_i(\theta) \).

To prove (3.3), suppose that for some \( s \in S \), \( \pi_i(s|\theta_i, \theta_{-i})x_i(\theta_i, \theta_{-i}) < \pi_i(s|\hat{\theta}_i, \theta_{-i})x_i(\hat{\theta}_i, \theta_{-i}) \). By the way \( f \) was chosen, we have \( p(f^*_i) = \hat{\theta}_i \) in that case, and thus

\[
u_i(\theta_i; f^*_i, e) = \eta\theta_i + (1 - \eta)v > \eta\hat{\theta}_i + (1 - \eta)v = u_i(\hat{\theta}_i; f^*_i, e).
\]
On the other hand, when \(\pi_i(s|\theta_i, \theta_{-i})x_i(\hat{\theta}_i, \theta_{-i}) > \pi_i(s|\hat{\theta}_i, \theta_{-i})x_i(\hat{\theta}_i, \theta_{-i})\), we have \(p(f^*_i) = \theta_i\), and thus

\[ u_i(\theta_i; f^*_s, e) = \eta \theta_i + (1 - \eta)v = \eta \max\{\hat{\theta}_i, p(f^*_i)\} + (1 - \eta)v = u_i(\hat{\theta}_i; f^*_i, e), \]

which yields condition (3.4).

Now suppose that we look at continuous distributions. Consider a sequence \(f^n_j\) of continuous distributions converging to the discrete distribution \(f_j\) described above, for any \(j\). By Remark 1 and direct inspection, the richness condition holds.

The proof that the post-mechanism interaction has monotone payoffs is very simple and thus omitted.

### B.5 Infinite signal spaces in cutoff mechanisms

This section presents supplementary results for Subsection 3.1.2. I extend Theorem 1 and Theorem 2 to mechanisms with infinite signal spaces, and prove an additional approximation result. Proofs are collected at the end of this appendix.

The conclusion of Theorem 1 extends to all cutoff mechanisms – a cutoff mechanism is always implementable if the aftermarket is monotone.

**Theorem 1’.** For any set of environments \(E\), if \(A\) is monotone under \(E\), a cutoff mechanism is implementable for any prior distribution \(f\), and any environment \(e \in E\).

Extending Theorem 2 is more difficult because with an infinite signal space \(S\), the richness condition cannot in general be expressed via equations (3.3) and (3.4). For a general signal function \(\pi\), Bayes’ rule may not be applicable. It is then difficult to associate each signal realization with a posterior belief. If the measure \(d\pi_i(\cdot|\theta)\) is continuous for all \(i\) and \(\theta\), equations (3.3) and (3.4) are well defined when we interpret each \(\pi_i\) as a density.

**Theorem 2’.** Suppose that a mechanism frame \((\mathbf{x}, \pi)\), whose \(\pi\) is a continuous distribution over \(S\) for all \(i\) and \(\theta\), is flexible with respect to \((\mathcal{F}, \mathcal{E})\). Further, suppose that for all \(i \in N\), types \(\theta_i > \hat{\theta}_i\) and \(\theta_{-i}\), there exists a prior distribution \(f \in \mathcal{F}\) and an environment \(e \in \mathcal{E}\) such that equations (3.3) and (3.4) hold, with each \(\pi_i\) interpreted as a density. Then, \((\mathbf{x}, \pi)\) is a cutoff mechanism.

The proof of Theorem 2’ is fully analogous to the proof of Theorem 2 (with sums replaced by integrals, and statements “for all s” replaced by “for almost all s”), and is
thus omitted.

Theorem 2′ holds well beyond the case of continuous distributions over signals. For example, the conclusion remains true when the signal is deterministic conditional on each type profile. What is needed is a version of Bayes’ rule, and a consistent interpretation of equations (3.3) and (3.4). Instead of pursuing the most general statement, I offer a result which provides a different justification for looking at cutoff mechanisms.

I say that a sequence of mechanism frames \{((x, \pi^n))\}_{n=1}^{\infty} on the same signal space \(S\) converges to \((x, \pi)\), if \(\pi^n(\cdot | \theta)x(\theta)\) converges to \(\pi(\cdot | \theta)x(\theta)\) in the weak* topology of measures on \(S\), for almost all \(\theta\). A mechanism frame with an infinite signal space but finite support of the signals is considered simple.

**Proposition 21.** A mechanism frame \((x, \pi)\) is a cutoff mechanism if and only if it is the limit of simple cutoff mechanisms with the same allocation function \(x\).

Proposition 21 implies that in rich settings only cutoff mechanisms can be approximated with flexible mechanisms admitting a finite signal space.

### B.5.1 Proof of Theorem 1′

The proof is analogous to the part of the proof of Lemma 2 which demonstrates that simple cutoff mechanisms are always implementable.

An analog of equation (B.1) is necessary and sufficient for implementability (by the same argument):

\[
\int_S \left[ u_i(\theta_i; f_i^s, e) - u_i(\hat{\theta}_i; f_i^s, e) \right] \left[ \frac{d\pi_i(s \mid \theta_i, \theta_{-i})x_i(\theta_i, \theta_{-i})}{\pi_i(s \mid \hat{\theta}_i, \theta_{-i})x_i(\hat{\theta}_i, \theta_{-i})} \right] \geq 0.
\]

(B.2)

Using the definition of cutoff mechanisms, we have

\[
\int_S u_i(\tau; f_i^s, e) d\pi_i(s \mid \tau, \theta_{-i})x_i(\tau, \theta_{-i}) = \int_S u_i(\tau; f_i^s, e) \int_0^{\theta_i} d\gamma_i(s \mid c, \theta_{-i})dx_i(c, \theta_{-i}).
\]

Thus, equation (B.2) becomes

\[
\int_S \int_{\hat{\theta}_i}^{\theta_i} \left[ u_i(\theta_i; f_i^s, e) - u_i(\hat{\theta}_i; f_i^s, e) \right] d\gamma_i(s \mid c, \theta_{-i})dx_i(c, \theta_{-i}) \geq 0,
\]

which is always true because the integrand is positive by the assumption that the post-mechanism game has monotone payoffs.
B.5.2 Proof of Proposition 21

First, suppose that a sequence of simple cutoff mechanisms \( \{(x, \pi^n)\}_{n=1}^{\infty} \) converges to some mechanism frame \((x, \pi)\). I have to show that \((x, \pi)\) is a cutoff mechanism.

Fix \( \theta \) and \( i \in \mathbb{N} \). Convergence in the weak* topology means that for any continuous bounded function \( g \) on \( S \), we have

\[
\lim_n \int_{S} g(s) d\pi^n_i(s|\theta_i, \theta_{-i}) x_i(\theta_i, \theta_{-i}) = \int_{S} g(s) d\pi_i(s|\theta_i, \theta_{-i}) x_i(\theta_i, \theta_{-i}).
\]

Because for each \( n \), \((x, \pi^n)\) is a (simple) cutoff mechanism, we have

\[
\int_{S} g(s) d\pi^n_i(s|\theta_i, \theta_{-i}) x_i(\theta_i, \theta_{-i}) = \int_{S} g(s) \int_{0}^{\theta_i} d\gamma^n_i(s|c, \theta_{-i}) dx_i(c, \theta_{-i}),
\]

for some probability measure \( \gamma^n_i \) on \( S \). By the Banach-Alaoglu theorem, the set of probability measures is compact in the weak* topology, so (after passing to a subsequence if necessary) we can assume that \( \gamma^n_i \) converges to some \( \gamma_i \). Thus

\[
\lim_n \int_{S} g(s) d\gamma^n_i(s|c, \theta_{-i}) = \int_{S} g(s) d\gamma_i(s|c, \theta_{-i}).
\]

By the Fubini’s theorem, and the Lebesgue dominated convergence theorem,

\[
\lim_n \int_{S} g(s) \int_{0}^{\theta_i} d\gamma^n_i(s|c, \theta_{-i}) dx_i(c, \theta_{-i}) = \lim_n \int_{0}^{\theta_i} \left( \int_{S} g(s) d\gamma^n_i(s|c, \theta_{-i}) \right) dx_i(c, \theta_{-i})
= \int_{0}^{\theta_i} \left( \int_{S} g(s) d\gamma_i(s|c, \theta_{-i}) \right) dx_i(c, \theta_{-i}) = \int_{S} g(s) \int_{0}^{\theta_i} d\gamma_i(s|c, \theta_{-i}) dx_i(c, \theta_{-i}).
\]

Combining the above equations,

\[
\int_{S} g(s) d\pi_i(s|\theta_i, \theta_{-i}) x_i(\theta_i, \theta_{-i}) = \int_{S} g(s) \int_{0}^{\theta_i} d\gamma_i(s|c, \theta_{-i}) dx_i(c, \theta_{-i}).
\]

Because the above equality is true for all continuous bounded functions \( g \), the two measures must be equal, i.e.

\[
\pi_i(S|\theta_i, \theta_{-i}) x_i(\theta_i, \theta_{-i}) = \int_{0}^{\theta_i} \gamma^n(S|c, \theta_{-i}) dx_i(c, \theta_{-i}),
\]

for all measurable \( S \subseteq S \), which shows that \((x, \pi)\) is a cutoff mechanism.
Conversely, suppose that \((x, \pi)\) is a cutoff mechanism. I have to find a sequence \(\{(x, \pi^n)\}_{n=1}^{\infty}\) of simple cutoff mechanisms that converges to \((x, \pi)\).

Fix \(\theta\) and \(i \in N\), and consider the measure \(d\gamma_i(\cdot \mid \theta_i, \theta_{-i})\) satisfying equation (3.5), defined on \(S \equiv \Theta\). Take a discrete approximation of \(d\gamma_i(\cdot \mid \theta_i, \theta_{-i})\), i.e. a sequence \(\{d\gamma^n_i(\cdot \mid \theta_i, \theta_{-i})\}_{n=1}^{\infty}\) of finite-support measures on \(S\) which converges in weak* topology to \(\gamma_i\) (such a discrete approximation always exists).

For each \(n\), define a mechanism frame \((x, \pi_n)\) by
\[
\pi^n_i(S \mid \theta_i, \theta_{-i}) = \int_{0}^{\theta_i} \gamma^n_i(S \mid c, \theta_{-i}) dx_i(c, \theta_{-i}),
\]
for all \(\theta_i, i \in N\), and measurable \(S \subseteq S\). Because \(\gamma^n_i\) has finite support, \((x, \pi^n)\) is a simple cutoff mechanism. By the same argument as in the first part of the proof, \((x, \pi)\) is a limit of \(\{(x, \pi^n)\}_{n=1}^{\infty}\).

**B.6 Proof of Proposition 9**

Take an arbitrary symmetric cutoff mechanism \((x, \pi)\), and a prior distribution \(f\). Because \(\pi(S \mid \theta_i, \theta_{-i}) x(\theta_i, \theta_{-i})\) is non-decreasing in \(\theta_i\) for each \(\theta_{-i}\), \(\pi_f(S \mid \theta)x_f(\theta)\) is non-decreasing in \(\theta\). Taking \(S = S\), we conclude that \(x_f(\theta)\) is non-decreasing.

To show existence of \(\gamma\) in the definition of a reduced-form cutoff mechanism, I will use part of the proof of Theorem 2 in Appendix B.3. Without loss of generality, I can take \(S \equiv \Theta \subseteq [0, 1]\). Denote \(\beta_s(\tau) \equiv \pi_f([0, s] \cap \Theta \mid \tau)x_f(\tau)\), for any \(s \in [0, 1]\). This time, \(\beta_s\) corresponds to the probability that the signal lies below \(s\), to account for the fact that \(S\) can be an infinite space (in which case it doesn’t make sense to talk about the probability of the event that the signal is equal to \(s\)). We know that \(\beta_s(\tau)\) is non-decreasing in \(\tau\). By the same argument as in the proof of Theorem 2, we obtain
\[
\pi_f([0, s] \cap \Theta \mid \theta)x_f(\theta) = \int_{0}^{\theta} g_{[0, s]}(c) dx_f(c),
\]
for some density function \(g_{[0, s]}(c)\), for any \(s \in [0, 1]\). Let \(G_c(s) \equiv g_{[0, s]}(c)\). By direct inspection of the above equality, \(G_c(0) = 0, G_c(1) = 1\), for \(dx_f\)–almost all \(c \in C\). I will show that \(G_c(s)\) is non-decreasing in \(s\), for \(dx_f\)–almost all \(c \in C\). To see that, consider
\(s < s',\) and note that
\[
\int_0^\theta g_{[0, s']} (c) dx_f (c) = \pi_f ([0, s] \cap \Theta) x_f (\theta) = \pi_f ([0, s] \cap \Theta) x_f (\theta) + \pi_f ((s, s'] \cap \Theta) x_f (\theta)
\]
\[
= \int_0^\theta g_{[0, s]} (c) dx_f (c) + \int_0^\theta g_{(s, s']} (c) dx_f (c),
\]
where \(g_{(s, s']} (c)\) is another non-negative density whose existence follows from the argument used above (with \(S = (s, s']\)). We obtain
\[
\int_0^\theta \left[ g_{[0, s']} (c) - g_{[0, s]} (c) - g_{(s, s']} (c) \right] dx_f (c) = 0,
\]
for all \(\theta\). It follows that \(g_{[0, s']} (c) = g_{[0, s]} (c) + g_{(s, s']} (c)\) for \(dx_f\)–almost all \(c\), and in particular, because \(g_{(s, s']} (c)\) is non-negative, \(g_{[0, s]} (c) \geq g_{[0, s]} (c)\), or \(G_c (s') \geq G_c (s)\).

Finally, using the monotonicity property established above and equation (B.3), \(G_c (s)\) is right-continuous, for \(dx_f\)–almost all \(c\).

Therefore, \(G_c (s)\) is a cumulative distribution function, for \(dx_f\)–almost all \(c\). We can thus define \(\gamma\), for \(dx_f\)–almost all \(c \in C\), by
\[
\gamma([0, s] \cap \Theta | c) = G_c (s),
\]
for any \(s \in [0, 1]\). (It is irrelevant how we define \(\gamma\) on the remaining \(dx_f\)–measure zero set of points \(c\).)

Because the distribution \(\gamma\) is uniquely determined by the value it assigns to all sets of the form \([0, s] \cap \Theta\) for all \(s \in [0, 1]\), by equation (B.3) we get
\[
\pi_f (S | \theta) x_f (\theta) = \int_0^\theta \gamma (S | c) dx_f (c),
\]
for all measurable \(S \subseteq S\). This finishes the proof.

**B.7 Proof of Theorem 3**

Fix a reduced-form cutoff mechanism \((\bar{x}, \bar{\pi})\) under prior distribution \(f\). First, suppose that \((\bar{x}, \bar{\pi})\) is simple, i.e. the signal space \(S\) is finite. By definition of reduced-form mechanisms, there exists a joint (symmetric) allocation function \(x\) such that \(\bar{x} = x_f\).
Define $\pi : \Theta^N \rightarrow \Delta(S)$ by

$$
\pi(s | \theta_i, \theta_{-i}) = \pi(s | \theta_i),
$$

for all $s \in S, \theta_i \in \Theta, \theta_{-i} \in \Theta^{N-1}$. Then, $(x, \pi)$ is a symmetric, Bayesian implementable mechanism frame such that $(x_f, \pi_f) = (\bar{x}, \bar{\pi})$.

The goal is to define a symmetric cutoff mechanism $(x^*, \pi^*)$ which induces the same reduced-form: $(x_f^*, \pi_f^*) = (\bar{x}, \bar{\pi})$.

To use the proof technique of Gershkov et al. (2013), I introduce the following notation. Let $\mathcal{K} = (N \cup \{0\}) \times S$ be the set of social alternatives, where an outcome $k = (i, s)$ is interpreted as player $i$ getting the object ($i = 0$ denotes the mechanism designer) and signal $s$ being sent. An allocation function in this setting is defined as an element of the set $\mathcal{Y} = \{\{y^{i,s}(\theta)\}_{i,s} : y^{i,s}(\theta) \geq 0, \sum_{i \in N,s \in S} y^{i,s}(\theta) \leq 1, \forall \theta\}$. Define an allocation

$$
x^{i,s}(\theta) = x(\theta_i, \theta_{-i}) \pi(s | \theta_i, \theta_{-i}),
$$

for all $i \in N$, and $\theta \in \Theta^N$, as the probability that outcome $(i, s)$ is implemented in the mechanism $(x, \pi)$ ($x^0$ is defined as the residual probability). Clearly, $\{x^{i,s}\} \in \mathcal{Y}$. The following lemma follows directly from the results of Gershkov et al. (2013).

**Lemma 3** (Gershkov, Goeree, Kushnir, Moldovanu and Shi, 2013). Suppose that for allocation $\{x^{i,s}\}$, $\sum_{\theta_{-i} \in \Theta^{N-1}} x^{i,s}(\theta_i, \theta_{-i}) f_{-i}(\theta_{-i})$ is non-decreasing in $\theta_i$, for all $i \in N$, $s \in S$. Define $\{y^{i,s}\}$ as the solution (which is guaranteed to exist) to the problem

$$
\min_{\{y^{i,s}\} \in \mathcal{D}} \sum_{\theta \in \Theta^N} \sum_{i \in N,s \in S} (y^{i,s}(\theta))^2,
$$

where

$$
\mathcal{D} = \left\{\{y^{i,s}\} \in \mathcal{Y} : \sum_{\theta_{-i} \in \Theta^{N-1}} y^{i,s}(\theta_i, \theta_{-i}) f_{-i}(\theta_{-i}) = \sum_{\theta_{-i} \in \Theta^{N-1}} x^{i,s}(\theta_i, \theta_{-i}) f_{-i}(\theta_{-i}), \forall i, \theta_i, s\right\}.
$$

Then, $y^{i,s}(\theta_i, \theta_{-i})$ is non-decreasing in $\theta_i$, for all $\theta_{-i}$, and all $i, s$.

The allocation function $\{x^{i,s}\}$ satisfies the assumption of Lemma 3 because

$$
\sum_{\theta_{-i} \in \Theta^{N-1}} x^{i,s}(\theta_i, \theta_{-i}) f_{-i}(\theta_{-i}) = \sum_{\theta_{-i} \in \Theta^{N-1}} \bar{\pi}(s | \theta_i) x(\theta_i, \theta_{-i}) f_{-i}(\theta_{-i}) = \bar{\pi}(s | \theta_i) \bar{x}(\theta_i),
$$

and the last expression is non-decreasing in $\theta_i$ because $(\bar{x}, \bar{\pi})$ is a reduced-form cutoff.
mechanism. Given the allocation \(\{y^{i,s}\}\) produced from \(\{x^{i,s}\}\) by Lemma 3, I now define a mechanism \((x^*, \pi^*)\) by

\[
x^*(\theta_i, \theta_{-i}) = \sum_{s \in S} y^{i,s}(\theta_i, \theta_{-i}),
\]

and

\[
\pi^*(s|\theta_i, \theta_{-i}) = \frac{y^{i,s}(\theta_i, \theta_{-i})}{x^*(\theta_i, \theta_{-i})},
\]

with \(\pi^*(s|\theta_i, \theta_{-i})\) defined in an arbitrary way for \(x^*(\theta_i, \theta_{-i}) = 0\). The pair \((x^*, \pi^*)\) is a well-defined mechanism, and it is symmetric, without loss of generality. To show that \((x^*, \pi^*)\) is a cutoff mechanism it is enough to invoke the proof of Lemma 2 and Theorem 2; because \(\pi^*(s|\theta_i, \theta_{-i})x^*(\theta_i, \theta_{-i}) \equiv y^{i,s}(\theta_i, \theta_{-i})\) is non-decreasing in \(\theta_i\), for all \(s \in S\) and \(\theta_{-i} \in \Theta^{N-1}\), it must be a cutoff mechanism. Finally, \((x_f^*, \pi_f^*) = (\bar{x}, \bar{\pi})\) follows from the fact that \(\{y^{i,s}\} \in \mathcal{D}\), and so

\[
\sum_{\theta_{-i} \in \Theta^{N-1}} \pi^*(s|\theta_i, \theta_{-i})x^*(\theta_i, \theta_{-i})f_{-i}(\theta_{-i}) = \sum_{\theta_{-i} \in \Theta^{N-1}} y^{i,s}(\theta_i, \theta_{-i})f_{-i}(\theta_{-i})
\]

\[
= \sum_{\theta_{-i} \in \Theta^{N-1}} x^{i,s}(\theta_i, \theta_{-i})f_{-i}(\theta_{-i}) = \sum_{\theta_{-i} \in \Theta^{N-1}} x(\theta_i, \theta_{-i})\pi(s|\theta_i, \theta_{-i})f_{-i}(\theta_{-i}),
\]

for all \(s\), and \(\theta_i\), and the same calculation can be done for \(x^*\) by summing over \(s\). This finishes the proof for simple \((\bar{x}, \bar{\pi})\).

Now consider a general \((\bar{x}, \bar{\pi})\). By Proposition 21 (which applies to reduced-form cutoff mechanisms), \((\bar{x}, \bar{\pi})\) can be represented as a limit of simple reduced-form cutoff mechanisms \(\{(\bar{x}, \bar{\pi}_n)\}_{n=1}^\infty\) (which can be take to be symmetric). By the first part of the proof, for each \(n\), there exists a (simple) symmetric cutoff mechanism \((x, \pi_n)\) such that \((x_f, \pi_f^n) = (\bar{x}, \bar{\pi}_n)\). Passing to a subsequence if necessary, we can assume that \((\bar{x}, \bar{\pi}_n)\) converges to some \((x, \pi)\). Applying Proposition 21 again, we conclude that \((x, \pi)\) is a cutoff mechanism. Moreover, \((x_f, \pi_f) = (\bar{x}, \bar{\pi})\) (because this equality holds along the sequence).

\(^{47}\) One can show that there always exists a symmetric solution to problem (B.4), or symmetry can be obtained by ex-ante uniformly random permutation of identities of agents.
C  Proofs and Supplementary Materials for Section 4

C.1 Proof of Proposition 11

By Corollary 10, optimization over disclosure rules can be performed directly in the space of posterior beliefs of the winner’s type. A distribution over posterior beliefs is feasible if and only if the posterior beliefs average out to the no-communication posterior $f^x$, and each posterior belief likelihood-ratio dominates the prior $f$ (see Section 2.5 for details). Recall that $\mathcal{W}(\bar{f})$ is the expected payoff to the mechanism designer conditional on posterior belief $\bar{f}$.

To calculate $\mathcal{W}(\bar{f})$, let $\bar{y} = \mathbb{E}_{\bar{f}}(\theta)$ denote the posterior mean. Then, negotiations happen in the post-mechanism game if the realized cost $k$ does not exceed $\eta(\Delta + (\delta - 1)\bar{y})$. Therefore,

$$\mathcal{W}(\bar{f}) = \int_0^1 V(\theta, \bar{y}) \bar{f}(\theta) d\theta,$$

where

$$V(\theta, \bar{y}) = H(\eta(\Delta + (\delta - 1)\bar{y}))(\Delta + \delta \theta) + (1 - H(\eta(\Delta + (\delta - 1)\bar{y})) \theta.$$

Thus, we get

$$\mathcal{W}(\bar{f}) = H(\eta(\Delta + (\delta - 1)\bar{y}))(\Delta + (\delta - 1)\bar{y}) + \int_0^1 \theta \bar{f}(\theta) d\theta.$$

In particular, $\mathcal{W}$ depends on $\bar{f}$ only through its mean $\bar{y}$.

When $H(x)x$ is convex, $\mathcal{W}$ is convex in $\bar{f}$ (as a composition of a convex function with a linear functional). By Corollary 10, the optimal payoff is given by the concave closure of $\mathcal{W}$ in the space of distributions that likelihood-ratio dominate the prior $f$. Because $\mathcal{W}$ is globally convex, the concave closure is obtained as a convex combination of extreme points of the feasible set. Extreme points of the set of distributions that likelihood-ratio dominate the prior are truncations of the prior. Thus, the optimal mechanism induces a distribution over beliefs whose support consists of truncations of the prior. Such information structure can be obtained by fulling revealing the cutoff. Thus, disclosing the cutoff (second highest type) is optimal.

In the opposite case when $H(x)x$ is concave, $\mathcal{W}$ is concave in $\bar{f}$. Thus, the concave closure of $\mathcal{W}$ is equal to $\mathcal{W}$ at every point. Therefore, no revelation is optimal.
\section*{C.2 Proof of Proposition 12}

I consider a relaxed problem, and then show that the solution is feasible. I work with a reduced-form \((\bar{x}, \bar{\pi})\). To simplify notation in the proof, I omit the bars on \((\bar{x}, \bar{\pi})\) and denote the reduced form by \((x, \pi)\). The action of the third party is binary because the optimal price is either \(h\) or \(l\). Thus, it is without loss of generality to focus on binary signals, with each signal labeled by the price it induces in the second stage, \(h\) or \(l\).

The mechanism is characterized by two functions: \(x(\theta)\) and \(y(\theta) = x(\theta)\pi(l|\theta)\), where \(y(\theta)\) is the probability that the good is allocated and a low price is recommended. Let \(\phi(\theta) \equiv (v - h) - (v - l)(1 - \theta)\).

The relaxed problem I consider takes the form:

\[
\max_{x, y} \left\{ v \int_0^1 x(\theta)f(\theta)d\theta - (v - h) \int_0^1 \theta y(\theta)f(\theta)d\theta \right\} \tag{C.1}
\]

subject to

\[
0 \leq y(\theta) \leq x(\theta) \leq 1, \ \forall \theta, \tag{C.2}
\]

\(x, y\) are non-decreasing, \tag{C.3}

\[
\int_0^1 x(\theta)\phi(\theta)f(\theta)d\theta \geq \int_0^1 y(\theta)\phi(\theta)f(\theta)d\theta, \tag{C.4}
\]

\[
\int_\tau^1 x(\theta)f(\theta)d\theta \leq \frac{1}{N}(1 - F_N(\tau)), \ \forall \tau \in [0, 1]. \tag{C.5}
\]

The objective function (C.1) is equal to the per-agent total expected surplus. If the third party buys, the ex-post surplus is \(v\). However, when the value of the agent is \(h\) (with probability \(\theta\)), and a low price is recommended (with probability \(y(\theta)\)), resale does not happen and the surplus is \(h\) instead of \(v\). As for constraints, condition (C.2) is obviously necessary, (C.3) follows from Lemma 2, (C.4) is the obedience constraint (when high price is recommended, the third party finds it optimal to quote a high price), and (C.5) is the Matthews-Border condition.

I will solve the problem (C.1)-(C.5) in two steps. In the first step, I optimize over \(y\) treating \(x\) as given. In the second, I optimize over \(x\). For a fixed non-decreasing function \(x\), the first-step problem is

\[
\min_y \int_0^1 \theta y(\theta)f(\theta)d\theta \tag{C.6}
\]
subject to
\[ 0 \leq y(\theta) \leq x(\theta), \quad \forall \theta, \tag{C.7} \]
y is non-decreasing, \( y \) is non-decreasing,
\[ \int_0^1 x(\theta)\phi(\theta)f(\theta)d\theta \geq \int_0^1 y(\theta)\phi(\theta)f(\theta)d\theta. \tag{C.9} \]

Consider function \( y \) satisfying conditions (C.7)-(C.9), and suppose that \( y' \) satisfies (C.7) and (C.8) and is “first-order stochastically dominated” by \( y \). That is, \( \int_0^1 y(\theta)f(\theta)d\theta = \int_0^1 y'(\theta)f(\theta)d\theta \), and \( \int_0^\tau y(\theta)f(\theta)d\theta \leq \int_0^\tau y'(\theta)f(\theta)d\theta \), for all \( \tau \in [0, 1] \). Then, for any increasing function \( g : \Theta \to \mathbb{R} \),
\[ \int_0^1 g(\theta)y(\theta)f(\theta)d\theta \geq \int_0^1 g(\theta)y'(\theta)f(\theta)d\theta. \]

This implies that \( y' \) satisfies (C.9) (becuase \( g(\theta) = \phi(t) \) is an increasing function) and achieves a weakly lower value of the objective function (C.6) (because \( g(\theta) = \theta \) is an increasing function).

The above argument means that in the class of functions \( y \) with \( \int_0^1 y(\theta)f(\theta)d\theta = c \) for some fixed \( c \), a solution to the problem is given by a function \( y \) which does not first-order stochastically dominate any function \( y' \) that satisfies (C.7) and (C.8). Because \( x \) is non-decreasing, \( y \) takes the form
\[ y(\theta) = \begin{cases} x(\theta) & \theta < \theta^* \smallskip \\ \alpha & \theta \geq \theta^* \end{cases}, \tag{C.10} \]
for some \( \theta^* \in [0, 1] \), and \( \alpha \in [x^{-}(\theta^*), x^{+}(\theta^*)] \), where \( x^{-}(\theta^*) \) and \( x^{+}(\theta^*) \) denote the left and the right limit of \( x \) at \( \theta^* \), respectively.

There are two cases. If (1) \( \int_0^1 x(\theta)\phi(\theta)f(\theta)d\theta \geq 0 \), then \( y \equiv 0 \) (\( \alpha = 0, \theta^* = 0 \)) achieves the global minimum. Otherwise, in case (2), \( \alpha \in [x^{-}(\theta^*), x^{+}(\theta^*)] \) is pinned down by a binding constraint (C.9):
\[ \int_{\theta^*}^1 (x(\theta) - \alpha)\phi(\theta)f(\theta)d\theta = 0. \]
If there are multiple \((\alpha, \theta^*)\) satisfying these restrictions, then it must be that \( x(\theta) \equiv \alpha \) in some (possibly one-sided) neighborhood of \( \theta^* \), so \( y \) is defined uniquely.

I will deal with case (1) first. Because in this case a high price is always quoted in
the second stage, the problem (C.1)-(C.5) becomes

\[ \max_{x, y} \int_0^1 x(\theta) f(\theta) d\theta \]  

subject to

\[ 0 \leq x(\theta) \leq 1, \forall \theta, \]  

\[ x \text{ is non-decreasing}, \]  

\[ \int_0^1 x(\theta) \phi(\theta) f(\theta) d\theta \geq 0, \]  

\[ \int_\tau^1 x(\theta) f(\theta) d\theta \leq \frac{1}{N} (1 - F^N(\tau)), \forall \tau \in [0, 1]. \]  

Using an analogous argument as above, we can show that an optimal \( x \) must not be first-order stochastically dominated by any \( x' \) satisfying conditions (C.12), (C.13), and (C.15). Thus, an optimal \( x \) satisfies (C.15) with equality for all \( \tau \geq \beta \), and is zero on \([0, \beta]\), where \( \beta \) is the smallest number such that constraint (C.14) holds. Thus, either

\[ \int_0^1 x(\theta) \phi(\theta) f(\theta) d\theta \geq 0, \]  

in which case \( \beta = 0 \), or \( \beta > 0 \) is defined by

\[ \int_\beta^1 x(\theta) \phi(\theta) f(\theta) d\theta = 0. \]  

Since \( x \) satisfies (C.15) with equality on \([\beta, 1]\), it means that it is induced by a joint rule that gives the good to the type with the highest value, conditional on at least one type being above \( \beta \). That is

\[ x(\theta) = \begin{cases} 
0 & \theta < \beta \\
F^{N-1}(\theta) & \theta \geq \beta.
\end{cases} \]  

With \( x \) as above, equation (C.16) is equivalent to \( \mathbb{E}_f[\theta_N^{(1)}] \geq (h-l)/(v-l) \). Thus, if this condition holds, \( \beta = 0 \), and the mechanism is an efficient auction with no information revelation. Otherwise, \( \beta > 0 \), and the mechanism is an auction with a positive reserve price and no information revelation.

This summarizes how optimal mechanisms look like in case (1). I now consider case
(2), and afterwards I compare the optimal mechanisms from both cases to find the globally optimal mechanism.

In case (2), problem (C.1)-(C.5), in a relaxed form, becomes

$$
\max_{x, \theta^*, \alpha} \left\{ \int_0^{\theta^*} [v - (v - h)\theta] x(\theta) f(\theta) d\theta + v \int_{\theta^*}^1 x(\theta) f(\theta) d\theta - \alpha(v - h) \int_{\theta^*}^1 \theta f(\theta) d\theta \right\}
$$

subject to

$$
\begin{align*}
0 &\leq x(\theta) \leq \alpha, \forall \theta \leq \theta^*, \\
\alpha &\leq x(\theta) \leq 1, \forall \theta \geq \theta^*, \\
x &\text{ is non-decreasing,} \\
\int_{\theta^*}^1 (x(\theta) - \alpha) \phi(\theta) f(\theta) d\theta &\geq 0, \\
\int_{\tau}^1 x(\theta) f(\theta) d\theta &\leq \frac{1}{N}(1 - F_N(\tau)), \forall \tau \in [0, 1],
\end{align*}
$$

where the relaxation is that condition (C.22) should in fact be an equality. For a fixed $\alpha$ and $\theta^*$, $x$ should be maximized point-wise on $[\theta^*, 1]$, which means that condition (C.23) will bind everywhere on $[\theta^*, 1]$ (point-wise maximization in this interval does not interact with any other constraint). Thus, $x(\theta) = F_N^{-1}(\theta)$ for $\theta \in [\theta^*, 1]$.

Now, consider $x$ on $[0, \theta^*]$. In the objective function, in this interval, $x$ multiplies a function that is positive decreasing. This means that $x$ will be constant, equal to some $\gamma \leq \alpha$ such that condition (C.23) is satisfied. Overall, the problem boils down to

$$
\max_{\gamma \leq \alpha, \theta^*} \left\{ \gamma \int_0^{\theta^*} [v - (v - h)\theta] f(\theta) d\theta - \alpha(v - h) \int_{\theta^*}^1 \theta f(\theta) d\theta \right\}
$$

subject to

$$
\begin{align*}
\int_{\theta^*}^1 (F_N^{-1}(\theta) - \alpha) \phi(\theta) f(\theta) d\theta &\geq 0, \\
\gamma(F(\theta^*) - t) &\leq \frac{1}{N}(F_N(\theta^*) - t^N), \forall t \in [0, F(\theta^*)].
\end{align*}
$$

Constraint (C.26) can only be binding at the ends of the interval because the function on the left hand side is affine in $t$, and the function on the right is concave in $t$. Thus, the condition becomes $\gamma \leq (1/N)F_N^{-1}(\theta^*)$. Because the objective function is increasing in $\gamma$, it is optimal to set $\gamma$ to its upper bound: $\gamma = \max(\alpha, (1/N)F_N^{-1}(\theta^*))$. The
objective is also strictly increasing in $\theta^*$. This means that constraint (C.25) must bind. Suppose that $\alpha > (1/N)F^{N-1}(\theta^*)$. Then, by decreasing $\alpha$ slightly, we increase the objective function, and preserve constraint (C.25). Thus, $\gamma = \alpha = (1/N)F^{N-1}(\theta^*)$. Finally, substituting the constraint (C.25) into the objective, we can define

$$\theta^* = \arg\max_{\tau} \left\{ \frac{\int_{\tau}^{1} F^{N-1}(\theta) \phi(\theta) f(\theta) d\theta}{\int_{\tau}^{1} \phi(\theta) f(\theta) d\theta} \right\} \left\{ vF(\tau) - (v - h) \int_{0}^{1} \theta f(\theta) d\theta \right\}. \quad (C.27)$$

Because we are in case (2), by assumption, (C.25) is violated with $\theta^* = 0$. Thus, $\theta^*$ is strictly positive, and so $\alpha > 0$ as well.

The above solution is implemented by

$$x(\theta) = \begin{cases} (1/N)F^{N-1}(\theta^*) & \theta < \theta^* \\ F^{N-1}(\theta) & \theta \geq \theta^* \end{cases},$$

and $y(\theta) \equiv (1/N)F^{N-1}(\theta^*)$ which are feasible for the unrelaxed problem.

The allocation rule $x$ is implemented by giving the good to the highest type if the highest type is above $\theta^*$, and allocating the object uniformly at random in the opposite case. Suppose that the only information revealed by the mechanism is whether the second highest type was below $\theta^*$ (low signal) or above $\theta^*$ (high signal). Then, from the point of view of an agent with type $\theta$, the ex-ante probability of winning the object and a low signal is equal to $(1/N)F^{N-1}(\theta^*)$ (this is obvious when $\theta < \theta^*$, and follows from an easy calculation otherwise). Therefore, the indirect implementation described in the second part of Proposition 12 corresponds to the optimal mechanism in case (2).

I have to show that when $E_{f}[\theta_{N}^{(1)}] \geq (h - l)/(v - l)$, the mechanism from case (1) is optimal, and when $E_{f}[\theta_{N}^{(1)}] < (h - l)/(v - l)$, the mechanism from case (2) is optimal (if a regularity condition holds).

Suppose that $E_{f}[\theta_{N}^{(1)}] \geq (h - l)/(v - l)$ so that in case (1) we have $\beta = 0$. Then, the optimal mechanism in case (1) achieves the upper bound on surplus because the per-agent total surplus (C.1) is equal to

$$v \int_{0}^{1} F^{N-1}(\theta) f(\theta) d\theta = \frac{v}{N} \int_{0}^{1} dF(\theta) = \frac{v}{N},$$

and no mechanism can yield more than $v$ in total. Therefore, if $E_{f}[\theta_{N}^{(1)}] \geq (h - l)/(v - l)$, the optimal solution is to run an efficient auction with no information revelation.
Now consider $\mathbb{E}_f[\theta_N^{(1)}] < (h - l)/(v - l)$. Then, $\beta > 0$ in case (1), so the optimal mechanism in case (1) can be implemented as an auction with a reserve price and no information revelation. This proves the third part of Proposition 12. To prove the second part, given the optimal mechanism for case (1), I will construct an alternative mechanism that is feasible and yields a strictly higher value of objective (C.1) under a regularity condition. This will mean that the mechanism from case (2) must be optimal.

Fix the optimal mechanism in case (1) with $\beta > 0$. Consider an alternative mechanism, indexed by $\epsilon \geq 0$ with $y_\epsilon \equiv \epsilon$, and

$$x_\epsilon(\theta) = \begin{cases} \epsilon & \theta < \beta_\epsilon \\ F_N^{N-1}(\theta) & \theta \geq \beta_\epsilon \end{cases},$$

where $\beta_\epsilon$ is defined by

$$\int_{\beta_\epsilon}^{1} (F_N^{N-1}(\theta) - \epsilon) \phi(\theta) f(\theta) d\theta = 0. \quad (C.28)$$

At $\epsilon = 0$, $\beta(0) = \beta > 0$ (because $\beta$ is defined by equation C.17), so for small $\epsilon$, there exists a strictly positive solution $\beta_\epsilon$ to equation (C.28). Intuitively, I constructed a mechanism that takes a small step $\epsilon$ towards the optimal mechanism in case (2). For small enough $\epsilon$, constraint (C.5) holds, and constraint (C.4) is satisfied with equality given that equation (C.28) holds. Thus, the pair $(x_\epsilon, y_\epsilon)$ is feasible for small $\epsilon$.

For $\epsilon = 0$, $(x_0, y_0)$ is the optimal solution for case (1). Therefore, it is enough to show that the objective function (C.1) is strictly increasing in $\epsilon$ in the neighborhood of $\epsilon = 0$. Because the objective function is differentiable in $\epsilon$ (in particular, $\beta_\epsilon$ is differentiable in $\epsilon$ by the implicit function theorem), it is enough to show that the derivative is strictly positive at 0. Using the implicit function theorem to differentiate $\beta_\epsilon$ using equation (C.28), the (one-sided) derivative of (C.1) under the mechanism $(x_\epsilon, y_\epsilon)$ at $\epsilon = 0$ can be shown to be

$$vF(\beta) + v \frac{\int_{\beta}^{1} \phi(\theta) f(\theta) d\theta}{\phi(\beta)} - (v - h) \int_0^1 \theta f(\theta) d\theta. \quad (C.29)$$

Equation (C.17) defining $\beta$ can be written as

$$\frac{v - h}{v - l} = 1 - \mathbb{E}_f[\theta_N^{(1)} | \theta_N^{(1)} \geq \beta] \equiv 1 - \theta^{(1)}_\beta.$$ 

Given that $l > 0$, we have $(v - h)/v < 1 - \theta^{(1)}_\beta$. Moreover, by the above, $\phi(\theta) = v[\theta - \theta^{(1)}_\beta]$. 
Using these relations, to show that (C.29) is strictly positive, it is enough to show that
\[ F(\beta) \geq \frac{\int_0^1 (\theta - \theta_\beta^{(1)}) f(\theta) d\theta}{\theta_\beta^{(1)} - \beta} + (1 - \theta_\beta^{(1)}) \int_0^1 \theta f(\theta) d\theta. \]

Rearranging terms, we get
\[ \theta_\beta^{(1)} - (\theta_\beta^{(1)} - \beta)(1 - \theta_\beta^{(1)}) \int_0^1 \theta f(\theta) d\theta - \beta F(\beta) \geq \int_0^1 \theta f(\theta) d\theta. \]

Using integration by parts, and rearranging again,
\[ (1 - \theta_\beta^{(1)}) \left[ 1 + (\theta_\beta^{(1)} - \beta) \int_0^1 \theta f(\theta) d\theta \right] \leq \int_0^1 F(\theta) d\theta. \quad (C.30) \]

If inequality (C.30) holds for all \( \beta \in [0, 1] \), I will say that the distribution \( F \) satisfies the regularity condition. Under the regularity condition, I have shown that the mechanism from case (1) cannot be optimal, therefore the mechanism from case (2) must be.

In the remainder of the proof, I show that \( F(\theta) = \theta^\kappa \) satisfies the regularity condition for any \( \kappa > 0 \). I will show that a more restrictive inequality holds:
\[ \int_0^1 F(\theta) d\theta - (1 - \theta_{\beta,2}^{(1)}) \left[ 1 + (1 - \beta) \int_0^1 \theta f(\theta) d\theta \right] \geq 0, \]

where \( \theta_{\beta,2}^{(1)} \) denotes the expectation of the first-order statistic conditional on exceeding \( \beta \) for \( N = 2 \) (the smaller \( N \), the harder it is to satisfy (C.30)). By brute-force calculation, one can check that the left hand side of the above inequality is a concave function of \( \beta \). Thus, it is enough to check that the inequality holds at the two endpoints. When \( \beta = 0 \), we have
\[ \int_0^1 F(\theta) d\theta - (1 - \theta_{0,2}^{(1)}) \left[ 1 + \int_0^1 \theta f(\theta) d\theta \right] = \frac{1}{1 + \kappa} - \left( 1 - \frac{2\kappa}{2\kappa + 1} \right) \left( 1 + \frac{\kappa}{\kappa + 1} \right) = 0. \]

On the other hand, for \( \beta = 1 \), we have \( \theta_{\beta,2}^{(1)} = 1 \), and the inequality is trivially satisfied.

**Discussion**

When the distribution \( F \) fails the regularity condition, a different mechanism may be optimal. Using two signals is beneficial because it allows to always allocate the object
while still inducing the high price under the high signal. However, in a cutoff mechanism, the low signal has to be sent for higher types with at least the probability that it is sent for lower types (by Lemma 2). Thus, the low signal is sent with positive probability for types above the threshold $\theta^*$ in which case they will not resell if their second-stage value is high (which happens with relatively high probability for high types). An alternative mechanism is to only send the high signal (and hence always induce a high price) at the cost of not allocating the good to low types. This mechanism may sometimes be optimal.

D Proofs and Supplementary Materials for Section 5

D.1 Implementability in the extended model of Subsection 5.1

A cutoff mechanism in the extended model is always implementable, as long as the aftermarket is monotone. To see that, note that the mechanism designer can disclose $k$ to the agent, so the problem of implementability can be considered for every $k$ separately. It is not at all clear that cutoff mechanisms, defined in this way, are the only mechanisms with this property (under a suitable richness condition). If the agent does not observe $k$, the designer can use the mechanism as a signaling device to influence incentive-compatibility constraints. Even the formulation of the problem requires additional notation and discussion. However, an affirmative answer can be provided in two easy cases. First, if the agent observes $k$, the result follows immediately under the assumptions of Theorem 2. Second, Theorem 2 goes through (even if the agent does not observe $k$) if we additionally require that $\{(x_k, \pi_k)\}_{k \in K}$ is implementable for all prior distributions $\alpha_0$.

D.2 Proof of Proposition 14 and an example

In the extended model, I define the function

$$V_{\text{eff}}(\theta; f^*, k) \equiv 1_{\{\theta \leq p(f^*)\}}(\lambda \theta + (1 - \lambda)\theta - k) + (1 - 1_{\{\theta \leq p(f^*)\}})(\theta - k),$$

where $p(f^*)$ is the highest price at which trade happens given posterior beliefs $f^*$,

$$p(f^*) = \max\{p \in [0, 1] : \int_0^p (v(\theta) - p)f^*(\theta)d\theta \geq 0\}.$$
Then, the function $U$, defined by equation (5.2), is given by the value of the optimization problem

$$U(\alpha) \equiv \max_{\{y_k\} \in \mathcal{X}^{|K|}, p \in \mathcal{K}} \sum_{k \in \mathcal{K}} \alpha(k) \int_0^p \left[ \lambda v(\theta) + (1 - \lambda)\theta - k \right] y_k(\theta)f(\theta)d\theta + \int_0^1 (\theta - k) y_k(\theta)f(\theta)d\theta$$

subject to

$$\int_0^p (v(\theta) - p) \left( \sum_{k \in \mathcal{K}} \alpha(k)y_k(\theta) \right) f(\theta)d\theta \geq 0.$$ 

Fix $p$ in the above problem, and a feasible vector of non-decreasing allocation functions $\{y_k\}$. Using a similar argument as in the proof of Proposition 12, consider an alternative function $y'_k$ for some $k \in \mathcal{K}$ such that $y'_k$ “first-order stochastically dominates” $y_k$. That is, $\int_0^1 y_k(\theta)f(\theta)d\theta = \int_0^1 y'_k(\theta)f(\theta)d\theta$, and $\int_0^\tau y'_k(\theta)f(\theta)d\theta \leq \int_0^\tau y_k(\theta)f(\theta)d\theta$, for all $\tau \in [0, 1]$. Then, for any increasing function $g: \Theta \rightarrow \mathbb{R},$

$$\int_0^1 g(\theta)y'_k(\theta)f(\theta)d\theta \geq \int_0^1 g(\theta)y_k(\theta)f(\theta)d\theta.$$

This implies that replacing $y_k$ with $y'_k$ weakly raises the value of the objective function while keeping the solution feasible.

It follows that the optimal $y_k$ is given as the maximal element in the FOSD order defined above. Because $y_k$ has to be non-decreasing, the maximal element in the FOSD order takes the form $y_k(\theta) = 1_{\{\theta \geq \tau_k\}}$ for some $\tau_k \in [0, 1]$, for each $k$. This conclusion holds for every $\alpha$. The thresholds $\tau_k$ can (and typically will) depend on $\alpha$.

By Proposition 13, the value of the problem is given by $\text{co}U(\alpha_0)$. In other words, the optimal mechanism induces a distribution over beliefs $\alpha$ over $k$ (the beliefs average out to $\alpha_0$), and conditional on posterior belief $\alpha$, the allocation is a threshold rule for every $k$. It follows that the optimal mechanism is a posted-price mechanism, with the price pinned down by the realization of $k$ and the signal sent by the mechanism (which induces a particular belief $\alpha$). By Proposition 13 and Corollary 11, the realization of $k$ is a sufficient statistic for the signal distribution.

\[\text{This does not imply that the unconditional allocation rule } x_k(\theta) \text{ is a threshold rule. If } \varsigma \text{ denotes the optimal distribution over beliefs } \alpha, \text{ and } \tau_\alpha^k \text{ is the optimal threshold when the cost is } k \text{ and the induced belief is } \alpha, \text{ then } x_k(\theta) = \sum_{\alpha \in \text{supp}(\varsigma)} \varsigma(\alpha) 1_{\{\theta \geq \tau_\alpha^k\}}.\]
Example. The optimization problem giving rise to function $U(\alpha)$ takes the form

$$\max_{\{\tau_k\} \in [0, 1]|K|, p \in [0, 1]} \left\{ \sum_{k \in K} \alpha(k) \int_{\tau_k}^{p \lor \tau_k} [\lambda v(\theta) + (1 - \lambda) \theta - k] f(\theta) d\theta + \int_{p \lor \tau_k}^{1} (\theta - k) f(\theta) d\theta \right\}$$

subject to

$$\sum_{k \in K} \alpha(k) \int_{\tau_k \land p}^{p} (v(\theta) - p) f(\theta) d\theta \geq 0.$$  

(D.1)

The problem consists in maximizing a function over $|K| + 1$ real variables subject to a constraint. It is difficult to obtain an analytical solution due to non-differentiability at the points where $\tau_k = p$ for some $k$, and because the solution will sometimes lie on the boundary (for example, we may have $p = 1$).

Below, I present a numerical solution for the case $\lambda = 1$, $v(\theta) = \delta \theta$, $K = \{0, 1/2\}$. Let $\alpha$ be the probability that the cost is 0, and suppose that the prior distribution is uniform, $\alpha_0 = 1/2$. I will consider $\delta = 1.25$ and $\delta = 1.5$. See Figure D.1.

Consider the case of low gains from trade ($\delta = 1.25$). First, I determine the shape of the function $U$ by solving the problem (D.1) for a fixed belief $\alpha$ that the cost is equal to zero. When $\alpha$ is low (expected cost is high), the optimal $\tau_0$ and $\tau_{1/2}$ are relatively high (see the graph in bottom left corner of Figure D.1). By excluding enough low types, the mechanism induces a price $p = 1$ in the resale game, overcoming the lemons problem. As the expected cost decreases ($\alpha$ increases), it becomes less and less beneficial to keep the price $p$ high at the cost of not allocating the asset in the first stage. At some point (roughly when $\alpha$ crosses $1/2$), it is no longer optimal to alleviate the lemons problem. The aftermarket collapses (price is 0), there is no resale, and the asset is allocated to all types above the cost.

The resulting function $U$ is convex in $\alpha$, and thus it is optimal to fully disclose the cost $k$, i.e. induce degenerate posterior beliefs (represented by the two white dots in the graph in the top left corner of Figure D.1). When the realization of $k$ is 0, the mechanism discloses that $k = 0$, and allocates the good to all types. In this case, there is no positive price $p$ at which trade can happen in the second stage, so the agent is the final owner of the asset. In the opposite case $k = 1/2$, the mechanism discloses that $k = 1/2$ and allocates only to type above 0.6 which ensures that the price is 1 in the aftermarket – resale always happens. In this example, the message sent by the mechanism plays a deciding role in shaping the outcome of the market interaction.

Intuitively, full disclosure is optimal because the form of the optimal mechanism changes...
with the expected cost.

Now consider the case of high gains from trade ($\delta = 1.5$). Compared to the previous case, it becomes more beneficial to allocate to the third party. The graph in the bottom right corner of Figure D.1 shows that it is always optimal (regardless of $\alpha$) to induce a price equal to 1 in the second stage by excluding enough low types in the first stage.

In this case, the function $U$ is concave (see the top right corner of Figure D.1), so it is optimal to reveal no information. Because the form of the optimal mechanism is the same for all $\alpha$, it is easier to ensure a high resale price by pooling the two realizations of $k$ into one posterior belief.

**Fig. D.1:** Function $U$ and the optimal prices and thresholds.

### D.3 Proof of Proposition 15

The proposition follows easily from other results in Subsection 5.1.

First, consider a feasible choice of Nature to send a fully revealing signal about the cost $k$. In this case, by a simplified version of Proposition 14, the optimal mechanism is a posted-price mechanism for every realization of $k$. The mechanism only reveals information about $k$ but the exact form of the disclosure policy is irrelevant because the third party observes the cost $k$ anyway. This provides an upper bound on the expected payoff of the mechanism designer, call it $\bar{V}$.

To show that this upper bound is achieved, consider the mechanism with the same allocation functions as in the above case and full disclosure of $k$. Since $k$ is fully dis-
closed, Nature’s choice is irrelevant. Thus, this mechanism achieves the same expected payoff $\bar{V}$. Because $\bar{V}$ is the upper bound on the designer’s payoff, no other mechanism gives a better welfare guarantee.

### D.4 Additional definitions and discussion for Subsection 5.2

In this appendix, I complete the analysis of Subsection 5.2 by formalizing the notions that were used, and providing additional discussion of the results.

Let $N^t$ denote the number of active bidders at the end of round $t$. A GCA is called Markov, if $P^t$ depends on $H^{t-1}$ only through $N^{t-1}$ and $Y^t$ depends on $H^t$ only through $(N^{t-1}, N^t)$, the number of active bidders at the beginning and at the end of the last round. If the auction ends at $t$, $N^t$ can be either 0 or 1, depending on whether the winner was declared active or inactive.

A pure strategy for an agent participating in a GCA is a mapping $a_i : \Theta \times T \times H \rightarrow \{0, 1\}$, i.e., for a type $\theta \in \Theta$, in round $t \in T$, given a partial history $H^{t-1} \in H^{t-1}$, $a_i(\theta, t, H^{t-1})$ specifies whether type $\theta$ should exit in round $t$ or not. A strategy for agent $i$ is monotone if for any two types $\theta > \hat{\theta}$, any $t \in T$ and $H^{t-1} \in H^{t-1}$, we have $a_i(\theta, t, H^{t-1}) \geq a_i(\hat{\theta}, t, H^{t-1})$. A strategy is Markov if $a_i(\theta, t, H^{t-1})$ depends on $H^{t-1}$ only through $N^{t-1}$. Mixed strategies $\sigma_i$ are defined in the usual way. I call a mixed-strategy $\sigma_i$ monotone if it is a randomization over monotone pure-strategies $a_i$.

An equilibrium is a Perfect Bayesian Equilibrium of the GCA with payoffs determined by the outcome of the auction and the aftermarket $A \equiv \{u(\theta; \bar{f}, e) : \theta \in \Theta, \bar{f} \in \Delta(\Theta), e \in \mathcal{E}\}$. Given a strategy profile $\sigma$, if an agent with type $\theta$ wins the auction and signal $s$ is released, I denote the posterior belief of the winner’s type by $f^s_{\sigma}$. In that case, the ex-post payoff of the winner is $u(\theta; f^s_{\sigma}, e)$. I assume that the aftermarket is monotone.

### D.5 Proof and discussion of Proposition 16

I first introduce some notation. A deterministic price path $p = (p^t)_{t \geq 1}$, a monotone pure-strategy profile $a$, and type profile $\theta$, together pin down a unique time of exit for every agent. I let $\Gamma^{(g, p, a)}(\theta_i, \theta_{-i})$ denote the exit time of agent $i$ with type $\theta_i$, when other types are $\theta_{-i}$. Because the underlying pure strategies are assumed to be monotone (by definition of a monotone mixed strategy), $\Gamma_i$ is non-decreasing in $\theta_i$. Let

\[49\text{Given the set of feasible actions and the definition of public history, it is irrelevant whether bidders base their strategies on private or public histories.}\]
\( H_0^\tau \) denote the history in which the final winner becomes inactive in the last round \( \tau \), and let \( H_1^\tau \) denote the history in which the final winner remains active in the last round \( \tau \). For the tuple \((y, p, a)\), the corresponding allocation and revelation rule are given by

\[
\pi_i^{(y,p,a)}(s|\theta_i, \theta_{-i})x_i^{(y,p,a)}(\theta_i, \theta_{-i}) = \begin{cases} 
\left( \frac{1}{N^\tau-1} \right) Y_\tau(H_0^\tau)(s) + \left( \frac{N^\tau-1}{N^\tau-1-1} \right) Y_\tau(H_1^\tau)(s) & \text{if } \Gamma_i^{(y,p,a)}(\theta) > \tau \equiv \max_{j \neq i} \Gamma_j^{(y,p,a)} \\
\left( \frac{1}{N^\tau-1} \right) Y_\tau(H_1^\tau)(s) & \text{if } \Gamma_i^{(y,p,a)}(\theta) = \tau \equiv \max_{j \neq i} \Gamma_j^{(y,p,a)} \\
0 & \text{otherwise},
\end{cases}
\]

for all \( \theta \in \Theta \) and \( s \in S \). In the above definition, \( \tau \) denotes the last round. If the winner is the only agent who decides to stay in round \( \tau \) (first case), there is a \( 1/N^\tau-1 \) probability that the all bidders will be announced inactive, in which case the signal is drawn from the distribution conditional on history \( H_0^\tau \). Otherwise, the winner is active, and the signal is drawn from distribution \( Y_\tau(H_1^\tau) \). If all bidders decide to exit in round \( \tau \) (second case), the signal is drawn from \( Y_\tau(H_0^\tau) \) conditional on winning.

Now, for for a Generalized Clock Auction \((Y, P) = \{(Y^t, P^t)\}_{t \geq 1}\), monotone mixed strategy profile \( \sigma \), and type profile \( \theta \) we can define

\[
\pi_i(s|\theta_i, \theta_{-i})x_i(\theta_i, \theta_{-i}) = E_{(y,p)\sim(Y,P), a\sim\sigma} \pi_i^{(y,p,a)}(s|\theta_i, \theta_{-i})x_i^{(y,p,a)}(\theta_i, \theta_{-i}).
\]

Each \( \pi_i^{(y,p,a)}(s|\theta_i, \theta_{-i})x_i^{(y,p,a)}(\theta_i, \theta_{-i}) \) is non-decreasing in \( \theta_i \), for any \( s \in S \). This can be seen by direct inspection of equation (D.2). Therefore, \( \pi_i(s|\theta_i, \theta_{-i})x_i(\theta_i, \theta_{-i}) \) is also non-decreasing in \( \theta_i \). This means that the conclusion of Lemma 2, i.e. condition \((M)\), holds. The proof of Theorem 2 implies that \((x, \pi)\) is a cutoff mechanism, and Proposition 16 is thus proven.

**Discussion** The restriction to monotone equilibria in Proposition 16 is almost without loss of generality in the sense that higher types always have a weakly stronger incentive to stay in the auction. Non-monotone equilibria are only possible when a set of types are indifferent between staying and exiting but higher types exit with higher probability.

The conclusion of Proposition 16 relies on the distinction of whether the winner is active or inactive at the end of the auction (this piece of information is included in the public history which determines the signal distribution). Informally, this is because the
auction must protect the privacy of the winner to a sufficient degree to guarantee that the disclosure rule will be always implementable. If the auction always disclosed the decision of the winner (whether she decided to exit or to stay in the last round), the implemented mechanism frame could fail to be a cutoff mechanism frame. For a fixed \((y, p, a)\) and \(\theta_{-i}\), the cutoff for agent \(i\) in GCA has a binary distribution on \(\{\theta^\tau, \theta^{\tau+1}\}\) with probabilities \(1/N^{\tau-1}\) and \((N^{\tau-1} - 1)/N^{\tau-1}\), respectively, where \(\theta^\tau\) is the smallest type of agent \(i\) who exits in round \(\tau\), and \(\theta^{\tau+1}\) is the smallest type of agent \(i\) who does not exit up to and including round \(\tau\). A cutoff mechanism only reveals the realization of the cutoff which means that the auction cannot always disclose that the type of the winner was above \(\bar{\theta}\). By introducing the first step in which a tentative winner is selected uniformly at random, I formally incorporated the determination of the cutoff into the definition of a GCA.

### D.6 Proof of Proposition 17

In the first step of the proof, given a cutoff mechanism frame \((x, \pi)\) with a decomposable allocation rule, I construct an equivalent mechanism frame \((x', \pi')\). In the second step, I show how to implement \((x', \pi')\) using a GCA.

Because \(x\) is decomposable, it can be represented as a convex combination of hierarchical allocation rules (the convex combination is finite because there are finitely many hierarchical auctions with a finite type space):

\[
x(\theta) = \sum_\alpha \lambda^\alpha x^{\kappa_1^{\alpha}...\kappa_k^{\alpha}}(\theta).
\]

Because \((x, \pi)\) is a symmetric cutoff mechanism frame, there exists a signal function \(\gamma\) such that for all \(s\), and \(\theta\),

\[
\pi(s| \theta_i, \theta_{-i})x(\theta_i, \theta_{-i}) = \sum_{c \leq \theta_i} \gamma(s|c, \theta_{-i})dx(c, \theta_{-i}),
\]

For a hierarchy \(\kappa_1, ..., \kappa_k\), define

\[
\kappa^{\kappa_1...\kappa_k}(\theta_{-i}) = \max\{\kappa_m : \kappa_m \leq \max_{j \neq i} \theta_j\},
\]

and

\[
n^{\kappa_1...\kappa_k}(\theta_{-i}) = |\{j \in N \setminus \{i\} : \theta_j \geq \kappa^{\kappa_1...\kappa_k}(\theta_{-i})\}|.
\]
In words, \( \kappa^{k_1 \cdots k_k}(\theta_{-i}) \) is the highest of the thresholds \( \kappa_{1 \cdots k} \) that at least one type in \( \theta_{-i} \) exceeds, and \( n^{k_1 \cdots k_k}(\theta_{-i}) \) is the number of types in \( \theta_{-i} \) that exceed \( \kappa^{k_1 \cdots k_k}(\theta_{-i}) \). The vector \( \nu^{k_1 \cdots k_k}(\theta_{-i}) \equiv (\kappa^{k_1 \cdots k_k}(\theta_{-i}), n^{k_1 \cdots k_k}(\theta_{-i})) \) is a sufficient statistic for \( \theta_{-i} \) needed to implement \( x_i^{k_1 \cdots k_k}(\theta_{i}, \theta_{-i}) \).

Define a symmetric cutoff mechanism frame \((x', \pi')\) by

\[
\pi'(s|\theta_i, \theta_{-i})x'(\theta_i, \theta_{-i}) = \sum_{\alpha} \lambda^\alpha \sum_{c \leq \theta_i} \gamma'(s|c, \nu^{k_1 \cdots k_k}(\theta_{-i}))dx^{k_1 \cdots k_k}(c, \theta_{-i}),
\]

for all \( s, \theta \), where the signal function \( \gamma' \) is defined by,

\[
\gamma'(s|c, \nu) = \frac{\sum_{\{\theta_{-i}: \nu=\nu^{k_1 \cdots k_k}(\theta_{-i})\}} \gamma(s|c, \theta_{-i})f_{-i}(\theta_{-i})}{\sum_{\{\theta_{-i}: \nu=\nu^{k_1 \cdots k_k}(\theta_{-i})\}} f_{-i}(\theta_{-i})},
\]

for any feasible vector \( \nu \). Note that \( x = x' \), and \( \pi' \) averages out the signal distribution under \( \pi \) across all \( \theta_{-i} \) which lead to the same allocation rule for agent \( i \). Thus, \( (x, \pi) \) and \((x', \pi')\) are equivalent. Moreover, \((x', \pi')\) can be decomposed into hierarchical mechanism frames in such a way that the allocation and signal distribution depend on \( \theta_{-i} \) only through the relevant \( \nu(\theta_{-i}) \).

In the second step of the proof, I show how to implement \((x', \pi')\) in a GCA. By definition of \((x', \pi')\), it is enough to show that the hierarchical mechanism frame

\[
\pi^{k_1 \cdots k_k}(s|\theta_i, \theta_{-i})x^{k_1 \cdots k_k}(\theta_i, \theta_{-i}) = \sum_{c \leq \theta_i} \gamma'(s|c, \nu^{k_1 \cdots k_k}(\theta_{-i}))dx^{k_1 \cdots k_k}(c, \theta_{-i}),
\]

can be implemented in a GCA with a price path that only depends on the number of active bidders, for any \( \alpha \). Indeed, the claim of Proposition 17 can then be obtained by randomizing over \( \alpha \) according to the distribution \( \{\lambda^\alpha\} \) in round 0 of the GCA.\(^50\)

In the remainder of the proof, I fix \( \alpha \) and omit it from the notation – I will denote the hierarchy to be implemented by \( \kappa_1, \ldots, \kappa_k \). The description of the auction and the equilibrium is kept informal to avoid additional notation.

First, I have to specify the signal distribution for every possible outcome of the bidding process. Without loss of generality, I can assume that the auction ends no later than in round \( k \) in equilibrium.\(^51\) If the auction ends in round \( \tau \leq k \), there are two cases.

\(^{50}\) I implicitly assume that the mechanism designer informs the bidders about the realization of \( \alpha \) in round 0, i.e. discloses which hierarchical auction will be used.

\(^{51}\) It is enough to set the price to a prohibitively high level in the subsequent round.
Either (i) all $N_{\tau-1}$ bidders become inactive in round $\tau$, or (ii) $N_{\tau-1} - 1$ bidders become inactive and exactly one bidder remains active. In case (i), the signal $s$ is drawn from distribution $\gamma'(s|\kappa_{\tau}, (\kappa_{\tau}, N_{\tau-1}-1))$. In case (ii), the signal $s$ is drawn from distribution $\gamma'(s|\kappa_{\tau+1}, (\kappa_{\tau}, N_{\tau-1} - 1))$. In particular, the signal distribution depends on the public history of the auction only through $N_{\tau-1}$ and $N_{\tau}$ (the latter variable determines which case, (i) or (ii), is used).

Second, I specify the price $p^t$, for each $t \leq k$. Proceeding recursively from the end, one can calculate the expected continuation payoff for each type $\theta$, conditional on the number of active bidders, under the assumption that in any round $t'$, exactly bidders with types above $\kappa_{t'}$ are active (this assumption pins down the posterior belief over the types of active bidders in any subgame). I set the price $p^t$ to be such that type $\kappa_{t+1}$ is indifferent between exiting and staying, given the set of active bidders. In particular, $p^t$ is only a function of $N^{t-1}$, the number of active bidders at the beginning or round $t$.

Third, I specify equilibrium strategies for bidders. In every round $t$, given the number of active bidders, type $\theta$ stays in the auction if the expected continuation payoff strictly exceeds the expected payoff from dropping out, and exits if the reverse strict inequality holds. In case of indifference type $\theta$ drops out in round $t$ if and only if $\theta < \kappa_{t+1}$.

Fourth, by the specification of the signal distribution and the fact that bidders with types $\theta \in [\kappa_t, \kappa_{t+1})$ exit in round $t$, if bidders follow the above strategies, the auction implements the desired mechanism frame $(x^{\kappa_1...\kappa_k}, \pi^{\kappa_1...\kappa_k})$. In particular, if a bidder considers a deviation, she faces a choice that is analogous to choosing a type to report given the mechanism frame, with the caveat that the agent might have access to some additional information about the types of other bidders. Because the mechanism frame is non-decreasing in $\theta_i$ conditional on every profile $\theta_{-i}$, it is also non-decreasing in $\theta_i$ given any belief about the profile $\theta_{-i}$.

Fifth, I argue why the above profile of strategies constitutes a Bayesian Perfect Equilibrium of the GCA. In every observable history of the game, bidder $i$’s beliefs about $\theta_{-i}$ coincide with the public belief, and therefore the expected continuation payoff of any $\theta_i$ can be correctly calculated given that belief. Because in round $t$ the price $p^t$ is set in such a way that type $\kappa_{t+1}$ should be indifferent between exiting or not, in any history, because of monotonicity of payoffs in the post-mechanism interaction, any type $\theta < \kappa_{t+1}$ will find it optimal to exit, and every type $\theta \geq \kappa_{t+1}$ will find it profitable to stay.
D.7 Proof of Proposition 18

I only prove the first part of the proposition because the second part (about implementing the no-revelation rule) follows from standard arguments (given that $v^\pi(\theta, \theta)$ is always non-decreasing in this case).

Suppose that $\pi$ is the full-disclosure rule. In all three designs considered in Proposition 18, a necessary condition for robust implementation of $(x, \pi)$ is that there exists an equilibrium in strictly increasing bidding strategies – otherwise, disclosing the second bid does not correspond to disclosing the second-highest value. Using the first-order condition, I can derive the unique candidate equilibrium bidding function.

In the SPA, it has to be that for the bidding function $\beta^{SPA}(\theta)$,

$$\theta \in \operatorname{argmax}_{\hat{\theta}} \int_0^{\hat{\theta}} (v^\pi(\theta, \tau) - \beta^{SPA}(\tau)) \, dF^{N-1}(\tau),$$

for any $\theta \in \Theta$. From the first-order condition,

$$\beta^{SPA}(\theta) = v^\pi(\theta, \theta).$$

Therefore, the bidding function is strictly increasing if and only if $v^\pi(\theta, \theta)$ is strictly increasing in $\theta$. Equation (D.3) holds with the above bidding function because $v^\pi(\theta, \tau)$ is non-decreasing in $\theta$. Thus, it is optimal for type $\theta$ to bid $\beta^{SPA}(\theta)$, given that other players do it as well. In this equilibrium, disclosing the price paid by the winner corresponds exactly to disclosing the value of the second-highest bidder. Thus, $(x, \pi)$ is robustly implemented.

In the FPA, for the bidding function $\beta^{FPA}(\theta)$, we must have

$$\theta \in \operatorname{argmax}_{\hat{\theta}} \int_0^{\hat{\theta}} (v^\pi(\theta, \tau) - \beta^{FPA}(\hat{\theta})) \, dF^{N-1}(\tau),$$

for any $\theta \in \Theta$. From the first-order condition,

$$\beta^{FPA}(\theta) = \int_0^{\theta} \frac{v^\pi(\tau, \tau)dF^{N-1}(\tau)}{F^{N-1}(\theta)}.$$

If $v^\pi(\theta, \theta)$ is not strictly increasing in $\theta$, there exists a distribution $f$ such that $\beta^{FPA}(\theta)$ is not strictly increasing in $\theta$. On the other hand, if $v^\pi(\theta, \theta)$ is strictly increasing in $\theta$, then the bidding function is strictly increasing for any $f$, and equation (D.4) holds – it
is optimal for type $\theta$ to bid $\beta_{FPA}(\theta)$, given that other players do it as well. Finally, in an all-pay auction, for the bidding function $\beta_{APA}(\theta)$, we must have

$$\theta \in \arg\max_{\hat{\theta}} \left\{ \int_0^{\hat{\theta}} v^*(\theta, \tau)dF^{N-1}(\tau) - \beta_{APA}(\hat{\theta}) \right\},$$  \hspace{1cm} (D.5)

for any $\theta \in \Theta$. From the first-order condition,

$$\beta_{APA}(\theta) = \int_0^\theta v^*(\tau, \tau)dF^{N-1}(\tau).$$

Because $v^*(\theta, \theta)$ is strictly positive, this bidding function is always strictly increasing. Equation (D.5) holds because $v^*(\theta, \tau)$ is non-decreasing in $\theta$ – it is optimal for type $\theta$ to bid $\beta_{APA}(\theta)$, given that other players do it as well. Thus, we have an equilibrium.

### D.8 Proof of Proposition 19 and 20

The proof closely resembles other proofs in this paper, so I will omit some details.

In the extended setting, a necessary and sufficient condition for implementability is that for all $\theta > \hat{\theta}$, distributions $f$, and monotone post-mechanism games $A_l$ and $A_w$,

$$\sum_{s \in S} \left[ u_l(\theta; f^s_l) - u_l(\hat{\theta}; f^s_l) \right] \left[ \pi_l(s|\theta)(1 - x(\theta)) - \pi_l(s|\hat{\theta})(1 - x(\hat{\theta})) \right]$$

$$\quad + \sum_{s \in S} \left[ u_w(\theta; f^s_w) - u_w(\hat{\theta}; f^s_w) \right] \left[ \pi_w(s|\theta)x(\theta) - \pi_w(s|\hat{\theta})x(\hat{\theta}) \right] \geq 0, \hspace{1cm} (D.6)$$

where $f^s_l$ is the posterior belief conditional on the agent not acquiring the object and signal $s$ being sent, and $f^s_w$ is the posterior belief conditional on the agent acquiring the object and signal $s$ being sent.

**Proof of Proposition 19.** I will show that condition (D.6) implies

$$\pi_l(s|\theta)(1 - x(\theta)) \text{ is non-decreasing,}$$  \hspace{1cm} (D.7)

and

$$\pi_w(s|\theta)x(\theta) \text{ is non-decreasing,}$$  \hspace{1cm} (D.8)
for all \( s \in S \). These two conditions imply that \( x \) has to be constant (by summing up over \( s \), we see that \( x(\theta) \) has to be non-decreasing and non-increasing).

First, set \( u_w(\theta; \bar{f}) = \theta \), for all \( \bar{f} \) and \( \theta \). Fix \( \theta > \hat{\theta} \), and let the game \( \mathcal{A}_l \) be such that \( u_l(\theta; f_i^s) = u_l(\hat{\theta}; f_i^s) \) for all \( s \in S_1 \), where

\[
S_1 = \{ s \in S : \pi_l(s|\theta)(1 - x(\theta)) > \pi_l(s|\hat{\theta})(1 - x(\hat{\theta})) \},
\]

and for \( s \notin S_1 \), let \( u_l(\theta; f_i^s) - u_l(\hat{\theta}; f_i^s) = \alpha(\theta - \hat{\theta}) \) for some \( \alpha > 0 \). Then, equation (D.6) becomes

\[
(\theta - \hat{\theta}) \left\{ \alpha \sum_{s \notin S_1} \left[ \pi_l(s|\theta)(1 - x(\theta)) - \pi_l(s|\hat{\theta})(1 - x(\hat{\theta})) \right] + x(\theta) - x(\hat{\theta}) \right\} \geq 0. \quad (D.9)
\]

Unless, \( S_1 = S \), so that the sum in the above condition is equal to zero, we have a contradiction for sufficiently large \( \alpha \). It follows that \( \pi_l(s|\theta)(1 - x(\theta)) > \pi_l(s|\hat{\theta})(1 - x(\hat{\theta})) \) so that condition (D.10) holds.

To show condition (D.11), take \( u_l \equiv 0 \), and use the argument from Section 2 (with \( u_l \equiv 0 \), the above framework is equivalent to the baseline model).

For the converse, a mechanism with constant allocation rule is always implementable.

**Proof of Proposition 20.** I will show that condition (D.6) is equivalent to

\[
\pi_l(s|\theta)(1 - x(\theta)) \text{ is non-increasing}, \quad (D.10)
\]

and

\[
\pi_w(s|\theta)x(\theta) \text{ is non-decreasing}, \quad (D.11)
\]

for all \( s \in S \).

The above conditions are analogous to condition (M) in Lemma 1. By the same (or analogous in case of condition D.10) argument that was used to prove that Lemma 1 implies Proposition 2, (D.10) and (D.11) imply the cutoff representation. Conversely, if \( (x, \pi_l, \pi_w) \) is a cutoff mechanism, conditions (D.10) and (D.11) are satisfied.

First, assume that conditions (D.10) and (D.11) hold. I will show that (D.6) holds. Under the condition that \( \mathcal{A}_w \) is single-crossing-separated from \( \mathcal{A}_l \), we have

\[
\sum_{s \in S} \left[ u_l(\theta; f_i^s) - u_l(\hat{\theta}; f_i^s) \right] \left[ \pi_l(s|\theta)(1 - x(\theta)) - \pi_l(s|\hat{\theta})(1 - x(\hat{\theta})) \right]
\]
The above condition can be rewritten as

\[
\sum_{s \in \mathcal{S}} \left[ \pi_w(s | \theta) x(\theta) - \pi_w(s | \hat{\theta}) x(\hat{\theta}) \right]
\]

\[
\geq d(\theta, \hat{\theta}) \sum_{s \in \mathcal{S}} \left[ \pi_l(s | \theta) (1 - x(\theta)) - \pi_l(s | \hat{\theta}) (1 - x(\hat{\theta})) \right] + d(\theta, \hat{\theta}) \sum_{s \in \mathcal{S}} \left[ \pi_w(s | \theta) x(\theta) - \pi_w(s | \hat{\theta}) x(\hat{\theta}) \right]
\]

\[
= d(\theta, \hat{\theta}) \left[ (1 - x(\theta)) - (1 - x(\hat{\theta})) \right] + d(\theta, \hat{\theta}) \left[ x(\theta) - x(\hat{\theta}) \right] = 0
\]

For the converse part, assume condition (D.6). To show condition (D.10), fixing \( \theta > \hat{\theta} \), take \( \mathcal{A}_w \) such that \( u_w(\theta; f^*_w) - u_w(\hat{\theta}; f^*_w) \equiv d(\theta, \hat{\theta}) \). Then, consider a game \( \mathcal{A}_l \) that gives differences in payoffs \( u_l(\theta; f^*_l) - u_l(\hat{\theta}; f^*_l) = d(\theta, \hat{\theta}) \) for \( s \notin \mathcal{S}_1 \) and \( u_l(\theta; f^*_l) - u_l(\hat{\theta}; f^*_l) = 0 \) otherwise.\(^{52}\) Then, condition (D.6) implies

\[
x(\theta) - x(\hat{\theta}) \geq \sum_{s \notin \mathcal{S}_1} \left[ \pi_l(s | \theta) (1 - x(\theta)) - \pi_l(s | \hat{\theta}) (1 - x(\hat{\theta})) \right]
\]

The above condition can be rewritten as

\[
x(\theta) - x(\hat{\theta}) \geq \sum_{s \notin \mathcal{S}_1} \pi_l(s | \theta) (1 - x(\theta)) - \sum_{s \notin \mathcal{S}_1} \pi_l(s | \hat{\theta}) (1 - x(\hat{\theta}))
\]

\[
= \left( 1 - \sum_{s \in \mathcal{S}_1} \pi_l(s | \hat{\theta}) \right) (1 - x(\hat{\theta})) - \left( 1 - \sum_{s \notin \mathcal{S}_1} \pi_l(s | \theta) \right) (1 - x(\theta)),
\]

which implies

\[
0 \geq \sum_{s \in \mathcal{S}_1} \left[ \pi_l(s | \theta) (1 - x(\theta)) - \pi_l(s | \hat{\theta}) (1 - x(\hat{\theta})) \right]
\]

The definition of \( \mathcal{S}_1 \) together with the above equation imply that \( \mathcal{S}_1 = \emptyset \). Because \( \theta > \hat{\theta} \) were arbitrary, condition (D.10) is proven.

To show condition (D.11), I take \( u_l \equiv 0 \), and specify the game \( \mathcal{A}_w \) (fixing \( \theta > \hat{\theta} \)) by \( u_w(\theta; f^*_w) - u_w(\hat{\theta}; f^*_w) = d(\theta, \hat{\theta}) \) if \( \pi_w(s | \theta)x(\theta) \geq \pi_w(s | \hat{\theta})x(\hat{\theta}) \), and \( u_w(\theta; f^*_w) - u_w(\hat{\theta}; f^*_w) = \alpha d(\theta, \hat{\theta}) \), for some \( \alpha > 0 \). Then, by direct calculation, for sufficiently high \( \alpha \) condition (D.6) is violated unless \( \pi_w(s | \theta)x(\theta) \geq \pi_w(s | \hat{\theta})x(\hat{\theta}) \), for all \( s \in \mathcal{S} \).

This finishes the proof of Proposition 20.

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\(^{52}\) Because I can choose \( f \), it is always possible to make sure that distinct signals lead to distinct posteriors. The only exception is when two signals are indistinguishable but in this case they can be merged into one signal without impacting the analysis.