Competing for Strategic Buyers*

Vincent Meisner†

September 20, 2016

Abstract

Although revenue management markets are rarely monopolistic, this assumption is typically made in the literature. In contrast, I consider multiple sellers who in total offer $K$ identical goods to $n > K$ buyers with private persistent valuations. Goods are traded in continuous time before some deadline. All buyers enter the market simultaneously, are fully forward-looking and do not discount. I find an equilibrium in which allocations, prices and payoffs are equivalent under monopoly and oligopoly, if a monopolist optimally sells all goods with probability one. In this equilibrium, all sellers set identical prices that decrease synchronously and continuously and jump after each sale. There is no incentive to undercut competitors’ prices, because each seller anticipates that, by letting her rivals sell out their goods, she will become a monopolist. However, if a monopolist prefers to withhold capacity, prices and industry profits under monopoly are higher compared to oligopoly.

JEL-Classification: D43, L13, D82, D4, L1, D45.
Keywords: Dynamic pricing, oligopoly, price posting, competition, revenue management, forward-looking buyers, strategic buyers.

VERY PRELIMINARY, PLEASE DO NOT CIRCULATE

*I thank Volker Nocke, Martin Peitz, Nicolas Schutz, Emanuele Tarantino, Thomas Tröger and Christoph Wolf for useful comments and suggestions. Latest version available at http://sites.google.com/site/vincentmeisner/ or via email.
†Technical University Berlin, vincent.meisner@gmail.com.
1 Introduction

In this paper, I investigate the interaction between forward-looking buyers and multiple sellers in a continuous-time revenue management setting. Perhaps surprisingly, allocations, prices, joint industry profits and buyer payoffs are equivalent under monopoly and oligopoly if a monopolist prefers to sell efficiently all her goods with probability one. For example, for uniformly distributed valuations, such a pricing strategy is optimal when sellers are unable to commit to future prices and goods are sufficiently scarce. In contrast, if a monopolist can commit to future prices, she only wants to sell her full capacity if she values the good sufficiently less than the lowest buyer type. The irrelevance of the distribution of goods over sellers is driven by the insight that a seller can let her competitors sell their entire stock, and then gain a monopoly continuation payoff. Hence, she is not willing to undercut every positive price. The results follow because intertemporal arbitrage of the forward-looking buyers entails martingale equilibrium prices. In equilibrium, a seller is, at each point in time, indifferent between selling at the current price and letting a competitor sell at that price and instead having the next trade at the same price in expectation. In contrast, if a monopolist in expectation profits from withholding some capacity with positive probability, the profit of a monopolist is higher than the industry profit of oligopolists. For example, this condition holds when sellers can commit to future prices in “no-gap cases”.

Since the Airline Deregulation Act of 1978, revenue management (RM) has been a standard business practice to price airline tickets and subsequently became a tool to price goods in a wide range of industries with similar characteristics, for example, cruise ships, hotels, rental cars, seasonal clothing, freight, electricity or sporting and entertainment events. Key business conditions conducive to RM are that (i) customers have heterogeneous valuations, (ii) (short-term) capacity is fixed and (iii) the goods lose their value after a deadline. Although none of the industries mentioned above is monopolistic, the literature on RM in oligopolistic settings is scant. In this paper, I ask how the interaction between forward-looking buyers and competing sellers shapes market outcomes.

I consider a RM environment in which $M \geq 2$ price-posting sellers desire to sell $K$ homogeneous goods, which are exogenously distributed among the sellers. Sellers can post prices at any point in a time continuum before the deadline. All $n$ buyers enter the market at the same time, privately draw a persistent valuation for the good, and strategically time their purchase decision. Importantly, the good is scarce, $n > K$. Sellers exit the market once they are stocked out and buyers exit the market once their single-unit demand is satisfied.

I find that for all model parameters such that a monopolist would optimally sell her goods with certainty, it is irrelevant for consumer rents and industry profits how the $K$ units of the good are distributed among sellers. Hence, a thorough understanding of the monopoly benchmark is essential for the analysis of the oligopoly setting. The monopoly benchmark for the case without price commit-
ment is provided by Hörner and Samuelson (2011). Their most important result for my setting is that a monopolist with $K$ goods replicates an efficient Dutch auction when facing $n > K + 5$ buyers with uniformly distributed value. Unfortunately, the analysis is quite involved, making it hard to expand this result qualitatively to other distributions.

To grasp the intuition behind the oligopoly prices, suppose that two sellers, each offering one good, jointly replicate sequential Dutch auctions without reserve prices: At first, they simultaneously post the choke price and then synchronously and continuously decrease the price until a sale occurs and the corresponding seller exits the market. Immediately after the sale, the remaining seller discontinuously raises the price to a choke level and continuously decreases it until the next sale occurs. The price must jump to avoid frenzies as in Bulow and Klemperer (1994), because supply decreased relative to demand. Because in sequential auctions forward-looking buyers arbitrage away any differences between current and expected future prices, both sellers are at each point in time indifferent between selling at the current price and letting the competitor sell and then replicating a Dutch auction in the monopoly continuation game. Consequently, the same price path arises, leading to the same allocation and the same expected payoffs per trade for all players as in a setting in which a monopolist sells two goods in an auction without reserve price.

However, a monopolist may profit from setting an exclusive reserve price implying that she may not sell her entire capacity. In my setting, only a monopolist with the ability to commit to future prices can replicate an exclusive optimal mechanism: Prices decrease continuously, jump to a choke price immediately after each sale and finally remain at an optimal reserve price. However, in the presence of a competitor, sellers have an incentive to decrease prices further than the optimal reserve price of a monopolist. In equilibrium under oligopoly, sellers sell out over time and the terminal price is determined by the last active seller once all competitors are stocked out. Although equilibrium prices decrease continuously as well, any buyer type who would get a good under both market conditions pays a lower price. Moreover, the price decreases below the level optimal under monopoly such that possibly more goods are sold in comparison. At the time a seller becomes a monopolist, she commits to (replicating) a Dutch auction that is optimal with respect to her updated prior about the remaining buyers’ valuations. The payoffs of players are bounded from above by the monopoly payoffs (the mechanism-design optimal profit) and bounded from below by the payoffs from sequential Dutch auctions without reserve price. Consequently, prices under oligopoly are lower and buyers are better off, while competing sellers are worse off compared to a situation in which they share a jointly maximized profit.

In traditional RM models, a monopolist faces sequentially arriving and perfectly impatient buyers, but there is survey evidence that buyers strategically time

\footnote{According to the consumer report “America’s Bargain-Hunting Habits”, Apr. 30th 2014, around 60% of consumers “wait for a sale to buy what they want.” See also the survey of American Research Group, Inc on “2014 Christmas Gift Spending Plans Stall”, Nov. 21st 2014.}
their purchase decision. For a review of dynamic pricing with forward-looking consumers, consult Gönsch et al. (2013) who report losses between 7% and 50% in the surveyed articles when sellers treat forward-looking consumers as myopic. In my model, the buyers' strategic purchase timing drives an important ingredient for the equivalence result, the martingale property of prices: In equilibrium, the expected sale price of the next unit of the good is at each time equal to the current price. As a consequence, it is important for antitrust authorities to know whether buyers are forward-looking or myopic. Note that buyers are only strategic in terms of their purchase timing. They have unit demand and hence do not need to strategically select sellers to preserve competition (as in, e.g., Anton et al. (2014)).

In terms of policy advice, my findings have to be interpreted with caution as they suggest that, under reasonable conditions, an industry with RM characteristics and forward-looking buyers does not require any merger control. There is no need to protect forward-looking consumers from a monopolistic price discrimination by breaking the monopoly into several smaller firms. This benchmark result, however, raises the question of what kind of additional features have to be included into the model to yield the more intuitive result that seller competition increases buyers' rents. The insight that the irrelevance result does not hold when sellers prefer to commit to excluding low buyer types sheds light on the role of commitment in RM markets which is valuable for evaluating antitrust issues. Because the martingale property of prices is key for the results, I discuss extensions for which it is known that prices do not follow a martingale process.

My oligopoly setting emphasizes results from the sequential auctions literature from a novel angle, and thereby links two seemingly unconnected insights: First, a price posting monopolist without price commitment replicates Dutch auctions by posting continuous price paths in equilibrium and, second, prices in sequential auctions are a martingale. The martingale property of prices in sequential auctions was derived by Milgrom and Weber (2000) and sparked the academic debate around the “declining price anomaly” discussed in Section 4. Settings with interdependent valuations, unknown size of inventory or background risk would be interesting to study as such models feature upward or downward trends in prices. In light of major applications such as airline tickets or hotel rooms, the role of sequentially arriving buyers is of great interest as well.

This RM model of multiple sellers facing buyers with private information fills an important gap in the literature. One reason why current research is paradoxically silent on competing sellers in a private value environment might be that it is not clear how the buyers’ selection strategies might look like if sellers do not post identical prices. One may think about correlated equilibria or alternatively introduce a coordination device or a search game. The approach taken here is to allow at most one good to be traded at each instant and this single good is traded at the lowest current price. Either all buyers reject the posted prices or a single buyer trades and the remaining buyers face new prices in the future. This procedure has convenient implications: First, sellers’ profits feature a discontinuity reminiscent of Bertrand (1883). Second, the buyers’ optimal dynamic strategy is
easy to characterize. Third, matching frictions such as those described in Burdett et al. (2001) are circumvented in a game-theoretically consistent way.

Only allowing a single transaction at each trading instant sounds more restrictive than it actually is as the number of possible trading instances approaches infinity. Primarily, it is a succinct way to capture the idea that, following a sale, sellers can adapt prices faster than buyers can react. Alternatively, I could put all buyers into a queue in random order. Neither sellers nor buyers have any knowledge about the positions in the queue except that they are drawn uniformly at random before each purchase decision. Then, at each time buyers are released sequentially from the queue and observe the prices and how many items were sold. Because buyers are released one-by-one and have single-unit demand, a buyer’s optimal strategy is to randomize among the cheapest sellers if he wants to buy. Sellers set a menu of prices contingent on how many sales have already occurred at that time. Consequently, with each price, sellers only compete for the first purchasing buyer in the queue and then the queue is redrawn. In equilibrium, each trade occurs between a randomly chosen interested buyer (the first accepting buyer in the random queue) and a randomly chosen cheapest seller (the one randomly selected by that buyer). This approach is similar to the model by Deneckere and Peck (2012) in which, however, the queue is not reformed in each period.

Another reason why the RM literature with competing seller is relatively sparse might be that it appears to be complicated to keep track of intertemporal arbitrage conditions of buyers and sellers simultaneously. In my setting, tractability can be sustained when prices are well-behaved. Importantly, prices are driven by continuation payoffs which makes the game easy to solve when the continuation payoffs are easy to solve for. In particular, I can incorporate the tractable solution to the problem of Hörner and Samuelson (2011) as the payoff of a monopoly continuation game of my richer oligopoly setting. To construct a well-defined game in continuous time, I discretize the game and consider a limit as the discretization grid becomes finer. The technical trick that allows me to incorporate seamlessly known monopoly results is to allow the mesh of the time grid to be dependent in market states such that I can take the continuous-time limit separately for each market state.

The following subsection relates my paper to the existing literature. I present the model in Section 2. The discussion in Section 4 serves the purpose to identify which assumptions are important for the main result, Proposition 1, and touches on a few interesting modifications of the model. Finally, I conclude in Section 5.

1.1 Literature

Initiating the literature on RM, Gallego and Van Ryzin (1994) consider a single seller facing demand by short-lived buyers, whose arrival is modeled as a Poisson process with intensity $\lambda(p)$. The take-away result of such models is that average prices fall over time as the approaching deadline diminishes the option value of
selling. The main focus of this literature has been to improve the modeling of buyer behavior (such as strategic buyers) or making the monopolist’s problem more complex by introducing additional resources to manage (network RM). Talluri and Van Ryzin (2005) provide an excellent overview of RM in their book that became the main reference of the field. There have been only few studies on RM with oligopoly. One reason might be that capacity constraints are a definitive characteristic of RM models, and equilibriums in a simple static benchmark model such as Bertrand-Edgeworth competition (Edgeworth (1897)) is widely unexplored beyond special cases. In such models, it is known that assumptions about how buyers are rationed are not innocuous. In my model, efficient rationing arises endogenously. Moreover, a static model obviously cannot quantify the value of commitment to future prices like my model is able to do.

Martínez-de Albéniz and Talluri (2011) generalize the model of Dudey (1992), who shows that a dynamic version of Bertrand-Edgeworth duopoly competition has a unique subgame-perfect equilibrium. They model sequentially and randomly arriving short-lived buyers with commonly known valuations. In contrast, the buyers in my setting are long-lived and forward-looking, and have private information. Similar to my results, Martínez-de Albéniz and Talluri (2011) find that continuation payoffs determine prices. Contrary to my results, the seller with the fewest goods sells her entire stock first, always priced at the reservation value of the next smallest seller, and the largest seller sells her goods at last and at a constant monopoly price. Gallego and Hu (2014) consider a similar framework with differentiated products.

Deneckere and Peck (2012) model a perfectly competitive dynamic market with a continuum of sellers, who have to produce output in advance, and a continuum of buyers who can costly delay their purchase. Moreover, demand uncertainty is innovatively modeled through a demand state. The unobserved demand state then determines the value distribution of a new batch of buyers that joins the remaining active buyers of the previous period. Sellers price under partial knowledge of the demand state: Prices within a period rise as sellers become more optimistic about the demand realization and then prices have to be corrected when demand dries up. Prices are dispersed as some sellers only want to sell when demand is sufficiently strong. However, as a consequence of intertemporal arbitrage conditions, lowest prices available are a martingale. My model differs in multiple respects: I model oligopolistic competition for (exogenously) scarce goods, there is no buyer entry and the possibility of being rationed is the only cost from delaying purchase.

The literature on the Coase conjecture (1972) was the first to investigate the role of a seller’s (lack of) commitment power. Surprisingly, Gul (1987) and Ausubel and Deneckere (1987) show that the competitive allocation result of durable-goods monopoly (e.g. Stokey (1981), Bulow (1982), Gul et al. (1986), Ausubel and Deneckere (1989)) is reversed when additional sellers populate the model: While

the monopolist prices at marginal cost, oligopolists can attain (total industry) profits arbitrarily close to the static monopoly profit. The reason why monopoly
is more competitive than oligopoly is that a competitor can help to sustain higher
goods prices through credible punishments, which is not possible when a monopolist only
competes with the future self. In comparison, my equivalence result does not stem
from punishment strategies. In fact, strategies only depend on a market state.
The equilibrium in this paper rather reflects that the market cannot become more
competitive when the good is scarce and buyers are forward-looking, because there
is no incentive to exert competitive pressure. Therefore, despite the similarities,
the durable-goods monopolist, who can offer as many goods as buyers are present,
is not the relevant monopoly benchmark of my RM setting. In contrast, my buyers
want to buy early to avoid being rationed and the good is paid and consumed at
a fixed date in the undiscounted future. Hence, the monopoly benchmark in a
setting with price commitment is given by an optimal Dutch auction that screens
types perfectly and can maintain an exclusive reserve price and the benchmark in
a setting without price commitment is explored by Hörner and Samuelson (2011).

2 The Model

Players: I consider a dynamic game with \( M \) sellers (she) and \( n \) buyers (he) over
the normalized time interval \( \mathcal{T} := [0, 1] \). Each seller \( m \in \mathcal{M} := \{m_1, ..., m_M\} \)
is endowed with \( K^m \in \mathbb{N} \) homogeneous goods, respectively. All players simulta-
neously enter the market at time 0. Each buyer \( i \in \mathcal{I} := \{i_1, ..., i_n\} \) demands
a single good and exits the market after purchasing. Similarly, a seller exits the
market after selling all her goods. Players who have not exited the market are
called active.

Environment: The good is scarce, \( n > K := \sum K^m \). A public market state
\( \kappa_t := (K^m_{t})_{m \in \mathcal{M}} \) informs all players about the distribution of goods at each time
and thereby also about the number of active sellers \((\sum \mathbb{1}_{K^m_t \neq 0})\) and buyers \((n-(K – \sum K^m_t))\). Let \( \mathcal{K} \subset \mathbb{N}^M \) be the set of all possible market states. In the beginning
of the game, buyer \( i \) is privately informed about his persistent valuation \( v_i \), an iid
draw from a commonly known distribution with cdf \( F \) and positive density \( f \) on
support \([v, \overline{v}]\). There is no discounting.

Actions: Each player \( i \in \mathcal{I} \cup \mathcal{M} \) has a corresponding action space \( \mathcal{A}_i \), and let
\( \mathcal{A} := \times_{i \in \mathcal{I} \cup \mathcal{M}} \mathcal{A}_i \). Each seller \( m \) posts a price \( p^m_i \in \mathcal{A}_m := \mathbb{R}_+ \) at each time \( t \).
Let \( p^M := (p^m_i)_{m \in \mathcal{M}} \) denote a price menu at time \( t \). Let \( p_t := \min\{p^m_i : m \in \mathcal{M}\} \)
denote a minimum price at \( t \). Each buyer \( i \) either decides to accept the current
minimum price or to delay purchase to the next purchasing opportunity, \( d_i \in \mathcal{A}_i := \{0, 1\} \). At each time \( t \), at most one good is traded, and the trading pair
is randomly selected by nature, \((i^t, m^t) \in \mathcal{A}_N := \mathcal{I} \cup \{0\} \times \mathcal{M} \cup \{0\} \). The seller
is randomly selected among the sellers posting the lowest price at the time, some
\( m \in \{m : p^m_i \leq p^m_{i'} \forall m' \in \mathcal{M}\} \), and the buyer is randomly selected among
the accepting buyers, some \( i \in \{ i : d_i^t = 1 \} \). If no buyer wants to purchase, \((i', m') = (0, 0)\).

**Outcomes and game histories:** A buyer outcome is a function \( o^i : \mathcal{T} \to A_i \). Analogously, a seller outcome is a function \( o^m : \mathcal{T} \to A_m \), and a Nature outcome is a function \( o^N : \mathcal{T} \to A_N \). Let the set of possible outcomes for a player \( i \) be denoted by \( \mathcal{O}_i \). A game outcome is denoted by \( o := (o_i)_{i \in I \cup M \cup \{N\}} \) and \( O := \times\mathcal{O}_i \) is the set of possible outcomes of the game. That is, an outcome links each point in time to all players’ corresponding actions and the corresponding realizations of Nature moves. Let a game history at time \( t \) be denoted by \( o_t := (o_t(t'))_{t' < t} \).

**Discretization and limit:** I define the continuous-time game as the limit of a sequence of discrete-time games. More precisely, the continuous-time game consists of limits of discretized continuation games and these limits are taken separately for each market state. Let \( \Gamma^*(o_t) \) be the continuous-time continuation game following game history \( o_t \). Let \( T : K \to \mathbb{N} \) be a function that maps a market state \( \kappa \) into a number \( T(\kappa) \) of remaining periods of of length \( \Delta^T(t) = \frac{1-t}{T(\kappa)} \). In a discretized continuation game \( \Gamma^T(o_t) \) players can only set finitely many actions: If no sale occurs, they set an action at each time in discretization set \( \mathcal{T}^T(t) := \{ t + \Delta^T(t), t + 2\Delta^T(t), \ldots, 1 \} \), and, once the market state changes, the game is discretized anew such that \( T(\kappa') \) equidistant periods remain. For all times outside the grid, \( t \not\in \mathcal{T}^T \), the actions of all players are fixed to be the previously set action, but trade is prohibited,

\[
\begin{align*}
d_i^v &= d_{i(v)} \quad \forall i \in \mathcal{I}, \\
p_m^v &= p_{m(v)} \quad \forall m \in \mathcal{M}, \\
(i', m') &= (0, 0),
\end{align*}
\]

where \( \tau(t') = \min\{ \hat{t} : \hat{t} \in \mathcal{T}^T, \hat{t} \leq t' \} \). I define the continuous-time game \( \Gamma^*(o_0) \) as the limit of a sequence of discrete-time games characterized by a sequence of functions \( T \) such that \( T(\kappa) \to \infty \) for all \( \kappa \in K \).

**Beliefs:** Let a posterior following player history \( h \) be denoted by \( F_h \), an updated cdf about (other) buyers’ valuations. Belief are updated according to Bayes’ rule or \( \varepsilon \)‑consistently according to Definition 1:

**Definition 1.** A distribution \( \hat{F} \) is \( \varepsilon \)‑consistent with a true Bayesian update \( F \) of prior \( \tilde{F} \), if and only if

\[
\begin{align*}
1 \geq \frac{\hat{F}(v)}{F(v)} & \geq 1 - \varepsilon \quad \forall v > \underline{v} \quad , \quad F(v) > F(v') \Rightarrow \hat{F}(v) > \hat{F}(v') \\
1 \geq \hat{F}(v) & \forall v, v' \in [\underline{v}, \overline{v}].
\end{align*}
\]

\footnote{This assumption is merely for simplicity of notation, and could arise as the equilibrium outcome of a more complicated game. As explained in the introduction, I could let queuing buyers decide which seller to select. Since only one trade can occur at each time, in equilibrium, a buyer would randomize over the cheapest sellers.}

\footnote{This assumption is just for ease of exposition. Since there is no discounting, the length of the periods is payoff irrelevant and only the number of remaining moves matters.}
Let a seller history be denoted by $h^m \in H^m$, and a buyer history be denoted by $h^i \in H^i$, where $H^i$ is the set of all possible histories of player $i$. Players with different player histories might have different posteriors given the same game history. Let the set of all players’ posteriors following game history $o_t$ be denoted by $\mathcal{F}_{o_t} := (F_{h^i})_{i \in I \cup M}$. If $\mathcal{F}_{o_t}$ is such that all players have the same belief, I speak of common priors.

**Player histories and states:** A seller history at time $t$ is given by

$$h^m_t := (T, \kappa_t' \cdot p^M_t)_{t' < t}, \kappa_t$$

that is, a seller knows the discretization function $T$, the current market state, and remembers all past prices and market states. A buyer history at time $t$ given by

$$h^i_t := (v_i, T, (d^i_t)_{t' < t}, (p^M_t, \kappa^i_t)_{t' \leq t}).$$

Thus, compared to a seller, a buyer additionally observes current prices, recalls all of his own actions and knows his own valuation. Given a player history $h^i$, let $H^i(h^i)$ be the set of all continuation histories of $h^i$, i.e., all histories that have identical elements as $h^i$ at each position, but might be longer.

Importantly, after a sale, no player observes which or how many (other) buyers tried to purchase. They only observe a change in the market state and, hence, can infer that a sale took place. The transition dynamics of the market state is the limit of the following discrete transition dynamics, for all $m \in M$ and all $t \in \mathcal{T}^T$.

$$K^m_{t+\Delta} = \begin{cases} K^m_t - 1 & \text{if } m^t = m, \\ K^m_t & \text{otherwise.} \end{cases}$$

Every buyer history $h^i$ terminates in a buyer state $\omega^i$ that consists of the discretization, a valuation, a current minimum price $p_t$, the market state $\kappa_t$ and a belief $F_{h^i}$

$$\omega^i(h^i) := (T, v_i, p_t, \kappa_t, F_{h^i}).$$

Similarly, a seller history translates into a seller state $\omega^m$ given by

$$\omega^m(h^m) := (T, \kappa_t, F_{h^m}).$$

**Payoffs:** At any time $t \in \mathcal{T}$, a type-$v_i$ buyer’s time-$t$ utility is given by

$$u^i_t(v_i, o(t)) = \begin{cases} v_i - p_t & \text{if } o^i(t) = 1; o^N(t) = (i, m); o^m(t) = p_t, \\ 0 & \text{otherwise.} \end{cases}$$

that is, a buyer only gains a nonzero payoff if he, at some point in time, accepted a price $p_t = \min_{m \in M} \{o^m(t)\}$ and also was selected for this trade. Obtaining a good

---

*The analysis in this paper remains the same if $F_h$ is extended to arbitrarily higher-order beliefs.*
is only valued before time 1, afterwards the good loses its value. From outcome \( o \), buyer \( i \) of type \( v_i \) gains payoff

\[
U_i^i(v_i, o) = \sum_{t \in T} u_i^i(v_i, o(t))
\]  

It is commonly known that sellers do not value the good and \( v \geq 0 \). Seller \( m \)'s revenue from outcome \( o \) given by

\[
U^m(o) = \sum_{t \in \text{Sale}_m} p^m_t,
\]

where \( \text{Sale}_m = \{ t' : o^N(t') = (\cdot, m) \} \) is the set of all times when seller \( m \) traded and \( p^m_t = o^m(t) \) is the corresponding sale price.

**Strategies:** Generally, a strategy for player \( i \in I \cup M \) is a mapping \( \sigma^i : \mathcal{H}^i \to A_i \), and a strategy profile is defined as \( \sigma := (\sigma^i)_{i \in I \cup M} \). In this paper, I restrict attention to Markov strategies that impose the restriction that \( \sigma^i(h) = \sigma^i(\hat{h}) \) for any two histories that terminate in the same player state, \( \omega^i(h) = \omega^i(\hat{h}) \). In the limit \( T(\kappa) \to \infty \) for all \( \kappa \in K \), all Markov strategies are essentially stationary Markov strategies.

**Equilibrium:** I restrict attention to a tractable class of equilibria: In a **Strongly Symmetric Markov Perfect \( \varepsilon \)-Bayesian Equilibrium (SSMP\( \varepsilon \)BE),**

1. sellers post identical prices at each time (SS),
2. all players use Markov strategies (M),
3. all actions are sequentially rational, given the player histories and anticipations of optimal continuation play (P),
4. beliefs are \( \varepsilon \)-consistent (Definition 1) with beliefs derived according to Bayes’ rule (\( \varepsilon \)B).

I call this equilibrium strongly symmetric, because sellers set the same price even if they do not have the same capacity.

I analyze the model under different assumptions regarding sellers’ ability to commit to future prices. In Section 3.8 I analyze the model with full price commitment, i.e., I look for semi-perfect equilibria. They are defined as above, but bullet point 3 is replaced with

3a. buyers’ purchase decisions are sequentially rational, given the buyer history and anticipations of optimal continuation play,

3b. each seller \( m \) commits to a revenue-maximizing price plan contingent on each possible seller state in the beginning of the game.

A SSMPBE is the limit of a sequence of SSMP\( \varepsilon \)BE strategy profiles with \( \varepsilon \to 0 \).
3 Analysis

This section is structured as follows: The first subsection serves the purpose to provide an upper bound of industry profits that turns out to be helpful over the course of the analysis. The reader familiar with basic auction design with single-unit demand following Myerson (1981), Riley and Samuelson (1981) and Maskin and Riley (1989) may want to skip to the definition of Condition 1 immediately. Subsection 3.2 studies the buyers’ game and Subsection 3.3 explains how I treat the beliefs about buyers’ valuations resulting from this game. After elaborating on the sellers’ game in Subsections 3.4 and 3.5, Subsection 3.6 specifies the buyers’ stopping strategy, i.e., at which price a type-$v$ buyer wants to purchase given a pricing strategy suggested by the analysis of the sellers’ game. Subsection 3.7 describes pricing under oligopoly without price commitment, and Subsection 3.8 deals with full price commitment.

3.1 The full-commitment monopoly benchmark

Lemma 1 establishes the first important benchmark: If sellers with the ability to commit to future prices collude and jointly maximize profits, each seller $m$ can get her fraction $K_m/K$ of the (mechanism-design) optimal industry profit.

Let the $i$-th highest order statistic of $n$ draws from distribution $F$ be denoted by $Y_i^{(n)}$ such that $Y_1^{(n)} \geq \ldots \geq Y_i^{(n)} \geq \ldots \geq Y_n^{(n)}$. Moreover, let the virtual valuation be denoted by

$$
\psi(v) = v - \frac{1 - F(v)}{f(v)},
$$

(10)

The literature on auctions speaks of a regular environment when $\psi$ is a strictly increasing function of the valuation.

Lemma 1 (Monopoly, full commitment). A monopolist with the ability to commit to future prices can replicate sequential Dutch auctions with any reserve price. In regular environments, this mechanism is optimal when the reserve price $r^*$ is such that $\psi(r^*) = \max\{0, \psi(v)\}$. With $K$ goods, this price path yields a profit of

$$
\mathbb{E} \left[ \sum_{i=1}^{K} \max\{\psi(Y_i^{(n)}), 0\} \right].
$$

(11)

I omit a formal proof and provide the intuition. Since there is no competition and the price path is not restricted to be sequentially rational, a monopolist can implement the optimal allocation (Maskin and Riley (1989)) by replicating an optimal Dutch auction: In continuous time, a seller has an infinite amount of pricing possibilities and, thus, she can optimally screen and exclude buyer types by setting a continuously decreasing price path that becomes flat at an optimal reserve price $r^*$ defined by $\psi(r^*) = \max\{0, \psi(v)\}$. It is irrelevant for the monopolist’s payoff whether the price decreases rapidly or slowly because there is no discounting. By
the revenue equivalence theorem the implemented allocation yields the optimal profit. The following statement follows immediately.

**Corollary 1** (Monopoly, no exclusion). In regular environments, the optimal allocation is efficient if and only if

\[ \psi(v) \geq 0 \iff \nu f(v) \geq 1. \]  

(12)

Replicating a sequential Dutch auctions without or with non-exclusive reserve price \( r \leq v \) is optimal for the seller.

**Condition 1** (No exclusion). A monopolist wants to sell all of her goods efficiently with probability one.

Under inequality (12) below, Condition 1 holds.

### 3.2 The buyers’ cutoff valuation \( x \)

For every potential purchase, an active buyer faces a stopping problem, i.e., he chooses a point along a path of minimum prices at which he wants to apply for a good, taking as given the stopping and pricing strategies of other players. If another buyer got to buy the good at some price, a similar stopping problem arises for the next sale and so on. In the following, fix some discretization for future play \([t, 1]\).

Consider buyer \( i \) with buyer history \( h^i \) (consistent with the player state) at time \( t \), and fix an arbitrary strategy profile for the other players \( \sigma^{-i} \). Define a buyer’s expected period-\( t \) payoff from decision \( d \) at buyer history \( h^i \) as

\[ w_i(h^i, d) := \mathbb{E}_{o(t)} [u_t(v_i, o(t)| h^i, \sigma^{-i}, o^i(t) = d] = \begin{cases} \phi_t(v_i - p_t) & \text{if } d = 1, \\ 0 & \text{otherwise}, \end{cases} \]  

(13)

where \( \phi_t \) denotes the probability of being selected for purchase at time \( t \) when accepting the minimum price. Obviously, this probability depends on how many other buyers accept the price which in turn depends on the given strategy profile \( \sigma^{-i} \) and all players’ states (including the current minimum price \( p_t \)). I ignore this dependence in the notation for ease of exposition. Note, however, that it is independent of the valuation \( v_i \).

Each period, buyers trade off trying to purchase against delaying purchase. If buyer \( i \) decides to buy at time \( t \), then either he gets the good and exits the market or another buyer was awarded the good. Similarly, if \( i \) decides to delay purchase, then either some buyer obtained a good or all other buyers declined as well, and he faces a new price with an updated belief about the other buyers. Let \( \rho_t \) be the probability that all other buyers \( j \neq i \) declined at \( t \), where again I dropped the dependence on histories and strategies for simplicity of notation.
A type-\(v\) buyer’s dynamic program is given by
\[
W_t(v, h^i) := \max\{w_t(h^i, 1) + (1 - \phi_t)W_{t+\Delta}^+(v, h^i, 1), 0 + \rho_tW_{t+\Delta}^+(v, h^i) + (1 - \rho_t)W_{t+\Delta}^-(v, h^i, 0)\}
\]
with
\[
W_{t+\Delta}^+(v, h^i, d) := \mathbb{E}_h [W_{t+\Delta}(v, h)| h^i, \sigma^{-i}, d^i_t = d, o^N(t) \neq (0, 0)],
\]
and
\[
W_{t+\Delta}^-(v, h^i) := \mathbb{E}_h [W_{t+\Delta}(v, h)| h^i, \sigma^{-i}, o^N(t) = (0, 0)].
\]
where the expectation is taken with respect to the history at time \(t + \Delta\) as it is uncertain which (if any) seller has been selected at \(t\) and, thus, how the market state and prices evolve. If no sale occurred, all buyers must have delayed purchase. All players are aware of the discretization and that nothing can happen for a time interval of \(\Delta\). Since the good loses its value after the deadline, \(W_{1+\Delta}(v, h) = 0\) for any history and any \(\Delta > 0\).

I can specify the probabilities \(\phi\) and \(\rho\) in equilibrium after the statement of Lemma 2. To find a critical type, who is indifferent between taking the current and the next price, the following expected payoff is crucial: From delaying purchase and accepting the minimum price at the next opportunity, time \(t + \Delta\), buyer \(i\) garners
\[
\rho_t [\phi_{t+\Delta}(v_i - p_{t+\Delta}) + (1 - \phi_t + \Delta)\mathbb{E}_{h^i_{t+\Delta}} [W_{t+2\Delta}^+(v_i, h^i_{t+\Delta}, 1)]] + (1 - \rho_t)W_{t+\Delta}^+(v_i, h^i, 0),
\]
where \(\phi_{t+\Delta}\) is the probability that buyer \(i\) is selected for purchase at time \(t + \Delta\).

Since the good is scarce and each sale reduces the supply further, a form of discounting arises endogenously through the probability that the good becomes more expensive or sells out. Consequently, higher types are more eager to buy.

**Lemma 2.** In equilibrium, there exists an \(\omega\)-dependent cutoff type \(x_t \in [v, \overline{v}]\) such that all types \(v \geq x_t\) decide to purchase and all types \(v < x_t\) delay purchase.

**Proof.** Fix some equilibrium and consider a buyer history \(h^i_t\). Suppose that some buyer \(i\) with valuation \(v_i\) prefers to buy at price \(p_t\) over delaying purchase to the next opportunity at price \(p_{t+\Delta}\). Then it must be that the first line in (14) is larger than (18). It remains to be shown that all types \(v > v_i\) decide to buy as well. I do this by showing that the derivative of (14) with respect to \(v\) is larger than the derivative of (18), i.e.,
\[
\phi_t + (1 - \phi_t)W_t'(v_i, h^i_t, 1) \geq \rho_t [\phi_{t+\Delta} + (1 - \phi_{t+\Delta})\mathbb{E}_{h^i_{t+\Delta}} [W_{t+2\Delta}^+(v_i, h^i_{t+\Delta}, 1)]] + (1 - \rho_t)W_t'(v_i, h^i_t, 0),
\]
where \(W'\) denotes the derivative of \(W^+\) with respect to the valuation \(v\).

By the envelope theorem, any derivative \(W'(\cdot)\) is a type-independent probability (the probability that buyer \(i\) gets selected for purchase before the good is sold out). Hence, \(W_{t+2\Delta}^+(v_i, h^i_{t+\Delta}, 1)\) is bounded from above by one and it follows.
that \( \phi_{t+\Delta} + (1 - \phi_{t+\Delta}) E_{h_{t+\Delta}} [W'_{t+2\Delta}(v_i, h_{t+\Delta}^i, 1)] \leq 1 \). Therefore a sufficient condition for the inequality above is

\[
\phi_t (1 - W'(v_i, h_{t}^i, 1)) \geq \rho_t (1 - W'(v_i, h_{t}^i, 0)).
\]

If

\[
W'(v_i, h, 0) \geq W'(v_i, h, 1)
\]

holds for all buyer histories \( h \) and all \( v_i \in [\underline{v}; \overline{v}] \), \((20)\) is clearly true, because the probability that no other buyer at all accepts the price cannot be larger than the probability that no other buyer is selected for purchase by definition, \( \phi_t \geq \rho_t \): If no other buyer \( j \neq i \) accepts the price, buyer \( i \) is selected with certainty, and even when other buyers \( j \neq i \) want to purchase as well, buyer \( i \) is still selected with positive probability.

The sufficient condition \((21)\) follows as a corollary from Lemma 3 which holds for arbitrary symmetric strategy profiles. The reason is that a declining buyer attaches a higher probability to obtaining a good than an accepting buyer. Hence, any symmetric equilibrium strategy for the first purchase is necessarily a cutoff strategy.

Define

\[
x_t := \min \{v_i : (14) \geq (18) \}.
\]

This cutoff is clearly history dependent. Moreover, the cutoff is not necessarily unique for a single buyer as the \( \phi_t \) and \( \rho_t \) depend on the corresponding cutoff type \( x_t \) as well. Moreover, different buyers might have different beliefs leading to different cutoff types.

Having established this lemma, I can express the probabilities \( \phi_t \) and \( \rho_t \) as functions of cutoff valuations \( x_t \). Straightforwardly, \( \rho_t \) is the probability that all \( (n-1) \) other buyers’ types are below the cutoff. Similarly with common priors, \( \phi_t \) also is a simple function of the cutoff valuation. Given that \( j \) other buyer types are above the cutoff, the probability that a buyer who is willing to purchase at price \( p_t \) gets to buy is given by \( 1/(j+1) \).

\footnote{Mathematically, probability \( \phi \) is identical to an allocation according to a queue as motivated in the introduction. Since the queue is unobserved and redrawn uniformly at random after each sale, the probability that exactly \( j \) buyers are in front in the queue is given by \( 1/n = (n-1)!/n! \) for all integer \( j \in [0, n-1] \). Probability \( \phi \) is thus a finite geometric series weighted by \( 1/n \), leading to an equivalent formulation of \((23)\).
}

**Corollary 2.** Suppose all players have common prior \( F_h \). The probabilities \( \phi \) (probability of getting a good when accepting price \( p_t \)) and \( \rho_t \) (probability that
no buyer accepted price \( p_t \) can be expressed as

\[
\phi_t = \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{1}{j+1} F_h(x_t)^{n-1-j}(1 - F_h(x_t))^j = \frac{1 - F_h(x_t)^n}{n(1 - F_h(x_t))}
\] (23)

and

\[
\rho_t = F_h(x_t)^{n-1}.
\] (24)

It remains to be formalized how \( F_h \) is formed for player general histories. The event that no sale occurred reveals that every buyer rejected the current prices and, hence, there is no asymmetric information about the buyer decisions in the market. As a consequence, starting with a common prior, an updated common prior is maintained: All buyers and sellers learn that all buyer types are below the cutoff level. However, if a sale occurred at \( t \), it is possible that types greater than cutoff \( x_t \) remain in the market because they were not (randomly) selected for purchase. Then the analysis is complicated by the fact that different players can have different beliefs on how likely it is that such types remained in the market. This issue is picked up in Subsection 3.3.

Even with common priors at each time, there still can be multiple cutoff values \( x_t \) that solve the corresponding equation. In a strongly symmetric equilibrium, given any history \( h_t \), the prices \( p_t \) and \( p_{t+\Delta} \) together with the continuation payoffs \( W \) determine critical types \( x_t \) in the equation (14) = (18) evaluated at valuation \( x_t \).

Solving for a sequence \( (x_t) \) corresponding to players’ strategies in a discretized game involves a higher-order difference equation with boundary conditions that the first cutoff is equal to \( v \) (the highest valuation, as players have not yet learned anything) and the last cutoff is equal to the final minimum price. If multiple sales have occurred, the critical types of all rounds in which a sale occurred along the history path enter the posterior and hence the difference equation is of higher order.

Like Hörner and Samuelson (2011), I face the issue of multiple solutions as well. For general distributions, such problems often feature multiple or even no solutions at all (see, for example, Agarwal (2000)). Following your economic intuition, you may have expected multiple solutions because buyers are strategic complements: If a buyer believes all other buyers are more likely to buy, he has more incentive to buy himself. Vice versa, a buyer’s incentive to delay increases if he believes that other buyers are more likely to delay. For this reason, it is unclear whether a general characterization of equilibria for all distributions exists. Hence, Hörner and Samuelson (2011) later restrict attention to the uniformly distributed valuations for which they find a unique equilibrium.

3.3 \( \varepsilon \)-consistent beliefs: Retaining common priors

After a sale occurred, buyers and sellers update their prior differently. Moreover, accepting buyers update their prior in a different way from declining buyers, as
Figure 1: An illustration of a prior $F_h$ and two different updates: $F_{s,h}$ (dashed) following a sale, $F_{no,h}$ (dash-dotted) following no sale. Moving $x_t$ to the right decreases $\varepsilon$ and makes the posteriors approach the prior.

depicted in Figure 1. The reason is that the information contained in individual histories differs. Suppose a common prior $F_h$. Let $F_{s,h}$ denote a seller’s update of $F_h$ upon observing a sale. Similarly, let $F_{a,h}$ and $F_{d,h}$ denote an accepting and a declining buyer’s update of $F_h$ upon observing a sale, respectively. Moreover, let $F_{no,h}$ denote a buyer or seller’s update of $F_h$ upon observing that all buyers declined the prices. The calculation of the true posteriors involves a straightforward application of Bayes’ Rule that is explained in greater detail in Appendix A, where the posteriors are formally stated in Corollary 1 of Lemma 2.

**Lemma 3.** The following first-order stochastic dominance results hold for any equilibrium in the buyers’ game

$$F_{no,h}(v) > F_{d,h}(v) > F_{a,h}(v) > F_h(v),$$

for all $v, x$ such that $F_h(v) \in (0, 1), F_h(x) \in (0, 1)$. For $v, x$ such that $F_h(v), F_h(x) \in \{0, 1\}$, all four probabilities are equal.

**Proof.** See Appendix A. □

When a sale occurred, sellers and buyers update their priors differently because a buyer has one piece of information more compared to the seller, i.e., he knows his own decision. To grasp the intuition, suppose there are three buyers and a sale occurs. A seller conducts the following thought experiment: The valuation of a remaining buyer can only be below the cutoff, if at least one of the other two buyers accepted the price, i.e., has a value above the cutoff. Otherwise, all buyers would have rejected the price and no sale would have occurred. Similarly, the valuation of a remaining buyer can only be above the cutoff, if at least one other buyer accepted the trade (has a valuation above the cutoff), and one of the two was selected for trade. In comparison, one of the two remaining buyers, $i$, updates.
his prior about the valuation of the other buyer \( j \) in a similar manner, but with the additional knowledge of his own decision. Whenever buyer \( i \) delayed purchase, he updates his prior like a seller, but he considers only \((n - 2)\) instead of \((n - 1)\) other buyers, as \( i \) knows he declined himself. However, if a sale occurred and buyer \( i \) had accepted the price, type \( v_j \) can only be above the cutoff, if the third buyer accepted the price as well and was selected for purchase. Put differently, if \( i \) accepted and was not selected for trade, it is more likely that \( j \) accepted as well, reducing the probability that \( i \) is selected. Therefore an accepting buyer \( i \) who was not selected attaches a higher weight to \( j \) having a value above the cutoff.

It is straightforward to show that any (buyer or seller) prior first-order stochastically dominates any (buyer or seller) posterior following any event for any cutoff valuation strictly within the support. If \( x_t \) is either the lower or the upper bound of the support, posterior and prior coincide, because either all buyers take the price or a sale happens with probability zero. Following the notion of the queue elaborated on in the introduction, the distribution of valuations of the buyer remaining in the market is the same as before, because either all buyers reject the price or the first buyer in the queue takes the price with probability one and the remaining buyers had no chance to buy. Nice cutoff paths give rise to this equivalence. For price paths that are not nice, the analysis leaves the realm of common priors. The individual posteriors give rise to individual cutoff valuations \( x \). However, for \( \epsilon \)-nice cutoff sequences, correct posteriors can be approximated with common \( \epsilon \)-consistent posteriors sense of Definition 1.

**Definition 2.** A cutoff sequence is \( \epsilon \)-nice over history \( h \) if, along history \( h \), at each point in time trade occurs with probability less than \( \epsilon \), i.e., for any truncation \( h_t \) of \( h \)

\[
F_{h_t}(x_t) \geq 1 - \epsilon. \tag{25}
\]

A cutoff sequence is nice when \( \epsilon = 0 \).

I later provide conditions such that nice cutoff sequences in the limit \( T(\kappa) \to \infty \) for all \( \kappa \) arise on equilibrium path. When the prices imply cutoffs within the support of the updated priors, only continuous price paths with discontinuous jumps upwards after each sale are consistent with nice cutoff sequences. The price jump has to be high enough such that a sale occurs with probability zero. Obviously, continuous price paths can only exist in the continuous-time game.

Nice cutoff sequences are analytically convenient: Because at each time a sale occurs with probability zero, all buyers’ and sellers’ posteriors are identical at each history along equilibrium path, and each buyer’s time-\( t \) strategy is characterized by the same state-specific cutoff type \( x_t \). In particular, any player’s posterior is equal to the prior of the previous instant. For the discretized game, I capture the notion that prior and posterior are “roughly the same” when the cutoff sequence is \( \epsilon \)-nice with the following lemma.

**Lemma 4.** If the cutoff sequence is \( \epsilon \)-nice over history \( h_t \), then for all truncations \( h \) of \( h_t \), the prior \( F_h \) is \( \epsilon \)-consistent with posteriors \( F_{s,h} \), \( F_{a,h} \) and \( F_{d,h} \) following
a sale at \( h \), and \( \hat{\varepsilon} \)-consistent with posterior \( F_{no,h} \) following no sale at the same history for some \( \hat{\varepsilon} \geq \varepsilon \).

**Proof.** See Appendix A. \( \square \)

Sloppily speaking, Lemma [4] says that, if the cutoff sequence is \( \varepsilon \)-nice, players only make a bounded mistake when they don’t update their prior at all, and this mistake vanishes as \( \varepsilon \to 0 \). In particular, Lemma [4] permits the following belief updating that is \( \varepsilon \)-consistent and retains common priors at each history.

**Corollary 3.** Let \( F_h \) be a common prior and let \( x \) be the corresponding cutoff type at history \( h \). If \( (1 - F_h(x)) \leq \varepsilon \), the prior \( F_h \) is an \( \varepsilon \)-consistent common posterior in case a sale occurred. It is also the most optimistic posterior. With \( \frac{F_{no}}{F_h(x)} \), a common posterior is formed in case no sale occurred, too.

This corollary implies that for any initially regular environment there exists an \( \varepsilon \)-Bayes consistent belief update such that the environment remains regular following any history over which the cutoff sequence is \( \varepsilon \)-nice. The reason is that regularity is conserved if it becomes common knowledge that there are no types above the cutoff, the Bayesian update after observing no sale. If a positive measure of types was to accept a price at some seller history \( h \), the correctly updated CDF corresponding to (42) has a kink at the cutoff type, as seen in Figure 1. This kink introduces a downward jump discontinuity in the updated virtual value function and thus the updated prior generically fails regularity - even when the initial prior was regular. The prior is the most optimistic \( \varepsilon \)-consistent posterior by definition of line (2). Let this updated virtual value following the belief updating described in Corollary 3 be denoted by

\[
\tilde{\psi}_h(v, x) := v - \frac{F(x) - F(v)}{f(v)} = v - \frac{1 - F_h(v)}{f_h(v)},
\]

where \( x \) is the cutoff type of the last previous instant at which all prices where rejected.

### 3.4 The oligopolists’ tradeoff: selling vs. not selling

As an implication of the assumption that, after a sale, sellers can adjust their price before buyers can buy again, sellers face a simple tradeoff. As buyers only patronize the cheapest sellers, seller \( m \)'s time-\( t \) expected profit has a discontinuity in the price at

\[
P^m_t = \min\{p^m_t' \}_{m' \in M \setminus \{m\}},
\]
i.e., at the minimum among the prices of \( m \)'s active competitors. The tradeoff in this section is present both for sellers with and without the ability to commit to future prices.

Without loss of generality, consider seller 1 as a deviator playing against strongly symmetric strategy profile \( \sigma^{-1} \) with \( \sigma^m(h) = p(h) \) for any seller-1 history \( h \) and all
sellers $m \in \mathcal{M} \setminus \{1\}$ that are still active at history $h$. In such a strongly symmetric equilibrium, seller 1 must not have a profitable one-shot deviation $p^1 \neq p(h)$ for any seller history $h$. Given $\sigma^{-1}$ (dropped from notation), define seller 1’s one-period continuation histories of seller history $h$ as

$$m h(p^1) := (h, (p^1, p(h), \ldots, p(h)), \kappa_t^m)$$

with $\kappa_{t+\Delta} = \kappa_t^m$ : $K_{t+\Delta}^m = K_t^m - 1, K_{t+\Delta}^{m'} = K_t^{m'} \forall m' \neq m$, (27)

$$0 h(p^1) := (h, (p^1, p(h), \ldots, p(h)), \kappa_t),$$

(28) i.e., the continuation histories in which a sale to seller $m$ or no sale occurs, respectively. When $m$ sells a good in market state $\kappa$, the market state is denoted by $\kappa^m$.

Seller $m$’s expected revenue, when offering $k$ units and setting price $p^m$ while all other sellers set price $p$, at time $t$ with seller history $h$ can be expressed piecewise as

$$R^1_t(\sigma^1, \sigma^{-1}, h) = \begin{cases} (30) & \text{if } p_t^1 > p(h) \\ (31) & \text{if } p_t^1 = p(h) \\ (32) & \text{if } p_t^1 < p(h) \end{cases}$$

where each line is explained below.

Let $F_h$ be the sellers’ prior, and let $n_h$ and $M_h$ be the number of active buyers and sellers at seller history $h$, respectively. The first line of the revenue is given by

$$(1 - F_h(x_t)^{n_h}) \mathbb{E}_{m \neq 1} \left[ R^1_{t+\Delta}(\cdot, \cdot, m h(p(h))) \right] + F_h(x_t)^{n_h} \cdot R^1_{t+\Delta}(\cdot, \cdot, 0 h(p(h))),$$

(30) representing the expected revenue from raising the price. Here, the cutoff type $x_t$ is unaffected by the deviator’s price, because only the minimum price is relevant for the buyer’s decisions. Consequently, $(1 - F_h(x_t)^{n_h})$ is the probability that one of 1’s competitors sells a good at $p(h)$. The expectation for the continuation payoff is needed, because the identity of the selected seller is uncertain, which affects the market state. With probability $F_h(x_t)^{n_h}$, no good is sold, leading to a continuation payoff of the the corresponding history.

Analogously, seller 1’s profit from complying, $p^1 = p(h)$, is given by

$$(1 - F_h(x_t)^{n_h}) \left( \frac{1}{M_h} (p^1 + R^1_{t+\Delta}(\cdot, \cdot, 1 h(p_t^1))) + \frac{M_h - 1}{M_h} \mathbb{E}_{m \neq 1} \left[ R^1_{t+\Delta}(\cdot, \cdot, m h(p_t^1)) \right] \right) + F_h(x_t)^{n_h} \cdot R^1_{t+\Delta}(\cdot, \cdot, 0 h(p_t^1)),$$

(31) because, if a sale occurs at $t$, seller 1 gets selected with probability $1/M_h$. In that case, she gets the price and the continuation payoff of having one good fewer following the corresponding history, $m h(p)$. With the complementary probability, the good is bought from another seller. Similarly, if no good is traded, the game continues as described in the previous case (30).

Finally, the payoff from undercutting the competitors’ price, $p^1 < p(h)$, is given by

$$(1 - F_h(y_t)^{n_h}) \left( p^1 + R^1_{t+\Delta}(\cdot, \cdot, 1 h(p_t^1)) \right) + F_h(y_t)^{n_h} \cdot R^1_{t+\Delta}(\cdot, \cdot, 0 h(p_t^1)),$$

(32)
where \( y_t \geq x_t \) is the cutoff type when \( p^t \) is the minimum price at \( t \). It cannot be below \( x_t \) because buying at \( t \) becomes more attractive. The continuation games differ from the two previous cases: If a good is traded, it is sold by the deviator 1 with certainty. If no good is traded, the continuation game is also different because a lower price got rejected and hence the belief about the remaining buyers’ valuation is updated more pessimistically from a seller’s point of view.

Verbalizing this profit function, a seller playing against strongly symmetric prices in essence has three choices:

1. She can raise the price, which means certainly abstaining from selling;
2. She can match the price, which means all sellers get selected with equal probability if a buyer accepts;
3. She can undercut the price, which means she gets selected with certainty if a buyer accepts.

The setup of the model has two implications on the continuation payoffs: First, because a seller exits the market when all her goods are sold, \( R_m^{t'}(\cdot, \cdot, h) = 0 \) for all \( h \in \mathcal{H} \) and \( t' \in T \) such that \( K_t^m = 0 \). Second, if \( t'' > 1 \), \( R_m^{t''}(\cdot, \cdot, h'') = 0 \) for any \( h'' \in \mathcal{H} \), the continuation payoff is 0 at the last trading opportunity.

Let

\[
\mu_t^m(h, p) = \mathbb{E}_{m' \neq m} \left[ R_{t+\Delta}(\sigma^m, \sigma^{-m}, m'h(p_t^m)) - R_{t+\Delta}(\sigma^m, \sigma^{-m}, m'h(p_t^m)) \right] \quad (33)
\]

with continuation histories defined as in (27) denote the seller \( m \)'s difference in continuation payoffs between having another seller sell at the same price and selling at this price herself. Catchy name needed...

**Lemma 5.** In any strongly symmetric equilibrium, following any seller history \( h \) at \( t \), each active seller \( m \) posts a strongly symmetric price

\[
p(h) = \mu_t^m(h, p(h)) = \mu_t(h, p(h)) \quad (34)
\]

when competing with at least one other seller.

**Proof.** In a strongly symmetric equilibrium, no seller has an incentive to deviate from symmetric price \( p_t \). A seller \( m \) does not want to raise the price if

\[
(30) \leq (31) \iff p_t \geq \mu_t^m(h, p_t).
\]

A seller \( m \) does not want to undercut price \( p_t \) with some price \( p^m < p_t \) if

\[
(32) \leq (31).
\]

In particular, the inequality above must hold for any arbitrarily small price cut \( p^m \approx p_t \), such that the inequality can be rewritten as

\[
p_t \leq \mu_t^m(h, p_t),
\]

19
because for any sequence of prices \( (l^t p^m)_{t=1} \ldots \) with \( (l^t p^m) \to p_t \), \( F(l^t y_t) \to F(x_t) \).

Combining the two conditions on \( p_t \) leads to a necessary condition for strongly symmetric equilibrium prices for any possible history, \( [34] \).

This lemma pins down a condition for strongly symmetric equilibria, namely, that continuation payoffs are symmetric over sellers, \( \mu_t^m(h, p(h)) = \mu_t(h, p(h)) \) for all active sellers, even when goods are asymmetrically distributed. In an environment in which all sellers supply a single unit, the continuation payoffs are clearly symmetric, because each seller either offers a single good or has exited the market.

### 3.5 Pricing under monopoly: No commitment

Now, I delineate the pricing strategy of a monopolist who lacks the ability to commit to future prices. In particular, I investigate in which aspects it differs from the monopolist’s full-commitment strategy discussed in Subsection 3.1. The monopolist’s game is relevant for two reasons: First, it establishes the benchmark outcome under collusion when sellers maximize joint profits and, second, it is a continuation game of the dynamic oligopoly game. Since continuation payoffs determine equilibrium oligopoly prices, the expected single-unit monopoly profit with an updated prior determines the price under duopoly when each seller offers a single good. Finally, I solve the game backwards sale-by-sale. Importantly, the assumption of no commitment is taken seriously here in a sense that the monopolist cannot (commit to) destroy any units of the good.

I say that a history \( h \) generates monopoly if it terminates in a monopolistic market state, i.e., if following history \( h \) only a single active seller remains in the market. By Lemma \( [5] \), the continuation payoffs pin down strongly symmetric equilibrium oligopoly prices. Since the good is scarce, every oligopoly can at some point become a monopoly when all but one sellers are stocked out. Under a duopoly in which both sellers offer a single good, \( \mu_t \) is a single-unit monopoly payoff. If I can determine the expected sequentially rational single-good monopoly continuation payoff for any for any seller continuation history \( m^h(\cdot) \) that changes the market state to a monopoly, \( [34] \) pins down the duopoly price for any corresponding history \( h \) when the market state is still duopolistic. With this insight, I can continue in a similar fashion starting from any other sequentially rational \( k \)-goods monopoly continuation payoff for any history that generates monopoly.

By taking limits of discretized continuation games for each time when the market state changes separately, I can exploit existing monopoly results. The market state can only change when a sale occurs, and in this case, following Corollary \( [3] \), the prior and posterior coincide. As a result, I can separately study the limit of a continuation game as the number of remaining pricing opportunities approaches

---

\( ^7 \)For example, for uniformly distributed valuations, disposal would be profitable as the seller’s revenue is only increasing in the amount of goods when there are at least twice as many buyers as goods.

---

20
infinity without changing the discretization of the game before this continuation game is reached. That is, the only impact of the discretization getting finer for a monopoly continuation game is that the continuation payoff converges to the continuous-time continuation payoff. This trick allows me to incorporate the convergence result of Hörner and Samuelson (2011) almost seamlessly.

In contrast to the full-commitment benchmark case outlined in the beginning, here, prices have to be sequentially rational. Intuitively, a monopolist without the ability to commit to future prices faces a tradeoff between perfect separation of buyer types and a positive terminal price that excludes low types. Hörner and Samuelson (2011) analyze the monopolist’s game and provide two lower bounds on her profit: The static monopoly profit (achieved by posting some price above the choke price at all histories that are not terminal and posting the static monopoly price at the final opportunity) and the profit of sequential Dutch auctions without reserve price. In contrast to the full-commitment case, the seller cannot sustain a positive terminal price while screening types perfectly. However, their multi-unit game is different in that they allow multiple sales at each time. I am especially interested in the latter bound, the Dutch auctions. Unfortunately, Hörner and Samuelson (2011) restrict attention to the uniform distribution and it is difficult to extend their results to all log-concave distributions.

Lemma 6. Suppose that $F$ is the uniform distribution and sellers are unable to commit to future prices. For $n \geq K + 5$, there exists a unique limit as $T(\kappa) \to \infty$ for all monopolistic $\kappa$. In equilibrium, the monopolist replicates sequential Dutch auctions without reserve price, Condition 1 holds.

Proof. See Hörner and Samuelson (2011, Proposition 6 and equation (26)). They show that, for sufficiently scarce goods, the seller prefers to screen types perfectly as the number of pricing opportunities approaches infinity. Cutoff types decrease continuously from the upper to the lower bound, and for $k$ remaining goods she charges $p_t = \frac{n-k}{n} x_t$ contingent on the current cutoff type $x_t$, the current upper bound of the updated prior.

An exemplary monopoly price path is depicted in Figure 2. In words, for sufficiently scarce goods, as soon as all competitors are sold out, the last remaining seller replicates a series of Dutch auctions by continuously decreasing the price. Once a sale occurs, the monopolist immediately raises the price to a choke level and again continuously lowers the price. The price has to jump after every sale, because buyers follow a more aggressive stopping strategy. The reason is that buyers observe the sale and hence they are aware that the relative supply of the scarce good has decreased. In Bulow and Klemperer (1994), sellers cannot immediately raise the price, but buyers are repeatedly allowed to buy at the same sale price. If excess demand occurs, the price in their model jumps. As a consequence either a frenzy occurs (several buyer buy at the same price) or the price “crashes”, i.e., drops discontinuously.

Lemma 6 provides the sequentially rational monopoly continuation payoff when
employing the \( \varepsilon \)-consistent belief updating of Corollary 3. The \( k \)-unit monopoly continuation payoff is given by

\[
\mathbb{E} \left[ \sum_{l=1}^{k} \tilde{\psi}(Y_i^{(n')}, x_t) \left| Y_i^{(n')} < x_t \right. \right],
\]

where \( Y_i^{(n)} \) is defined as the \( i \)-th highest order statistic of \( n \) draws from distribution \( F \) and \( n' = n - (K_m - k) - \sum_{m' \neq m} K_{m'} \) is the number of remaining buyers.

3.6 The buyers’ stopping strategy

A sequence of decreasing prices corresponds to a sequence of decreasing cutoff types, and a buyer accepts a price if and only if his valuations is above the corresponding cutoff type. A cutoff type can be translated into a cutoff price at which this type decides to purchase. That is, taking a symmetric strategy profile \( \sigma \) as given, a type-\( v \) buyers accepts a price \( \min\{p_t : x_t \leq v\} \) along a sequence of prices.

When the cutoff sequence is nice, a buyer faces a strategic tradeoff exactly as in sequential Dutch auctions with or without reserve price. To compute an optimal stopping strategy at the start of one of the sequential auctions, a buyer only needs to know his valuation, a prior about the other buyers’ valuations and how many goods remain. Let

\[
\beta_{k,r} \text{ be a symmetric stopping strategy in the } k\text{-th auction,}
\]

when the reserve price is \( r \), and let \( \chi_k \) be the buyer type who was awarded the good in auction \( k \). If the cutoff sequence is only \( \varepsilon \)-nice, there exists an \( \varepsilon \)-consistent posterior such that the strategic tradeoffs are “close” to sequential Dutch auctions.
Claim 1. If cutoffs sequences are nice, the stopping strategies \((\beta_{k,r})_{k \leq K}\) of all auctions are strictly increasing and differentiable in the valuation.

This claim is a standard method in auction theory. It simplifies the analysis, and is easy to verify ex-post. By inverting \(\beta_{k,r}\), buyers and sellers can infer a purchasing buyer’s valuation \(\chi\) from the trading price. By Claim 1, buyers buy the goods in order of their valuations. Since a Dutch auction price path is continuous and, thus, nice, the updated prior at the beginning of any \(k\)-th auction is given by \(F(v)/F(\chi_{k-1})\) and any types of earlier buyers, \(\chi_s\) with \(s < k - 1\), are irrelevant for the belief update.

Imposing a seller strategy profile that replicates sequential Dutch auctions with reserve price \(r\), \(\beta_{k,r}(v)\) is the price at which a type-\(v\) buyer wants to purchase in the \(k\)-th auction. It turns out that the type of the buyer who purchased in the previous auction cancels out and hence the stopping strategy is also independent of the previous buyer’s valuation, \(\chi_{k-1}\).

Lemma 7. Suppose \(K\) units are offered in \(K\) sequential Dutch auctions with reserve price \(r\). The desired purchase price of a type-\(v\) buyer in the \(k\)-th auction is given by

\[
\beta_{k,r}(v) = \mathbb{E} \left[ \beta_{k+1,r}(Y_1^{(n-k)}) \mid Y_1^{(n-k)} < v \right] = \mathbb{E} \left[ \max\{Y_K^{(n-1)}, r\} \mid Y_k^{(n-1)} < v < Y_{k-1}^{(n-1)} \right].
\] (37)

Now, Claim 1 now can be verified easily. The proof of this statement proceeds along the lines of, e.g., Krishna (2009, Proposition 15.2). I provide some details in Appendix B. Following the lines of the proof of Lemma 8 also in Appendix B, with \(r(x) = r\) for all \(x\) establishes the result as well.

In equilibrium of a \(K\)-unit sequential Dutch auction, forward-looking buyers arbitrage away the gains from preponing or postponing purchase. Consequently, sale prices are a martingale. If sale prices had, say, an upward trend, a buyer would benefit from employing a more aggressive stopping strategy in the current auction as the next item will be more expensive in expectation. If sale prices had a downward trend, a buyer would want to shade his current bid more, because the option value of the following auction is higher. On the one hand side, a buyer is willing to pay more for the good when a sale occurred, because relative supply decreased. On the other hand side, fewer buyers are active and they have a lower value than the buyer who purchased the previous item. In equilibrium in a symmetric independent private value environment, both effects exactly offset each other. The martingale property in sequential auctions was first derived by Milgrom and Weber (2000, written in 1982). In the following subsection, I show a perhaps surprising implication of the martingale property in oligopolistic settings: Proposition 1.
3.7 Pricing under oligopoly yields monopoly profits

The main result below appears to be counterintuitive at first glance: Why does a competing seller not have incentives to undercut a monopoly price path at any time? The underlying reason is the martingale property derived in the previous section: Because the monopolist without price commitment sells with probability one, the sellers’ expected payoff from selling to the currently highest type is at each point in time equal to the expected payoff from selling to the next highest type after the highest type purchased. Under Condition, buyers and sellers know that as soon as all but one sellers are stocked out, the remaining single seller replicates a non-exclusive Dutch auction. Because all players anticipate this sequentially rational continuation play, the proposed price path offers no opportunity for intertemporal arbitrage on the seller side as well. The following proposition establishes that, under Condition, sellers post identical prices that decrease synchronously and continuously and jump immediately after each sale.

**Proposition 1.** Under Condition, there exists a SSMPBE in which the outcome of efficient sequential Dutch auctions is replicated, which is independent of the distribution of the K goods.

**Proof.** I begin with the analysis on equilibrium path: Lemma describes the buyers’ best response given the sellers use the pricing strategy proposed. I now show that the sellers have no incentive to deviate from replicating sequential Dutch auctions.

The cutoff sequence is nice over any history along equilibrium path. By Condition, the last remaining seller replicates a non-exclusive sequential Dutch auction, r ≤ v, for any number of goods k remaining, as it is sequentially rational at each monopolistic market state. By scarcity of the good, the K-th good is provided under monopoly by construction.

Suppose the market for the (K - 1)-th good is duopolistic, wlog seller 1 and 2 offer one good. By Lemma, wlog seller 1 wants to deviate from any price that differs from the monopoly profit she can make when her competitor sells at this price, pt = µ1t(h, pt) = R1t+(2h(pt)).

Let Gi(n)(v, x) be defined as

$$G_i^{(n)}(v, x) := \sum_{l=0}^{i-1} \binom{n}{l} \left( \frac{F(v)}{F(x)} \right)^{n-l} \left( 1 - \frac{F(v)}{F(x)} \right)^l,$$

the cdf of Y_i(n) < x, and let gi(n)(v, x) be the corresponding density.

Let h_t be some continuation history of h with (K - 2) sales so far, in which the price decreased continuously and the (K - 1)-th trade yet has to occur, i.e., all prices for the (K - 1)-th good have been rejected so far. The following term is rearranged in Appendix C. For any price pt posted at any such history h_t, there
exists a corresponding buyer type $\chi = x_t$ who wants to accept the price. The price paid for the $(K - 1)$-th good by this buyer is given by (37) of Lemma 7,

$$\tilde{p}_t = \beta_{K-1,\omega}(\chi) = v + \int_v^{\chi} g_1^{(n-K+1)}(v, \chi) \beta_{K-1,\omega}(v) dv$$

$$= \mathbb{E}[\tilde{\psi}(Y_1^{(n-K+1)}, \chi)|Y_1^{(n-K+1)} < \chi] \quad \text{(38)}$$

which is exactly (35), the expected profit of a monopolist selling the last good when the penultimate good got sold to type $\chi$: The expected virtual valuation of the highest buyer type left in the market. Therefore both sellers 1 and 2 are indifferent between selling the $(K - 1)$-th good at any price $\tilde{p}_t = \beta_{K-1,\omega}(x_t)$ along equilibrium path and letting the competitor sell and gaining the corresponding monopoly continuation payoff.

Next, consider the $k$-th sale for any $k < K - 1$. From Lemma 7,

$$\beta_{k,\omega}(\chi_k) = \mathbb{E} \left[ \beta_{k+1,\omega}(Y_1^{(n-k)})) | Y_1^{(n-k)} < \chi_k \right].$$

Hence, a seller is indifferent between selling the $k$-th traded good in the market to some type $\chi_k$ or trading the $(k+1)$-th good in the market with the highest type at the time, given that the $k$-th good was sold to type $\chi_k$. This statements also holds true when the market is asymmetric. Because of the martingale property, the expected prices of future sales are equal. Then the symmetry of continuation payoffs pin down equilibrium prices at each time, that is, for any history $h$ and the corresponding cutoff type $x_t = \chi_k$. Hence,

$$p_t = \beta_{k,\omega}(\chi_k) = k\beta_{k,\omega}(\chi_k) - (k - 1)\beta_{k,\omega}(\chi_k) = \mu_{n}^{n}(h, p_t)$$

and the necessary condition of Lemma 5 holds. This condition is also sufficient: Deviating to a higher price at any time is not profitable as it is equivalent with letting the competitor sell if a corresponding buyer type is present. Deviating to a lower price $p'_t < \tilde{p}_t$ at any time is not profitable as the deviator either makes no sale which she also would not do when following the equilibrium strategy, or the deviator makes a sale and thereby garners a lower payoff than when selling or not at the equilibrium price.

Suppose that a seller at some time $t$ with history $h$ sets a lower price $p' < p_t$ than she is supposed to set. As a consequence, the cutoff sequence off equilibrium path is only $\varepsilon$-nice, where $\varepsilon$ depends on the size of the price cut. Then, there exists a continuum of $\varepsilon$-consistent beliefs approximating correct off-path beliefs $F_{no,h}$ and $F_{s,h}$ and all of them are first-order stochastically dominated by the prior, the on-path belief. Therefore not only the current price, also the off-path continuation payoff is weakly below the on-path continuation payoff. The highest continuation payoff is reached when adapting the belief update following Corollary 3, the most optimistic posterior Definition 1 allows. This maximum continuation payoff is equal to the on-path continuation payoff. In total, the deviation is strictly nonprofitable as $p' < p_t$. $\square$
In words, when sequential Dutch auctions are replicated, any seller has, at any price along the continuous price path, no incentive to deviate. The reason is that, in equilibrium, the marginal continuation payoff is at each point in time equal to the current price as prices are a martingale. Under Condition 1, all goods gets traded with certainty, and hence the price is equal to the marginal continuation payoff.

Proposition 1 only holds when the monopoly continuation game induces an efficient allocation. From earlier analysis, it is known that a seller with price commitment may want to exclude low buyer types, and thus Condition 1 fails. I address sellers with price commitment in the following section. However, even for the case of no price commitment, Hörner and Samuelson (2011) show that a monopolist only prefers to implement an efficient allocation when the good is sufficiently scarce. To illustrate, a monopolist facing a single buyer maximizes her profit by always posting unacceptable prices up to the deadline, at which she posts the static monopoly price. Similarly, for few buyers and few goods, the monopolist also posts unacceptable prices until few pricing opportunities before the deadline.

3.8 Full price commitment

In this section, I shed light on the role of the ability to commit to future prices. Although the no-commitment case can be more suitable for applications, the full-commitment solution is a relevant benchmark case to quantify the value of commitment. In this subsection, I consider the same model as in the previous one, but I relax the no-commitment restriction on the sellers’ behavior: In the beginning of the game, each seller commits to a price plan contingent on each possible seller state (full price commitment).

By Corollary 1, Proposition 1 continues to hold if (12) is true. However, if \( v f(\bar{v}) < 1 \), the optimal allocation excludes low buyer types and hence the good is not sold with probability one. In any no-gap case, \( \bar{v} = 0 \), a monopolist prefers to exclude low-type buyers for any posterior. Hence, if sellers’ strategies are not restricted by sequential rationality, the remaining seller at any history that generates monopoly would not want to sell with probability one. However, the measure of excluded types with respect to the initial type distribution is smaller.

Suppose, all sellers replicate the monopoly price path of an optimal Dutch auction. Then, a single seller can profitably deviate by decreasing the price further than the optimal reserve price \( r^* \). On the one hand, the deviator gains in case she becomes the only seller that actively reduces the price because she exploits the revealed information that active buyers’ valuations are low. On the other hand, the deviator loses from the fact that buyers employ a less aggressive stopping strategy as they anticipate that with some probability (in the case in which only the deviator remains) the terminal price will be lower than the seller-optimal reserve price. For a monopolist, the loss from the second effect exceeds the gain from the first effect. Under oligopoly, however, the second effect is shared with all
other sellers while the gains of the first effect are solely pocketed by the deviator. In other words, the buyers shade their bid less compared to the monopoly case because the reserve price is lower once the deviator becomes the monopolist which occurs with a positive probability. By a Bertrand argument, posting a positive price at the final trading possibility cannot form an equilibrium when the market is oligopolistic at that time.

The following proposition pins down an equilibrium price path: All sellers post identical synchronously and continuously decreasing prices that jump to the choke price after each sale if at least two sellers are active. When only a single seller remains active, she sells her goods by replicating sequential Dutch auctions with an exclusive reserve price that is optimal with respect to the updated prior of the history that generated the monopoly. By Claim 4, the type of the buyer who purchased the last good traded in oligopoly is learned from the corresponding price paid. Figure 3 shows (a) an exemplary oligopoly price path, when sellers can commit, juxtaposed with (b) the corresponding monopoly price path. Since the proposed price path is nice, the updated virtual valuation is given by (26). Let \( \psi^{-1}(0, x_t) \) be the inverse of \( \psi \) with respect to the first argument evaluated at 0 and \( x_t \).

**Proposition 2.** When sellers can commit to a prices contingent on seller states, the price continuously decreases with upward jumps whenever a sale occurs. For any history \( h_t \) that generates monopoly, the monopolist commits to a Dutch auction with reserve price \( r^*(x_t) = \max\{\psi^{-1}(0, x_t), v\} \). Prices and expected industry profits are higher under monopoly than under oligopoly.

**Proof.** The reserve price \( r(\chi) \) set by a monopolist who emerged endogenously at some time \( t \) is determined by valuation \( \chi = x_t \), the type of the last buyer who purchased under oligopoly. Function \( r \) maps a buyer type \( x \) into a reserve price such that \( \tilde{\psi}(r(x), x) = 0 \) with \( \tilde{\psi} \) given in (26). From then on, the buyers’ best response to the monopolist’s strategy is given by Lemma 7 with \( r = r^*(\chi), \beta_{k,r^*(\chi)}(v) \).

Next consider the buyers’ best response to the sellers’ proposed pricing strategy under oligopoly. Let \( \beta_k^\kappa(v) \) be stopping price of type \( v \) when the \( k \)-th sale takes place under an oligopolistic market state \( \kappa \). The strategy depends on \( \kappa \) because the distribution of goods determines how likely it is that the market becomes monopolistic for the next sale(s).

The intertemporal arbitrage condition of a buyer implies that he must be indifferent between getting the \( k \)-th good at some price \( \hat{p} \) and entering an auction for the remaining \((K - k)\) goods when the \( k \)-th good was sold at price \( \hat{p} \). In particular, this is true for any sale that could potentially be the last sale under oligopoly. In Lemma 8 in Appendix B, I show that

\[
\beta_k^\kappa(v) = \sum_{m \in M_h} \Pr(\kappa' = \kappa''| v, Y^{(n-1)}_{k+1}) \cdot \mathbb{E}[\beta_{k+1}^\kappa(Y_k^{(n-1)}) \mid Y_k^{(n-1)} < v < Y_k^{(n-1)}], \tag{40}
\]
where $\mathcal{M}_h$ is the set of active sellers following history $h$ and $\kappa^m$ is the market state arising from $\kappa$ after $m$ sells a good. Similar to the procedure before, this formulation allows me to solve the game backwards from the $K$-th sale on, which by construction occurs in a monopolistic market. For details, see Lemma 8 in Appendix B.

I now show that the proposed seller behavior is indeed the best reply to the buyers’ strategy.

Consider the penultimate sale $(K - 1)$ with a duopolistic market state $\kappa$ at some history $h$ such that after a sale the market is monopolistic with certainty. Suppose all players have behaved as proposed so far. Let $\tilde{h}_t$ be some continuation of $h$ in which all prices for the penultimate good along the continuous price path until time $t$ were rejected.

The stopping strategy $\beta_{K-1}^\kappa(\chi)$ (see (54) in Appendix B) of a buyer type $\chi$ is given by

$$
\beta_{K-1}^\kappa(\chi) = v + (n - K - 1) \int_\chi^x \left( \frac{F(z)}{F(\chi)} \right)^{n-K} \frac{f(z)}{F(\chi)} \cdot \left[ z - \int_z^\chi \left( \frac{F(y)}{F(z)} \right)^{n-K} dy \right] dz.
$$

I can rewrite this term (see the details in Appendix B) as

$$
\beta_{K-1}^\kappa(\chi) = \int_{r(\chi)}^\chi (n - K + 1) f(y) \left( \frac{F(y)}{F(\chi)} \right)^{n-K} \left( y - \frac{F(\chi) - F(y)}{f(y)} \right) dy
$$

where the second line holds for exactly one function $r: r^*(x)$ such that $\tilde{\psi}(r^*(x), x) = 0$, the reserve price function proposed in this statement. Each $\chi$ corresponds to a history $\tilde{h}_t$ with $x_t = \chi$ and $p_t = \beta_{K-1}^\kappa(\chi) = R_{m+\Delta}(\cdot, m\tilde{h}(p_t))$

That is, both sellers are indifferent between selling to type $\chi$ at price $\beta_{K-1}^\kappa(\chi)$ and obtaining the expected monopoly profit of the final sale when type $\chi$ purchased the penultimate good. The equality of the price and the corresponding monopoly continuation payoff holds at every point in time when players follow the proposed strategy profile. The resulting price path is nice everywhere.

The off-equilibrium path analysis is more involved compared to Proposition 1 since after deviations continuation play is not restricted to be sequentially rational. One-shot discontinuous price cuts are not profitable deviations following the same argument as in the proof of Proposition 1. It strictly reduces the payoff from the current price and it weakly decreases the continuation payoff which is maximized under the proposed rule.
Suppose that, for a monopolistic market state, some seller commits to some price path other than a sequential Dutch auction with reserve price rule $r^*(x)$. Still the same types as on equilibrium path buy at the oligopolistic market states and hence the same monopoly posterior is induced. By definition the monopoly continuation payoff decreases, as it is maximized under the proposed rule. The intertemporal arbitrage condition of the buyers requires that the marginal type $\chi$ that accepts the last price posted under oligopoly is indifferent between buying at this price and entering the monopoly continuation game. Suppose the deviator increases the monopoly continuation payoff of type $\chi$ compared to the proposed equilibrium. By incentive compatibility, it also increases the payoff of all types larger than $\chi$. As a consequence the stopping strategy of all types that buy in equilibrium becomes less aggressive such that the accepted prices are lower, making this deviation non-profitable for the deviating seller.

Suppose the deviating monopoly continuation game features a lower continuation utility of type $\chi$. The idea of this deviation is to increase the price to be gained under oligopoly at the cost of decreasing the seller’s monopoly continuation payoff. An upper bound of this deviation is the profit of the same deviation while also decreasing the oligopoly price slightly faster than the other sellers. That way it is guaranteed that the deviator sells all her goods under oligopoly without reaching the monopoly continuation game which yields less payoff than the equilibrium payoff. However, in this case the continuation game after the deviator is sold out is exactly the same as in equilibrium. Hence, the bidding strategies are the same and the deviator does not gain from this deviation.

The proposed monopoly continuation play is the only strategy that ensures that the buyers’ and the sellers’ intertemporal arbitrage conditions hold simultaneously. That is, only when sellers, who endogenously become monopolists, commit to conducting sequential Dutch auctions with the given reserve price function, the sellers’ marginal continuation payoffs have the martingale property as well. In comparison with the no-commitment (monopoly or oligopoly) case, buyers purchase at higher prices, but in expectation fewer goods are sold because the prices do not decrease as much. The opposite is true in comparison with the full-commitment monopoly case.

Figure 3 illustrates the difference between monopoly and oligopoly prices with price commitment. Although the first three units are sold to the same buyer types, monopoly prices are higher. The reason is that an ab-initio $K$-unit monopolist commits to a higher terminal price than an endogenously emerging monopolist. The latter’s reserve price is ex-ante unknown, but lower than the ab-initio monopolist’s reserve price with probability one.
4 Discussion

In this model, the strategic interaction between forward-looking buyers and sellers without price commitment in continuous time suggests that it is irrelevant for profits and buyer surplus how goods are distributed among sellers, because monopolistic market power can be sustained anyway. Proposition 1 is a counterintuitive result because economists instinctively promote competition in standard settings (without innovation, synergies, natural monopoly cost structures etc). From any real world angle, it appears to be a questionable policy advice to ignore market conditions in the industries mentioned in the introduction. To provide a better understanding of Proposition 1 as a benchmark result, I suggest some modifications which may overturn the result.

In standard oligopoly models, sellers’ incentives to undercut competitors provide benefits to consumers. In the equilibrium of Proposition 1 these incentives are not present because the intertemporal arbitrage conditions of buyers and sellers jointly result in martingale prices and martingale per-unit profits. Since the benchmark result is driven by the martingale property of equilibrium prices, it opens the door for research investigating a similar setting in which prices do not follow a martingale.

Interdependent values: The martingale property of prices in sequential auctions in was derived by Milgrom and Weber (2000, written in 1982). In addition, they show that prices tend to drift upwards in a model with interdependent values with affiliated signals. For example, a reasonable application of my model with affiliated signals are fashion fads: There is a sales season and the previously pro-
duced goods are only fashionable for a given time after which the market dries up. Because the sale price of items bought earlier reveals information about the value of the good to other buyers, remaining buyers are willing to pay higher prices for the next items. Then, however, a seller prefers to trade later rather than earlier. Extending my model in this extension would be interesting.

Risk: Empirically, prices in real world sequential auctions appear to show a downward trend, a stylized fact known as the “declining price anomaly”. This term was popularized by [Ashenfelter (1989)] who notes such a trend in prices of sequential art and wine auctions. Since then many empirical papers (e.g., [Van den Berg et al. (2001)]) reported declining prices and many theoretical papers provided possible explanations for the finding. A natural explanation for declining prices is risk aversion. [McAfee (1993)] can explain the discrepancy between theory and empirics with nondecreasing absolute risk aversion. More recently, [Hu and Zou (2015)] set up a model with ”background risk”, i.e., bidders participate in auctions not only to seek profits, but also to avoid losses. Buyers’ background risk can easily be incorporated into my model and the implications of background risk in a setting with competition on the seller side remain to be investigated.

Unobserved inventory: I assume that buyers are always aware of how many goods are left to allocate. Internet platforms often reveal the inventory (e.g., number of seats, rooms or tickets left), but in many settings, especially in bigger markets, this assumption might be implausible. Because airlines sometimes reserve seats for special passengers, the number of remaining seats observed online is only an informative proxy for real inventory. In [Jeitschko (1999)] prices in sequential auctions can have a trend because the number of units is unknown. Due to the uncertainty whether a next auction takes place, the option value of participating in the next auction declines which drives up the price in the current auction. Similarly, an increasing expected price path can be found when information arrives that fewer units than anticipated will be sold.

Arriving buyers: Another intriguing extension would be to allow additional buyer entry over time. A dynamic buyer population could be incorporated into my setting by dividing the continuous time interval into several continuous time intervals which start with the arrival of additional buyers. This extension is particularly interesting in the context of airline tickets as, say, business travelers find out about their need to travel much later than leisure travelers. The vast majority of theory papers predict falling prices as the deadline approaches, contradicting the data (see [McAfee and Te Velde (2006)] for stylized facts about pricing in the airline industry). [Board and Skrzypacz (2015)] consider such a model with a single seller. Remarkably, they show that the optimal allocation in the continuous-time limit of their setting can be implemented with an optimal path of posted prices. However, this result heavily hinges on their assumption of discounting. For several applications, discounting is of second order importance. To illustrate, a hotel room is consumed and paid at the day of arrival and hence the time of purchase is only indirectly relevant through the price and the probability that there still is a hotel room available. This indirect form of discounting is endogenously part of my
model and an explicit discount factor may only reflect a reduced form approach to model an urge to buy early. Without a discount factor, their monopolist would simply wait until the deadline when all buyers have arrived and conduct an optimal auction following [Myerson (1981)]. This strategy is clearly not an equilibrium if additional sellers were present. Competition gives rise to interesting dynamics of preponing sales to attract already present high-value buyers versus postponing sales to include buyers entering in the future.

**Heterogeneous goods:** Although many typical applications, such as low-cost bus and plane travel or small-sized rental cars, do not display significant brand or product differentiation, my assumption of homogeneous goods limits the scope for reasonable applications. Here, I want to stress that good homogeneity is not driving the results qualitatively per se. Suppose there is a quality difference between two goods offered by two different sellers. For example, two flights with the same destination departing on the same day, one leaving at 4 am and the other at 11 am. If the quality difference is modeled as a shift of the distribution, qualitatively the analysis remains the same. That is, if for any buyer whose willingness to pay for the good of seller A amounts to $v$, then this seller values the good of seller B $v + \delta$. Because the quality difference is assessed unanimously, incentives are not distorted and prices continuously decrease at the same speed, but at different levels.

As discussed in the introduction, strategic buyers are prevalent in many markets. However, the purchase of some goods is rather the result of impulsive decision making instead of fully strategic considerations. The proofs of Propositions 1 and 2 hinge on the buyers’ objective to optimally time their purchase. Strategic buyer arbitrage away any expected intertemporal price differences. When, in contrast, buyers are fully myopic, i.e., when they have a discount factor of zero, they buy as soon the price is below their valuation, $x_t = p_t$ for any history $h_t$.

**Myopic buyers:** I consider the extreme case of fully-forward-looking buyers. For a better understanding of this assumption, it is helpful to study the opposite extreme assumption, fully myopic buyers as in [Lazear (1986)]. Assume the environment of Section 2 but suppose buyers have a discount factor of zero. As a result, a $K$-good monopolist maximizes profit by continuously decreasing prices to make a profit of

$$E \left[ \sum_{k=1}^{K} Y_{k(t)}^{(n)} \right]$$

and all buyers obtain zero utility. Clearly, prices are decreasing over time.

Under oligopoly, a competitor has incentive to undercut prices to attract the highest-type buyer. Consequently, a positive measure of buyer types accepts the first price and, hence, the price path is not nice in equilibrium under oligopoly. Lemma 5 continues to hold and, hence, monopoly continuation payoffs determine oligopoly prices.

**Production:** Another obvious extension would be to introduce a production
stage, in which sellers individually produce capacity a la Kreps and Scheinkman (1983).

5 Conclusion

In this paper, I contribute to filling a gap in the RM literature by analyzing oligopolistic competition. Virtually none of the industries characterized by RM business conditions is monopolistic and hence this paper adds to a better understanding of real world RM industries. By taking the limit of discretized games separately for each market state, I establish a simple technique to solve the game backwards sale-by-sale. My setting features equal allocations, prices, joint industry profits and buyer payoffs under monopoly and oligopoly if under monopoly, an efficient allocation arises with probability one. With uniformly distributed values, the latter is true when sellers are unable to commit to future prices and goods are sufficiently scarce. With commitment, it holds when sellers value the good sufficiently less than the lowest buyer type. This result is driven by forward-looking buyers and the scarcity of the good, because the buyers’ intertemporal optimization entails martingale equilibrium prices and hence, in equilibrium, sellers have no incentive to deviate from the monopoly price path. If, however, a single seller optimally want to commit to excluding low buyer types from trade, competition on the seller side leads to lower prices accompanied by higher consumer surplus and lower industry profits.

The main result of this paper is puzzling: Why do price paths observed in reality differ when competition is introduced? I discuss modifications of the model that produce the more intuitive result that competition on the seller side benefits the consumers and harms the sellers. Nevertheless, this model is a relevant benchmark that contributes to a better understanding of oligopolistic RM markets. It highlights the role of commitment and forward-looking buyers. Moreover, I derive sharp predictions about the behavior of prices and thereby open the door for intriguing empirical research. For example, the observation that competition beats down prices in the airline industry suggests that buyers are myopic instead of forward-looking.
Appendix

A The Posterior

**Corollary 4.** Suppose history $h$ of time $t$ features no sale so far. Following a sale or no sale, sellers update their prior according to Bayes’ Rule as follows:

$$f_{no,h}(v) = \frac{f_h(v)}{F_h(x)} \text{ and}$$

$$f_{s,h}(v) = \begin{cases} \frac{f_h(v)(1-\rho_t)}{(1-F_h(x)^n)\frac{n}{n-1}} & \text{if } v < x \\ \frac{f_h(v)(1-\phi_t)}{(1-F_h(x)^n)\frac{n}{n-1}} & \text{if } v \geq x \end{cases} \text{ for any } m \in M,$$

where $x$ is given by (22), $\rho_t$ is given by (24) and $\phi_t$ is given by (23).

Similarly, buyers update their prior after a sale as follows:

$$f_{d,h}(v) = \begin{cases} \frac{f_h(v)(1-\rho'_t)}{(1-F_h(x)^{n-1})\frac{n-1}{n-2}} & \text{if } v < x \\ \frac{f_h(v)(1-\phi'_t)}{(1-F_h(x)^{n-1})\frac{n-1}{n-2}} & \text{if } v \geq x \end{cases} \text{ for any } m \in M,$$

$$f_{a,h}(v) = \begin{cases} \frac{f_h(v)(1-\phi''_t)}{(1-\phi_t)^2} & \text{if } v < x \\ \frac{f_h(v)(1-\phi''_t)}{(1-\phi_t)^2} & \text{if } v \geq x \end{cases} \text{ for any } m \in M,$$

with $\rho'_t = F_h(x)^{n-2}$ and $\phi'_t = \frac{1-F_h(x)^{n-1}}{(n-1)(1-F_h(x))}$ and $\phi''_t = \frac{2(n-1-nF(x)+F(x)^n)}{(n-1)n(1-F_h(x)^n)}$.

Let $h'$ be a continuation of history $h$ with one additional trading opportunity Bayes’ Rule states

$$f_h(v_1|h') = \frac{f_h(v)\Pr(h'|h,v_1)}{\Pr(h'|h)}.$$

The posterior when no sale occurred is straightforward to derive. Suppose a sale occurred. Let a seller consider buyer 1 wlog.

If $v_1 < x_t$, a sale to some buyer $i \neq i_1$ could only occur when at least one of the other $(n-1)$ buyers has a value greater than $x_t$. Hence,

$$\Pr(h'|v_1,h) = 1 - F_h(x_t)^{(n-1)}.$$

If $v_1 \geq x_t$, buyer 1 only remains in the market following a sale when 1 was not selected to trade. There must have been another buyer type larger than the cutoff and this buyer was selected instead. Otherwise, buyer 1 would have bought the good and would have exited the market. Hence, $\Pr(h'|h,v_i) = 1 - \phi_t$.

The denominator is the probability that a good gets traded, but not with buyer $i$:

$$\int_0^{x_t} f(v)(1-F(x_t)^{n-1})dv + \int_{x_t}^1 f(v) \left(1 - \frac{\phi}{(1-F(x_t))^n} \right)dv.$$
Next, suppose a sale occurred following a buyer history \( h \), and wlog consider buyer 2 forming a posterior about buyer 1. Suppose 2 declined purchase. As 2 declined himself, 1 only could have declined as well (\( v_1 < x_t \)) if one of the other \((n - 2)\) buyers accepted the price. Similarly for valuations \( v_1 \geq x \), buyer 2 updates his belief exactly as a seller, but considers only \((n - 2)\) other buyers.

Now, suppose 2 accepted the price. In case 1 declined (\( v_1 < x \)), it must have been that some buyer other than 2 was selected for purchase. If 1 tried to purchase (\( v_1 \geq x \)) as well, it must have been that one buyer other than 1 or 2 was selected,

\[
1 - \sum_{j=0}^{n-2} \binom{n-2}{j} \frac{2}{j+2} F(x_t)^{n-2-j}(1 - F(x_t))^j = 1 - \phi''_t.
\]

The following lemma holds for all symmetric equilibria, even when no cutoff strategies are played:

**Lemma 3.** The following first-order stochastic dominance results hold for any equilibrium in the buyers’ game

\[
F_{no,h}(v) > F_{d,h}(v) > F_{a,h}(v) > F_h(v),
\]

for all \( v, x \) such that \( F_h(v) \in (0,1) \), \( F_h(x) \in (0,1) \). For \( v, x \) such that \( F_h(v), F_h(x) \in \{0,1\} \), all four probabilities are equal.

**Proof.** For ease of exposition, I suppress the subscripts. I proof this inequality for symmetric cutoff strategies, so that the posteriors look like the ones in Corollary 4. In any equilibrium not in cutoff strategies, it must be that higher types are more likely to accept a price. Otherwise, the equilibrium candidate would violate the principle of incentive compatibility. The following extends to mixed strategies that are weakly increasing in the type.

The last inequality is obviously true. I start with the first in inequality. It holds for \( v < x \) if and only if

\[
\frac{1}{F(x)} > \frac{(1 - \rho'_t)F(v)}{(1 - \rho'_t)F(x) + (1 - F(x))(1 - \phi'_t)},
\]

which is obviously true. For \( v > x \), \( F_{no,h}(v) = 1 \) such that the inequality holds, too.

The second inequality holds for \( v < x \) if and only if

\[
\frac{(1 - \rho'_t)F(v)}{(1 - \rho'_t)F(x) + (1 - F(x))(1 - \phi'_t)} > \frac{(1 - \phi'_t)F(v)}{(1 - \phi'_t)F(x) + (1 - F(x))(1 - \phi''_t)}
\]

\[
\frac{1 - \rho'_t}{1 - \phi'_t} > \frac{1 - \rho'_t}{1 - \phi'_t} \cdot \frac{F(x) + (1 - F(x))^{1-\phi'_t}}{F(x) + (1 - F(x))^{1-\phi''_t}}
\]

35
which holds when the second term on the RHS is $\leq 1$. Hence, it is to show that

$$\left(\sum_{j=1}^{n-2} a_j\right) \left(\sum_{k=1}^{n-2} a_k\right) < \left(\sum_{j=1}^{n-2} b_j\right) \left(\sum_{k=1}^{n-2} c_k\right)$$

with

$$2a_j a_k = 2\left(\frac{j}{j+1}\right) \left(\frac{k}{k+1}\right) (n-2)^2 F(x)^2(n-1-j) (1-F(x))^{j+k} <$$

$$b_j c_k + b_k c_j = \left(\frac{j}{j+2} + \frac{k}{k+2}\right) (n-2)^2 F(x)^2(n-1-j) (1-F(x))^{j+k}$$

for all $j \neq k \in \{1, \ldots, n-2\}$ and

$$a_j a_k = \left(\frac{j}{j+1}\right)^2 (n-2)^2 F(x)^2(n-1-j) (1-F(x))^{2j} <$$

$$b_j c_k = \left(\frac{j}{j+2}\right) (n-2)^2 F(x)^2(n-1-j) (1-F(x))^{2j}$$

for all $j = k \in \{1, \ldots, n-2\}$.

The second inequality holds for $v \geq x$ if and only if

$$\frac{(1-\phi'_t)F(x) + (F(v) - F(x))(1 - \phi'_t)}{(1-\phi'_t)F(x) + (1 - F(x))(1 - \phi'_t)} > \frac{(1-\phi'_t)F(x) + (F(v) - F(x))(1 - \phi'_t)}{(1-\phi'_t)F(x) + (1 - F(x))(1 - \phi'_t)}$$

$$\frac{(1-\phi'_t)f(x) + (F(v) - F(x))(1 - \phi'_t)}{(1-\phi'_t)f(x) + (1 - F(x))(1 - \phi'_t)} > \frac{(1-\phi'_t)f(x) + (F(v) - F(x))(1 - \phi'_t)}{(1-\phi'_t)f(x) + (1 - F(x))(1 - \phi'_t)}$$

which is true because of $[45]$. \qed

**Corollary 5.**

$$W'(v_i, h, 0) \geq W'(v_i, h, 1) \quad \text{for a.e. } v_i \in [v, \bar{v}]$$

We have shown that a declining buyer is more optimistic about getting a good than an accepting buyer the Lemma 3.

**Lemma 4.** If the cutoff sequence $\varepsilon$-nice over history $h_t$, then for all truncations $h$ of $h_t$, the prior $F_h$ is $\varepsilon$-consistent with posteriors $F_{s,h}$, $F_{a,h}$ and $F_{d,h}$ following a sale at $h$, and $\hat{\varepsilon}$-consistent with posterior $F_{no,h}$ following no sale at the same history for some $\hat{\varepsilon} \geq \varepsilon$.

**Proof.** Let $x$ be the corresponding cutoff type. Since the (buyer or seller) posterior after a sale, $\frac{F_{s,h}(x)}{F_{s_h}(x)}$ is first-order stochastically dominated by the prior, and
it first-order stochastically dominates the posterior following no sale, for all $G \in \{F_{s,h}, F_{a,h}, F_{d,h}\}$,

$$\frac{F_h(v)}{F_h(x)} \geq G(v) \geq F_h(v) \quad \text{(46)}$$

$$1 \geq \frac{F_h(v)}{G(v)} \geq F_h(x) \geq 1 - \varepsilon, \quad \text{(47)}$$

which means that $F_h$ is $\varepsilon$-consistent with any true Bayesian update $G$.

From inequality (46), it also follows that any such $G$ is $\varepsilon$-consistent with $\frac{F_h}{F_h(x)}$,

$$1 \geq \frac{G(v)}{F_h(v)/F_h(x)} \geq F_h(x) \geq 1 - \varepsilon.$$

As a consequence $F_h$ is $\hat{\varepsilon}$-consistent with $\frac{F_h}{F_h(x)}$, because

$$\frac{F_h(v)}{F_h(x)} \geq G(v) \geq (1 - \varepsilon) \frac{F_h(v)}{F_h(x)}$$

together with (47) implies

$$1 \geq \frac{F_h(v)}{G(v)} \geq \frac{F_h(v)}{F_h(v)/F_h(x)} \quad \text{and}$$

$$1 - \varepsilon \leq \frac{F_h(v)}{G(v)} \leq \frac{F_h(v)}{(1 - \varepsilon)F_h(v)/F_h(x)}.$$

Hence,

$$1 \geq \frac{F_h(v)}{F_h(v)/F_h(x)} \geq (1 - \hat{\varepsilon}) = (1 - \varepsilon)^2.$$
B Stopping Strategies $\beta$

Proof of Lemma 7

**Proof.** Suppose that Claim 1 is true. Then buyers purchase in order of their values and players can infer the valuation of a buyer from the price he paid. Let $h$ be a history in which the penultimate good got sold to type $\chi$. Then, the (unique, see, e.g., [Maskin and Riley (2000)]) equilibrium stopping strategy in the final Dutch auction with $n_h$ buyers and reserve price $r \geq v$ is given by

$$
\beta_{K,r}(v) = r \frac{G_i^{(n_h-1)}(r)}{G_i^{(n_h-1)}(v,\chi)} + \frac{1}{G_i^{(n_h-1)}(v,\chi)} \int_r^v y g_i^{(n_h-1)}(y,\chi) dy
$$

(48)

Types $v < r$ abstain from buying and types above $v > \chi$ purchase at $\beta_1(\chi)$ which only happens off path. The strategy $\beta_{K,r}$ is independent of $\chi$.

The iterative arguments behind Krishna (2009, Proposition 15.2) or Lemma 8 (see the proof below) straightforwardly extend to a non-zero and fixed reserve price,

$$\beta_{k,r}(v) = E \left[ \beta_{k+1,r}(Y_1^{(n-k)}) \middle| Y_1^{(n-k)} < v \right]$$

(50)

which is (37).

**Lemma 8.** Suppose $K$ units are offered and prices behave as suggested in Proposition 2. When the market has an oligopolisitc market state $\omega$ at history $h$, the desired purchase price of a type-$v$ buyer when the $k$-th good is offered is given by

$$\beta_k^o(v) = \sum_{m \in M_h} Pr(\kappa' = \kappa^m) \cdot E \left[ \beta_{K+1,r}(Y_1^{(n-1)}) \middle| Y_1^{(n-1)} < v < Y_K^{(n-1)} \right]$$

(40)

where $\kappa^m$ is the market state evolving from $\kappa$ after a sale of seller $m$ and $M_h$ is the set of active sellers at $m$.

**Proof.** Consider the sale of the $K$-th good. Suppose that sellers have acted according to Proposition 2 so far. At history $h$, the market was duopolistic and then the $(K-1)$-th good was the last good traded in oligopoly, leading to history $m_h(p)$. Let the corresponding buyer type be denoted by $\chi = x_t$ and he bought
at price \( p = \beta_{K-1}^r(\chi) \). Then, by the Revenue Equivalence Theorem, a buyer’s expected utility from entering the last auction at history \( m \tilde{h}(\beta_{K-1}^r(\chi)) \) is given by

\[
u_K(v, r(\chi), m \tilde{h}) = \begin{cases} 
\int_{r(\chi)}^v G_1^{(n-K)}(y, \chi) dy & \text{for } v \geq r(\chi) \\
0 & \text{for } v < r(\chi),
\end{cases}
\]  

(52)

with \( r(x) \) such that \( \tilde{\psi}(r(x), x) = 0 \), because \( G_1^{(n-K)} \) is the probability of winning when \( n - K + 1 \) buyers are active.

Next, consider the penultimate sale at history \( h \) (with \( \chi_2 \) as upper bound of the support of \( F_h \)) and suppose two sellers are active, each offers one good. The expected utility of a type-\( v \) buyer from disguising as another type \( z \) in the penultimate auction is given by

\[
G_1^{(n-K+1)}(z, \chi_2)(v - \beta_{K-1}^v(z)) + 
\int_{z}^{\chi_2} g_1^{(n-K+1)}(x, \chi_2) u_K(v, r(x), m \tilde{h}(\beta_{K-1}^v(x))) dx
\]  

(53)

where \( G_1^{(n-K+1)}(z, \chi_2) \) is the probability of winning the penultimate good and \( \beta_{K-1}^v(z) \) is the price to be paid when disguising as type \( z \). Expression (52) is the expected utility when a type-\( \chi \) buyer snatched the penultimate good (\( \chi > z \)).

Dropping some super- and subscripts for convenience, the FOC wrt \( z \) is given by

\[
g(z)(v - \beta(z)) - \beta'(z)G(z) - g(z)u_K(v, r(z), m \tilde{h}(\beta(z))) = 0.
\]

Imposing \( z = v \), rearranging and then integrating yields

\[
g(z)(z - u_K(z, r(z), m \tilde{h}(\beta(z)))) = g(z)\beta(z) + G(z)\beta'(z)
\]

which can be rewritten as

\[
\beta_{K-1}^v(v) = v + \frac{1}{G(v)} \int_{Y_{1}}^{v} g(z) \left[ z - u_K(z, r(z), \tilde{h}^m(\beta(z))) \right] dz  
\]  

(54)

\[
= v + \frac{1}{G(v)} \int_{Y_{1}}^{v} g(z) \cdot \beta_{K,r(z)}(z)dz
\]

\[
= \mathbb{E} \left[ \beta_{K,r(Y_{1}^{(n-K+1)}, Y_{1}^{(n-K+1)})} \left| Y_{1}^{(n-K+1)} < v \right. \right] 
\]

\[
= \mathbb{E} \left[ \beta_{K,r(Y_{K-1}^{(n-1)}, Y_{K-1}^{(n-1)})} \left| Y_{K-1}^{(n-1)} < v < Y_{K-1}^{(n-1)} \right. \right],
\]  

(55)

where the second line follows from the fact that type \( z \), as the highest type of the support of \( F_{m \tilde{h}(\beta(z))} \), wins the last auction with certainty and pays \( \beta_{K,r(z)}(z) \).

The penultimate trade is the last trade that could possibly occur in an oligopolistic market state. Thus, the next sale is monopolistic and \( \chi \), the type of the penultimate buyer, determines the reserve price of the final auction, \( r(\chi) \). For earlier sales, however, the probability mass function that assigns a probability with which
any of the following auctions is monopolistic depends on how the goods are distributed, which depends on the history.

Consider another history \( h' \) and suppose there are three active sellers offering the last three goods, the \((K-2)\)-th sale. Then, with probability one, the corresponding continuation game is the duopoly analyzed above. If type \( \chi_2 \) buys the \((K-2)\)-th good, type \( v \) either gets the \((K-1)\)-th good, when all other types are lower, or he gets the \( K \)-th good when there exists only one active type \( \chi \geq v \) and all other active types less than \( v \), conditional on \( v \geq r(\chi) \).

Consider another history \( h'' \) and suppose there are two sellers offering the last three goods, and suppose that seller 1 has one good and seller 2 has two goods. Because I look for symmetric equilibria, it is irrelevant which of the two sellers supplied one and two goods, respectively. Then, the continuation game is either duopolistic or monopolistic. Both continuation games are equally likely because both sellers sell with equal probability,

\[
\mathbb{E}_H[u_{K-1}(v, r_h, H)] = \frac{1}{2} u_{K-1}^{mon}(v, r(\chi_2), h^{mon}) + \frac{1}{2} u_{K-1}^{pol}(v, 0, h),
\]

where \( h^{mon} \) generates monopoly and \( h \) generates duopoly (as analyzed above). If type \( \chi_2 \) buys from seller 1, seller 2 becomes a monopolist and replicates a sequential Dutch auction with a reserve price \( r(\chi_2) \). A type-\( v \) buyer gets to buy the good if and only if \( v \) is among the highest two valuations and \( v \geq r(\chi_2) \). If type \( \chi_2 \) buys from seller 2, type \( v \) gets the next good if and only if all other types are lower or he gets to buy the last good when \( v \) has the highest valuation among the then active buyers, conditional on \( v \geq r(\chi) \).

From (53), it follows that

\[
\mathbb{E}_h[u_{K-1}(z, r(z), m_h(\beta(z))] = \frac{1}{2} (z - \beta_{K-1}(z, r(z))) + \frac{1}{2} (z - \beta_{K-1}^{duopoly}(z)),
\]

because \( z \) is the highest type and, on equilibrium path, wins the good with certainty.

To find \( \beta_{K-2}^\prime \). I maximize the expected utility of a buyer of type \( v \) masking as a type \( z \) when stopping along a price path for the \((K-2)\)-th good. The objective looks just like (53) and the FOC corresponds to (54). It can be rearranged to

\[
\beta_{K-2}^\prime(v) = v + \frac{1}{G(v)} \int_v^\infty g(z) \left[ z - \mathbb{E}_h[u_{K-1}(z, r(z), m_h(\beta(z))))] \right] dz \tag{56}
\]

\[
= v + \frac{1}{G(v)} \int_v^\infty g(z) \frac{1}{2} \left[ \beta_{K-1,r(z)}(z) + \beta_{K-1}^\prime(z) \right] dz
\]

\[
= \frac{1}{2} \mathbb{E} \left[ \beta_{K-1,r(Y_{K-2}^{(n-1)})}(Y_{K-2}^{(n-1)}) \big| Y_{K-2}^{(n-1)} < v < Y_{K-3}^{(n-1)} \right]
\]

\[
+ \frac{1}{2} \mathbb{E} \left[ \beta_{K-1}(Y_{K-2}^{(n-1)}, \omega) \big| Y_{K-2}^{(n-1)} < v < Y_{K-3}^{(n-1)} \right]
\]

\[
= \frac{1}{2} \mathbb{E} \left[ \beta_{K,r(Y_{K-2}^{(n-1)})}(Y_{K-2}^{(n-1)}) \big| Y_{K-2}^{(n-1)} < v < Y_{K-3}^{(n-1)} \right]
\]

\[
+ \frac{1}{2} \mathbb{E} \left[ \beta_{K,r(Y_{K-2}^{(n-1)})}(Y_{K-2}^{(n-1)}) \big| Y_{K-2}^{(n-1)} < v < Y_{K-3}^{(n-1)} \right].
\]
where I plugged in \( (55) \).
Iteratively, I arrive at \( (40) \).

\[ \square \]

C Details of Proof of Proposition \( \�\]

\[
\beta_{K-1,v}(\chi) = v + \int_v^\chi g_1^{(n-K+1)}(v, \chi) \beta_{K,v}(v) dv
= v + \int_v^\chi (n-K+1) \left( \frac{F(v)}{F(\chi)} \right)^{n-K} f(v) \left[ v - \int_v^\chi \left( \frac{F(y)}{F(v)} \right)^{n-K} dy \right] dv.
\]

Then change the order of integration in the second term and cancel \( F(v) \) once

\[
\int_0^\chi \int_0^v (n-K+1) \left( \frac{F(v)}{F(\chi)} \right)^{n-K} f(v) \left( \frac{F(y)}{F(v)} \right)^{n-K} dy dv
= \int_0^\chi \int_y^\chi (n-K+1) \left( \frac{F(y)}{F(\chi)} \right)^{n-K} f(v) dv dy
= \int_0^\chi (n-K+1) \left( \frac{F(y)}{F(\chi)} \right)^{n-K} \int_y^\chi f(v) dv dy.
\]

Then (after swapping \( v \) and \( y \) wlog) plug this term back into the original term to yield

\[
\beta_{K-1,v}(\chi) = v + \int_v^\chi (n-K+1) \left( \frac{F(v)}{F(\chi)} \right)^{n-K} f(v) \left[ v - \int_v^\chi \left( \frac{F(y)}{F(v)} \right)^{n-K} \frac{F(\chi) - F(v)}{f(v)} \right] dv
= v + \int_v^\chi (n-K+1) \left( \frac{F(v)}{F(\chi)} \right)^{n-K} f(v) \left[ v - \int_v^\chi \frac{F(\chi) - F(v)}{f(v)} \right] dv
= \mathbb{E} \left[ \tilde{\psi}(Y_1^{(n-K+1)}, \chi) \bigg| Y_1^{(n-K+1)} \leq \chi \right].
\]
D Details of Proof of Proposition 2

Dropping some subscripts, I can rewrite the second part of term (54) as

\[
\int_\chi^x \int_{r(z)}^z (n - K + 1) \frac{f(z)}{F(\chi)} \left( \frac{F(y)}{F(\chi)} \right)^{(n-K)} dydz
= \int_\chi^x \int_{y}^{\min\{r^{-1}(y), x\}} (n - K + 1) \frac{f(z)}{F(\chi)} \left( \frac{F(y)}{F(\chi)} \right)^{(n-K)} dzdy
= \int_{r(x)}^x (n - K + 1) \left( \frac{F(y)}{F(\chi)} \right)^{(n-K)} \int_y^{r^{-1}(y)} f(z) dzdy
+ \int_{r(x)}^x (n - K + 1) \left( \frac{F(y)}{F(\chi)} \right)^{(n-K)} \int_y^{F(y)} f(z) dzdy
= \int_{r(x)}^x (n - K + 1) \left( \frac{F(y)}{F(\chi)} \right)^{(n-K)} \frac{F(r^{-1}(y)) - F(y)}{F(\chi)} dy
+ \int_{r(x)}^x (n - K + 1) \left( \frac{F(y)}{F(\chi)} \right)^{(n-K)} \frac{F(\chi) - F(y)}{F(\chi)} dy.
\]

The integral of the first term of (54) can be split into \(\int_{r(x)}^x \ldots \int_{r(x)}^x\) and the integration variable can be renamed as \(y\). Adding the two parts of (54) again yields

\[
\beta_{K-1}^\kappa (\chi, \omega_h) = \int_{r(x)}^x (n - K + 1) \left( \frac{F(y)}{F(\chi)} \right)^{(n-K)} \left( y - \frac{F(\chi) - F(y)}{f(y)} \right) dy
+ \int_{r(x)}^x (n - K + 1) \left( \frac{F(y)}{F(\chi)} \right)^{(n-K)} \left( y \frac{f(y)}{F(\chi)} - \frac{F(r^{-1}(y)) - F(y)}{F(\chi)} \right) dy
= \int_{r(x)}^x y^{(n-K+1)}(y, \chi) \cdot \bar{\psi}(y, \chi) dy,
\]

because the additive term in the second line is zero as the inverse of \(r\) is given by \(r^{-1}(y) = F^{-1}[y f(y) + F(y)]\):

\[
\bar{\psi}(v, y) = v - \frac{F(y) - F(v)}{f(v)} = 0 \iff y = F^{-1}[v f(v) + F(v)].
\]

References


43


