Uninsured Idiosyncratic Production Risks, Dynamics of Income Distribution and Fiscal Policies*

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Abstract

This paper addresses the cyclical behavior of income distribution and the evaluation of fiscal policies considering equity motives in the business cycle. Assuming CRRA preferences, I first show that either overaccumulation or underaccumulation of capital takes place in the market equilibrium with idiosyncratic production risks, depending on the risk aversion degree. Then the model generates realistic income inequality and dynamics of income distribution during the cycle compared to the US data. A fiscal policy experiment illustrates that considering the equity motive the government may optimally tax capital less during recessions given the fixed long-run level of tax rate, which differs from the result of optimal capital tax prescription in the basic business cycle framework.

JEL classification: E25; E32; E62

Keywords: idiosyncratic production risks; dynamics of income distribution; business cycle; fiscal policies; incomplete markets

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1 Introduction

The income distribution of the US exhibits right skewness. Table 1 shows that the income of top 5% of households accounts for over 18% of the US total income, even more than the summed income of 40% of households from the bottom. As for the wealth distribution, inequality is even larger. As Cagetti and De Nardi (2006) document, the richest 20 percent of the US population hold 81% the total US net worth. Macroeconomists have studied the income inequality for decades and many models they construct are able to generate a quantitatively reasonable concentration on the right tail of the income and wealth distributions.

Table 1: Cyclical Property of Income Distribution

<table>
<thead>
<tr>
<th>Income groups</th>
<th>Corr. with output</th>
</tr>
</thead>
<tbody>
<tr>
<td>bottom-20%</td>
<td>0.29</td>
</tr>
<tr>
<td>20-40%</td>
<td>0.30</td>
</tr>
<tr>
<td>40-60%</td>
<td>0.15</td>
</tr>
<tr>
<td>60-80%</td>
<td>-0.19</td>
</tr>
<tr>
<td>80-95%</td>
<td>-0.44</td>
</tr>
<tr>
<td>Top 5%</td>
<td>0.07</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ratios</th>
<th>Corr. with output</th>
</tr>
</thead>
<tbody>
<tr>
<td>95/50 ratio</td>
<td>-0.24</td>
</tr>
<tr>
<td>50/20 ratio</td>
<td>-0.06</td>
</tr>
</tbody>
</table>

All the values of income shares use corresponding data from CPS from 1947 to 2013. CPS provides the statistics of households only since 1967 so I pick up the income share of families from 1947 to 1966. The ratios are computed using income limits for each fifth and top 5 percent of all households and median income of all households from 1975 since the latter started to be reported from the year 1975. The procedures of computing the correlations of income shares with output are explained in the Subsection Cyclical Properties of Income Distribution.

The income distribution, however, is time-varying. In fact, it evolves with the business cycle. Guvenen, Ozkan and Song (2014) find that an aggregate shock hits people of different income groups differently. But the dynamics of income distribution have not drawn enough attention from economists. Even fewer models have successfully captured the cyclicity of the income distribution. The exceptions, such as Castañeda et al (1998), fail to feature the cyclical behaviors quantitatively. For instance, the correlations with output with the income shares in their model are either close to -1 or 1, which overstates the procyclical or countercyclical behavior of different income groups. The top earners show very much countercyclicality in the model while data show acyclicality.

Moreover, the varying income distribution implies that fiscal policies dealing with the business cycle, especially during recessions, may exert diverse impacts on different income groups. The government should balance efficiency and equity to evaluate a policy, which requires a tractable framework to model business cycle and heterogeneous agents simultaneously. Yet the fiscal policy literature often assumes representative agents in a business cycle. This framework only emphasizes efficiency as the unique standard to assess policies. I claim it an incomplete evaluation since equity is ignored.
In addition, idiosyncratic production (investment) risks matter. On one hand, empirical studies show the importance of these risks by reporting the poor diversification of investment\(^1\). Moskowitz and Vissing-Jørgensen (2002) present the average percent of net worth invested in private equity across all households with some private equity holdings and positive net worth. The average household in this group invests 41 percent of its wealth (45 percent when weighting by net worth) in private equity. Moreover, this investment is typically devoted to a single private firm in which the household has an active management interest. A poorly diversified portfolio also implies a high risk in the personal investment. These facts remind us that we should consider the firm ownership or entrepreneurship and idiosyncratic investment risks since they affect not only personal income or wealth for entrepreneurs but also the aggregate economy given concentrated wealth owned by rich entrepreneurs. On the other hand, the literature on such topics has demonstrated theoretically that both entrepreneurship and idiosyncratic investment risks may influence the aggregate economy. An example is that Angeletos (2007) shows that under plausible calibration his model with idiosyncratic investment risks generates underaccumulation of capital in incomplete markets compared to complete markets. It further gives implications on fiscal policies aiming at stimulating the economy or redistributing allocations.

The majority of literature on the business cycle, however, fails to feature idiosyncratic production risks partly because researchers start to consider such risks only from recent years and most of their works investigate the effect of idiosyncratic production risks on the general equilibrium in steady state or on the stationary income or wealth distribution. Angeletos (2007) discusses the transitional dynamics by mimicking a negative productivity shock starting from steady state in his framework. His finding highlights that uninsured idiosyncratic investment risk is likely to amplify the transitional dynamics by reducing the demand for investment during a recession whereas labor income risk is likely to have a mitigating effect by contributing to higher savings during a recession, which implies the importance of studying idiosyncratic investment risks in the business cycle. Few subsequent researches leave room for studies on the effect of idiosyncratic production risks on the business cycle properties or transition behaviors and on the prescription of fiscal policies for the dynamic stochastic environment.

I attempt to first study cyclical properties of income distribution under the context of idiosyncratic production risks. Then I use the model to carry on a series of policy experiments. In particular, I ask when the time to tax capital is better, during a boom or a recession. To achieve these objectives, I build up a business cycle model adapted from the framework of Angeletos (2007) including entrepreneurship and idiosyncratic production risks. First, a model with entrepreneurs reflects a type of important economic agents who own a large amount of wealth and income and the model has mathematically demonstrated that it is able to generate a realistic income and wealth distribution. Second, the particular Angeletos’ framework provides a tractable way to analyze the mechanism of how idiosyncratic production risks affect the economy even with aggregate shocks.

My model features two types of risk-averse agents: entrepreneurs and hand-to-mouth workers. Both types have CRRA preferences. I restrict entrepreneurs to only invest in their firms or alternatively in non-state-contingent government bonds. Aggregate shocks hit the economy every period. Together with

\(^1\)I use the term "idiosyncratic production risks" for the rest of the paper because it affects the individual firm through production. It corresponds to "idiosyncratic investment risks" used in a large number of paper.
i.i.d idiosyncratic production risks, the aggregate shocks exert an impact on the decision of how much to consume and to save (saving choice) and of how much investment on risky but productive assets and on riskfree assets (portfolio choice).

I study the effect of idiosyncratic risks on the steady state by a numerical analysis. I calibrate the model by targeting the US data. A choice of risk aversion degree does matter for the behavior of the model at the steady state. A risk aversion degree equal to 1 (logarithmic preferences) induces a constant saving rate independent of idiosyncratic risks and a decreasing proportion of risky assets to the whole portfolio when idiosyncratic risks increase. The former finding results from the property of the logarithmic utility function and linear budget constraints. The latter is due to risk aversion: entrepreneurs fear that their consumption may drop a lot by holding risky assets.

More interestingly, a risk aversion degree larger than 1 causes increasing saving rate and ratio of risky assets with increasing idiosyncratic risks. Entrepreneurs save more because of precautionary saving motives. A tradeoff appears in the portfolio choice. On one hand, to save more means to invest both assets more and the risky asset can be more attractive since it is productive. On the other hand, risk aversion prevents investors from risky assets. Under the current value of risk aversion degree, the former dominates the latter.

My model generates the dynamics of income distribution with small computational costs. The result shows a reasonable income inequality and fits well the cyclical property of different income groups as in the data.

I then carry on a series of fiscal policy experiments based on the model. Particularly, I fix the steady state level of tax rate and let the capital tax rate correlated to log-deviation of output. In the normal business cycle framework, a seminal paper by Chari, Christiano and Kehoe (1994) states that the optimal capital tax should be procyclical. The logic derives from proportional taxes. Consider that an economy has two states: high and low. The government decides to either subsidize 5% to capital income or tax 5% from capital income. It chooses whether to choose tax in the low state. Then procyclical policies make a large economy even larger. The gain from a subsidy during the boom could compensate the loss from the same proportion of tax during the recession. But at the same time, such a policy renders smoothing consumption even less possible, which creates a tradeoff between the mean effect and the fluctuation effect. In that case, the mean effect dominates the fluctuation effect.

I find that the capital tax could be less to obtain a higher social welfare with idiosyncratic risks and with reasonable relative size of two agents. A procyclical tax policy benefits entrepreneurs while harms workers’ welfare and vice versa. An optimal capital tax conditional on the steady state level finally depends on the relative size or the weight the government puts on each group.

I also find that entrepreneurs advocate a more volatile debt policy because it implies a higher mean level compared to stabler policy, which provides safe assets to entrepreneurs. In addition, a countercyclical debt policy benefits entrepreneurs more than a procyclical one. On the contrary, workers prefer stabler debt policy, thus again the best to raise debt relies on the comparison of welfare weight on each group.

Other interesting results from policy experiments are as follows. (1). A low capital tax rate may not heighten the welfare of entrepreneurs if the government has to lower the debt holding to balance
the budget. The reduction in the riskfree assets indeed pulls down the available wealth despite low tax burden. (2). Income taxation under which the government does not distinguish the labor or capital tax lowers the welfare of both entrepreneurs and workers. Compared to the previous differentiated taxes, entrepreneurs suffer from much more tax burden so they are worse off. Besides, the declined capital stock drags down the wage, which makes workers worse off as well.

My paper is in line with the literature of uninsured idiosyncratic risks and their effects on the welfare. The Bewley-Aiyagari framework considers idiosyncratic labor-income risks and incomplete markets. Most mechanisms following this framework cannot generate a realistic concentration of the income and wealth distributions. A few exceptions such as Castañeda et al (2003) do capture the distributions, yet they do not involve the discussion of undiversified and risky investment.

The paper is also close to the studies regarding entrepreneurial savings and investment, and questions based on the implications. The modelling of entrepreneurship can effectively generate the high concentration of income we observe in data as shown in Quadrini (2000) and Cagetti and De Nardi (2006, 2009). Yet they do not study such issues over the business cycle.

This paper is even closer to the recent literature focusing on uninsured idiosyncratic investment risks and entrepreneurship. These studies imply that market incompleteness may lead to underaccumulation or overaccumulation of capital (Angeletos (2007); Angeletos and Calvet (2006); Covas (2006); Meh and Quadrini (2006)). Some researchers develop some more full-fledged mechanisms to explain specific questions on fiscal policies (Angeletos and Panousi (2009), Panousi (2010)), international difference of growth (Angeletos and Panousi (2011) and the like. But they do not specify a model combining idiosyncratic and aggregate shocks.

This paper is closest to Goldberg (2014), who sets up a framework of modelling idiosyncratic investment risks in the business cycle. We differ in a few ways. First, I model incomplete markets as a unique non-state-contingent bond to be purchased; Goldberg provides state-contingent promises but with moral hazard. Second, he aims to theoretically construct such a model while I build up this model to answer specific questions.

The paper is related to studies on fiscal policies as well. The studies, nevertheless, either concentrate on the redistributional effect of fiscal policies on the welfare based on a stationary distribution, or confine the scope to representative agents to examine policies in a business cycle. Few studies include a comprehensive discussion of both dimensions.

The paper is organized as follows. I develop a simple Angeletos’ framework in the business cycle in Section 2 and study the characterization of equilibrium in Section 3. Section 4 is devoted to the steady state analysis, where I stress the effect of idiosyncratic production risks on the size of the aggregate economy. In Section 5, I quantitatively study the income shares of different income groups and their cyclical properties generated by my model. I also make a comparison with the data. Section 6 carries on a set of experiments on fiscal policies. Section 7 concludes.
2 A Simple Angeletos’ Model in Business Cycle

2.1 Economy

The economy is populated with two types of agents: entrepreneurs and workers. I normalize the measure of entrepreneurs to 1 and the measure of workers, $\lambda$. I interpret $\lambda$ as the ratio of numbers of workers over entrepreneurs or the worker/entrepreneur ratio. There is a continuum of individuals for each type, indexed by $i$ and $j$, respectively. Workers are endowed with $\frac{1}{\lambda}$ unit of time so that the aggregate labor time is 1. I denote $u_t$ as instantaneous utility and each agent maximizes her expected lifetime utility

$$U = E_0 \sum_{t=0}^{\infty} \beta^t u_t$$

subject to her own budget constraint and borrowing constraints to be specified. The aggregate productivity $z_t$ affects the production of each firm in the economy and follows an AR(1) process

$$\log z_{t+1} = \rho \log z_t + \epsilon_{t+1},$$

(1)

where $\epsilon_{t+1}$ is normally distributed, $\epsilon_{t+1} \sim N(0, \sigma^2)$. 

2.2 Entrepreneurs

I assume that entrepreneurs only care about consumption. I specify a CRRA utility function $u(c_i^t) = c_i^t \frac{1}{1-\gamma}$, where $\gamma$ denotes the degree of risk aversion. Each entrepreneur owns a private firm. At $t$, an entrepreneur can invest capital $k_{i,t+1}$ in the firm owned by herself, but not in other private firms; she can purchase riskfree government bonds $b_{i,t+1}$ as an alternative financial asset. Entrepreneurs do not work and receive asset gains as the unique source of income. $\pi_i^t$ denotes the profit of her firm together with remaining capital stock net of depreciation to be specified in the next subsection while $\tau^a_t$, the asset tax rate. The disposable income consists of gross income from capital and bond holdings net of taxation. The budget constraint and nonnegativity constrains are

$$c_i^t + k_{i,t+1} + b_{i,t+1} = (1 - \tau^a_t)(\pi_i^t + R_t b_i^t),$$

$$c_i^t \geq 0 \text{ and } k_{i,t+1} \geq 0,$$

(2)

where $R_t$ represents the riskfree rate from $t-1$ to $t$. Entrepreneurs are possible to borrow from the government but the borrowing amount has to fulfill the No-Ponzi condition

$$\lim_{T \to \infty} \mathbb{E}_t \prod_{s=0}^{T-1} (1 - \tau^a_{t+s})^{-1} R_{t+s}^{-1} b_i^t = 0.$$ 

(3)

2.3 Firms

A firm hires labor in a completely competitive labor market, employs its owner’s capital and produces consumption goods. I name the firm run by the entrepreneur $i$ also with $i$. The firm-level productivity specific to firm $i$, $A^i_t$ affects the final output $y^i_t$. Particularly, $A^i_t$ consists of two components, an idiosyncratic production risk $e^i_t$ and aggregate productivity $z_t$: $\log A^i_t = \log z_t - \frac{\sigma_{e,t}^2}{2} + e^i_t$, where $e^i_t$ is
independently and identically distributed among firms and across time while \( z \) allows for persistence as defined before. The idiosyncratic risk \( e_t \) is modelled as a normal, \( e_t \sim \mathcal{N}(0, \sigma_{e,t}^2) \), where \( \sigma_{e,t} \) represents the standard deviation of idiosyncratic production risks at \( t \), which may vary across periods following

\[
\sigma_{e,t}^2 = \sigma^2_e (1 - \eta \log z_t),
\]

where \( \eta \) denotes the response of volatility to productivity. Bloom et al (2014) justify it by measuring the dispersion of TFP shocks for a panel of plants during the two years before the recent recession (2005 to 2006) and two years during the recession (2008 to 2009). The idiosyncratic risk captures the position of a specific firm ranked by the firm-level productivity. I assume the neoclassical production technology

\[
y_{it} = F(k_{it}, n_{it}, A_{it}) = A_{it}^\alpha k_{it}^\alpha n_{it}^{1-\alpha}
\]

which exhibits constant returns to scale with respect to \( k \) and \( n \). I define the pre-tax wealth value of entrepreneur \( i \) as the firm revenue and capital stock net of labor costs and depreciation

\[
\pi_{it}(k_{it}, n_{it}, A_{it}) = y_{it} - w_t n_{it} + (1 - \delta) k_{it},
\]

where \( w_t \) reprents the wage rate at \( t \). The competitive labor market ensures a universal wage rate at a particular period and only the wage depends on the aggregate productivity and aggregate allocations.

For the expositional convenience, I assume that the remaining capital stock after depreciation \((1 - \delta)k_{it}'\) and the income from government bonds are also taxed. To tax the remaining capital stock is nonstandard. However, this assumption does not affect any qualitative results and influences little quantitative results since first it does not distort further the decision of entrepreneurs and second I can map the capital tax \( \tau_k \) commonly used in the literature.

### 2.4 Workers

A worker has preferences on consumption, which is specified as \( u(c_{jt}) = c_{jt}^{-\gamma} \). I assume that a worker shares the same curvature of consumption as an entrepreneur. Workers supply their labor in the competitive labor market, work in firms owned by entrepreneurs and consume all of their earnings in the manner of hand-to-mouth workers. It reflects that quite a number of households hardly own any wealth other than a house. Workers provide identical working hours but differ in their labor efficiency \( e_{jt} \), which is independently and identically distributed across workers. So the effective labor differs across workers. A worker’s personal labor income depends on idiosyncratic labor efficiency and the wage rate. A worker is taxed by proportional labor taxation with the rate of \( \tau_{nt} \). The assumption of the hand-to-mouth feature implies that the aggregate consumption and income for workers do not depend on the distribution of workers. I assume that the labor efficiency follows a persistent stochastic process,

\[
\log e_{jt+1} = \rho_w \log e_{jt} + \epsilon_{jt+1},
\]

where \( \rho_w \) denotes the autocorrelation of labor efficiency and \( \epsilon_{jt+1} \) is normally distributed, \( \epsilon_{jt+1} \sim \mathcal{N}(0, \sigma^2_w) \).

The budget constraint for Worker \( j \) is

\[
c_{jt} = (1 - \tau_{nt}) w_t \frac{1}{\lambda} e_{jt}.
\]
2.5 Government

The government spending is assumed to be exogenous, following an AR(1) process.

\[
\log g_{t+1} = (1 - \rho_g) \log \bar{g} + \rho_g \log g_t + \epsilon_{t+1}^g, \tag{6}
\]

where the steady state level of government consumption is \(\bar{g}\) and \(\epsilon_{t+1}^g\) is normally distributed, \(\epsilon_{t+1}^g \sim \mathcal{N}(0, \sigma_g^2)\).

Each period the government balances its budget by levying the proportional taxes \(\{\tau^n_t, \tau^a_t\}\) and issuing bonds \(\{B_{t+1}\}\).

\[
\int \tau^n_t \pi_t + \int \tau^a_t w_t \frac{1}{\lambda} \epsilon^a_t + B_{t+1} = g_t + (1 - \tau^a_t) R_t B_t, \tag{7}
\]

where \(B_{t+1}\) denotes the total amount of bonds issued at \(t\) and paid off at \(t+1\). I assume that the government does not distinguish the gains from risky assets and from riskfree assets so that it taxes the asset income at the same rate. In this paper, I intend to analyze the optimal policies of capital taxation and debt conditional on the long-run levels, or the effect on the social welfare of cyclical properties of taxation and debt.

\[
\log \left( \frac{\tau^a_t}{\bar{\tau}^a} \right) = m_{Y^a} \log \left( \frac{Y_t}{\bar{Y}} \right); \tag{8}
\]

\[
\log \left( \frac{B_{t+1}}{\bar{B}} \right) = \rho_B \log \left( \frac{B_t}{\bar{B}} \right) + m_{Y_B} \log \left( \frac{Y_t}{\bar{Y}} \right) \tag{9}
\]

\(\bar{\tau}^a\) represents the steady state level of tax rate, \(\bar{Y}\) denotes the steady state value of output and \(\bar{B}\) shows the steady state level of debt. \(m_{Y^a}\) and \(m_{Y_B}\) are two coefficients indicating the responses of current tax rates and debt to output deviation, respectively. \(\rho_B\) denotes the autoregressive property of government debt\(^2\). For instance, the government taxes more capital income and issues more government bonds during recessions if \(m_{Y^a} < 0\) and \(m_{Y_B} < 0\). In a word, the government sets the rules of debt and asset tax, and applies labor tax to balance the budget.

Finally the No-Ponzi condition holds

\[
\lim_{T \to \infty} \mathbb{E}_t \prod_{s=0}^{T-1} (1 - \tau^a_{t+s})^{-1} R_{t+s}^{-1} B_{t+T} = 0. \tag{10}
\]

2.6 Timing

At every period, the aggregate shocks and the idiosyncratic risks hit the economy. The government then announces fiscal policies based on the current state. After noticing the shocks and the news of policies, workers supply labor while the firm optimally chooses the demand of labor. The firm then takes labor and predetermined capital as input to produce. The entrepreneur consumes, accumulates capital and purchases bonds after she receives income from interest and bond payment, and pays taxes. The worker consumes all the labor income.

\(^2\)I attempt the specifications: \(\log \left( \frac{B_{t+1}}{\bar{B}} \right) = \rho_B \log \left( \frac{B_t}{\bar{B}} \right) + m_{Y_B} \log \left( \frac{Y_t}{\bar{Y}} \right)\) and \(\log \left( \frac{B_{t+1}}{\bar{B}} \right) = m_{Y_B} \log \left( \frac{Y_t}{\bar{Y}} \right).\) I use FRED dataset to estimate the coefficients and only the former specification shows significance. As for tax policy, I also consider other specifications. Yet when I use OECD tax dataset to estimate, none of specifications I proposed shows significance. Hence I apply the above simple model.
3 Equilibrium

3.1 Equilibrium Definition

I define an equilibrium as a stochastic sequence of prices \( \{w_t, R_t\}_{t=0}^{\infty} \), a stochastic sequence of individual allocations \( \{c_{it}, k_{it+1}, b_{it+1}\}_{t=0}^{\infty}, i \in [0, 1] \) for entrepreneurs and \( \{y_{jt}, n_{jt}\}_{t=0}^{\infty}, j \in [0, \lambda] \) for workers, \( \{g_t, \lambda_t\}_{t=0}^{\infty} \) for firms, and aggregate allocations \( \{C_t^E, C_t^W, K_{t+1}, B_{t+1}, Y_t\}_{t=0}^{\infty} \), such that

1. Given prices \( \{w_t, R_t\}_{t=0}^{\infty} \), the fiscal policy \( \{\tau_t, \tau_t, B_{t+1}\} \) and the distribution of initial assets \( k_0 \) and \( b_0 \), every entrepreneur \( i \) and every worker \( j \) maximize their respective lifetime utility by choosing \( \{c_{it}, y_{jt}, k_{it+1}, b_{it+1}\}_{t=0}^{\infty} \) and \( \{c_{it}, n_{it}\}_{t=0}^{\infty} \), and every firm \( i \) maximizes its profit by choosing \( \{n_{it}\} \).

2. Aggregation: \( C_t^E = \int c_t^i, Y_t = \int y_t^i, K_t = \int k_t^i \) and \( C_t^W = \int c_t^j \) for all \( t \).

3. Labor market clearing: \( \int n_t^i = \int n_t^j c_t^j \) for all \( t \).

4. Bond market clearing: \( \int b_t^i = B_t \) for all \( t \).

5. Goods market clearing: \( C_t^E + C_t^W + K_{t+1} + g_t = Y_t + (1 - \delta)K_t \).

6. The government budget keeps balanced given capital tax and bond specifications for all \( t \), i.e. (7), (8) and (9) hold.

3.2 Individual Behavior

Because the firm chooses employment \( n_{it} \) after observing the shock and after determining the capital stock, \( n_{it} \) is the only control variable to maximize the profit. By constant returns to scale, optimal firm employment and capital income are linear in capital following Angeletos (2007):

\[
n_{it} = \left( 1 - \frac{1}{w_t} \right)^{\frac{1}{\alpha}} \frac{1}{\alpha} A_t^i k_t^i = n(A_t^i, w_t)k_t^i, \tag{11}
\]

\[
\pi_t^i = \alpha \left( 1 - \frac{1}{w_t} \right)^{\frac{1}{\alpha}} A_t^i + \frac{1}{\alpha} \delta \] \[ k_t^i = (r(A_t^i, w_t) + 1 - \delta)k_t^i. \tag{12}
\]

It indicates that the firm experiences linear returns to investment by adjusting its employment linearly to the capital stock.

Denote the effective wealth of entrepreneur \( i \) in period \( t \) by

\[
x_t^i \equiv (1 - \tau_t^i)(\pi_t^i + R_t b_t^i) = (1 - \tau_t^i) \left[ (r(A_t^i, w_t) + 1 - \delta)k_t^i + R_t b_t^i \right].
\]

I rewrite the budget constraint as \( c_t^i + k_{t+1}^i + b_{t+1} = x_t^i \).

I characterize the solution to the optimization problem for the entrepreneur following Angeletos (2007):

\[
c_t^i = (1 - \nu_t)x_t^i, \tag{13}
\]

\[
k_{t+1}^i = \nu_t \phi_t x_t^i, \tag{14}
\]

\[
b_{t+1}^i = \nu_t (1 - \phi_t)x_t^i, \tag{15}
\]
where the saving rate $\nu_t$ and capital share in the portfolio $\phi_t$ are two stochastic coefficients, depending on only aggregate states and satisfying

$$
\phi_t = \arg \max_{\phi \in [0,1]} CE_t \left\{ \phi(r(A_{t+1}^i, w_{t+1}) + 1 - \delta) + (1 - \phi)R_{t+1} \right\},
$$

(16)

$$(1 - \nu_t)^{-\gamma} = \beta \nu_t^{-\gamma} CE_t \left\{ (1 - \nu_{t+1})^{-\gamma} (1 - \nu_t^{-\gamma} [\phi_t(r(A_{t+1}^i, w_{t+1}) + 1 - \delta) + (1 - \phi_t)R_{t+1}] \right\}^{-\gamma}.
$$

(17)

CE represents the certainty equivalent of an entrepreneur.

Furthermore, I define the value function for entrepreneurs as $V(x^i_t)$. By guess and verify, the value function can be written as

$$
V(x^i_t) = \frac{(1 - \nu_t)^{-\gamma} x^i_t^{-\gamma}}{1 - \gamma}.
$$

(18)

I summarize the characterization of individual behaviors in the following lemma.

**Lemma 1** Given prices and fiscal policies, optimal consumption, capital stock and bond holdings are linear in current wealth as shown in (13), (14) and (15); the saving rate $\nu_t$, the capital proportion in the portfolio $\phi_t$ and the value function for entrepreneurs $V(x^i_t)$ depend solely on the current aggregate state, expressed by (16), (17) and (18).

The proof is included in Appendix A. The entrepreneur makes the optimal portfolio choice $\phi_t$ by maximizing risk-adjusted portfolio returns, expressed by the certainty equivalent $CE$ of the portfolio return given the saving choice $\nu_t$. If the return of capital is surely greater than the return of bonds, she will invest all her savings on capital. The uncertainty of capital return, though, incurs her to divide her investment on both assets, which ensures an interior point of portfolio decisions. The current specification of tax policies implies that a change in the asset tax rate fails to directly influence the portfolio choice since both types of assets are taxed at the same rate. Nonetheless, the asset tax rate distorts the investment choice and asset stock, indirectly affecting the composition of portfolio. Given the portfolio choice, the entrepreneur chooses the saving rate according to the intertemporal condition to maximize the summed utility from consumption. Because of the i.i.d idiosyncratic production risk, entrepreneurs make the saving and portfolio decisions independent of one period’s income distribution. Hence the saving rate and the risky-asset-to-wealth ratio only depend on the aggregate state.

The hand-to-mouth worker consumes all her labor income every period. So the consumption path is completely characterized by the budget constraint of workers.

### 3.3 General Equilibrium

Recall that aggregate productivity $z_t$ and idiosyncratic risk $e_t^i$ are orthogonal. Aggregate labor demand and firm profit are given by

$$
N_t^D = \bar{n}(w_t, z_t)K_t \quad \text{and} \quad \Pi_t = \bar{r}(w_t, z_t)K_t,
$$

where

$$
\bar{n}(w_t, z_t) = \int_{-\infty}^{\infty} n(A_{t+1}^i, w_t) \, dF(e_t^i) = \left(\frac{1 - \alpha}{w_t}\right)^{-\alpha} z_t
$$

and

$$
\bar{r}(w_t, z_t) = \int_{-\infty}^{\infty} r(A_{t+1}^i, w_t) \, dF(e_t^i) = \alpha \left(\frac{1 - \alpha}{w_t}\right)^{-\alpha - 1} z_t.
$$

$F(\cdot)$ represents the distribution function of i.i.d idiosyncratic shocks. Thus $\bar{n}(w_t, z_t)$ denotes the average labor employed
by a firm and \( \hat{r}(w_t, z_t) \), the average capital return of an entrepreneur. Let \( N_t^S = \int_1^\phi \frac{1}{\lambda} e^t = 1 \) represent aggregate labor supply and labor market clearing requires that \( N_t^D = \tilde{n}(w_t, z_t)K_t = 1 \) and then \( w_t = w(K_t, z_t) = (1 - \alpha)(z_tK_t)^{\alpha} \), \( r_t = \hat{r}(w_t, z_t) = \alpha z_t \alpha K_t^{\alpha - 1} \). I write aggregate output as \( Y_t = \Pi_t + w_tN_t^S = r_tK_t + w_t = z_t^\alpha K_t^\alpha \). Aggregate allocations are also independent of the wealth distribution because consumption, bond holdings, and private investment are linear in individual wealth and idiosyncratic risks at one period are i.i.d across firms and periods. Then the general equilibrium is expressed as follows, where I drop the determination of \( z_t \) and \( g_t \) as they are assumed to be exogenous. I denote \( C_t^E \) as the aggregate consumption assigned to the group of entrepreneurs and \( C_t^W \), to the group of workers.

\[
C_t^E + C_t^W + K_{t+1} + g_t = z_t^\alpha K_t^\alpha + (1 - \delta)K_t, \tag{19}
\]

\[
C_t^W = (1 - \tau_t^a)w_t, \tag{20}
\]

\[
C_t^E = (1 - \nu_t)(1 - \tau_t^a)[(r(t, z_t) + 1 - \delta)K_t + R_tB_t], \tag{21}
\]

\[
K_{t+1} = \nu_t \phi_t(1 - \tau_t^a)[(r_t + 1 - \delta)K_t + R_tB_t], \tag{22}
\]

\[
B_{t+1} = \nu_t(1 - \phi_t)(1 - \tau_t^a)[(r_t + 1 - \delta)K_t + R_tB_t], \tag{23}
\]

\[
n(w_t)K_t = 1, \tag{24}
\]

These six equations, together with (7), (16) and (17) solve for nine variables \( C_t^E, C_t^W, K_{t+1}, \tau_t^a, \phi_t, w_t, R_{t+1}, \nu_t \) and \( \phi_t \), given \( K_t, B_t, z_t \) and \( g_t \).

To solve the individual’s problem and general equilibrium in this framework, I separate the computation into two steps: first, obtain the aggregate levels of variables and then, divide allocations via specific rules of distribution. I am able to accomplish it because, economically speaking, every entrepreneur makes exactly identical saving and portfolio decisions regardless of their individual wealth. In another word, I can always find a representative entrepreneur to study the behaviors of all entrepreneurs.

Notice that \( \frac{K_{t+1}}{B_{t+1}} = \frac{\phi_t}{1 - \phi_t} \). Since the steady state level of bonds has been exogenously given, the capital stock can be uniquely determined by the capital share of portfolio, \( \phi_t \). Fixing bonds, the capital stock monotonically increases with \( \phi_t \). Then increased capital enhances next period’s output, wages and the consumption of workers. But the return of capital stock diminishes at next period, thus it is ambiguous to see the direction of the change in effective wealth. Then the effects on consumption of entrepreneurs and the saving choice after one period are also difficult to judge in the analytical result. I will discuss it in the following quantitative analysis.

### 4 Steady State

I exhibit the steady state of the general equilibrium in the subsection. First I set down the steady state counterparts of above equilibrium characterizations. Then I show the steady state results of the benchmark model after calibrating the model.
4.1 Steady State of Equilibrium Characterizations

The key to understanding the model lies on saving and portfolio choices. Given capital stock and the interest rate of riskfree assets, the portfolio choice $\bar{\phi}$ is pinned down by (47), the solution to the (16) at the steady state. After that, the saving choice $\bar{\nu}$ can be computed by inserting the expression of $\bar{\phi}$ into Equation (17) evaluated at the steady state,

$$\bar{\nu}^\gamma = \beta (1 - \bar{\tau}^a)(1 - \gamma)\bar{C}E^{1-\gamma}, \quad \text{(25)}$$

where $\bar{C}E$ is the certainty equivalent at the steady state decided by (46).

Furthermore, the rest of equations which characterize the general equilibrium at the steady state are as follows:

$$\bar{C}^E + \bar{C}^W + \bar{g} = z^\alpha K^\alpha - \delta \bar{K}, \quad \text{(26)}$$

$$\bar{C}^W = (1 - \bar{w})\bar{w}, \quad \text{(27)}$$

$$\bar{C}^E = (1 - \bar{\nu})(1 - \bar{\tau}^a)[(\bar{r} + 1 - \delta)\bar{K} + \bar{R}\bar{B}], \quad \text{(28)}$$

$$\bar{K} = \bar{\nu}\bar{\phi}(1 - \bar{\tau}^a)[(\bar{r} + 1 - \delta)\bar{K} + \bar{R}\bar{B}], \quad \text{(29)}$$

$$\bar{B} = \bar{\nu}(1 - \bar{\phi})(1 - \bar{\tau}^a)[(\bar{r} + 1 - \delta)\bar{K} + \bar{R}\bar{B}], \quad \text{(30)}$$

$$n(\bar{w})\bar{K} = 1, \quad \text{(31)}$$

Thus, above mentioned equations solve the equilibrium at the steady state.

4.2 Calibration

Numerical analysis is needed to evaluate the effect of idiosyncratic production risks on the steady state. I first describe the details of calibrating the model in this subsection.

I assume that a period in my model corresponds to a year. I use some parameter values which is common in the literature of business cycle: The capital share of output $\alpha$ is assumed to be 0.36; the discount rate $\beta$, 0.96; the depreciation rate $\delta$, 0.08. The steady state allocations do not rely on the calibrated value of the worker/entrepreneur ratio, $\lambda$, although its value matters for the results on dynamics of income distribution and welfare. I will discuss the calibration of it in the following sections.

The choice of risk aversion degree matters for my study. I attempt the risk aversion degree equal to 1 or 4 to see its effect together with idiosyncratic risks on steady state. The literature of asset pricing or the like often applies such a method and authors even choose a higher risk aversion degree.

I calibrate the capital tax rate to guarantee the after-tax gross return to capital in my model equivalent to the counterpart defined as $(1 - \tau^k)r + 1 - \delta$ applying the realistic corporate income tax rate of the US for $\tau^k$ given the same pre-tax rental rate of capital $r$ with no idiosyncratic productivity risks and the value is 0.0252, which I keep for different specifications of idiosyncratic risks and government debt levels to see the effect of these risks on steady state. To be specific, the pre-tax capital return is $r$. The
equivalence requires that \((1 - \tau^a)(r + 1 - \delta) = (1 - \tau^k)r + 1 - \delta\). The steady state Euler equation implies that, \(r = \frac{1}{\beta} - 1 + \delta\). I obtain the pre-tax rental rate given the realistic corporate income tax rate of the US. Then I back up the asset tax rate in my model. I pick the labor income tax rate to balance the government budget. In the benchmark calibration, I set the steady state levels of debt as 60 percent of output and government consumption as 18 percent of output, both of which depict the U.S. situation before the recent crisis.

I could hardly find an exact measurement of idiosyncratic production risks in the empirical studies. DeBacker et al find that the standard deviation of uninsurable idiosyncratic income risk from privately held businesses accounts for 45 percent of the average business income. They employ individual income tax returns data from the Internal Revenue Service over 23 years, compute for each household the time series standard deviation of its business income normalized by the household’s average total income over time, and then combine those business income "coefficients of variation" into one cross sectional average. It is the most convincing empirical research I have known so far on the volatility of idiosyncratic production (investment) risks. Thus I calibrate my model to match their finding; specifically the cross sectional standard deviation of individual firm’s return is 45% of the average return. Notice that a number of macroeconomic studies apply the volatility of firms’ returns as 50% of the mean return, thus I assure my calibration close to their choice and my results comparable. All above parameter values for the benchmark are included in Table 2.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
<th>Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha)</td>
<td>0.36</td>
<td>capital share of output</td>
</tr>
<tr>
<td>(\beta)</td>
<td>0.96</td>
<td>discount rate</td>
</tr>
<tr>
<td>(\delta)</td>
<td>0.08</td>
<td>depreciation rate</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>1 or 4</td>
<td>risk aversion degree</td>
</tr>
<tr>
<td>(\bar{b})</td>
<td>60% output</td>
<td>steady state level of bonds</td>
</tr>
<tr>
<td>(\bar{g})</td>
<td>18% output</td>
<td>steady state level of government consumption</td>
</tr>
<tr>
<td>(\bar{\tau}^a)</td>
<td>0.0252</td>
<td>steady state asset income tax rate</td>
</tr>
<tr>
<td>(\sigma_e)</td>
<td>45% average return</td>
<td>standard deviation of idiosyncratic risks</td>
</tr>
</tbody>
</table>

### 4.3 Effect of Idiosyncratic Production Risks on Steady State

I report the steady state result of my model with the benchmark risks together with no idiosyncratic risks and high risks in Table 3. Note that without idiosyncratic risks the allocation is efficient. The current calibration in my model generates overaccumulation of capital.

If I consider the logarithmic preferences, idiosyncratic production risks have four major effects:

First, the saving choice with the logarithm preferences keeps the same value as the discount rate. This is because of the mathematical behavior of the logarithmic utility and linear budget constraint.

Second, the capital share of portfolio decreases in risks. Recall that idiosyncratic risks fail to change the saving choice with \(\gamma = 1\). Then more risks force entrepreneurs to reduce risky assets because of the fear that their consumption may drop drastically by holding the same portfolio.
Table 3: Comparison of Steady State

<table>
<thead>
<tr>
<th>Variables</th>
<th>No risks</th>
<th>Idio. risks $\gamma=1$</th>
<th>Idio. risks $\gamma=4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Saving Rate</td>
<td>0.96</td>
<td>0.96</td>
<td>0.97</td>
</tr>
<tr>
<td>Cap. Ratio</td>
<td>0.802</td>
<td>0.801</td>
<td>0.81</td>
</tr>
<tr>
<td>Capital</td>
<td>3.99</td>
<td>3.96</td>
<td>4.47</td>
</tr>
<tr>
<td>Output</td>
<td>1.65</td>
<td>1.64</td>
<td>1.71</td>
</tr>
<tr>
<td>Ave. Rental Rate</td>
<td>0.1486</td>
<td>0.1492</td>
<td>0.14</td>
</tr>
<tr>
<td>Interest Rate</td>
<td>1.069</td>
<td>1.066</td>
<td>1.05</td>
</tr>
<tr>
<td>Wage</td>
<td>1.053</td>
<td>1.050</td>
<td>1.10</td>
</tr>
<tr>
<td>Ent. Consumption</td>
<td>0.207</td>
<td>0.206</td>
<td>0.16</td>
</tr>
<tr>
<td>Wor. Consumption</td>
<td>0.8229</td>
<td>0.8230</td>
<td>0.89</td>
</tr>
<tr>
<td>Equity Premium</td>
<td>0</td>
<td>0.003</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Third, due to the effect on saving and portfolio choices, risky capital stock declines with risks. The decreased capital stock causes an increase in the average capital return and a decrease in the wage. The consumption of entrepreneurs diminishes since reduced productive assets lead to less output and less consumption of entrepreneurs.

Last, the interest rate of riskfree assets decreases with risks. More entrepreneurs prefer to own riskfree assets, pushing up the price of bonds so that the interest rate declines.

I turn to the case when $\gamma = 4$, whose results exhibit a number of disparities compared with the previous case.

First, $\bar{\nu}$ rises with idiosyncratic risks. It results from precautionary saving motives with which entrepreneurs sacrifice part of current consumption to augment capital stock and smooth consumption, which is confirmed by the change of entrepreneurs’ consumption $\bar{C}^E$.

Second, the capital ratio in the portfolio $\bar{\phi}$ increases in risks. It seems counterintuitive since entrepreneurs are supposed to save more but hold more safe assets due to risk aversion. The basic intuition is that in the aggregate level, when the total saving goes up relative to consumption, the willingness to raise the safe assets reduces the interest rate and bonds are not as attractive as risky assets. So the capital stock goes up. Then in the individual level, every entrepreneur makes the same decision so they all increase the proportion of capital.

If I consider the entrepreneur’s problem solely, I interpret the intuition as follows. On one hand, more savings help smooth the consumption so that entrepreneurs are able to hold even more risky but more productive assets. On the other hand, risk averse entrepreneurs fear to hold risky assets as idiosyncratic risks show up. Precautionary saving motives and risk premium exist simultaneously and in the case of $\gamma = 4$, the former overweights the latter, the same mechanism discussed by Covas (2006).

I point that my model with idiosyncratic production risks could potentially produce realistic equity premium. With the risk aversion degree equal to 4, a relatively small value compared to many other exercises, I generate a sizable equity premium as 0.01 considering realistic idiosyncratic risks. Without idiosyncratic risks, non-arbitrage condition requires that $r + 1 - \delta = R$. The existence of idiosyncratic risks modifies it to $r \exp\left(-\frac{\sigma^2}{2}\right) + 1 - \delta = R$. Thus the equity premium defined by $r + 1 - \delta - R$ would be greater than 0 with a positive risk. A higher risk aversion degree or Epstein-Zin preferences are two possible methods to generate a larger difference.
5 Business Cycle Statistics of Income Distribution

This section first constructs a stationary income distribution to generate a thick tail in both ends. The model, with aggregate shocks, can analyze the cyclicality of income distribution with small computational costs. For most quantiles, the model fits the data well in the quantitative sense.

5.1 Construct Stationary Income Distribution

The income of an entrepreneur $i$ is defined as the asset gains, $I_i^t = r_i^t k_i^t + (R_t - 1)b_i^t$ as Castañeda et al (1998) do, while the income of a worker $j$ is the labor income, $\frac{1}{\lambda} w_t e^j_t$.

The current model implies that the income occupied by entrepreneurs will go to infinity. I add an exogenous death probability $Pr_d$ equal to all entrepreneurs to guarantee a stationary distribution without aggregate shocks. Specifically the death shock is assumed to happen after every entrepreneur makes her own decision each time. Some entrepreneurs die while the same number of newborns enter into the economy. Then every newborn inherits the average amount of risky and riskfree assets previously owned by the dead. From Section 3, we know that entrepreneurs make saving and portfolio choices independent of the income distribution, only according to the aggregate state. Hence this assumption does not change the original choices before the death and facilitates the computation.

5.2 Calibrate Business Cycle Properties

I assume the worker/entrepreneur ratio as 1.5 so that hand-to-mouth workers account for 60% of the total population. The value seems much higher than the estimated proportion of hand-to-mouth workers, and also much lower than the estimation of entrepreneurs’ composition. It follows the logic that about 60% of households in the US own no wealth other than home according to the 2007 Survey of Consumer Finances, since the assumption of hand-to-mouth workers aims to shut down saving from workers who face labor income risks.

I set the autocorrelation of idiosyncratic labor income risks equal to 0.9989 and the standard deviation, 0.0166 using the estimation of Storesletten, Telmer and Yaron (2004).

The probability of death is assumed to be 0.025 as Krueger, Mitman and Perri (2015). They argue that the death probability would be 1/40 for an expected working lifetime of 40 years. I modify the discount rate so that the subjective discount rate of entrepreneurs after considering death still remains $0.96^3$.

Recall that I correlate the volatility of idiosyncratic production risks with aggregate productivity, $c_i^t \sim \mathcal{N}(0, \sigma^2_i (1 - \eta \log z_t))$. I calibrate the value of $\eta$ to match the empirical finding of Bloom et al (2014) that plant-level TFP shocks increased in variance by 76% during the recession. I calibrate the sensitivity of the variance of idiosyncratic production risks to productivity shocks, $\eta$, to feature the observation.

The calibration for exogenous technology process and government spending follows Chari, Christiano and Kehoe (1994). I use FRED dataset and OECD tax dataset to pin down the parameters in the above specified fiscal policies. I regress debt on HP-filtered log of output and lagged debt following the fiscal

\[\text{The discount rate becomes } 0.96 \div (1 - 0.025) = 0.9846.\]
policy specification. I estimate the response of asset tax rate to output gap with a similar method except that the regressor only contains HP-filtered log of output. Worth to mention, the tax policy amplifies (reduces) the business cycle effect by taxing more (less) in the recession (expansion). Meanwhile, the government reduces debt in the expansion and vice versa. The results are summarized in Table 4. I keep other parameter values the same as the analysis of steady state; in particular, I choose the risk aversion degree equal to 4 to generate a sizable equity premium. I call this set of parameters as the benchmark calibration in the experiment of income distribution.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
<th>Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>1.5</td>
<td>worker/entrepreneur ratio</td>
</tr>
<tr>
<td>$\rho_w$</td>
<td>0.9989</td>
<td>autocorrelation of labor efficiency</td>
</tr>
<tr>
<td>$\sigma_w$</td>
<td>0.0166</td>
<td>cross-sectional standard deviation of labor income</td>
</tr>
<tr>
<td>$Pr_{d}$</td>
<td>0.025</td>
<td>probability of death for entrepreneurs</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.96</td>
<td>sensitivity of the variance of idiosyncratic production risks to productivity shocks</td>
</tr>
<tr>
<td>$\rho_z$</td>
<td>0.81</td>
<td>autocorrelation of technology process</td>
</tr>
<tr>
<td>$\sigma_z$</td>
<td>0.04</td>
<td>standard deviation of technology shock</td>
</tr>
<tr>
<td>$\rho_g$</td>
<td>0.89</td>
<td>autocorrelation of government spending</td>
</tr>
<tr>
<td>$\sigma_g$</td>
<td>0.07</td>
<td>standard deviation of government spending shock</td>
</tr>
<tr>
<td>$\rho_B$</td>
<td>0.84</td>
<td>autocorrelation of government debt</td>
</tr>
<tr>
<td>$m_{YB}$</td>
<td>-0.62</td>
<td>response of debt to output gap</td>
</tr>
<tr>
<td>$m_{YT}$</td>
<td>-0.08</td>
<td>response of tax to output gap</td>
</tr>
</tbody>
</table>

5.3 Computation Process

The framework allows me to separate the computation of general equilibrium and income distribution. In the beginning, I simulate a time series of equilibrium at the aggregate level in the business cycle starting from the steady state. Then I divide aggregate allocations into the individual level. The next period’s capital and bonds of entrepreneur $i$ can be calculated as a product of saving choice, portfolio choice and effective wealth as Equation (14) and (15). The first two components are obtained from the time series of general equilibrium. I separate the last part into idiosyncratic capital returns, interest rate of bonds and capital and bonds stock held from last period. The time series of equilibrium contains the values of interest rate and the average capital returns, then from (43) we get idiosyncratic capital returns by inputting a stochastic process of idiosyncratic risks. Thus the amount of assets forms a recursion given asset prices in the aggregate level and idiosyncratic risks. Workers share the total labor income determined by the general equilibrium with different labor efficiency.

I create 10000 grids as the number of entrepreneurs and as a consequence 15000 is the number of workers. I allocate equally the steady state level of capital and bonds to 10000 entrepreneurs as the initial point. Since individual capital returns are diverse across entrepreneurs, lucky entrepreneurs accumulate more and more assets so that the gap of capital income has been enlarged. I record the income of every agent, including entrepreneurs and workers, order all values and then compute the quantile statistics and the like.
5.4 Cyclical Behavior of General Equilibrium

Before presenting the results of income distribution, I briefly show the cyclical behavior of general equilibrium in my baseline model in the following table. Particularly, I emphasize the saving choice $\nu$ and the portfolio choice $\phi$ since these two variables summarize the key property of the entire equilibrium allocations and prices. I compute the relative standard deviation by dividing the standard deviation of each variable by its mean and report the result of relative standard deviation in percentage.

<table>
<thead>
<tr>
<th>Variables</th>
<th>Relative Standard Deviation</th>
<th>Correlation with Output</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No risks</td>
<td>Idio. risks</td>
</tr>
<tr>
<td>Saving Rate</td>
<td>0.04</td>
<td>0.03</td>
</tr>
<tr>
<td>Cap. Ratio</td>
<td>2.73</td>
<td>2.59</td>
</tr>
<tr>
<td>Capital</td>
<td>3.64</td>
<td>3.61</td>
</tr>
<tr>
<td>Output</td>
<td>3.39</td>
<td>3.37</td>
</tr>
<tr>
<td>Ave. Rental Rate</td>
<td>2.19</td>
<td>2.22</td>
</tr>
<tr>
<td>Interest Rate</td>
<td>0.23</td>
<td>0.22</td>
</tr>
<tr>
<td>Wage</td>
<td>3.39</td>
<td>3.37</td>
</tr>
<tr>
<td>Ent. Consumption</td>
<td>0.33</td>
<td>0.29</td>
</tr>
<tr>
<td>Wor. Consumption</td>
<td>7.15</td>
<td>6.79</td>
</tr>
</tbody>
</table>

The result of relative standard deviation is in percentage.

The saving rate $\nu$ and the capital share in the portfolio $\phi$ are positively correlated with output, which is a common result in the business cycle framework shown by the second column. Investors tend to save more assets in the expansion because the motive of smoothing consumption implies a minor variation of consumption in the business cycle. The higher output mainly converts to more savings. Besides, agents invest in more productive but risky assets during booms since they expect higher returns to capital. Due to the same argument, the recession engenders a reduction in capital and the variation of capital in the entire cycle boosts the mean output and consumption.

The most apparent difference in the comparison of cyclical properties between idiosyncratic production risks and no risks lies in the interest rate. With only aggregate shocks, agents demand more riskfree assets in the bust; meanwhile, the government supplies more bonds according to the policy specification. A higher supply relative to demand results in a lower price of government bonds, or equivalently, a higher interest rate. When idiosyncratic production risks are added, a more volatile production leads to more demand of safe assets which surpasses the supply. It overturns the countercyclical behavior of interest rate.

5.5 Income Inequality: Level

My model manifests the income equality as the data show in Table 5. For the convenience of comparison, I duplicate the US income distribution in the second column. I report the calibration with idiosyncratic shocks.

My model fits the data in the sense that it generates the thick tails in both ends of income distribution although the rich own less income and the poor share more compared with the data. I propose several
Table 6: Income Inequality: Level

<table>
<thead>
<tr>
<th>Quantiles</th>
<th>Share of income</th>
<th>Share of income</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(Data)</td>
<td>Model</td>
</tr>
<tr>
<td>1st (bottom 20%)</td>
<td>4.1%</td>
<td>10.37%</td>
</tr>
<tr>
<td>2nd (20-40%)</td>
<td>10.3%</td>
<td>14.96%</td>
</tr>
<tr>
<td>3rd (40-60%)</td>
<td>16.4%</td>
<td>18.53%</td>
</tr>
<tr>
<td>4th (60-80%)</td>
<td>23.9%</td>
<td>22.91%</td>
</tr>
<tr>
<td>5th (80-100%)</td>
<td>45.4%</td>
<td>33.22%</td>
</tr>
<tr>
<td>Top 5%</td>
<td>18.4%</td>
<td>10.94%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ratios</th>
<th>Value (Data)</th>
<th>Value (High risks)</th>
</tr>
</thead>
<tbody>
<tr>
<td>95/50 ratio</td>
<td>3.29</td>
<td>1.93</td>
</tr>
<tr>
<td>50/20 ratio</td>
<td>2.40</td>
<td>1.41</td>
</tr>
</tbody>
</table>

explanations of the gap. First, I assume that idiosyncratic production risks, due to computational convenience, are i.i.d. A persistent shock, a more realistic case, may improve the quantitative result. Second, the assumption of either hand-to-mouth workers or entrepreneurs may be too strict since a number of households hold net worth such as shares of public corporations. Third, I simplify the types of assets to two kinds: risky capital stock of one’s own firm and riskfree government bonds, which may lose the feature of households’ portfolio, especially for the rich.

I count the number of entrepreneurs in the top 5% and the bottom 20% earners. Entrepreneurs account for a proportion large than their ratio in the population. It indicates that the thick tails in both ends are obtained by modelling entrepreneurs with idiosyncratic production risks, which is confirmed by other researches, such as Benhabib, Bisin and Zhu (2014) and Nirei and Aoki (2014).

5.6 Income Inequality: Cyclical Behavior

I report the correlations with output of the income shares earned by income groups in the data and in my model in Table 6. I choose the same data from CPS as in Table 1 and choose seasonally adjusted real GDP measured by 2009 billion dollars during 1948-2013. I detrend the shares, ratios and the log of real GDP by HP-filter and calculate the correlations. For the model, I pick the time series of income shares of corresponding income groups as the data and see the correlations with the output in the business cycle. I run a simulation with 2500 periods and drop the first 1500 periods to guarantee the stability.

Table 7: Correlation with Output

<table>
<thead>
<tr>
<th>Income groups</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>bottom 20%</td>
<td>0.29</td>
<td>0.48</td>
</tr>
<tr>
<td>20-40%</td>
<td>0.30</td>
<td>0.59</td>
</tr>
<tr>
<td>40-60%</td>
<td>0.15</td>
<td>0.60</td>
</tr>
<tr>
<td>60-80%</td>
<td>-0.19</td>
<td>0.15</td>
</tr>
<tr>
<td>80-95%</td>
<td>-0.44</td>
<td>-0.48</td>
</tr>
<tr>
<td>Top 5%</td>
<td>0.07</td>
<td>-0.70</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ratios</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>95/50 ratio</td>
<td>-0.24</td>
<td>-0.62</td>
</tr>
<tr>
<td>50/20 ratio</td>
<td>-0.06</td>
<td>-0.34</td>
</tr>
</tbody>
</table>

My model with the baseline calibration features most of the realistic correlation with output of income shares qualitatively except the top 5% group and 60-80% although the model still differs from reality.
quantitatively. The model overstates the procyclicality for from bottom to 60% and the countercyclicality for ratios. My model produces quantitatively precise results of correlation for the group 80-95%.

To uncover the mechanism how my model generates the dynamics of income distribution, I build up two more statistics to see the impact of business cycle on individual income: coefficients of variation of $t+1$’s income for entrepreneurs and workers given $t$’s information, Coef.Var.,($I^1_{t+1}$) and Coef.Var.,($I^1_{t+1}$). Notice that for simplicity I do not consider the death risk in the subsequent discussion of mechanism since to add death does not affect any qualitative result. I express the two statistics as follows, whose derivation lies in Appendix C.

$$\text{Coef.Var.}_t(I^1_{t+1}) = \frac{\phi_t \alpha z_t^{\alpha \rho_x} K_t^{\alpha - 1} \sqrt{\exp \left[ \sigma_z^2 (1 - \eta \rho_x \log z_t) + \frac{1}{2} (2 \alpha - \eta \sigma_x^2) \right] - \exp (\alpha^2 \sigma_x^2)}}{\phi_t \alpha z_t^{\alpha \rho_x} \exp \left( \frac{1}{2} \alpha^2 \sigma_x^2 \right) K_{t+1}^{\alpha - 1} + (1 - \phi_t) (R_{t+1} - 1)}.$$  \hspace{1cm} (32)

$$\text{Coef.Var.}_t(I^1_{t+1}) = \frac{\sqrt{\exp (\alpha^2 \sigma_x^2 + \sigma_w^2) - 1}}{1 + (\frac{1}{\phi_t} - 1) \frac{R_{t+1} - 1}{\alpha z_t^{\alpha \rho_x} \exp \left( \frac{1}{2} \alpha^2 \sigma_x^2 \right) K_{t+1}^{\alpha - 1}}}.$$  \hspace{1cm} (33)

The coefficient of variation of workers’ next period income is constant. It results from the assumption of hand-to-mouth workers. The wage inherits the cyclical behavior from the output; in particular, its coefficient of variation equates the one of output. I.i.d idiosyncratic labor income risks with constant variance enlarge the total variance of workers’ income so that eventually the summed volatility appears in the coefficient of variation, indicating the function of both aggregate shocks and idiosyncratic labor income risks on the cross-sectional disparity of workers’ income.

The personal difference of entrepreneurs’ income during the cycle looks more complicated. I start from the discussion on the denominator or the conditional mean of income. Idiosyncratic production risks, as assumed to be i.i.d, exert no effect on expected next period’s income. Only aggregate shocks influence the conditional expectation through the channel of asset taxes and returns on capital. When a negative shock happens at $t$, persistence implies a low productivity next period, resulting in lower saving rate, smaller share of capital in the portfolio and lower expected returns on capital. Besides, heavier asset taxes further weaken the willingness to save and invest. As a consequence, the conditional expected income diminishes during the recession.

When investigating the cyclical property of conditional standard deviation, I first suppose that the dispersion of idiosyncratic risks does not change with business cycle ($\eta = 0$) to simplify the analysis. Then the nominator reduces to $\phi_t \alpha z_t^{\alpha \rho_x} \exp \left( \frac{1}{2} \alpha^2 \sigma_x^2 \right) K_{t+1}^{\alpha - 1} \sqrt{\exp (\alpha^2 \sigma_x^2 + \sigma_w^2) - 1}$, which decreases in the recession and increases in the expansion. I rewrite the coefficient of variation by dividing $\phi_t \alpha z_t^{\alpha \rho_x} \exp \left( \frac{1}{2} \alpha^2 \sigma_x^2 \right) K_{t+1}^{\alpha - 1}$ from the nominator and the denominator,
The capital share $\phi$ declines in the recession. The expected return on capital decreases while the interest rate $R$ decreases as well but less. The denominator, thus, increases in the recession, pulling down the fraction value and generating procyclicality.

If the variance of idiosyncratic production risks increase in the recession as considered in the benchmark ($\eta = 0.96$), then the effect of aggregate shocks shows differently even in the qualitative sense. Specifically the coefficient of variation increases in the recession due to the increased nominator deriving from the countercyclical variance of idiosyncratic risks. It implies that the calibration of the sensitivity of variance of idiosyncratic production risks to productivity shocks contribute to generating the realistic dynamics of income distribution. To give readers direct evidence I run another simulation with $\eta = 0$ and obtain completely different results of positive correlation of income with output for top earners and negative correlation for bottom earners.

When the high productivity is realized, the global wage and the average income from asset gains go up. Then workers and entrepreneurs with mean income follow the same trend. The rich workers and entrepreneurs in the high state experience a larger increase in their income while the poor agents see a smaller increase or even a decrease in the income, depending on the size of idiosyncratic risks. In particular, almost all workers are highly likely to obtain higher income, though the extent varies, in the high state because idiosyncratic labor income risks are much smaller than aggregate shocks in the benchmark simulation.

The low productivity produces qualitatively reverse results compared with the preceding ones. Since the coefficient of variation of workers' income remains constant during the cycle, the difference of income in the low state for rich and poor workers keep the same as in the high state. Summarizing both states concludes that income of all three types of workers shows procyclical behavior.

Income of entrepreneurs shows a different pattern. Recall that idiosyncratic production risks are higher than productivity shocks and more importantly, idiosyncratic production risks rise in the recession. So the low state enlarges the gap of income between rich and poor entrepreneurs. To be specific, the increased individual difference magnifies the damage of low state to the poor, but reduces it to the rich. Even a number of the rich benefit from the recession. The expansion witnesses a similar change but with an opposite direction and a smaller scale. Thus the possibility of obtaining lower income in the expansion is much less than that of higher income in the recession. To sum up, income of rich entrepreneurs indicates countercyclical while income of poor entrepreneurs shows procyclicality.

When I add up the income distribution of workers and entrepreneurs, income of poor agents must show procyclicality, yet income of rich agents becomes indeterminate. Recall that entrepreneurs account for a large number of top earners, so rich agents in my model see countercyclical income. I plot the histogram for entrepreneurs, workers and total agents when one standard deviation of negative/positive aggregate shock happens and the 60th period after the shock to give readers direct evidence.

### 5.7 Sensitivity of Worker/entrepreneur Ratio

I test the sensitivity of worker/entrepreneur ratio $\lambda$ by either lowering it to 1 or raising it to 5. Notice that any change in the ratio does not affect the total labor supply, but the relative number of workers.
Figure 1: Histogram of Different Groups: Recession

Note: The top panel shows the histogram for entrepreneurs; the middle panel, for workers; and the bottom for total agents.
Figure 2: Histogram of Different Groups: Expansion

Note: The top panel shows the histogram for entrepreneurs; the middle panel, for workers; and the bottom for total agents.
Thus if capital stock remains unchanged, a higher ratio requires more workers and they share the fixed total labor income so that an individual worker earns less.

A lower value particularly diminishes the countercyclicality of top 5% earners’ income since this income groups contains more workers. The procyclicality of labor income dilutes the entire countercyclical behavior of the pool. As a result, the correlation with output of 95/50 ratio turns to a small positive number. Yet the proportion of workers in the top income group even surpasses workers’ composition in the population. Obviously it contradicts the finding of other studies, such as Cagetti and De Nardi (2006).

In contrast, $\lambda = 5$ strengthens the countercyclical property of the top 5% earners while the income of the rest all shows procyclicality because this time the top income group only consists of entrepreneurs and other groups are mainly or all composed of workers. The above tests manifest that the parameter value of the worker/entrepreneur ratio is important to generate realistic dynamics in my model.

6 Fiscal Policy Experiments

This section explores some fiscal experiments to see if idiosyncratic production risks impact the choice of fiscal policies in a normative perspective. I put emphasis on the best time to tax capital more given long-run tax rate among all the experiments. I find that under some calibration consistent with empirical studies, the government should tax more during the expansion.

6.1 Time to Tax Assets

Chari, Christiano and Kehoe (1994) investigate optimal fiscal policies in the business cycle. They find that the correlation of an optimal capital income tax rate with technology shock is negative with uncontingent debt in their baseline model. In addition, when there is a negative innovation to the technology shock or a positive innovation to government consumption, there is a positive innovation in the tax on private assets. They argue that it is efficient for these shocks which affect the government budget constraint to be absorbed mainly by the tax on private assets. The logic results from the fact that income taxes are proportional to a certain type of income. Suppose that an economy contains two states: high productivity and low productivity. The government either taxes capital income with the rate of 5% or subsidizes it by 5%. Then the government decides when to tax capital, during the high state or the low state. A high productivity enlarges the economy and a subsidy makes a large economy even larger. The gain from subsidizing capital during the boom can compensate the loss from taxing capital in the recession. From another point of view, however, to smooth consumption becomes less possible by such a tax. So a tradeoff appears. Since they merely consider the aggregate shocks, then the mean effect dominates the fluctuation effect.

Yet considering idiosyncratic production risks may obtain a different result. A heavier tax on capital during the recession may even exacerbate the low expected capital return, which discourages entrepreneurs from investing in risky assets, worsening the aggregate economy. Besides, it may create a conflict between the welfare of entrepreneurs and workers since workers rely on the wage which increases with the capital
So this subsection answers a specific question of when it is advisable to tax capital more given the long-run level of tax rate.

I propose the welfare criterion as the ex-ante utilitarian social utility by which the social planner sums up the life-time individual utility across agents,

\[ U^0_{SP} = \int_i \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{C_{i,t}^{1-\gamma}}{1-\gamma} + \int_j \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{C_{j,t}^{1-\gamma}}{1-\gamma} = \int_i V(x^0_i) + \int_j V(x^0_j). \]

I denote the value function of entrepreneurs and workers as \( \int_i V(x^0_i) \) and \( \int_j V(x^0_j) \), respectively. Recall that the value function of entrepreneurs is showed as (18),

\[ V(x^0_i) = \frac{(1 - \nu_0)^{-\gamma} x^{1-\gamma}_{i,0}}{1 - \gamma}. \]

Besides, Appendix D derives the value function of workers and I replicate the result:

\[ V(x^0_j) = \lambda^\gamma \sum_{t=0}^{\infty} \beta^t \mathbb{E}_0 \left\{ \frac{[(1 - \tau^n_t) w_t]^{1-\gamma}}{1-\gamma} \right\} \exp \left( \frac{\sigma^2_w (1 - \gamma)^2}{2(1 - \rho^2_w)} \right). \]

Given that the initial conditions of capital and bond holdings are identical under different policy comparison, the value function of entrepreneurs maps one-to-one to the saving rate. Moreover, since the risk aversion degree \( \gamma \) is larger than one in my welfare analysis, the life-time utility for entrepreneurs decrease with the saving rate. In another word, more consumption conditional on a certain available wealth heightens the welfare of entrepreneurs.

I search for the best time to tax capital more by changing the correlation of tax rate to output and comparing the welfare measurement. I plot the welfare change in terms of consumption equivalent variation in percentage as the response of tax rate to output gap ranges from -2.2 to 2.2 with the same calibration previously used to produce the dynamics of income distribution.

I find that when \( m_{Y,\tau} = -2.07 \), the welfare measurement reaches the maximum if I assume idiosyncratic risks. It seems to confirm the conclusion of Chari, Christiano and Kehoe (1994). Yet when we turn to look at the aggregate consumption in the economy, the value \( m_{Y,\tau} = 0.09 \) maximizes the mean of aggregate consumption. It implies that the distribution of consumption goods among the groups of workers and entrepreneurs plays a crucial role for the social welfare. Thus, apart from the mean and fluctuation effect I mentioned above, the analysis of welfare in my model takes into account the distribution effect, or in a general picture, the equity motive.

I compute and exhibit the consumption equivalent variation of the aggregate life-time utility across entrepreneurs, across workers and across agents in the second and third columns of the following table. I report in the same table as well the variation of capital, wage, debt holdings, labor tax rate, wage and consumption for each type of agents and total agents to facilitate the analysis of mechanism. I set the base as the steady state level for each variable with idiosyncratic risks. All values are reported in percentage.

If capital tax rate is negatively correlated with output, the declined total productivity, together
Figure 3: Welfare Change As $m_{Yr}$ Changes (a)
with enlarged individual difference among entrepreneurs plus even higher tax burden, downgrades the capital accumulation in the recession. The expansion witnesses an increase in capital stock compared to the steady state level, yet in a smaller margin than in the recession due to asymmetric volatility of idiosyncratic production risks during the cycle. In total, the mean capital stock is less than the steady state level, leading to a slightly higher output and wage. The policy specifications imply a higher level of bond and a higher capital tax rate than the steady state. Although the government bonds crowd out more capital, more income from safe assets counterbalances the loss from less productive assets and more tax burden, resulting in more consumption for entrepreneurs. The government budget balance requires a higher labor tax rate, which causes a decrease in consumption for workers and for all agents.

On the contrary, if capital tax rate has positive correlation with output, the mean capital, output and wage increase. Following the above argument, the bond increases less and capital tax rate declines. Entrepreneurs increase their consumption compared with the steady state, but less than under procyclical tax policy. It confirms again that the welfare benefit from safe assets surpasses the adverse crowding-out effect. The labor tax rate, still higher than the steady state, induces a lower workers' consumption and total consumption even if the wage increases in a larger margin.

When comparing these two tax policies, procyclical tax policy entails higher entrepreneurs' consumption and lower workers' consumption so that entrepreneurs are better off, workers are worse off and the total welfare improves, though the countercyclical policy induces a maximal total consumption. The benchmark calibration guarantees that workers earn about two thirds of output and assumes the worker/entrepreneur ratio as 1.5, implying that an entrepreneur with average asset gains consumes less than an average worker. The concavity in the utility function produces a much lower summed utility for entrepreneurs than for workers. Thus an improvement in the entrepreneurs' consumption affects the total welfare more than the same or an even higher extent of the improvement in the workers' consumption.

Simply put, the total welfare sees a tradeoff in the respective welfare of two types of agents. It provides a platform for fiscal policies to maximize the social welfare via several channels. First, more government bonds promote the income of entrepreneurs. Meanwhile, however, safe assets crowd out
risky but productive assets, capital, which lowers the wage and workers’ welfare. The quantitative results partially count on tax policy specification. Second, different capital tax policies imply qualitatively different long-run means of tax rate compared to steady state, reducing or exacerbating the tax burden for entrepreneurs. Third, the government demands more tax revenue from labor income as it issues more debt. It worsens the workers’ welfare. All the channels intertwine with each other and the optimal capital tax policy conditional on the state state level is obtained by the balance of all forces.

Yet the benchmark calibration implies a higher composition of entrepreneurs relative to the findings of Cagetti and De Nardi (2006) and others. If I interpret the worker/entrepreneur ratio in my model as the number of business owners and self-employed dividing the rest. Then what Cagetti and De Nardi (2006) find indicates the ratio equal to 5. The modified ratio exerts two impacts. On one hand, more workers share the fixed total labor income so the change reduces the income of individual workers as argued in the last section. On the other hand, the government enhances the weight on workers to compute the total welfare according to the utilitarian social welfare criterion. Thus the effect of the worker/entrepreneur ratio on the welfare is reflected on the first term in the expression of value function of workers. A higher ratio increases the summed welfare of all workers and the social welfare. Hence, a different calibrated value of worker/entrepreneur ratio directly affects the distribution effect.

I plot another time the welfare change of different groups under this calibration. I also present in the rest two columns the result based on the new calibration of the worker/entrepreneur ratio. The fourth column shows the results using the realistic tax policy while the last records the results with the conditionally optimal policy when the response of capital tax rate to output \( m_{Yr} = 0.1 \). Apparently since the government emphasizes the welfare of workers, a countercyclical policy maximizes the welfare because it relatively cuts down the labor income tax rate and pushes up the capital stock.

The result manifests that with empirically confirmed idiosyncratic production risks and realistic composition of entrepreneurs, the classical result from Chari, Christiano and Kehoe (1994) is possible to be overturned: it is reasonable to state that to tax capital less or to even subsidize it during the recession is a preferred choice for the government. It implies that neglecting idiosyncratic production risks and the motive of equity may underestimate the total risks faced by firms and the entire economy, which further induces a suboptimal fiscal policy.

Following the current calibration of the worker/entrepreneur ratio, I study the impulse response of two tax policies showed above after one standard deviation of negative shock under idiosynratic risks: with \( m_{Yr} = -0.08 \) and with \( m_{Yr} = 0.1 \). Appendix E plots the result. The government should cut the tax during the recession taking into account idiosyncratic production risks, resulting in a faster recovery of output. To tax less during the recession prevents the decline of investment in the beginning. Meanwhile, it improves entrepreneurs’ consumption but reduces workers’ consumption because less taxation income from capital owners requires heavier tax burden on workers. Yet the summed consumption for the group of workers will be higher after Period 9 since more capital resulting from countercyclical taxes instead of procyclical taxes compensates the loss from taxation. The total consumption shows the same pattern as the consumption for workers since workers receive about two third of output.

In a word, given the debt policy and steady state level of capital tax rate, a procyclical tax benefits
Figure 4: Welfare Change As $m_{Y_t}$ Changes (b)
entrepreneurs while a countercyclical tax improves the welfare of workers. The time to tax more capital depends on the relative size of two groups. Qualitatively, it actually obeys a unique principle: the government should choose a policy which favours the poorer representative entrepreneur or worker with average asset income or labor income. In a broader horizon, the prescription demonstrates that equity, in addition to efficiency, should be considered in the policy assessment.

6.2 Time to Raise Debt

Next I conduct a similar exercise to study the best time to raise debt given the steady state level of debt by fixing the asset tax policy and modifying the response of debt to output gap $m_{YB}$ to maximize the welfare measurement.

Figure 5 and 6 indicate that a volatile debt policy improves the welfare of entrepreneurs while worsens that of workers. The following table furthermore shows the change of allocations, prices and welfare measurements when the government raises debt at the best time indicated by the response of debt to output under two different calibrations on the worker/entrepreneur ratio, respectively. Since higher volatility in debt raises the mean level of debt. The increased safe assets heighten the income of entrepreneurs though crowding out the capital and reduce the risk in consumption faced by entrepreneurs. Yet more government expenditure relies more on labor taxes; besides less capital means lower wage in the model, so workers are worse off. In addition, a countercyclical debt policy shows an advantage in improving the welfare of entrepreneurs compared with a procyclical one. Finally, the best time to raise debt depends on the comparison of population of different groups.

<table>
<thead>
<tr>
<th>Variables</th>
<th>$\lambda = 1.5$</th>
<th>$\lambda = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capital</td>
<td>-0.04</td>
<td>0.05</td>
</tr>
<tr>
<td>Bond</td>
<td>12.29</td>
<td>0</td>
</tr>
<tr>
<td>Cap. Tax</td>
<td>0.01</td>
<td>0</td>
</tr>
<tr>
<td>Wage</td>
<td>-0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>Lab. Tax</td>
<td>4.51</td>
<td>1.75</td>
</tr>
<tr>
<td>Entr. Con.</td>
<td>3.65</td>
<td>0.08</td>
</tr>
<tr>
<td>Wor. Con.</td>
<td>-1.16</td>
<td>-0.34</td>
</tr>
<tr>
<td>Ttl. Con.</td>
<td>-0.42</td>
<td>-0.28</td>
</tr>
</tbody>
</table>

Welf. Var.

Entr. 3.90 0.07
Wor. -2.88 -1.19
Total 3.50 -1.03

The base values for all variables are calculated in steady state with idiosyncratic production risks.

6.3 Low Capital Tax

This section explores the consequence of lowering capital tax rate. In particular, I conduct an experiment in which the government cuts the capital tax while keeping the same level of labor tax rate but reducing the debt holding to balance the government budget. I find that low capital tax and low debt leads to
Figure 5: Welfare Change As $m_{YB}$ Changes (a)
Figure 6: Welfare Change As $m_{Y,B}$ Changes (b)
lower social welfare, lower welfare for entrepreneurs but higher welfare for workers.

I reduce the capital tax rate from 0.0252 to 0.0135 so that the debt/output ratio declines to 1.89%, a sufficiently low level. The reason why I do not shut down the capital tax is that the government then owns assets instead of debt, a less possible scenario in reality. Other parameter values remain unchanged. I report the result in the following table. Specifically the welfare variation expressed by consumption equivalent variation is presented in the bottom panel in two cases: the worker/entrepreneur ratio equal to 1.5 or 5. All the values of welfare variation are expressed in percentage based on the benchmark with the same low capital tax.

Table 10: Comparison between Standard and Low Capital Tax

<table>
<thead>
<tr>
<th>Variables</th>
<th>Benchmark</th>
<th>Low Tax</th>
</tr>
</thead>
<tbody>
<tr>
<td>Saving Rate</td>
<td>0.971</td>
<td>0.974</td>
</tr>
<tr>
<td>Cap. Ratio</td>
<td>0.81</td>
<td>0.99</td>
</tr>
<tr>
<td>Capital</td>
<td>4.47</td>
<td>5.45</td>
</tr>
<tr>
<td>Debt</td>
<td>1.03</td>
<td>0.03</td>
</tr>
<tr>
<td>Output</td>
<td>1.71</td>
<td>1.80</td>
</tr>
<tr>
<td>Wealth</td>
<td>5.66</td>
<td>5.13</td>
</tr>
<tr>
<td>Wage</td>
<td>1.10</td>
<td>1.15</td>
</tr>
<tr>
<td>Entr. Cons.</td>
<td>0.16</td>
<td>0.14</td>
</tr>
<tr>
<td>Work. Cons</td>
<td>0.89</td>
<td>0.93</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Welf. Var.</th>
<th>( \lambda = 1.5 )</th>
<th>( \lambda = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entr.</td>
<td>-16.21%</td>
<td>-16.21%</td>
</tr>
<tr>
<td>Work.</td>
<td>5.07%</td>
<td>5.07%</td>
</tr>
<tr>
<td>Total</td>
<td>-15.48%</td>
<td>1.03%</td>
</tr>
</tbody>
</table>

The base values for welfare comparison are calculated in steady state with idiosyncratic production risks.

The most striking feature is that although the tax cut stimulates the capital accumulation, the simultaneous reduction in riskfree assets lowers the available wealth held by entrepreneurs. Then the consumption and consequent welfare of entrepreneurs decline with the available wealth. The increase in capital stock pushes up the wage so that workers are better off. At last, the total welfare depends on the value of the worker/entrepreneur ratio.

I conduct the exercise only based on the comparison of steady state. Ideally, I should also consider the transition dynamics from one steady state to the other. If so, then entrepreneurs have to sacrifice consumption to accumulate capital before reaching the steady state level. Thus the consideration of transition dynamics will strengthen my result because the welfare of entrepreneurs decreases even further.

6.4 Income Taxation

This section studies the effect of idiosyncratic production risks on the steady state and the welfare under the stationary distribution if the government levies income taxes, or equivalently to say, equalizes the labor income tax rate and the capital income tax rate.

I adjust the steady state income tax rate to maintain the debt/output ratio as 60% and balance the government budget. I recalculate the steady state values of allocations and prices with idiosyncratic risks.
under income taxation and show the results together with the steady state of the benchmark.

Table 11: Comparison between Standard and Income Taxation

<table>
<thead>
<tr>
<th>Variables</th>
<th>Benchmark</th>
<th>Income Tax</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cap. Tax</td>
<td>0.0252</td>
<td>0.0667</td>
</tr>
<tr>
<td>Lab. Tax</td>
<td>0.1933</td>
<td>0.0667</td>
</tr>
<tr>
<td>Saving Rate</td>
<td>0.971</td>
<td>0.976</td>
</tr>
<tr>
<td>Cap. Ratio</td>
<td>0.81</td>
<td>0.77</td>
</tr>
<tr>
<td>Capital</td>
<td>4.47</td>
<td>2.92</td>
</tr>
<tr>
<td>Output</td>
<td>1.71</td>
<td>1.47</td>
</tr>
<tr>
<td>Wealth</td>
<td>5.66</td>
<td>3.89</td>
</tr>
<tr>
<td>Wage</td>
<td>1.10</td>
<td>0.94</td>
</tr>
<tr>
<td>Entr. Cons</td>
<td>0.16</td>
<td>0.09</td>
</tr>
<tr>
<td>Work. Cons</td>
<td>0.89</td>
<td>0.88</td>
</tr>
<tr>
<td>Welf. Var.</td>
<td>$\lambda = 1.5$</td>
<td>$\lambda = 5$</td>
</tr>
<tr>
<td>Entr.</td>
<td>-42.40%</td>
<td>-42.40%</td>
</tr>
<tr>
<td>Work.</td>
<td>-0.80%</td>
<td>-0.80%</td>
</tr>
<tr>
<td>Total</td>
<td>-41.57%</td>
<td>-13.86%</td>
</tr>
</tbody>
</table>

The result shows that at the identical level of risks, the income taxation downgrades the welfare for both entrepreneurs and workers. The unique tax rate requires the rate, 0.07 between the capital income tax rate and the labor income tax rate under differentiated tax regime. It exerts heavy burden to entrepreneurs and distorts the investment to a large extent. Thus, the available wealth diminishes a lot and so does the wage. Then the utility for both groups of agents decreases.

7 Conclusion

This paper focuses on modelling idiosyncratic production risks in the business cycle, studying the cyclical behavior of income distribution and fiscal policy under heterogeneous agents and stochastic general equilibrium. All these topics are less developed in the literature. The framework of Angeletos (2007) allows the exact aggregation which renders the model highly tractable analytically and numerically. The dynamics of income distribution under this context turns out to be easy to obtain.

A higher risk causes more investment on risky assets with the risk aversion degree equal to 4, which is consistent with overaccumulation of capital in Covas (2006) and other works applying idiosyncratic labor income risks. Next I use my model to numerically explore the dynamics of income distribution. The model with baseline calibration can generate a reasonable income distribution featuring think tails in both ends and also fits the realistic cyclicality of different income groups to some extent.

At last I turn to carry on a series of experiments on fiscal policies. In particular I answer the question of what the best time is to tax capital income. The experiment I carry out illustrates that under realistic conditions the tax rate should be cut during the recession, a result qualitatively different from Chari, Christiano and Kehoe (1994).
References


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Appendix A

Proof of Lemma 1

I start from showing the characterization of allocations and choices in the individual level. The Euler equations derived from the entrepreneur optimization problem are

$$ (c_t^i)^{-\gamma} = \beta E_t^{i} [(c_{t+1}^{i})^{-\gamma}(1 - \tau_{t+1}^{a})(r(A_{t+1}^{i}, w_{t+1}) + 1 - \delta)] , $$ \hspace{1cm} (34)

$$ (c_t^i)^{-\gamma} = \beta E_t^{i} [(c_{t+1}^{i})^{-\gamma}(1 - \tau_{t+1}^{a}R_{t+1})] . $$ \hspace{1cm} (35)

I guess that the solution to the entrepreneur optimization problem is as follows:

$$ c_t^i = (1 - \nu_t)x_t^i, $$ and

$$ k_{t+1}^i = \nu_t \phi_t x_t^i, $$ so

$$ b_t^i = \nu_t(1 - \phi_t)x_t^i $$ from the budget constraint of the entrepreneur. To simplify the notation, I define the aggregate state at time $t$, $s_t = (K_t, B_t, z_t, g_t)$. Then I write $\nu_t = \nu_t(s_t)$ and $\phi_t = \phi_t(s_t)$. With some algebra,

$$ x_t^i = (1 - \tau_{t+1}^{a})[(r(A_{t+1}^{i}, w_{t+1}) + 1 - \delta)k_{t+1}^i + R_{t+1}b_{t+1}^i] = (1 - \tau_{t+1}^{a})[\phi_t(s_t)(r(A_{t+1}^{i}, w_{t+1}) + 1 - \delta) + (1 - \phi_t(s_t))R_{t+1}\nu_t(s_t)x_t^i]. $$ \hspace{1cm} (36)

Then the first Euler equation becomes

$$ (1 - \nu_t(s_t))^{-\gamma}x_t^{-\gamma} = \beta E_t^i([(1 - \nu_t(s_t+1))x_{t+1}^i]^{-\gamma}(1 - \tau_{t+1}^{a})(r(A_{t+1}^{i}, w_{t+1}) + 1 - \delta)) $$$$ = \beta \nu_t(s_t)^{-\gamma}x_t^{-\gamma}E_t^i((1 - \nu_t(s_t+1))^{-\gamma}(1 - \tau_{t+1}^{a})^{1-\gamma} $$$$ [\phi_t(s_t)(r(A_{t+1}^{i}, w_{t+1}) + 1 - \delta) + (1 - \phi_t(s_t))R_{t+1}]^{-\gamma}(r(A_{t+1}^{i}, w_{t+1}) + 1 - \delta)) . $$

Then we cross out the same factors from both handsides,

$$ (1 - \nu_t(s_t))^{-\gamma} = \beta \nu_t(s_t)^{-\gamma}E_t^i((1 - \nu_t(s_t+1))^{-\gamma}(1 - \tau_{t+1}^{a})^{1-\gamma} $$$$ [\phi_t(s_t)(r(A_{t+1}^{i}, w_{t+1}) + 1 - \delta) + (1 - \phi_t(s_t))R_{t+1}]^{-\gamma}(r(A_{t+1}^{i}, w_{t+1}) + 1 - \delta)) . $$ \hspace{1cm} (37)

Likewise, the second Euler equation can be transformed as

$$ (1 - \nu_t(s_t))^{-\gamma} = \beta \nu_t(s_t)^{-\gamma}E_t^i((1 - \nu_t(s_t+1))^{-\gamma}(1 - \tau_{t+1}^{a})^{1-\gamma} $$$$ [\phi_t(s_t)(r(A_{t+1}^{i}, w_{t+1}) + 1 + \delta) + (1 - \phi_t(s_t))R_{t+1}]^{-\gamma}R_{t+1}) . $$ \hspace{1cm} (38)

Combining these two equations, we obtain the equality between the gross returns of risky and risk-free assets:
0 = \mathbb{E}_t\{(1 - \nu_{t+1}(s_{t+1}))^{-\gamma}(1 - \tau_{t+1}^\nu)^{1-\gamma}\phi_t(s_t)(r(A_{t+1}^i, w_{t+1}) + 1 - \delta) + \\
(1 - \phi_t(s_t))R_{t+1}]^{1-\gamma}(r(A_{t+1}^i, w_{t+1}) + 1 - \delta - R_{t+1})\}

= \int_0^\infty \left( \int_0^\infty (1 - \nu_{t+1}(s_{t+1}))^{-\gamma}(1 - \tau_{t+1}^\nu)^{1-\gamma}\phi_t(s_t)(r(e_{t+1}^i, \epsilon_{t+1}^i, w(\epsilon_{t+1}^i); z_t) + 1 - \delta) + \\
+ (1 - \phi_t(s_t))R_{t+1}]^{1-\gamma}(r(e_{t+1}^i, \epsilon_{t+1}^i, w(\epsilon_{t+1}^i); z_t) + 1 - \delta - R_{t+1}) dF(e_{t+1}^i) \right) dF(\epsilon_{t+1}^i).

(39)

Multiply the above Equations 26 and 27 with \(\phi_t\) and 1 - \(\phi_t\), respectively, and sum up to get:

\[ (1 - \nu_t(s_t))^{-\gamma} = \beta \nu_t(s_t)^{-\gamma}\mathbb{E}_t\{(1 - \nu_{t+1}(s_{t+1}))^{-\gamma}(1 - \tau_{t+1}^\nu)^{1-\gamma} \phi_t(s_t)(r(A_{t+1}^i, w_{t+1}) + 1 - \delta) + (1 - \phi_t(s_t))R_{t+1}]^{1-\gamma}\}. \]

(40)

Define \(\phi_t = \arg\max_{\phi \in [0, 1]} \mathbb{C}_t\{(\phi_t(r(A_{t+1}^i, w_{t+1}) + 1 - \delta) + (1 - \phi_t)R_{t+1}\} \). In particular,

\[ \mathbb{C}_t\{(\phi_t(r(A_{t+1}^i, w_{t+1}) + 1 - \delta) + (1 - \phi_t)R_{t+1}\} \]

\[ = \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\phi_t(s_t)(r(e_{t+1}^i, \epsilon_{t+1}^i, w(\epsilon_{t+1}^i); z_t) + 1 - \delta) + (1 - \phi_t(s_t))R_{t+1}]^{1-\gamma} dF(e_{t+1}^i) \right] dF(\epsilon_{t+1}^i) \frac{1}{1 - \gamma}. \]

(41)

The process requires numerical solutions. The preceding integral demonstrates that the optimal saving and portfolio choices depend only on the aggregate state. I drop the notation of aggregate state since now.

Given \(\phi_t\) and \(\tau_{t+1}^\nu\), \(\nu_t\) can be computed by (38).

Next I derive the functional form of the value function for entrepreneurs. I first guess that the value function \(V(x_{t+1}^i) = \psi_t u(x_{t+1}^i)\). From the envelope theorem, \(V'(x_{t+1}^i) = u'(x_{t+1}^i)\), which is

\[ \psi_t x_{t+1}^{-\gamma} = c_{t+1}^{-\gamma} = [(1 - \nu_t)x_{t+1}^{-\gamma}]^{-\gamma} \]

I simplify the above equation by crossing out the common factor so that

\[ \psi_t = (1 - \nu_t)^{-\gamma}. \]

(42)

Hence, \(V(x_{t+1}^i) = \frac{(1 - \nu_t)^{-\gamma}}{1 - \gamma} x_{t+1}^{-\gamma}. \)

Now I verify whether the value function obtained fits the entrepreneur’s optimization problem. Notice that \(V(x_{t+1}^i) = \frac{c_{t+1}^{-\gamma}}{1 - \gamma} + \beta \mathbb{E}_t V(x_{t+1}^i)\). I insert the expressions of the value function and consumption, Eq. (13), into the equation.

\[ \frac{(1 - \nu_t)^{-\gamma}}{1 - \gamma} x_{t+1}^{-\gamma} = \frac{[(1 - \nu_t)x_{t+1}^{-\gamma}]^{1-\gamma}}{1 - \gamma} + \beta \mathbb{E}_t \left[ \frac{(1 - \nu_{t+1})^{-\gamma} x_{t+1}^{-\gamma}}{1 - \gamma} \right] \]

I multiply both handsides by \((1 - \gamma)\), remove the first term on the right handside to the left, and combine
the two terms on the left handside,

\[ \nu_t(1 - \nu_t)^{-\gamma} x_{i,t}^{1-\gamma} \]

\[ = \beta \mathbb{E}_t \left[ (1 - \nu_{t+1})^{-\gamma} x_{i,t+1}^{1-\gamma} \right] \]

\[ = \beta \mathbb{E}_t \left[ (1 - \nu_{t+1})^{-\gamma} \{ (1 - \tau_{t+1}^a) \phi_t (r(A_{i,t+1}, w_{t+1}) + 1 - \delta) + \right. \]

\[ \left. + (1 - \phi_t) R_{t+1} \nu_t x_{i,t} \}^{1-\gamma} \right]. \]

Then I eliminate the common factors \( \nu_t \) and \( x_{i,t}^{1-\gamma} \) from both handsides,

\[ (1 - \nu_t)^{-\gamma} = \beta \nu_t^{-\gamma} \mathbb{E}_t \left[ (1 - \nu_{t+1})^{-\gamma} \{ (1 - \tau_{t+1}^a) \phi_t (r(A_{i,t+1}, w_{t+1}) + 1 - \delta) + \right. \]

\[ \left. + (1 - \phi_t) R_{t+1} \nu_t x_{i,t} \}^{1-\gamma} \right], \]

which is validated by (38). \( \Box \)
Appendix B

Effect of Idiosyncratic Production Risks

This section shows how idiosyncratic production risks affect the portfolio choice at the steady state. For the expositional convenience, I simply assume the risk aversion degree \( \gamma = 1 \), that is, the logarithm case.

From the wage determination \( w_t = (1 - \alpha) (z_tK_t)^\alpha \) and the specification of risks, I rewrite the expression of individual capital investment return

\[
x_{t+1}^i = \alpha \left( \frac{1 - \alpha}{w_{t+1}} \right)^{-1} A_{t+1}^i
= \alpha z_{t+1}^\alpha K_{t+1}^{\alpha - 1} A_{t+1}^i
= \alpha z_{t+1}^\alpha K_{t+1}^{\alpha - 1} \exp \left[ -\frac{\sigma_e^2}{2} + \epsilon_{t+1}^i \right].
\]

I use the second order Taylor series with respect to \( \epsilon_{t+1}^i \) around 0 to approximate the certainty equivalent.

\[
\begin{align*}
&\text{CE}_t \{ \phi_t(x_{t+1}^i + 1 - \delta) + (1 - \phi_t)R_{t+1} \} \\
&\approx \left( \int_{-\infty}^\infty \left[ \phi_t(\alpha z_{t+1}^\alpha K_{t+1}^{\alpha - 1} \exp \left[ -\frac{\sigma_e^2}{2} \right] + 1 - \delta) + (1 - \phi_t)R_{t+1} \right]^{1-\gamma} + \frac{1 - \gamma}{2} \sigma_e^2 \\
&- \gamma \phi_t(\alpha z_{t+1}^\alpha K_{t+1}^{\alpha - 1} \exp \left[ -\frac{\sigma_e^2}{2} \right] + 1 - \delta) + (1 - \phi_t)R_{t+1} \right]^{-\gamma-1} \left( \phi_t \alpha z_{t+1}^\alpha K_{t+1}^{\alpha - 1} \exp \left[ -\frac{\sigma_e^2}{2} \right] \right)^2 \\
&+ \phi_t(\alpha z_{t+1}^\alpha K_{t+1}^{\alpha - 1} \exp \left[ -\frac{\sigma_e^2}{2} \right] + 1 - \delta) + (1 - \phi_t)R_{t+1} \right]^{-\gamma-1} \phi_t \alpha z_{t+1}^\alpha K_{t+1}^{\alpha - 1} \exp \left[ -\frac{\sigma_e^2}{2} \right] \right) \text{d}F(\epsilon_{t+1}^i) \right)^{\frac{1}{1-\gamma}} \\
&\int_{-\infty}^\infty \left\{ \left( \alpha z_{t+1}^\alpha K_{t+1}^{\alpha - 1} \exp \left[ -\frac{\sigma_e^2}{2} \right] + 1 - \delta - R_{t+1} \right)^3 + \frac{1 - \gamma}{2} \sigma_e^2 \left( \alpha z_{t+1}^\alpha K_{t+1}^{\alpha - 1} \exp \left[ -\frac{\sigma_e^2}{2} \right] \right)^2 \\
&+ (1 - \gamma) + (1 - \delta - R_{t+1}) \alpha z_{t+1}^\alpha K_{t+1}^{\alpha - 1} \exp \left[ -\frac{\sigma_e^2}{2} \right] \left( \alpha z_{t+1}^\alpha K_{t+1}^{\alpha - 1} \exp \left[ -\frac{\sigma_e^2}{2} \right] \right) \right\} \text{d}F(\epsilon_{t+1}^i) = 0
\end{align*}
\]

I take the first order condition with respect to \( \phi_t \) to pin down the optimal portfolio choice.

\[
\begin{align*}
&\int_{-\infty}^\infty \left\{ (1 - \gamma) + (1 - \delta - R_{t+1}) \alpha z_{t+1}^\alpha K_{t+1}^{\alpha - 1} \exp \left[ -\frac{\sigma_e^2}{2} \right] \right\} \text{d}F(\epsilon_{t+1}^i) = 0
\end{align*}
\]

The deterministic steady state rules out the aggregate shocks. I list the certainty equivalent and its derivative with respect to \( \phi_t \) at the steady state in the following.
\[ \mathcal{C} \varepsilon \approx \left( \gamma(\alpha K^{-1} \exp \left( -\frac{\sigma^2}{2} \right) + 1 - \delta) + (1 - \gamma)\bar{R} \right)^{1 - \gamma} + \frac{1 - \gamma}{2} \sigma^2 \\
\{ -\gamma(\alpha K^{-1} \exp \left( -\frac{\sigma^2}{2} \right) + 1 - \delta) + (1 - \gamma)\bar{R} \}^{\gamma - 1} \left( \phi \alpha K^{-1} \exp \left( -\frac{\sigma^2}{2} \right) \right)^2 + \\
\{ \phi \alpha K^{-1} \exp \left( -\frac{\sigma^2}{2} \right) + 1 - \delta \} \left( 1 - \gamma \right) \left( \bar{R} \right)^{1 - \gamma} \left( \phi \alpha K^{-1} \exp \left( -\frac{\sigma^2}{2} \right) \right)^{2} \]

\[ (\alpha K^{-1} \exp \left( -\frac{\sigma^2}{2} \right) + 1 - \delta - \bar{R})^{3} + \frac{1 - \gamma}{2} \sigma^2 (1 - \gamma)(\alpha K^{-1} \exp \left( -\frac{\sigma^2}{2} \right) )^2 + \\
(1 - \delta - \bar{R})\alpha K^{-1} \exp \left( -\frac{\sigma^2}{2} \right)(\alpha K^{-1} \exp \left( -\frac{\sigma^2}{2} \right) + 1 - \delta - \bar{R}) \gamma^2 + \\
\{ 2(\alpha K^{-1} \exp \left( -\frac{\sigma^2}{2} \right) + 1 - \delta - \bar{R})^{2} + \sigma^2 (1 - \gamma)(\alpha K^{-1} \exp \left( -\frac{\sigma^2}{2} \right) )^2 \} + \\
\alpha K^{-1} \exp \left( -\frac{\sigma^2}{2} \right)(1 - \delta - \bar{R}) \gamma^2 (1 - \gamma)(\alpha K^{-1} \exp \left( -\frac{\sigma^2}{2} \right) ) - \\
\exp \left[ -\frac{\sigma^2}{2} \right] + 1 - \delta - \bar{R}) \bar{R} \phi + \bar{R}^2 (1 + \frac{\sigma^2}{2}) \alpha K^{-1} \exp \left[ -\frac{\sigma^2}{2} \right] + 1 - \delta - \bar{R} = 0 \]
Appendix C

Coefficient of Variation of Income

I first report the coefficient of variation of entrepreneurs’ income at \( t+1 \) conditional on \( t \)'s information \( \text{Coef}. \text{Var}_t(I_{i+1}) \).

Conditional expectation of \( t+1 \)'s income of entrepreneurs at \( t \) is

\[
E_t(I_{i+1}) = \mathbb{E}_t \left[ r_{i+1} k_{i+1} + (R_{i+1} - 1)b_{i+1}^t \right] = \mathbb{E}_t \left\{ \nu_t x_i^t \left[ \phi_t r_{i+1}^t + (1 - \phi_t)(R_{i+1} - 1) \right] \right\} = \nu_t x_i^t \left[ \phi_t \alpha z^{\alpha \rho z} \exp \left( \frac{1}{2} \alpha^2 \sigma_z^2 \right) K_{i+1}^{\alpha - 1} + (1 - \phi_t)(R_{i+1} - 1) \right];
\]

Conditional variance of \( t+1 \)'s income of entrepreneurs at \( t \) is

\[
\text{Var}_t(I_{i+1}) = \mathbb{V}_t \left[ r_{i+1} k_{i+1} + (R_{i+1} - 1)b_{i+1}^t \right] = \mathbb{V}_t \left[ \nu_t x_i^t \phi_t r_{i+1}^t + (1 - \phi_t)(R_{i+1} - 1) \right] = \nu_t^2 x_i^t \phi_t^2 \alpha^2 z^{2\alpha \rho z} K_{i+1}^{2(\alpha - 1)} \left\{ \exp \left[ \sigma_z^2 (1 - \eta \rho z \log z_t) + \frac{1}{2} (2 \alpha - \eta \sigma_z^2) \sigma_z^2 \right] - \exp (\alpha^2 \sigma_z^2) \right\}.
\]

Thus, the coefficient of variation is computed as the dividend of the standard deviation and the expectation.

\[
\text{Coef}. \text{Var}_t(I_{i+1}) = \frac{\phi_t \alpha z^{\alpha \rho z} K_{i+1}^{\alpha - 1} \sqrt{\exp \left[ \sigma_z^2 (1 - \eta \rho z \log z_t) + \frac{1}{2} (2 \alpha - \eta \sigma_z^2) \sigma_z^2 \right] - \exp (\alpha^2 \sigma_z^2)}}{\phi_t \alpha z^{\alpha \rho z} \exp \left( \frac{1}{2} \alpha^2 \sigma_z^2 \right) K_{i+1}^{\alpha - 1} + (1 - \phi_t)(R_{i+1} - 1)}.
\]

Next I derive the coefficient of variation of workers’ income at \( t+1 \) conditional on \( t \)'s information \( \text{Coef}. \text{Var}_t(I_{j+1}) \).

Conditional expectation of \( t+1 \)'s income of workers at \( t \) is

\[
E_t(I_{j+1}) = \mathbb{E}_t \left[ w_{i+1} e_{i+1}^j \right] = \mathbb{E}_t \left[ (1 - \alpha) z_{i+1}^{\alpha} K_{i+1}^{\alpha} e_{j+1}^j \right] = (1 - \alpha) z_{i+1}^{\alpha} K_{i+1}^{\alpha} e_{j+1}^j \exp \left[ \frac{1}{2} (\alpha^2 \sigma_z^2 + \sigma_{\omega}^2) \right];
\]
Conditional variance of $t + 1$'s income of workers at $t$ is

$$\text{Var}_t(I_{t+1}^j) = \text{Var}_t\left[w_{t+1}e_{t+1}^j\right] = \text{Var}_t\left[(1 - \alpha)Z_{t+1}^{\alpha}K_{t+1}^{\alpha}e_{t+1}^j\right] = (1 - \alpha)^2 z_t^{2\gamma_\alpha} K_t^{2\gamma_\alpha} e_{j,t}^{2\rho_\alpha} \left[\exp\left(2\alpha^2 \sigma_z^2 + 2\sigma_w^2\right) - \exp\left(\alpha^2 \sigma_z^2 + \sigma_w^2\right)\right].$$

Then the coefficient of variation is

$$\text{Coef. Var}_t(I_{t+1}^j) = \sqrt{\exp\left(\alpha^2 \sigma_z^2 + \sigma_w^2\right) - 1}.$$
Appendix D

Welfare for Hand-to-mouth Workers

From (4)

$$\log e_{t+1}^j = \rho_w \log e_t^j + \epsilon_{t+1}^j,$$

we have that

$$E_0 (\log e_{j,t}) = E_0 (\rho_w \log e_{j,t-1}) + E_0 (\epsilon_{j,t}) = 0;$$

$$\text{Var}_0 (\log e_{j,t}) = \frac{\sigma_w^2}{1 - \rho_w^2};$$

$$E_0 (e_{j,t}) = \exp \left( \frac{\sigma_w^2 (1 - \gamma)^2}{2(1 - \rho_w^2)} \right), \forall t;$$

and furthermore,

$$E_0 (e_{j,t}^{1 - \gamma}) = \exp \left( \frac{\sigma_w^2 (1 - \gamma)^2}{2(1 - \rho_w^2)} \right), \forall t.$$

Then the summed unconditional mean of utility at $t$ across workers is

$$\int_j E_0 \left\{ \frac{c_{j,t}^{1 - \gamma}}{1 - \gamma} \right\} = \int_j E_0 \left\{ \frac{[(1 - \tau^n_t)^{\frac{1}{1 - \gamma}} \lambda w_t e_{j,t}]^{1 - \gamma}}{1 - \gamma} \right\}$$

$$= \lambda^\gamma \int_j E_0 \left\{ \frac{[(1 - \tau^n_t)w_t]^{1 - \gamma}}{1 - \gamma} \right\} \exp \left( \frac{\sigma_w^2 (1 - \gamma)^2}{2(1 - \rho_w^2)} \right). \tag{48}$$

At last, the value function of workers is expressed as

$$\int_j V(x_0^j) = \int_j \sum_{t=0}^{\infty} \beta^t \frac{c_{j,t}^{1 - \gamma}}{1 - \gamma}$$

$$= \sum_{t=0}^{\infty} \beta^t \int_j E_0 \left\{ \frac{c_{j,t}^{1 - \gamma}}{1 - \gamma} \right\}$$

$$= \lambda^\gamma \sum_{t=0}^{\infty} \beta^t E_0 \left\{ \frac{[(1 - \tau^n_t)w_t]^{1 - \gamma}}{1 - \gamma} \right\} \exp \left( \frac{\sigma_w^2 (1 - \gamma)^2}{2(1 - \rho_w^2)} \right). \tag{49}$$
Appendix E

Impulse Responses to Negative Shocks under Different Tax Policies