Quantitative Easing and The Portfolio Balance Channel of Monetary Policy: A New Approach

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Abstract

Fed officials argue that quantitative easing works through the portfolio balance channel, and the preferred habitat theory has been widely applied to justify the significance of the portfolio balance channel at the zero lower bound. In this paper, I propose a new model with financial market limited participation and heterogeneous households to analyze the portfolio balance channel of monetary policy. Similar to the preferred habitat theory, different households trade different assets in equilibrium, which makes the purchases of long-term bonds be effective on decreasing long-term interest rates when short-term interest rate is zero. Different from the preferred habitat theory, the markets for assets of different maturity are not segmented, and a household’s portfolio is endogenously determined. Thus, the substitutability between long- and short-term bonds is preserved, and the zero short-term interest rate has impacts on the effectiveness of long-term security purchases. Specifically, when the short-term interest rate is at the zero lower bound, long-term security purchases become less effective than the case of above the zero lower bound.

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1 Introduction

Prior to the financial crisis of 2008, the U.S. Federal Reserve primarily influence the econ-
omy through conventional monetary policy: it adjusted the target short-term nominal
interest rate and achieved this target rate through open market operation in short-term
Tresury securities and overnight repos. Since 2008, owing to a series of interest rate cuts
implemented by the Fed, the short-term nominal interest rate has been close to its effec-
tive zero lower bound. At the zero lower bound, one encounters the liquidity trap prob-
lem, and thus, there was little scope left for conventional monetary policy. However,
the nominal interest rates of long-term Treasury securities remained significantly higher
than zero. Therefore, central bankers turned their attention to long-term yields and con-
ducted a series of quantitative easing policies in order to "ease" by putting downward
pressure on the long-term interest rates.¹

The transmission mechanism for quantitative easing, asserted by the policy mak-
ners, has been the portfolio balance channel. The portfolio balance channel relies on the
presumption that different financial assets are not perfect substitutes in investors’ port-
folios, so that changes in the supply of an asset available affect its yield and those of
similar assets (see Bernanke, 2010).² The preferred habitat theory has been widely ap-
plied to justify the significance of the portfolio balance channel at the zero lower bound,
which posits that markets for assets of different maturities are segmented, or different
financial investors have different preferences for assets of particular maturities.³ The
arbitrage limited under those assumptions, and the interest rate for a given maturity is
determined by the demand and supply specific to that maturity; thus, a zero short-term
interest rate has little influence on the effectiveness of a change in long-term security
supply.⁴

However, the preferred habitat theory lacks a rationale for why the markets for long-
and short-maturities security are segmented, or why an investor prefers assets of some
particular maturities. Moreover, the preferred habitat theory disengages the connection
between long-term yields and short-term yields, so the shape of yield curves should

¹For example, under LSAPs, the Federal Reserve purchased longer-term securities from the market.
Under the MEP, the Federal Reserve sold shorter-term treasury securities and used the proceeds to buy
longer-term Treasury securities.
²The portfolio balance channel was initially described by Friedman and Schwartz (1963). A related
idea is the imperfect asset substitutability of Tobin (1969)
³The preferred habitat theory is introduced by Modigliani and Sutch (1996). Recent development
includes Andres et al. (2004), and Vayanos and Vila (2009).
⁴By employing various methods and data, most empirical studies agree that quantitative easing has
helped to reduce long-term interest rates. Among others, D’Amico and King (2011), Gagnon et al. (2011)
be arbitrary. However, in Figure 1, one can observe a rigid pattern of yield curves and a comovement of long- and short-term yields, which demonstrates a strong connection between short-term and long-term securities, and the strong connection must change the effectiveness of long-term asset purchases when the short-term yield hits zero.

I fill this gap by proposing a new model of the portfolio balance channel in this paper. Unlike the preferred habitat theory, I do not assume markets segmentation for different maturities or impose ad hoc preferences over maturities on households. Instead, an investor faces limited participation in the financial market as a whole. The limited participation obstructs arbitrage trade, so long- and short-term yields can be different. The literature has widely considered the potential for limited participation in financial markets. For example, Alvarez et al. (2002) analyze the effects of money injections when the goods market and asset market are segmented. An agent who wants to transfer assets must pay a Baumol-Tobin-style fixed cost (Baumol, 1952; Tobin, 1956). In Aiyagari and Williamson (2000), limited participation in financial sector occurs at random for individuals, and money is held to insure against random circumstances.\(^5\)

In this paper, the financial friction takes the form of a random participation limitation. I assume that a household can enter the financial market only with a certain probability, and once a household enters the financial market, it is allowed to trade securities at all maturities. The financial friction along with the mismatch between the arrival of liquidity shocks and the maturity of assets generates two risks regarding the holding of assets. The first is liquidation risk: households may not be able to sell their assets when they need cash. The second is reinvestment risk: bonds become cash when they mature.

\(^5\)Among others, the limited participation in financial market is also applied in Alvarez et al. (2009), Duffie and Sun (1990), Gabaix and Laibson (2001), and Alvarez et al. (2004).
and the asset holders may fail to repurchase assets when they do not need cash, and therefore lose the yields. In principle, long-term bonds bear more liquidation risk, while short-term bonds bear more reinvestment risk. As a result, a yield curve can be upward or downward sloping depending on the relative strength of these two risks; notably, this feature results from the natural differences between long- and short-term bonds, and no maturity-specific friction or cost is imposed.

Similar to the preferred habitat theory, my model features heterogeneous households. Different households value asset differently depending on their individual characteristic such as the time discount rates, the arrival rate of liquidity demand, and the limited participation they face. In equilibrium, households with different characteristics trade different securities. Unlike the preferred habitat theory, the diversity of households portfolio is an endogenous outcome, and a households’ portfolio may change depending on the market interest rates. The heterogeneity of households generates flexibility between equilibrium long- and short-term yields. A zero short-term interest rate means that the short-term yield reaches the short-term security traders’ reservation yield. However, because the traders of short-term and long-term securities can be different, long-term yield may still higher than long-term security traders’ reservation long-term yields. As a result, the purchases of long-term securities can effectively decrease the long-term interest rate despite the short-term interest rate being zero.

Moreover, because there is no segmentation between the markets of long- and short-term securities in my model, a stronger substitution effect between long- and short-term securities are generated. Thus, the shapes of yield curves are not arbitrary. More important, the zero lower bound problem for short-term yields has influence on the effectiveness of long-term security purchases. Although the purchases of long-term securities can decrease the long-term interest rate, such purchases are less effective in doing so when the short-term interest rate reaches zero.

This paper is closely related to the literature which analyzes quantitative easing in monetary models. In Williamson (2016), quantitative easing increases the quantity of outstanding money, and thus, decreases the inflation rate, which in turn increases the real interest rates at the zero lower bound. Similar results are obtained in Braun and Oda (2010), and Herrenbrueck (2014). In my model, when short-term interest rate is zero, the reduction in the long-term bond supply also results in an increase in the short-term real interest rate because of a decrease in the inflation rate. However, if households are sufficiently diverse, the decrease in the long-term nominal interest rates dominates the decrease in inflation following a reduction in long-term bond supply, and thus, real long-term interest rates may decrease even if short rate reaches the zero lower bound.
The remainder of the paper is organized as follows. In Section 2, I describe a baseline model with representative households that face random limited participating. In Section 3, I incorporate monetary policies into the model. In Section 4, I introduce a complete model with households that face heterogeneous asset market frictions, time discount rates and liquidity shocks. Section 5 concludes.

2 The Representative Household Model

2.1 Environment

Time \( t = 1, 2, \ldots \) is discrete and infinite. There is a continuum of infinitely lived households indexed by \( \omega \in [0, 1] \). There exists a consumption good, which generates utility and is perfectly divisible and perishable. There are two types of assets in the economy: cash, \( M \), and nominal bonds with various maturities, \( B = (B_1, B_2, \ldots, B_k) \). Both cash and nominal bonds are issued by the government, and they are perfectly divisible and non-perishable. Asset quantities are defined as of the end of each period. For tractability, I use the "random maturity" structure of Leland (1994). Let \( \lambda_i > 0 \) denote the maturity rate of bond \( B_i \); then, in the beginning of each period, with probability \( \lambda_i \), one unit of nominal bonds matures, and the household receive one unit of cash from the government. With probability \( 1 - \lambda_i \), one unit of nominal bonds remains in the bond holder’s account. I denote by \( \theta_i = \frac{1}{\lambda_i} \) the average maturity of bonds, and hence, longer-term bonds have larger \( \theta_i \) (smaller \( \lambda_i \)), and shorter-term bonds have smaller \( \theta_i \) (larger \( \lambda_i \)).

Each period is divided into two subperiods, and the subperiods are identified by their markets. In the first subperiod, the asset market opens after bonds mature. Households exchange cash and bonds in the asset market, and the government sells new bonds in the asset market. In the second subperiod, the goods market opens, and the trade of goods takes place. Figure 2 depicts the timing of the markets. Further details are described below.

A household can be in one of two states \( J = \{c, n\} \): a consumer \( (j = c) \) or a non-consumer \( (j = n) \). A consumer can produce and consume in the goods market; a non-consumer can only produce but cannot consume. A household does not consume its own output, meaning that it must exchange goods in the goods market. At the beginning of each period, before the asset market opens, the states of households are realized. A household is in the consumer state with probability \( \alpha \) and in a non-consumer state with probability \( 1 - \alpha \), and the state is iid. Let \( C_t \subset [0, 1] \) denote the set of consumers and \( N_t \subset [0, 1] \) denote the set of non-consumers at time \( t \). Household \( \omega \) is a consumer at
time $t$ if $\omega \in C_t$ and is a non-consumer if $\omega \in N_t$. I assume that utility is linear in consumption: consuming $c$ units of goods generates $c$ units of utility. Producing $h$ units of consumption goods generates convex disutility $l(h)$ with $l(0) = 0, l'(0) = 0$. Households discount future at the rate $\frac{1}{1+\rho}, \rho > 0$. A Household’ lifetime utility can be expressed as follows:

$$E_0 \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} [x_t c_t - l(h_t)].$$

Where $x_t$ denotes the realization of a preference shock. $x_t = 1$ represents the consumer state, and $x_t = 0$ represents the non-consumer state.

To focus on the role of cash as a medium of exchange, I assume that households are anonymous, meaning that credit arrangements are not feasible in the goods market and the asset market. Added to the assumption of perishable consumption goods, households that wish to buy consumption goods must pay with media of exchange. I assume that cash is the only asset that can be recognized as a medium of exchange in the goods market, meaning that bonds cannot be so used.

The asset market is subject to random limited participation. With probability $\mu$, a household is able to trade in the asset market; with probability $1 - \mu$, the household fails to trade in the asset market. A household realizes whether it can trade in the asset market at the beginning of each period. In the electronic world, bonds are automatically redeemed at maturity: when one unit of bonds matures, one unit of cash is directly transferred to the bond holder’s account, and the unit of bonds simultaneously disappears from its account. Note that a household does not have to participate the financial market to find the government or to contact the dealer to redeem their bonds, meaning that the maturity of bonds is not subject to limited participation.

In the goods market, the price of cash in terms of goods is denoted by $\phi_t$. The inflation rate is denoted by

$$1 + \pi_t = \frac{\phi_t - 1}{\phi_t}.$$

I denote cash in real terms by $m_t = \phi_t M_t$, and nominal bonds in real terms by $b_{i,t} = \phi_t B_{i,t}$. For the sake of conciseness, $m_t$ and $b_t = (b_{1,t}, b_{2,t}, \ldots, b_{k,t})$ will be called cash and bonds hereafter.

It is worthwhile to compare this model with models which embed the OTC market into monetary models. For example, in Lagos and Zhang (2013), Geromichalos and Herrenbrueck (2014), and Geromichalos et al. (2013), households face idiosyncratic demand shocks, and the shocks are the key factors that generate demand and supply in the OTC market. In their models, there is a periodic walrasian market where households can
exchange goods and all assets; the walrasian market facilitates portfolio rebalance and helps degenerate the distribution of individual’s asset holdings. However, the matching frictions in the over-the-counter market also become unsustainable because households can always engage in arbitrage trading in the walrasian market.

In this paper, I do not assume a periodic walrasian market for goods and assets, and hence, the only market in which households can exchange cash and bonds is the frictional asset market. Therefore, the matching friction in the asset market will be sustained into the future, and arbitrage trading is obstructed. The distribution of asset holdings is not degenerate in this model; thus, households may hold different portfolios depending on the history of their states. However, the linearity of the consumption utility results in constant marginal values of assets, meaning that asset holdings have no effect on households’ decisions. This property helps analyze households’ behavior under a non-degenerate asset distribution.

2.2 Households’ Problems

Households’ problems will be discussed in reverse order of time. I first analyze the goods market, and then consider the asset market.
2.2.1 The Goods Market

A household’s state variables are cash, $m_t$, and bonds, $b_t$, that the household brings into each market. The government imposes a lump-sum tax, $\tau_t$, in the form of cash once the goods market closes. The tax is independent of the household’s state and asset holdings. I assume that \( \frac{1}{(1+\rho)(1+\pi_t)} < 1 \), or a household will defer its consumption. Let $\Phi_t^j(m_t, b_t)$ and $U_t^j(m_t, b_t)$ denote the value of a state $j$ household in the beginning of the asset market and the goods market, respectively. Since a household is a consumer with probability $\alpha$ and a non-consumer with probability $(1-\alpha)$, I denote $\Phi_{t+1}(m_{t+1}, b_{t+1}) \equiv a\Phi_{t+1}^c(m_{t+1}, b_{t+1}) + (1-a)\Phi_{t+1}^n(m_{t+1}, b_{t+1})$ to be a household’s expected value at the beginning of the next period. $U_t^j(m_t, b_t)$ satisfies the following Bellman equation:

\[
U_t^j(m_t, b_t) = \max_{c_t, h_t, m_t, m_{t+1}, \tilde{b}_{i,t}} \left\{ x_t c_t - l(h_t) + \frac{1}{1+\rho} \Phi_{t+1}(m_{t+1}, b_{t+1}) \right\}
\]

subject to

\[
\begin{align*}
\tilde{m}_t &= h_t + m_t - \tau_t - c_t \\
c_t &\leq m_t \\
\tilde{b}_{i,t} &= b_{i,t} \\
m_{t+1} &= \frac{1}{1+\pi_t} (\tilde{m}_t + \sum_i \lambda_i \tilde{b}_{i,t}) \\
b_{i,t+1} &= \frac{1}{1+\pi_t} (1-\lambda_i) \tilde{b}_{i,t} \\
c_t &\geq 0
\end{align*}
\]

I first analyze the problem for consumers. Consumers can produce and consume in the goods market ($x_t = 1$). The first constraint is the budget constraint: consumers’ end-of-period cash holdings, $\tilde{m}_t$, must be equal to the initial cash holdings, $m_t$, plus labor income, $h_t$, minus the tax charged by the government, $\tau_t$, and minus consumption, $c_t$. The second constraint is the cash-in-advance constraint: as there is no credit in the goods market, consumption is subject to the cash holdings at the beginning of goods market trading, $c_t \leq m_t$. The third constraint states that consumers simply retain bonds until the goods market closes, because bonds cannot be employed as a medium of exchange in the goods market; thus, the end-of-period bond holdings, $\tilde{b}_{i,t}$, are equal to the initial bond holdings. Finally, a $\lambda_i$ proportion of bond $i$ matures at the beginning of the next period, and thus, after being discounted by the inflation rate $(1 + \pi_t)$, the cash holdings at the beginning of the subsequent asset market phase is $\frac{1}{1+\pi_t} (\tilde{m}_t + \sum_i \lambda_i \tilde{b}_{i,t})$, and the holding of bond $i$ is $\frac{1}{1+\pi_t} (1-\lambda_i) \tilde{b}_{i,t}$.

I denote by $v_t^j$ and $w_t^j$ the marginal value of cash and bonds when the goods market phase begins. That is,
\[ v^j_i = \frac{\partial U^j_i}{\partial m}(m_t, b_t), \]
\[ w^j_{i,t} = \frac{\partial U^j_i}{\partial b_i}(m_t, b_t). \]

I first analyze the goods market problem for households in the consumers state. Let \( \frac{1}{1 + d_i} = \frac{1}{(1 + \rho)(1 + \pi_i)} < 1 \) denote the discount factor. The necessary conditions for consumers’ optimal decision are as follows:\(^6\)

\[ \frac{dl(h)}{dh} = \frac{1}{1 + d_i} \frac{\partial \Phi_{t+1}}{\partial m}(m_{t+1}, b_{t+1}), \] \[ v^c_i = \max \left\{ 1, \frac{1}{1 + d_i} \frac{\partial \Phi_{t+1}}{\partial m}(m_{t+1}, b_{t+1}) \right\}, \] \[ w^c_{i,t} = \frac{1}{1 + d_i} \left[ \lambda_i \frac{\partial \Phi_{t+1}}{\partial m}(m_{t+1}, b_{t+1}) + (1 - \lambda_i) \frac{\partial \Phi_{t+1}}{\partial b_i}(m_{t+1}, b_{t+1}) \right]. \]

These conditions have straightforward interpretations. Equation (1) states that the marginal disutility of working must be equal to its marginal benefit, which is the discounted expected marginal value of cash in the next period, \( \frac{1}{1 + d_i} \frac{\partial \Phi_{t+1}}{\partial m} \). Equation (2) states that consumers make decisions concerning spending cash by comparing the marginal utility of consumption, 1, with the discounted expected marginal value of cash, \( \frac{1}{1 + d_i} \frac{\partial \Phi_{t+1}}{\partial m} \). If \( \frac{1}{1 + d_i} \frac{\partial \Phi_{t+1}}{\partial m} < 1 \), consumers spend all of their cash on consumption \( (c_t = m_t) \); if \( \frac{1}{1 + d_i} \frac{\partial \Phi_{t+1}}{\partial m} > 1 \), consumers do not consume and instead save their cash for the next period \( (c_t = 0) \). Moreover, because households cannot use bonds as a medium of exchange in the goods market, the marginal value of bonds in the goods market \( w^c_{i,t} \) is equal to their discounted expected marginal value of bonds.

Non-consumers’ problem is similar to the consumers’ problem, but non-consumers cannot consume \( (x_t = 0) \), so the values of assets for non-consumers are as follows:

\[ v^n_i = \frac{1}{1 + d_i} \frac{\partial \Phi_{t+1}}{\partial m}(m_{t+1}, b_{t+1}), \] \[ w^n_{i,t} = \frac{1}{1 + d_i} \left[ \lambda_i \frac{\partial \Phi_{t+1}}{\partial m}(m_{t+1}, b_{t+1}) + (1 - \lambda_i) \frac{\partial \Phi_{t+1}}{\partial b_i}(m_{t+1}, b_{t+1}) \right]. \]

Because the utility function is linear in consumption, households may play Ponzi games when the return on saving is too high: they may not spend cash on goods and

\(^6\)See the Appendix for the proof of the necessary conditions.
instead hold all of their cash into the next period. These Ponzi game paths are excluded by the transversality conditions.

\[
\lim_{t \to \infty} \left( \frac{1}{1+r} \right)^t v^j m_{t+1} = 0, \\
\lim_{t \to \infty} \left( \frac{1}{1+r} \right)^t w^j_i b_{i,t+1} = 0.
\]

2.2.2 The Asset Market

In the first subperiod, the asset market opens, which allows a segment of the households to exchange bonds and cash. Limited participation occurs at random: a household can only participate in the asset market with probability \( \mu \), which is independent of the history. Households can exchange different bonds with cash in different submarkets, but there is no market for households to exchange different maturity bonds with each other directly; thus, there exist \( k \) submarkets for bonds \( i = 1, 2, \ldots, k \). Consumers and non-consumers face the same problem in the asset market. For a household that cannot trade in the asset market, its value in the first subperiod is equal to its value in the goods market, \( U^j_t(m_t, b_t) \). Let \( \Omega^j_t(m_t, b_t) \) denote the value of a household that can trade in the asset market. Then

\[
\Phi^j_t(m_t, b_t) = \mu \Omega^j_t(m_t, b_t) + (1 - \mu) U^j_t(m_t, b_t). 
\]

\( \Omega^j_t(m_t, b_t) \) can be written as the following optimization problem:

\[
\Omega^j_t(m_t, b_t) = \max_{m^\dagger_t, b^\dagger_t, \tilde{m}_t, \tilde{b}_t} U^j_t(\tilde{m}_t, \tilde{b}_t) \\
\text{Subject to} \\
\begin{align*}
\tilde{m}_t &= m_t - \sum_i m^\dagger_{i,t} + \sum_i q_{i,t} b^\dagger_{i,t} \\
\tilde{b}_{i,t} &= b_{i,t} + \frac{m^\dagger_{i,t}}{q_{i,t}} - b^\dagger_{i,t} \\
\sum_i m^\dagger_{i,t} &\leq m_t \\
b^\dagger_{i,t} &\leq b_{i,t} \\
m^\dagger_{i,t} &\geq 0 \\
b^\dagger_{i,t} &\geq 0
\end{align*}
\]

In the asset market, households can sell cash for bond \( i \) in submarket \( i \), denoted by \( m^\dagger_{i,t} \), and they can also sell bond \( i \) in submarket \( i \) for cash, denoted by \( b^\dagger_{i,t} \). Let \( q_{i,t} \) denote the price of bond \( i \) in terms of cash. The end-of-period cash holdings \( \tilde{m}_t \) are equal to the initial cash holdings, \( m_t \), minus the amount of cash sold, \( \sum m^\dagger_{i,t} \), plus the amount of cash gained from selling bonds, \( \sum q_{i,t} b^\dagger_{i,t} \). The end of period bond holdings \( \tilde{b}_{i,t} \) are
equal to the initial bond holdings, \( b_{i,t} \), plus the amount of bonds gained from selling cash, \( \frac{m^\dagger_{i,t}}{q_{i,t}} \), minus the amount bonds sold, \( b^\dagger_{i,t} \). The amount of assets for sale is subject to the assets-in-advance constraint; thus, \( \sum_i m^\dagger_{i,t} \leq m_t \) and \( b^\dagger_{i,t} \leq b_{i,t} \).

Similar to the households’ problem in the goods market, we have the following equations for the marginal value of cash and bonds in the asset market.\(^7\)

1. \[
\frac{\partial \Omega^j_t}{\partial m_t} (m_t, b_t) = \max \left\{ \frac{w^j_{1,t}}{q_1}, \ldots, \frac{w^j_{k,t}}{q_k} \right\} \tag{7}
\]

2. \[
\frac{\partial \Omega^j_t}{\partial b^i_t} (m_t, b_t) = \max \left\{ q_i v^i_t, w^j_{i,t} \right\} \tag{8}
\]

These conditions state that a household sells its assets to the submarkets that provides the best terms of trade, or the household simply holds the asset into the goods market. I first consider the households’ decision of selling cash. For example, suppose there is a bond \( i \), such that \( \frac{w^j_{i,t}}{q_i} > v^i_t \) and \( \frac{w^j_{k,t}}{q_k} > \frac{w^j_{i,t}}{q_i} \) for all \( k \neq i \), then the bond \( i \) submarket provides the best terms of trade. In this case, the household sells all it its cash for bond \( i (m^\dagger_{i,t} = m_t) \). Similarly, if \( v^j_t > \frac{w^j_{i,t}}{q_i} \) for all \( i \), the prices of bonds in all submarkets are all too high; thus the household simply retains its cash and brings it to the goods market \( (m^\dagger_{i,t} = 0 \text{ for all } i) \).

Households’ decisions regarding selling bonds have similar interpretations. If \( w^j_{i,t} < \frac{v^j_t q_i}{q_i} \), the household sells bond \( i (b^\dagger_{i,t} = b_{i,t}) \) for cash; if \( w^j_{i,t} > q_i v^j_t \), the household does not sell bond \( i \) in the asset market but retain bond \( i \) to the goods market \( (b^\dagger_{i,t} = 0) \).

Let \( \hat{v}^j_t \) and \( \hat{w}^j_{i,t} \) denote the marginal value of assets at the beginning of each period, before households realize whether they can trade in the asset market. Combining (6), (7), and (8), we have the following equations:

\[
\hat{v}^j_t = \frac{\partial \Phi^j_t}{\partial m_t} (m^\dagger_t, b^j_t) = \mu \max \left\{ \frac{w^j_{1,t}}{q_1} - v^j_t, \ldots, \frac{w^j_{k,t}}{q_k} - v^j_t, 0 \right\} + v^j_t, \tag{9}
\]

\[
\hat{w}^j_{i,t} = \frac{\partial \Phi^j_t}{\partial b^i_t} (m^\dagger_t, b^j_t) = \mu \max \left\{ q_i v^j_t - w^j_{i,t}, 0 \right\} + w^j_{i,t}. \tag{10}
\]

Equations (9) and (10) state that the value of assets in the asset market is equal to their value in the goods market plus the trading surplus in the asset market, and the trading

\(^7\)See appendix for the prove of the necessary conditions
surplus is subject to asset market frictions, \( \mu \).

Finally, the transversality conditions in the asset market also need to be satisfied:

\[
\lim_{t \to \infty} \left( \frac{1}{1+\rho} \right)^t \varphi_t \eta_{t+1} = 0,
\]
\[
\lim_{t \to \infty} \left( \frac{1}{1+\rho} \right)^t \varphi_t \omega_{i,t} b_{i,t+1} = 0.
\]

2.3 The Government

The government is the only issuer of cash and bonds. Let \( \bar{M}_t^g \) and \( B_{i,t}^g \) denote the aggregate bond supply at the end of each period. The supply of cash and bonds has growth rate

\[
1 + \gamma_t = \frac{\bar{M}_t^g}{\bar{M}_{t-1}^g} = \frac{B_{i,t}^g}{B_{i,t-1}^g}.
\]

To achieve the target quantity of bonds, \( \bar{B}_{i,t}^g \), the government sells new bonds in the asset market. Because \( \lambda_i \) proportion of old bonds \( i \) mature at the beginning of each period, the government issues \( B_{i,t}^g - (1 - \lambda_i) B_{i,t-1}^g \) units of new bond \( i \). I restrict my attention to the case that the government is always a bond seller in the asset market; that is,

\[
\bar{B}_{i,t}^g - (1 - \lambda_i) \bar{B}_{i,t-1}^g > 0,
\]

Because the government is the only issuer of cash and bonds and faces no asset-in-advance constraint, I assume for simplicity that the government has no limited participation problem and can always sell the new bonds in the asset market.

Each period the government consumes \( g_t \) unit of consumption goods, and I assume that \( g_t \) is exogenously determined. The following government’s balanced budget constraint must hold:

\[
g_t = \phi_t T_t + \{ \phi_t \bar{M}_t^g - \phi_t \bar{M}_{t-1}^g \} + \left\{ \sum_i q_{i,t} \phi_t \left[ B_{i,t}^g - (1 - \lambda_i) \bar{B}_{i,t-1}^g \right] - \sum_i \lambda_i \phi_t B_{i,t-1}^g \right\}.
\]

The government’s balanced budget constraint states that the government expenditure is financed with lump-sum tax, \( \phi_t T_t \), seigniorage revenue, \( \phi_t \bar{M}_t^g - \phi_t \bar{M}_{t-1}^g \), and debt management surplus, \( \sum_i q_{i,t} \phi_t \left[ B_{i,t}^g - (1 - \lambda_i) B_{i,t-1}^g \right] - \sum_i \lambda_i \phi_t B_{i,t-1}^g \). I rewrite the gov-
ernment’s budget balance in real term as follows:

\[ g_t = \tau_t + \left\{ \bar{m}_t^g - \frac{1}{1 + \pi_t} \bar{m}_t^g \right\} + \left\{ \sum_i q_{i,t} \left[ \bar{B}_{i,t}^g - \frac{(1 - \lambda_i)}{1 + \pi_t} \bar{B}_{i,t-1}^g \right] - \sum_i \frac{1}{1 + \pi_t} \lambda_i \bar{b}_{i,t-1} \right\}. \tag{11} \]

Generally speaking, a quantitative monetary policy instrument is an adjustment of the composition of central bank’s balance sheet and the outstanding government liabilities through open market operations. In this paper, I do not explicitly differentiate the roles of central banks and fiscal authorities, and an open market operation conducted by the central bank is equivalent to a management of government liabilities; thus, the monetary policy tool in this paper is an adjustment of the quantity of the outstanding bonds, \( \bar{b}_{i,t}^g \).

An adjustment of outstanding bonds directly changes the debt management surplus, and from the government’s budget constraint (11), the change in the debt management surplus can be rebalanced through either seigniorage revenue or taxes. A long line of research emphasizes that how the government meets the budget constraint, or the interaction between fiscal and monetary policy, is important to determine the policy effects.\(^8\) In order to evaluate the policy effects in a more comprehensive perspective, I follow Aiyagari and Gertler (1985) and distinguish between the polar Ricardian and non-Ricardian fiscal regimes.

In the polar Ricardian regime, the fiscal authority fully accommodates monetary policy by balance the government budget through tax levies, but the monetary policy does not react to the government deficit with money creation. Thus, an adjustment in outstanding debt implies an adjustment in direct taxes, and the seigniorage revenue stay unchanged. That is, there exist reference quantities of taxation and debt obligations, \( \Lambda_\tau \), \( \Lambda_m \) and \( \Lambda_b \), such that

\[
\begin{align*}
\tau_t - \Lambda_\tau &= \left\{ \sum_i \frac{1}{1 + \pi_t} \lambda_i \bar{B}_{i,t-1}^g - \sum_i q_{i,t} \left[ \bar{B}_{i,t}^g - \frac{(1 - \lambda_i)}{1 + \pi_t} \bar{B}_{i,t-1}^g \right] \right\} - \Lambda_b \\
\frac{1}{1 + \pi_t} \bar{m}_t^g - \frac{1}{1 + \pi_t} \bar{m}_t^g &= \Lambda_m = 0
\end{align*}
\] \tag{12}

In the polar non-Ricardian regime, the central bank fully accommodates a fiscal deficit by financing the government debt and expenditure with money creation, but the fiscal authority would not commit itself to meet government budget with taxes. Thus, a change in the debt management surplus implies a change in seigniorage revenue, but

\^8\text{Examples include Sargent and Wallace (1981), Wallace (1981), Aiyagari and Gertler (1985), and Leeper (1991).}
taxes stay the same. That is,

\[
\left\{ \begin{array}{l}
\left\{ \bar{m}_t^g - \frac{1}{1+\pi_t} \bar{m}_{t-1}^g \right\} - \Lambda_m = \left\{ \sum_i \frac{1}{1+\pi_i} \lambda_i \bar{b}_{t,i}^g \bar{m}_{i,t-1} - \sum i q_i \bar{b}_{i,t}^g \right\} - \Lambda_b \\
\tau_t - \Lambda_\tau = 0
\end{array} \right. \] (13)

In between the two polar regimes lies a continuum of regimes in which monetary and fiscal policies accommodate one another to varying degrees.\(^9\) The policy effect in these two polar regimes will be discussed in Section 3.

### 2.4 Market Clearing Conditions

As noted in Section 2.1, the linearity of utility functions results in no wealth effect, and hence, it is sufficient to analyze the aggregate asset holdings without considering the asset distributions among households.

Let \(\bar{m}_t^j\) and \(\bar{b}_t^j\) denote the total amount of cash and bonds held by state \(j\) households at the beginning of the asset market phase and \(\bar{m}_t^{ij}\) and \(\bar{b}_t^{ij}\) denote the total amount of assets holdings at the beginning of the goods market phase. I also denote by \(\bar{m}_{i,t}^j\) and \(\bar{b}_{i,t}^j\) the total amount of cash and bonds sold by state \(j\) households in the bond \(i\) market.

The following asset market clearing condition is thus obtained for the bond \(i\) submarket:

\[
\frac{1}{q_i \bar{b}_{i,t}} \left[ \bar{m}_{i,t}^{tc} + \bar{m}_{i,t}^{tn} \right] = \bar{b}_{i,t}^{tc} + \bar{b}_{i,t}^{tn} + \bar{b}_{i,t}^g - \left( \frac{1 - \lambda_i}{1+\pi} \right) \bar{b}_{i,t-1}^g. 
\] (14)

The left-hand side is the demand for bond \(i\), which consists of the cash sold by consumers and non-consumers in the bond \(i\) submarket divided by the price of bond \(i\). The right-hand side is the supply of bond \(i\), which consists of the quantity of bond \(i\) sold by consumers and non-consumers plus the new quantity of bond \(i\) issued by the government, \(\bar{b}_{i,t} - \left( \frac{1 - \lambda_i}{1+\pi} \right) \bar{b}_{i,t-1}\). Notice that only proportion \(\mu\) of households can trade in the asset market, and thus, it must be the case that \(\bar{m}_{i,t}^{ij} \leq \mu \bar{m}_{i,t}^j\) and \(\bar{b}_{i,t}^{ij} \leq \mu \bar{b}_{i,t}^j\).

In the goods market, the resource constraint must hold: the goods produced by the
households must be equal to the amount of goods consumed by consumers. Thus,

\[ h_t = \int c_t(\omega)1_{C_t}(\omega)d\omega, \quad (15) \]

where \( 1_A(\omega) \) is the indicator function

\[ 1_A(\omega) := \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}. \]

### 2.5 Assets Distributions

At the beginning of each period, before the asset market opens, the existing bonds mature, and households draw their new states. Let \( \bar{m}_t^\prime \) and \( \bar{b}_{i,t}^\prime \) denote the total amount of cash and bonds held by the households at the end of the period. Then, the aggregate cash holdings at the beginning of period \( t \), after bonds mature, is \( \frac{1}{1+\pi_t}(\bar{m}_t^{\prime \prime} + \sum_i \lambda_i \bar{b}_{i,t}^{\prime \prime}) \), and the aggregate real bond \( i \) holdings equal \( \frac{1}{1+\pi_t}(1-\lambda_i)\bar{b}_{i,t}^{\prime \prime} \). A proportion \( \alpha \) of assets are held by consumers, and a proportion \( (1-\alpha) \) of assets are held by non-consumers. Then

\[
\begin{align*}
\bar{m}_t^c &= \frac{\alpha}{1+\pi_t}(\bar{m}_t^{\prime \prime} + \sum_i \lambda_i \bar{b}_{i,t}^{\prime \prime}); \\
\bar{b}_{i,t}^c &= \frac{\alpha}{1+\pi_t}(1-\lambda_i)\bar{b}_{i,t}^{\prime \prime}; \\
\bar{m}_t^n &= \frac{1-\alpha}{1+\pi_t}(\bar{m}_t^{\prime \prime} + \sum_i \lambda_i \bar{b}_{i,t}^{\prime \prime}); \\
\bar{b}_{i,t}^n &= \frac{1-\alpha}{1+\pi_t}(1-\lambda_i)\bar{b}_{i,t}^{\prime \prime}.
\end{align*}
\]

In the asset market, households exchange cash and bonds. Cash holdings at the close of the asset market should be equal to cash holdings at the opening of the asset market, \( \bar{m}_t^\prime \), minus the aggregate amount of cash sold in all bonds submarkets, \( \sum_i \bar{m}_{i,t}^{\prime \prime} \), plus the amount of cash gained from selling bonds, \( \sum_i q_{i,t} \bar{b}_{i,t}^{\prime \prime} \). The bond \( i \) holdings at the opening of the goods market have similar interpretations. Let \( \bar{m}_{i,t}^\prime \) and \( \bar{b}_{i,t}^\prime \) denote the cash and bond holdings at the end of the asset market phase; then,

\[
\begin{align*}
\bar{m}_{i,t}^{\prime \prime} &= \bar{m}_{i,t}^c - \sum_i \bar{m}_{i,t}^{c \prime} + \sum_i q_{i,t} \bar{b}_{i,t}^{c \prime} + \frac{1}{q_{i,t}} \bar{b}_{i,t}^{c \prime} - \bar{b}_{i,t}^{c \prime} ; \\
\bar{b}_{i,t}^{\prime \prime} &= \bar{b}_{i,t}^c + \frac{1}{q_{i,t}} \bar{m}_{i,t}^{c \prime} - \bar{b}_{i,t}^{c \prime} ; \\
\bar{m}_{i,t}^{\prime \prime} &= \bar{m}_{i,t}^n - \sum_i \bar{m}_{i,t}^{n \prime} + \sum_i q_{i,t} \bar{b}_{i,t}^{n \prime} + \frac{1}{q_{i,t}} \bar{b}_{i,t}^{n \prime} - \bar{b}_{i,t}^{n \prime} .
\end{align*}
\]

At the beginning of the goods market phase, households’ asset holdings are equal to the amount held at the close of the asset market. In the goods market, households’ cash holdings increase because of the sale of consumption goods, \( h_t \). Moreover, consumers’ cash holdings decrease because of the purchases of consumption goods, \( -\int c_t(\omega)1_{C_t}(\omega)d\omega \), and tax, \( \tau \). \( \bar{m}_i^S \) represents the amount of cash issued by the government in the end of the goods market.

\[
\bar{m}_i^{\prime \prime} + \bar{m}_i^c + h_t - \tau_t - \int c_t(\omega)1_{C_t}(\omega)d\omega = \bar{m}_i^{\prime \prime}. 
\]
Because households cannot exchange bonds in the goods market, the end-of-period bond holdings are equal to the bond holdings at the opening of the goods market. Thus,

$$\bar{b}_{i,t}^x + \bar{b}_{i,t}^c = \bar{b}_{i,t-1}^c.$$  

Finally, the total amount of cash and bonds held by the households must be consistent with the total amount of outstanding cash and bonds issued by the government. Thus

$$\bar{m}_{i,t}^" = \bar{m}_{i}^S,$$

$$\bar{b}_{i,t}^c = \bar{b}_{i,t}^S.$$  

### 2.6 Equilibrium

I focus on the stationary equilibrium in which asset prices, asset values, and the inflation rate are constant over time. Recall that the distribution of asset holdings is not degenerate in this model. Moreover, bonds with different maturities are perfect substitutes for households, and thus, the equilibrium distribution of households’ asset allocations may not be unique. The stationary monetary equilibrium is defined as follows.

**Definition 1** A stationary monetary equilibrium is a vector $x = (q_i, c, h, m^S, \bar{b}_{i}^S, \pi, \tau, v^j, w^j, \hat{v}^j, \hat{w}^j)$ with corresponding aggregate asset allocations $(\bar{m}^j, \bar{m}^0j, \bar{m}^xj, \bar{b}_{i}^j, \bar{b}_{i}^0j, \bar{b}_{i}^xj)$ such that the following conditions are satisfied:

1. Households’ optimal decision
2. Consistent asset values
3. Market clearing conditions
4. Consistent asset allocations
5. Government’s budget is balanced
6. Government’s monetary policy

In a stationary equilibrium, all of the time subscripts can be dropped. To exclude the Ponzi game paths, the continuum value of cash in the goods market must be smaller than one. That is,

$$\frac{1}{1 + d} [\alpha \hat{v}^c + (1 - \alpha) \hat{w}^c] < 1,$$

(\star)
which guarantees that consumers do not accumulate cash for the future but spend all of their cash on consumption in the goods market; therefore,

$$\int c(\omega)1_c(\omega)d\omega, = \bar{m}c.$$ \hspace{1cm} (16)

Moreover, combining (2) and (16), we have

$$v^c = 1$$

This is because the value of cash for a consumer comes from the purchases of consumption goods, and one unit of consumption goods generates one unit of utility, so the value of cash for a consumer must be one.\footnote{The first-best allocation is achieved if and only if \( \frac{dl(h)}{dh} = 1 \). That is, the marginal cost of production is equal to the marginal benefit of consumption. Under the Friedman rule, \( \frac{dl(h)}{dh} = 1 \) can be achieved under the Friedman rule, \( 1 + \pi^* = \frac{1}{1 + \rho} \). Under the Friedman rule, the timing friction is irrelevant, so the issuance of bonds has no effects on social welfare. As a result, bonds are not essential for social optimal in this model. For the essensity of illiquid bonds on improving social welfare, the readers are referred to Kocherlakota (2003), Shi (2005), Boel and Camera (2006).} First, (3) and (5) imply that consumers and non-consumers value bonds the same in the goods market, \( w^c_i = w^n_i \equiv w_i \). Second, in this model, the benefit of selling cash in different bond markets for the representative household must be equal. That is

$$\frac{w_i}{q_i} = \frac{w_{i'}}{q_{i'}} \text{ for all } i, i' \in 1, \ldots, k.$$  

The proof is intuitive. Suppose that there exist bond \( i \) and bond \( i' \) such that \( \frac{w_i}{q_i} > \frac{w_{i'}}{q_{i'}} \); then, because bond \( i \) market provides better term of trade for the representative household than bond \( i' \) market, there will be no demand for bond \( i' \). However, there is positive supply of bond \( i' \) because of the new bond \( i' \) issued by the government. Thus, the market clearing condition of the bond \( i' \) market does not hold. In other words, a household must be indifferent with respect to selling their cash in different bond markets.\footnote{Note that this argument is only true in the representative household model. The heterogeneous household model is discussed in Section 4, and different households may value bonds differently and therefore trade different bonds. As a result, the surpluses of selling cash in different bond market may not be equal for a household.} Therefore, I assume that

$$\frac{w_i}{q_i} = \frac{w_{i'}}{q_{i'}} = V \text{ for all } i, i' \in 1, \ldots, k,$$

so households’ value toward assets can be simplified as the following equations:
1. Asset values in the asset market (9), (10) can be rewritten as:

\[
\hat{v}_c = \mu \max \{v^c, V\} + (1 - \mu)v^c \\
\hat{v}_n = \mu \max \{v^n, V\} + (1 - \mu)v^n \\
\hat{w}_i^c = \mu \max \{v^c, V\} + (1 - \mu)V \\
\hat{w}_i^n = \mu \max \{v^n, V\} + (1 - \mu)V
\]

2. Asset values in the goods market (2), (3), (4), (5) can be rewritten as:

\[
v^c = 1 \\
v^n = \frac{1}{1 + d} [\alpha \hat{v}_c + (1 - \alpha)\hat{v}_n] \\
V = \frac{1}{1 + d} \left\{ \frac{\lambda_i}{q_i} [\alpha \hat{v}_c + (1 - \alpha)\hat{v}_n] + (1 - \lambda_i) \left[ \frac{\hat{w}_i^c}{q_i} + (1 - \alpha)\frac{\hat{w}_i^n}{q_i} \right] \right\}
\]

Where

\[
\frac{w_i}{q_i} = V
\]

Note that $V = \frac{w_i}{q_i}$ represents the exchange value of cash for a household in the asset market: on the one hand, a household can sell one unit of cash for $\frac{1}{q_i}$ units of bond $i$ in the asset market and obtain $\frac{w_i}{q_i} = V$ units of utility; on the other hand, a household can also obtain one unit of cash by selling $\frac{1}{q_i}$ units of bond $i$ in the asset market, which costs the household $\frac{w_i}{q_i} = V$ units of utility. As a result, households’ decisions on asset exchanges are determined by the relative values between the exchange value of cash $V$ and the use value of cash in the goods market, $v^n$ and $v^c$.

I first take the inflation rate, $\pi$, and the exchange value of cash, $V$, as given, and the following variables $v^n$, $v^c$, $\hat{v}^n$, $\hat{v}^c$, $w_i^n$, $w_i^c$, $\hat{w}_i^n$, $\hat{w}_i^c$, $q_i$ can be solved from (V1) to (V8) as functions of $V$ and $\pi$.

As $v^c > v^n$, we can divide (V1) to (V8) into three cases:

1. $V < v^n < v^c$: In this case, the exchange value of cash in the asset market $V$ is lower than both $v^c$ and $v^n$, meaning that a household does not purchase bonds in either the consumer state or the non-consumer state. Therefore, the asset market provides no value added for cash, and the value of cash in the asset market is equal.
to its value in the goods market; so

\[ \hat{\sigma}^c = \sigma^c, \]
\[ \hat{\sigma}^n = \sigma^n. \]  

(17)

Combining (V6) and (17), one can find that \( \sigma^n = \frac{\alpha}{\alpha + d} \) and \( \sigma^c = 1 \). Because it requiring that \( V < \sigma^n \), this case hold if and only if \( V < \frac{\alpha}{\alpha + d} \). Note that in this case, there is no demand for any bonds in the asset market but there is always a positive supply of bonds issued by the government. Therefore, the market clearing conditions cannot be satisfied, and this case cannot occur in a stationary equilibrium.

2. \( \sigma^n < V < \sigma^c \): In this case, the exchange value of cash \( V \) is higher than non-consumers’ reservation value of cash but lower than consumers’ reservation value of cash. As a result, when a household is in the non-consumer state, it sells all of its cash for bonds if it can participate in the asset market. When the household is in the consumer state, it sells all of its bonds for cash if it is able to trade in the asset market. Note that \( \sigma^n < V < \sigma^c \) implies \( \frac{w^n'}{v^n'} > q_i > \frac{w^c'}{v^c'} \), where \( \frac{w^n'}{v^n'} \) and \( \frac{w^c'}{v^c'} \) are the reservation bond prices for non-consumers and consumers. As \( \frac{w^n'}{v^n'} > \frac{w^c'}{v^c'} \), non-consumers value bonds relatively higher than consumers do, so the asset market facilitates asset reallocations between the high-liquidity demanders (the consumers) and the low-liquidity demanders (the non-consumers). (V1) and (V2) can be rewritten as

\[ \hat{\sigma}^c = \sigma^c \]
\[ \hat{\sigma}^n = \mu V + (1 - \mu) \sigma^n \]

3. \( V > \sigma^c > \sigma^n \): In this case, the exchange value of cash \( V \) is higher than the value of cash in the goods market for both consumers and non-consumers. Therefore,

\[ \hat{\sigma}^c = \mu V + (1 - \mu) \sigma^c \]
\[ \hat{\sigma}^n = \mu V + (1 - \mu) \sigma^n \]  

(18)

Note that \( V > \sigma^c > \sigma^n \) implies \( \frac{w^n'}{v^n'} > \frac{w^c'}{v^c'} > q_i \), and thus, it is also possible interpret this case as meaning that the prices of bonds are lower than the reservation bond prices for consumers and non-consumers. Both interpretations imply that a household sells all of its cash for bonds whenever it can trade in the asset market. Moreover, a household never sells bonds but holds to maturity. Because \( \sigma^c = 1 \), this case holds only if \( V > 1 \). Note that it is also necessary to verify that (\( \ast \)) holds,
or the transversality condition will be violated. Combining (18), (V6), and (\star), the transversality condition hold if only if \( V < \frac{\mu + d}{\mu} \). In summary, for \( V > 1 \), the return in the asset market are sufficiently high that even a consumer would desire to sell cash for bonds in the asset market. However, there is a probability \( \mu \) that a consumer fails to participate in the asset market and holds a positive amount of cash when entering the goods market. If \( V < \frac{\mu + d}{\mu} \), the continuum value of cash in the goods market is not excessively high ((\star) holds), and hence, consumers in the goods market will spend all their cash on consumption in the goods market. If \( V > \frac{\mu + d}{\mu} \), the continuum value of cash is larger than one, and thus, consumers defer their consumption in the goods market, and the transversality condition is violated.

The following lemma summarizes the discussion above.

**Lemma 1** There exists \( V = \frac{\alpha}{\alpha + d} \) and \( \tilde{V} = \frac{\mu + d}{\mu} \) such that

1. For \( V \in (0, \tilde{V}) : V < v^n(V) < v^c(V) \). (Region 0)
2. For \( V \in [\tilde{V}, 1) : v^n(V) < V < v^c(V) \). (Region I)
3. For \( V \in [1, \tilde{V}) : v^n(V) < v^c(V) < V \). (Region II)
4. For \( V \in (\tilde{V}, \infty) : \) Transversality conditions are violated. (Region III)

**Proof.** See the Appendix. □

Figure 3 illustrates the relationship between the exchange value of cash \( V \) and households’ decision in the asset market. Recall that Region 0 and Region III are excluded from the stationary equilibrium because of violations of market clearing conditions and the transversality conditions, so we focus on the case in which \( V \in \left[ \frac{\alpha}{\alpha + d}, \frac{\mu + d}{\mu} \right] \). The following lemma depicts the relationship between the bond prices, \( q_i \), the net trading surplus of selling cash, \( V - v^n \), and the exchange value of cash, \( V \).

**Lemma 2** For \( V \in \left[ \frac{\alpha}{\alpha + d}, \frac{\mu + d}{\mu} \right] \) and \( \lambda_i \in (0, 1) \), there is a unique \((v^n, v^c, \hat{v}^n, \hat{v}^c, w_i^n, w_i^c, \hat{w}_i^n, \hat{w}_i^c, q_i)\) that solves (V1) to (V8).

1. \( \frac{dq_i}{dV} > 0 \)
2. \( \frac{d(V - v^n)}{dV} > 0, \frac{d(V - v^c)}{dV} > 0. \)
3. \( \frac{\partial v^n}{dV} > 0, \frac{\partial v^c}{dV} \geq 0 \)
(Region 0) Households Never buy bonds

(Region I) Consumers sell bonds Non-consumers buy bonds

(Region II) Households buy bonds in both states and never sell TVCs are violated

\[
\begin{align*}
\alpha & = \frac{a}{a+d} \\
V & < \nu^n < \nu^c \\
\nu^n &= V < \nu^c \\
\nu^n &< V = \nu^c \\
\nu^n &< V < \nu^c \\
\nu^n &< V < \nu^c \\
\nu^n &< V < \nu^c
\end{align*}
\]

Figure 3: Exchange value of cash and households’ trading decision

**Proof.** See the Appendix.

First, the lemma shows that \( q_i \) and \( V \) are negatively related. When the asset prices are lower, households can purchase more bonds in the asset market by selling cash; thus, a higher exchange value of cash, \( V \), implies lower bond prices. Second, the net surplus from selling cash in the asset market, \( V - \nu^i \) (a negative value represents the surplus of purchasing cash), increases as \( V \) increases. Combining Lemma 2.1 and Lemma 2.2 indicates that the increase of bond prices, \( q_i \), increases the surplus of cash selling in the asset market, \( V - \nu^i \), and decrease the surplus of cash purchases in the asset market, \( \nu^i - V \). Finally, the lemma states that the value of cash in the asset market increases as \( V \) increases. This is because \( V \) is positively related to the surplus of selling cash in the asset market, \( V - \nu^i \), and a larger trading surplus in the asset market generates a higher value of cash.

### 2.7 The Personal Yield Curves

According to Lemma 2, we can solve for the prices of bonds from equations (V1) to (V8) as functions of the exchange value of cash, \( V \), the inflation rate \( \pi \), and the maturity rate \( \lambda_i \):

\[ q = q(\lambda; V, \pi). \]
Figure 4: Yield curves: $\mu = 0.8, \rho = 0.05, \pi = 0.05$. $R$ is the yield curve. $R^c \equiv \frac{\lambda}{\mu} - \lambda$ is the reservation yield curve for consumers. $R^n \equiv \frac{\lambda}{\pi} - \lambda$ is the reservation yield curve for non-consumers.

Given a bond price, its implied nominal interest rate, $R$, can be obtained from $q = \frac{1}{1+R} [\lambda + (1-\lambda)q]$. Therefore, $R$ can also be expressed as a function of $V, \pi$ and $\lambda_i$:

$$R(\lambda; V, \pi) = \frac{\lambda}{q(\lambda; V, \pi)} - \lambda.$$  

Because yield curves are arranged as functions of the expected maturities of bonds, which are inverse functions of maturity rates, I define

$$\tilde{R}(\theta; V, \pi) \equiv R\left(\frac{1}{\theta}; V, \pi\right).$$

Given $V$ and $\pi$, the function $\tilde{R}(\theta; V, \pi)$ collects the yields across bonds with different maturities that generate the same exchange value of cash, $V$, for the representative household. I refer to $\tilde{R}(\theta; V, \pi)$ as a household’s personal yield curves. In the representative household model, the representative household’s personal yield curves are the candidates for the market yield curve, and the market yield curve is determined when $V$ and $\pi$ are specified from the market clearing conditions, the government budget constraint, and the policy regime.

Figure 4 illustrates $\tilde{R}(\theta; V, \pi)$ under different values of $V$. Observe that the personal yield curves are upward sloping when the yields are low and downward sloping when the yields are high. This feature results from the asset market friction, and the following lemma helps analyze the mechanism driving these results.

**Lemma 3** For $V \in \left[\frac{\alpha}{\alpha+d}, \frac{\mu+d}{\mu}\right]$, 

1. Region I: \( v^c \geq V \geq v^n \)

\[
\frac{\partial R(\lambda; V, \pi)}{\partial \lambda} = (1 - \mu) \frac{(1 - \alpha)[V - v^n] + \alpha[V - v^c]}{V - \{(1 - \mu)(1 - \alpha)[V - v^n] + \alpha[V - v^c]\}}
\]

2. Region II: \( V \geq v^c > v^n \)

\[
\frac{\partial R(\lambda; V, \pi)}{\partial \lambda} = (1 - \mu) \frac{(1 - \alpha)[V - v^n] + \alpha[V - v^c]}{V - \{(1 - \mu)(1 - \alpha)[V - v^n] + (1 - \mu)\alpha[V - v^c]\}}
\]

**Proof.** See the Appendix. □

Lemma 3 states that the slopes of personal yield curves are determined by the asset market friction, \( \mu \), and asset market trading surplus, \( V - v^n \) and \( V - v^c \). When the asset market is frictionless (\( \mu = 1 \)), the interest rates are irrelevant to the maturity of bonds (\( \frac{\partial R}{\partial \lambda} = 0 \)), meaning that the yield curves are flat lines. This result is straightforward because a household can always engage in arbitrage trading if it faces no friction, and therefore, its required yields for different maturities must be the same.

When a household faces financial frictions (\( \mu < 1 \)), its personal yield curves can be upward sloping, downward sloping, or flat. From the discussion at the end of Section 2.7, when the bond yields are close to their lower bound (\( V \rightarrow v^n \)), \( V - v^n \) is close to zero, and \( V - v^c \) is negative. In this case, according to Lemma 3, \( \frac{\partial R}{\partial \lambda} < 0 \), and the yield curve is upward sloping. When bond yields are high (\( V \) high), \( V - v^c \) and \( V - v^c \) are both positive; thus, according to Lemma 3, \( \frac{\partial R}{\partial \lambda} > 0 \), and the slope of the personal yield curve becomes downward sloping. This feature is consistent with the pattern of yield curves in reality (Figure 1.)

The feature is attributed to two risks generated by the limited participation in the financial market. The first is liquidation risk: households may not be able to sell their bonds for cash when they need cash to consume. The second is reinvestment risk: households may not need cash when their bonds mature, and the households may fail to repurchase bonds and therefore forgo the opportunity to gain the bond yields. In principle, short-term bonds bear greater reinvestment risk, and long-term bonds bear greater liquidation risk. For example, in this paper, one unit of short-term bonds with maturity rate \( \lambda = 1 \) becomes one unit of cash in the next period for certain; thus, the short-term bond bears no liquidation risk but bears the greatest reinvestment risk. However, one unit of long-term bonds (small \( \lambda \)) is highly unlikely to be converted to cash in a given period. As a result, the holders of the long-term bonds are not concerned about reinvestment risk but face greater liquidation risk.
I first discuss the case of region I \((v^c \geq V \geq v^n)\), in which households purchase bonds in the non-consumer state and sell bonds in the consumer state. When the exchange value of cash is low \((V = v^n, \text{Figure 4 Case 1})\), the bond yields are low. The surplus from bond purchasing \((V - v^n)\) is zero but the surplus from bond selling \((v^c - V)\) is positive. On the one hand, the zero surplus from bond purchasing implies that a non-consumer is not concerned about whether they can sell cash in the asset market, and thus, reinvestment risk does not matter. On the other hand, the positive surplus from bond selling implies that households are concerned about whether they can sell bonds in the market. As a household faces greater liquidation risk when it holds long-term bonds, it requires higher long-term yields than short-term yields, and its personal yield curve is upward sloping.

When the bond yields increase, the surplus from selling cash increases and the surplus from selling bond decreases. Consequently, the importance of liquidation risk decreases, and the importance of reinvestment risk increases. When the bond yields are close to the reservation prices of consumers \((V \rightarrow v^c, \text{Figure 4 Case 3})\), the surplus from bond selling \((v^c - V)\) is low but the surplus from bond purchasing \((V - v^n)\) is high. On the one hand, a household is indifferent with respect to selling bonds or not, and thus, liquidation risk does not matter. On the other hand, the surplus from selling cash \((V - v^n)\) is high, and households are therefore substantially concerned about whether they can sell cash in the market. As a household faces greater reinvestment risk when they hold short-term bonds, it requires higher short-term yields than long-term yields, and its personal yield curve is downward sloping.

When bond yields are higher than the reservation yields of consumers \((V \geq v^c > v^n)\), region II is reached (Figure 4 Case 4). Households purchase bonds in both the consumer state and the non-consumer state and never sell bonds. As a result, households are influenced by reinvestment risk in both states, and liquidation risk is irrelevant. Therefore, its personal yield curve is downward sloping. These properties of yield curves are summarized in Proposition 1.

**Proposition 1** For \(V \in \left[\frac{\alpha}{\alpha + d}, \frac{\mu + d}{\mu}\right]\)

1. If \(\mu = 1\) : \(\frac{\partial R(\theta; V, \pi)}{\partial \theta} = 0\)

2. If \(\mu < 1\) : There exists \(\tilde{V} \in \left(\frac{\alpha}{\alpha + d}, 1\right)\) such that

   \(a\) \(\frac{\partial \tilde{R}(\theta; V, \pi)}{\partial \lambda} > 0\) for \(V < \tilde{V}\)

   \(b\) \(\frac{\partial \tilde{R}(\theta; V, \pi)}{\partial \lambda} < 0\) for \(V > \tilde{V}\)
Proof. See the Appendix. ■

2.8 The Zero Lower Bound

In this section, I study the relationship between asset supplies and equilibrium bond yields. Recall that households must gain the same surplus from selling cash in different bond markets, and thus, in equilibrium, households allocate their cash to different bond markets until the equilibrium bond prices make them indifferent. Therefore, the bond prices are determined by the overall quantity of cash and bonds; I thus summarize the market clearing conditions for all bond markets:

$$ \sum_i \left[ \bar{m}^{tc}_i + \bar{m}^{tn}_i \right] = \sum_i q_i \left[ \bar{b}^{tc}_i + \bar{b}^{tn}_i + \left( \frac{\lambda_i + \pi}{1 + \pi} \right) \bar{b}^{\pi}_i \right]. $$

(19) states that the equilibrium bond prices are determined by the relative quantity of households’ cash holdings to households’ bond holdings plus the new bonds issued by the government. I first consider the regime in which the equilibrium bond prices are between the reservation prices of consumers and non-consumers: $q_i \in \left( \frac{w^c_i}{\delta c}, \frac{w^n_i}{\delta n} \right)$. The consumers sell all of their bonds and do not sell cash in the asset market, so $\bar{b}^{tc}_i = \mu \bar{b}^c_i$ and $\bar{m}^{tc}_i = 0$; the non-consumers sell all of their cash and do not sell bonds in the asset market, so $\sum_i \bar{m}^{tn}_i = \mu \bar{m}^n$ and $\bar{b}^{tn}_i = 0$. Therefore, the aggregate market clearing condition (19) becomes

$$ \mu \bar{m}^n = \sum_i q_i \left[ \bar{b}^c_i + \left( \frac{\lambda_i + \pi}{1 + \pi} \right) \bar{b}^{\pi}_i \right], $$

(20) and the equilibrium bond prices solve (20).

However, when the amount of cash is sufficient large relative to the amount of bonds such that

$$ \mu \bar{m}^n > \sum_i \frac{w^n_i}{\delta n} \left[ \bar{b}^c_i + \left( \frac{\lambda_i + \pi}{1 + \pi} \right) \bar{b}^{\pi}_i \right], $$

(20) can not hold for $q_i < \frac{w^n_i}{\delta n}$. The equilibrium bond prices are equal to their upper bound, the reservation prices for the non-consumers ($q_i = \frac{w^n_i}{\delta n}$). The non-consumers are indifferent between buying bonds and not, and they only sell part of their cash to the bond markets and get rationed ($\bar{m}^{tn}_i < \mu \bar{m}^n$). In this case, the short-term yield become zero, and it is the case of the zero lower bound.

Similarly, when the aggregate amount of cash is sufficient small relative to the aggre-
gate bonds such that
\[ \mu \bar{m}^n < \sum_i \frac{w^c_i}{\bar{v}^c} \left[ \tilde{b}^c_i + \left( \lambda_i + \pi \right) \bar{b}^c_i \right], \]
the equilibrium bond prices are equal or smaller than the reservation prices for the consumers, \( q_i \leq \frac{w^c_i}{\bar{v}^c} \). Specifically, if \( q_i = \frac{w^c_i}{\bar{v}^c} \), the bond prices are equal to the reservation prices of consumers, and the consumers feel indifferent to buy bonds and sell bonds and are rationed. This case holds requiring that \( \mu \bar{m}^n + \mu \bar{m}^c \geq \sum_i \frac{w^c_i}{\bar{v}^c} \left( \frac{\lambda_i + \pi}{1 + \pi} \right) \bar{b}^c_i \). If \( \mu \bar{m}^n + \mu \bar{m}^c < \sum_i \frac{w^c_i}{\bar{v}^c} \left( \frac{\lambda_i + \pi}{1 + \pi} \right) \bar{b}^c_i \), the equilibrium bond prices will be smaller than the reservation prices of consumers (\( q_i < \frac{w^c_i}{\bar{v}^c} < \frac{w^c_i}{\bar{v}^c} \)). Thus, households are bond buyers in both the consumer and the non-consumer state, and the bond prices solve \( \mu \bar{m}^n + \mu \bar{m}^c = \sum_i q_i \left( \frac{\lambda_i + \pi}{1 + \pi} \right) \bar{b}^c_i \).

I denote the following cutoff amounts of cash, \( M_1 \equiv \frac{1}{\mu} \sum_i \frac{w^c_i}{\bar{v}^c} \left[ \mu \tilde{b}^c_i + \left( \frac{\lambda_i + \pi}{1 + \pi} \right) \bar{b}^c_i \right] \) and \( M_2 \equiv \frac{1}{\mu} \sum_i \frac{w^c_i}{\bar{v}^c} \left[ \mu \tilde{b}^c_i + \left( \frac{\lambda_i + \pi}{1 + \pi} \right) \bar{b}^c_i \right] \). Lemma 4 summarizes the above discussions.

**Lemma 4** The equilibrium bond prices satisfy

\[
q_i \begin{cases} 
  = \frac{w^c_i}{\bar{v}^c} & \text{if } \bar{m}^n \geq M_2 \\
  \in \left( \frac{w^c_i}{\bar{v}^c}, \frac{w^c_i}{\bar{v}^c} \right) & \text{if } \bar{m}^n \in (M_1, M_2) \\
  \leq \frac{w^c_i}{\bar{v}^c} & \text{if } \bar{m}^n \leq M_1 
\end{cases}
\]

### 3 Comparative Statistics With Respect to the Bond Supply

In this section, I analyze the impact of a change of the bond supply. For the sake of simplicity, I assume that the government issues only two types of bonds hereafter: the short-term bond \( (\lambda_s = 1) \) and the long-term bond \( (\lambda_l < 1) \). Moreover, since my research interest is the policy effects at the zero lower bound, I focus on the case of region I.

#### 3.1 The Non-Ricardian Regime

In the non-Ricardian regime, the tax is not affected by a change of bond supply. Combining the government’s budget constraint (11) and (13), the following equation must hold:

\[
g - \left[ q_l - \frac{1}{1 + \pi} \right] \bar{b}^l - \left[ q_l \left( 1 - \frac{1 - \lambda_l}{1 + \pi} \right) - \frac{\lambda_l}{1 + \pi} \right] \bar{b}^l - \left[ 1 - \frac{1}{1 + \pi} \right] \bar{m}^l = \Lambda_{\tau} \quad (21)
\]

Recall that in region I, consumers sell all their bonds for cash at the zero lower bound whenever they are able to participate in the asset market, and thus, we can rewrite the
goods market resource constraint (15) as

$$\frac{\alpha}{1 + \pi}[\bar{m}^g + \bar{b}^g_s + \lambda_l \bar{b}^g_l] + \frac{\alpha \mu}{1 + \pi}q_l(1 - \lambda_l)\bar{b}^g_l = h,$$

(22)

where the first part on the left-hand side represents the cash holding of consumers at the beginning of the asset market, and the second part represents the consumers’ cash gain from selling long-term bonds in the asset market.

Figures 11 and 12 illustrate how the supply of short- and long-term bonds affects nominal interest rates, real interest rates, and inflation. When short-term interest rate is greater than zero, bond prices are smaller than non-consumers’ reservation prices, and thus, from the asset market clearing condition (20), a reduction in bond supply can decrease both long- and short-term nominal interest rates; This is referred to as the "scarcity channel".

When the short-term yield reaches zero ($q_s = 1$), bond prices are equal to non-consumers’ reservation prices, and the scarcity channel becomes ineffective. In this case, according to (21) and (22), the short-term bond, $\bar{b}^g_s$, and cash, $\bar{m}^g$, are equivalent. Therefore, a decrease in the outstanding short-term bond only increases outstanding cash on a one-to-one basis, and the nominal interest rate, the real interest rate, and the inflation rate do not change. Specifically, at the zero lower bound, a decrease of short-term bond supply only decreases non-consumers’ short-term bond purchases and increases the cash they retain to the goods market, but consumers’ cash holdings in the goods market is not effected. Since only consumers are engaged in goods purchases, a change in the short-term bond supply has no effect on price level and therefore is neutral at the zero lower bound.

However, although the long-term yield also reaches its lower bound when the short-term yield is zero, its lower bound is nevertheless greater than zero. When the yield of a bond is greater than zero, the bond is an inferior asset than cash and is more costly for the government to issue. As a result, a reduction in the bond supply improves the government’s deficit and allows the government to decrease the inflation rate to balance its budget constraint. Moreover, the deflation decreases households’ cost of holding bonds and therefore decreases the nominal interest rate if it is larger than zero; This channel is referred to as the "deflation channel". Thus, although the scarcity channel is not effective, a reduction in the long-term bond supply still decreases long-term interest rate through the deflation channel. Notice that the deflation channel has only slight effects on nominal interest rate, and thus, at the zero lower bound, both long- and short-term real interest rates increase because of the decrease in the inflation rate.
Although the government is assumed to issue bonds with only two different maturities, one can characterize the implied interest rate on bonds at all maturities. The implied interest rates represent the interest rates of the bonds if they are additionally issued for a very small amount, and the implied market yields curves are equal to the personal yield curve for the representative households in equilibrium. Figure 13 depicts how implied market yield curves change with respect to a reduction in the long-term bond supply when the short-term interest rate is zero. The reduction in long-term bonds only slightly shifts down the yield curve, but the real yield curve shifts up because of the decrease in the inflation rate.

3.2 The Ricardian Regime

In the Ricardian regime, the seigniorage revenue is not effected by government debt issuance, so

$$1 - \frac{1}{1 + \pi} \bar{m}_m = \Lambda_m.$$  \hspace{1cm} (23)

I focus on the case in which the fiscal authority runs a net deficit; that is,

$$q_s - \frac{1}{1 + \pi} \bar{b}_s^g - \left[ q_l \left(1 - \frac{1 - \lambda_l}{1 + \pi}\right) - \frac{\lambda_l}{1 + \pi}\right] \bar{b}_l^g > 0.$$

Thus, the seigniorage revenue should be positive to balance the government’s budget (\(\Lambda_m > 0\)), and a positive inflation rate is required.

I demonstrate the policy effect in the Ricardian regime in Figure 14 and 15. First, when the short-term interest rate is above zero, reductions of long- and short-term bond supplies decrease nominal interest rate, real interest rates, and the inflation rate. This result is similar to the case of the non-Ricardian regime. However, unlike the non-Ricardian regime, when the short-term interest rate reaches zero, a reduction of short-term bond supply is not neutral but deflationary in the Ricardian regime. This is because a reduction in bond supplies increases the quantity of cash households are willing to hold. According to (23), the increase in outstanding cash allows the monetary authority to decreases the inflation rate regardless of whether the short-term interest rate is above or equal to zero.\(^{12}\) As a result, at the zero lower bound, because of the deflation effect, a reduction in long- or short-term bond supply slightly decreases the nominal long-term interest rates, and the real long- and short-term interest rates increase.

\(^{12}\)When the government is running a net surplus, the seigniorage revenue is negative, and the conclusion is reversed.
In summary, under both regimes, a reduction in the long-term bond supply has limited downward effects on the yield curve at the zero lower bound, and it shifts up the real yield curve because of the deflation effect. The key reason behind the ineffectiveness of long-term bond reduction is the perfect substitutability between long- and short-term bonds: when the short-term yield reaches the zero lower bound, the long-term yield also reaches its effective lower bound. In Section 4, the rigid relationship between long- and short-term yields is relaxed by introducing heterogeneous types of households. The scarcity channel can be effective even if the short-term interest rate is zero, and thus, a reduction in the supply of long-term bonds can more significantly decrease the long-term nominal interest rate and therefore can decrease the long-term real interest rate.

### 3.3 Monetary Policy and Welfare

In this model, the social planner’s problem is simple:

$$\begin{align*}
\max & \quad \sum_t \alpha c_t - l(h_t) \\
\text{s.t.} & \quad \alpha c_t = h_t
\end{align*}$$

Therefore, the first-best allocation is achieved when $\frac{dl(h)}{dh} = 1$. By (1) and (2), the marginal disutility of labor is equal to the expected discounted value of cash, $\nu^n$:

$$\frac{dl(h)}{dh} = \nu^n$$

Therefore, an increase in $\nu^n$ implies an increase in social welfare as long as the No-Ponzi condition is not violated ($\nu^n \leq 1$), and the first-best allocation is achieved when $\nu^n = 1$.

First, the first-best allocation can be generated by the Friedman rule. Under the Friedman rule, the timing friction is irrelevant to the individual’s decision, and thus, the issuance of bonds has no effects on social welfare. However, the Friedman rule may not be achievable. Depending on the fiscal policy regime, bond supply affects welfare differently. Figure 5 illustrates the relationship between long-term bond supply and social welfare in the non-Ricardian and Ricardian fiscal regime. In the Ricardian fiscal regime, bonds issuance is fully supported by government’s tax levies. In the case of above the zero lower bound, an increase in bond supply increases bond yields and increases the exchange value of cash in the asset market. As a result, a larger bond supply generates higher cash value, $\nu^n$, so social welfare is larger. However, at the zero lower bound, cash is abundant related to bonds and is valued in its fundamental value, $\nu^n = \frac{\alpha}{\alpha + d}$. An
increase in the bond supply does not increasing cash’s exchange value in the margin. However, as discussed in Section 3.2, an increase in bond supply increases inflation, and therefore decrease the fundamental value of cash and therefore decreases the social welfare at the zero lower bound.

In the non-Ricardian regime, the taxation is predetermined, and the issuances of both cash and bonds are entirely funded by the fixed tax. Recall that bonds cannot be used as a medium of exchange in this model, and when the nominal interest rate of a bond is above zero, the bond becomes an inferior asset than cash. As the issuance of cash and bonds is entirely funded by the government’s tax/transfer, zero nominal interest rates is optimal, and a reduction in the issuance of bonds with positive interest rate improves welfare. Because of the asset market friction, a zero long-term nominal interest rate is not achievable, and thus, the second-best allocation will be achieved when the long-term bond supply is zero and the short-term bond supply is small enough that the short-term nominal interest rate hits the zero lower bound. In the non-Ricardian regime, the largest achievable social welfare is restricted by the government’s ability to raise taxes, and the Friedman rule can only be achieved if the tax is large enough.\(^\text{13}\)

### 4 The Heterogeneous Household Model

In this section, I introduce households with different characteristics. A household’s characteristics include the asset market friction, \(\mu\), the time discount rate, \(\rho\), and the arrival rate of liquidity demand, \(\alpha\), and I denote the vector of household characteristics by \(A = (\mu, \rho, \alpha)\). The shapes of a household’s personal yield curves are determined by

\(^{13}\)Note that bonds are not essential to achieve social optimal in this model. For the essensity of illiquid bonds on improving social welfare, readers are referred to Kocherlakota (2003), Shi (2005), and Paola and Camera (2006).
the aggregate variables (for example, the inflation rate, $\pi$) as well as individual household’s characteristics; therefore, different households may have different personal yield curves. Given a combination of equilibrium bond yields, a household would purchase the bonds that offer it the highest value, and thus, different households may trade bonds with different maturities. Consequently, even if the short-term bond yield reaches the reservation short-term yield of short-term bond buyers, the long-term yield may not reach the reservation long-term yield for long-term bond buyers, so a reduction in the long-term bond supply can still effectively decrease the nominal long-term interest rate even if short-term interest rate hits the zero lower bound.

4.1 Comparison of The Personal Yield Curves

I first analyze how a household’s characteristics changes its personal yield curves. Let $\tilde{R}_l(R_s; A)$ denote the long-term yield that makes the household indifferent to purchasing long-term bonds and short-term bonds when the short-term yield is $R_s$. Suppose that there are two types of households with characteristics $A_1, A_2$ and $\tilde{R}_l(R_s^*; A_1) < \tilde{R}_l(R_s^*; A_2)$. Then, when the short-term interest rate is $R_s^*$, a type $A_2$ household requires a higher long-term interest rate than a type $A_1$ household to make it willing to purchase long-term bonds.

First of all, I compare the personal yield curves for households with different time discount rates, $\rho$. In Figure 8, the left and the middle panels illustrate the personal yield curves for impatient households ($\rho_1 = 0.075$) and patient households ($\rho_2 = 0.025$), respectively; the right panel is the function $\tilde{R}_l(R_s; \rho_i)$ for these two households.\(^{14}\) Note that $\tilde{R}_l(R_s; \rho_1) > \tilde{R}_l(R_s; \rho_2)$ for all $R_s$, and thus, given a short-term interest rate, $R_s$, impatient households require higher long-term yields to make them willing to purchasing long-term bonds. As a result, impatient households are short-term bond buyers, and patient households are long-term bond buyers. Moreover, the reservation bond yields for patient households are lower than the impatient households in both consumer and non-consumer state. That is to say, the patient households require lower yield to enter Region II, so they tend to hold a bond until it matures. This result explains the observation that insurance companies and pension funds tend to purchase long-term securities and hold assets to maturity.

Asset market frictions influence households’ personal yield curves in a different way. Figure 9 illustrates the personal yield curves for frictional households ($\mu_1 = 0.75$) and

\(^{14}\)The notation is slightly abused. I denote $\tilde{R}_l(R_s; \rho_i)$ by $\tilde{R}_l(R_s; (\mu, \rho_i, \alpha))$, where $\mu$ and $\alpha$ are the same across households.
frictionless households \((\mu_2 = 1)\). As Proposition 1 states, the personal yield curves are upward sloping when yields are low and downward sloping when yields are high, but the personal yield curves are flat for households that face small asset market frictions. Therefore, \(\tilde{R}_l(R_s; \mu_1) > \tilde{R}_l(R_s; \mu_2)\) when \(R_s\) is small and \(\tilde{R}_l(R_s; \mu_1) < \tilde{R}_l(R_s; \mu_2)\) when \(R_s\) is large. Thus, when yields are low, frictional households prefer short-term bonds more than frictionless households; when yields are high, frictional households prefer long-term bonds more than frictional households.

Finally, Figure 10 illustrates the personal yield curves for households facing different arrival rate of liquidity shocks, \(\alpha\). Observe from the right panel that an increase in \(\alpha\) increases the slope of \(\bar{R}_l(R_s; \alpha)\) but has no effect on the intercept. Therefore, \(\tilde{R}_l(R_s; \alpha_1) > \tilde{R}_l(R_s; \alpha_2)\) for all \(R_s > 0\). That is, \(\alpha\) has no influence on the personal yield curve at the zero lower bound, but when the short-term yield is above the zero lower bound, households that have higher \(\alpha\) demand higher long-term yields.

To conclude, first of all, the assets a household would like to purchase is endogenously determined; second, households’ preferred assets may change depending on the market interest rates. Finally, a household is allowed to trade bonds at all maturities once it enters the asset market, so the substitution between different bonds is enhanced. These features distinguish this model with the preferred habitat theory.

**Lemma 5** Given \(\pi\), let \(R_s\) and \(R_l\) be the equilibrium yields of short-term and long-term bonds, then \(R_l \in [\bar{R}_l(R_s, \alpha), \tilde{R}_l(R_s)]\), where \(\bar{R}_l(R_s) = \min_k \{ \bar{R}_l(R_s; A_k) \}\) and \(\tilde{R}_l(R_s) = \max_k \{ \tilde{R}_l(R_s; A_k) \}\).

The proof is straightforward. If \(R_l > \bar{R}_l(R_s)\), there will be no demand for short-term bonds. If \(R_l < \bar{R}_l(R_s)\), there will be no demand for long-term bonds. Note that in a representative household model, \(\bar{R}_l(R_s) = \tilde{R}_l(R_s)\), so the relationship between short-term and long-term bonds is one-to-one. However, in the model with heterogeneous types of households, given a short-term interest rate, any interest rate located between \(\bar{R}_l(R_s)\) and \(\tilde{R}_l(R_s)\) can be an equilibrium long-term interest rate. Consequently, the rigid relationship between long-term and short-term bonds is relaxed in a heterogeneous households model.

### 4.2 The Model with Two Types of Households

In this section, I assume that there are two types of households in the economy, and their characteristics are \(A_1\) and \(A_2\). Let \(R_s^*\) and \(R_l^*\) be the equilibrium short-term and long-term yields, and without loss of generality, I assume that \(\tilde{R}_l(R_s^*; A_1) > \tilde{R}_l(R_s^*; A_2)\). According to Lemma 5, the equilibrium long-term yield, \(R_l^*\), must satisfy \(\tilde{R}_l(R_s^*; A_1) \geq \bar{R}_l(R_s^*)\).
Therefore, given an equilibrium short-term yield, I can classify the equilibrium long-term yields into three cases.

In the first case, $\tilde{R}_l(R_s^*; A) = R^*_I > \tilde{R}_l(R_s^*; A_2)$ (Figure 6 left), the long-term yield is equal to its highest possible value, and this is referred to as the "High $R_l$" case. In this case, type 1 households are indifferent between purchasing short- and long-term bonds; type 2 households only purchase long-term bonds because purchasing long-term bonds provides them with higher marginal utility than purchasing short-term bonds. In the second case, $\tilde{R}_l(R_s^*; A_1) > R^*_I > \tilde{R}_l(R_s^*; A_2)$ (Figure 6 middle). Type 1 households only purchase short-term bonds, and type 2 households only purchase long-term bonds. This case is referred to as the transition regime. In the last case, $\tilde{R}_l(R_s^*; A_1) > R^*_I = \tilde{R}_l(R_s^*; A_2)$ (Figure 6 right). The long-term yield is equal to its lowest possible value, and this is referred to as the "Low $R_l$" case. In this case, type 1 households only purchase short-term bonds, but type 2 households are indifferent between purchasing short- and long-term bonds.

In the rest of this section, I analyze the policy effects when the short-term interest rate hits the zero lower bond. Because the change in the short-term bond supply has similar effects as in the representative household model, I focus on the policy effects from the change in the long-term bond supply. In the following policy experiment, I assume that households face two different degrees of limited participation: half of the households are frictionless households ($\mu = 0.95$), and half of the households are frictional households ($\mu = 0.75$).

Figures 16 and 17 illustrates the effects of a change in long-term bond supply in the non-Ricardian and Ricardian regime. First, when the short-term interest rate reaches zero, if the long-term bond supply is relatively high, the long-term yield will be equal to the reservation long-term yields of frictional households. This is the case of the high $R_l$.
region. In this case, The frictionless households gain positive surplus from purchasing long-term bonds, so they sell all of their cash in the asset market for long-term bonds only. However, the frictional households are indifferent among purchasing long-term bonds, purchasing short-term bonds and retaining their cash holding, and they are buyers of both long- and short-term bonds. In this region, the reduction in the long-term bond supply only reduces the amount of cash frictional households sell to the long-term bond market, and the scarcity channel is ineffective on decreasing long-term interest rate. The long-term interest rate only slightly decrease through deflation channel.

When the frictional households’ long-term bond purchases have been reduced to zero, the transition region is reached. In the transition region, frictionless households are the only long-term bond buyers, and frictional households are the only short-term bond buyers. The reduction in the long-term bond supply significantly decreases the long-term yield through the scarcity channel, and the trading surplus for frictionless households is diminished at the mean time.

Finally, when the long-term yield reaches frictionless households’ reservation yield for long-term bonds, the low $R_l$ region is reached. In the low $R_l$ region, frictionless households are indifferent among purchasing long-term bonds, purchasing short-term bonds and retaining their cash holding. In this region, the scarcity channel is, again, not effective. If the supply of long-term bonds decreases further, frictionless households will reduce the amount of cash they sell in the long-term bond market, and the long-term interest rate only decreases slightly through the deflation channel.

To summarize, at the zero lower bound, a reduction in the long-term bond supply has a limited effect on nominal interest rates in the high $R_l$ and the low $R_l$ regions. This policy effect is similar to a reduction in the long-term bond supply at the zero lower bound in the representative household model, and these two regions are also referred to as the "ineffective" regions. However, there also exists a transition region in which a reduction in the long-term bond effectively decreases long-term interest rate even if the short-term interest rate is zero. This transition region results from the heterogeneity of households and is referred to as the "effective" region.

4.3 The Model with Multiple Types of Households

In this section, I introduce households that face various financial frictions and time discount rates. To capture the observation that the majority of the households face limited asset market frictions, I assume that the limited participation that households face follows a left-skewed beta distribution; the distribution of time discount rates is assumed
to be uniform. The joint distribution of households’ characteristics is depicted in Figure 7. The mechanism here is similar to that discussed in Section 4.2. The only difference is that there are multiple types of households, meaning that monetary policy encounters a mixture of multiple effective and ineffective regions.

Figures 18 and 19 illustrate the policy effect of a change in the amount of outstanding long-term bonds. Unlike the representative household model, a reduction in the long-term bond supply significantly decreases the nominal long-term interest rate when the short-term interest rate reaches zero. This is because, the heterogeneity of households loosens the one-to-one relationship between the long-term yield and the short-term yield. Although the short-term yield is at the zero lower bound, the long-term interest rate may not reach its effective lower bound. There exist long-term bond buyers who gain positive trading surplus from purchasing long-term bonds, and thus, a reduction in the long-term bond supply can effectively diminish the trading surplus of those households and decrease the nominal long-term interest rate.

However, the substitution effect between long- and short-term bonds for individual households makes the reduction in the supply of long-term bonds less effective at the zero lower bound. When the short-term rate reaches zero, there exist marginal traders whose reservation long-term interest rate is equal to the equilibrium long-term rate. Those marginal traders are indifferent among trading long- and short-term bonds and retaining their cash. When the long-term bond supply decreases, the marginal traders are unwilling to pay higher prices for the long-term bonds, but they decrease the amount of cash they sell in the long-term bond market. The existence of these marginal traders places pressure on the rise in long-term bond prices. As a result, the impact of a reduction in the long-term bond supply on the long-term interest rate is less effective when the short-term rate reaches zero.

I also concern with whether a reduction in the long-term bond supply can decrease real interest rates given a zero short-term rate. In this policy experiment, the households are highly diversified in both market frictions and time discount rates, so the effective region is large. As a result, the substitution effect between long- and short-term yields is small, meaning that long-term bond purchases can more effectively decrease the nominal long-term interest rate. The decrease in the nominal long-term interest rate dominates the decrease in the inflation rate, and thus the long-term real interest rate decreases. Notice that since short-term interest rate has already reached the zero lower bound, the reduction in the long-term bond supply still increases real short-term interest rate because of deflation.

In Figure 20, I characterize the impact of a long-term bond reduction on implied
market yield curve. The implied market yield curve equal the personal yield curve for the marginal traders. The reduction in the long-term bond supply at the zero lower bound results in significant downward shift of the yield curve. Moreover, the long-term real interest rates shift down, while the shorter-term real interest rates shift down. Consequently, the reduction in the long-term bond supply results in a clockwise rotation of the real yield curve.

5 Conclusion

Bernanke (2010) justifies quantitative easing by applying the portfolio channel. In this paper, I accomplish this “imperfect substitutability” by using two key features: limited market participation and household heterogeneity. These features eliminate the one-to-one relationship between the yields of short- and long-term bonds: despite that the short-term nominal interest rate hits the effective lower bound, the long-term nominal interest rate may not. As a result, even if the short-term nominal interest rate is zero, long-term asset purchases can still effectively decrease the nominal long-term interest rate, and if the level is higher than the decrease in the inflation rate, the real long-term interest rate may also decrease. Moreover, the substitution effect between long- and short-term bonds is preserved. The shape of yield curves is monotonic, and long-term asset purchases become less effective when the short-term interest rate hits the zero lower bound.

This model has a potential to be modified for future research. First, one could in-
introduce a liquidity premium into this model as in Williamson (2016), and the nominal long-term interest rate could be higher in the case of a zero short-term rate. Second, one could include capital in the model to analyze the transmission of quantitative easing from the Treasury bond market to the capital market and the labor market. Finally, the model in this paper features sustained asset market frictions, and this would help analyze the issues related to asset market frictions such as the on-the-run and off-the-run phenomenon. I leave these issues for future studies.

References


6 Figures

6.1 Individual Characteristics and Personal Yield Curves

Figure 8: Personal yield curves: \( \pi = 0.05, \mu = 0.75, \alpha = 0.5 \)

Figure 9: Personal yield curves: \( \pi = 0.05, \rho = 0.05, \alpha = 0.5 \)

Figure 10: Personal yield curves: \( \pi = 0.05, \mu = 0.75, \rho = 0.05 \)
6.2 Comparative Statistics with respect to Bond Supply

In the following policy experiment, I assume $\lambda_s = 1$ and $\lambda_l = 0.025$. The disutility of working is assumed to be isoelastic: $l(h) = \frac{h^{1/\gamma}}{1+\frac{1}{\gamma}}, \gamma = 1$.

6.2.1 The Representative Household Model – Non-Ricardian Regime

Figure 11: Non-Ricardian regime (fixed tax): $\rho = 0.05, \mu = 0.8, b_l = 0.34$

Figure 12: Non-Ricardian regime (fixed tax): $\rho = 0.05, \mu = 0.8, b_s = 0.04$

Figure 13: The impact of long-term bond reduction at the zero lower bound.
6.2.2 The Representative Household Model – Ricardian Regime

Figure 14: Ricardian Regime (Fixed Seigniorage): \( \rho = 0.05, \mu = 0.8, b_l = 0.34 \)

Figure 15: Ricardian Regime (Fixed seigniorage): \( \rho = 0.05, \mu = 0.8, b_s = 0.04 \)
6.2.3 The Heterogeneous Household Model – Two Types

Figure 16: Non-Ricardian Regime (Fixed tax): $\mu_1 = 0.75, \mu_2 = 0.95, \rho = 0.05, b_s = 0.14$

Figure 17: Ricardian Regime (Fixed seigniorage): $\mu_1 = 0.75, \mu_2 = 0.95, \rho = 0.05, b_s = 0.03$
6.2.4 The Heterogeneous Household Model – Multiple Types

Figure 18: Non-Ricardian Regime: $\rho \sim U(0.02,0.08), \mu \sim Beta(13.5,1.5), b_s = 0.03$

Figure 19: Ricardian Regime: $\rho \sim U(0.02,0.08), \mu \sim Beta(13.5,1.5), b_s = 0.03$

Figure 20: The impact of long-term bond reduction at the zero lower bound.
7 Proofs

7.1 Consumers’ Problem in The Goods Market

\[ U^c_t(m_t, b_t) = \max_{c_t, m_t, m_{t+1}} \left( c_t - l(h) + \frac{1}{1+\rho} \left\{ \alpha \Phi^c_{t+1}(m_{t+1}, b_{t+1}) - (1 - \alpha) \Phi^p_{t+1}(m_{t+1}, b_{t+1}) \right\} \right) \]

subject to

\[
\begin{align*}
\dot{m}_t &= m_t - c_t \\
\ddot{m}_t &= \tau + h_t + \dot{m}_t \\
\tilde{b}_{i,t} &= b_{i,t} \\
m_{t+1} &= \frac{1}{1+\pi_t} \left( \dot{m}_t + \sum_i \lambda_i \tilde{b}_{i,t} \right) \\
b_{i,t+1} &= \frac{1}{1+\pi_t} (1 - \lambda_i) \tilde{b}_{i,t} \\
\dot{m}_t &\geq 0 \\
c_t &\geq 0
\end{align*}
\]

The Lagrangian:

\[
L = c_t - l(h) + \frac{1}{1+\rho} \left\{ \alpha \Phi^c(m_{t+1}, b_{t+1}) + (1 - \alpha) \Phi^p(m_{t+1}, b_{t+1}) \right\} \\
+ \gamma_{m_t} (m_t - \dot{m}_t - c_t) + \sum_i \gamma_{b_{i,t}} (b_{i,t} - \tilde{b}_{i,t}) \\
+ \psi_{m_t} \left[ \frac{\tau + h_t + \dot{m}_t + \sum_i \lambda_i \tilde{b}_{i,t}}{(1+\pi_t)} - m_{t+1} \right] \\
+ \sum_i \psi_{b_{i,t}} \left[ \frac{(1 - \lambda_i) \tilde{b}_{i,t}}{(1+\pi_t)} - b_{t+1} \right] \\
+ \delta_t c_t + \delta_t \dot{m}_t
\]

First order conditions imply

\[
\begin{align*}
\gamma_{m_t} &= 1 + \delta_t \\
l'(h) &= \frac{1}{(1+\rho)(1+\pi_t)} \left[ \alpha \frac{\partial \Phi^c}{\partial m}(m_{t+1}, b_{t+1}) + (1 - \alpha) \frac{\partial \Phi^p}{\partial m}(m_{t+1}, b_{t+1}) \right] \\
\gamma_{m_t} &= \frac{1}{(1+\rho)(1+\pi_t)} \left[ \alpha \frac{\partial \Phi^c}{\partial m}(m_{t+1}, b_{t+1}) + (1 - \alpha) \frac{\partial \Phi^p}{\partial m}(m_{t+1}, b_{t+1}) \right] + \tilde{\delta}_t \\
\gamma_{b_{i,t}} &= \frac{1}{(1+\rho)(1+\pi_t)} \left\{ \lambda_i \left[ \alpha \frac{\partial \Phi^c}{\partial b_{i,t}}(m_{t+1}, b_{t+1}) + (1 - \alpha) \frac{\partial \Phi^p}{\partial b_{i,t}}(m_{t+1}, b_{t+1}) \right] \right\} \\
&\quad + (1 - \lambda_i) \left[ \alpha \frac{\partial \Phi^c}{\partial b_{i,t}}(m_{t+1}, b_{t+1}) + (1 - \alpha) \frac{\partial \Phi^p}{\partial b_{i,t}}(m_{t+1}, b_{t+1}) \right]
\end{align*}
\]
Kuhn-Tucker conditions:
\[
\delta_t c_t = 0, \delta_t \geq 0 \\
\bar{\delta}_t m'_t = 0, \bar{\delta}_t \geq 0
\]

The envelope theorem implies:
\[
\frac{\partial U^c_i(m_t, b_t)}{\partial m} = \gamma_{m_i} \\
\frac{\partial U^c_i(m_t, b_t)}{\partial b_i} = \gamma_{b_{i,t}}
\]

**Claim 2** \( \delta_t \cdot \bar{\delta}_t = 0 \)

**Proof.** Suppose \( \delta_t \cdot \bar{\delta}_t > 0 \), then \( \delta_t > 0 \) and \( \bar{\delta}_t > 0 \), so \( c_t = 0 \) and \( \hat{m}_t = 0 \). Then \( \hat{m}_t = m_t - c_t \) is violated. ■

\( \delta_t \cdot \bar{\delta}_t = 0 \) and \( \delta_t \geq 0, \bar{\delta}_t \geq 0 \) implies that \( \gamma_{m_t} = \max \left\{ x_{i,t} \frac{1}{1+\pi_{i,t}} \psi^m_t \right\} \), so \( \frac{\partial U^c_i(m_t, b_t)}{\partial m} = \max \left\{ x_{i,t} \frac{1}{1+\pi_{i,t}} \frac{\partial \Phi}{\partial m} (m_{t+1}, b_{t+1}) \right\} \).

### 7.2 Households' Problem in Asset Market

\[
\Omega^i(m_t, b_t) = \max_{m'_t, b'_t, \hat{m}_{i,t}, \hat{b}_{i,t}} U^i(\hat{m}_t, \hat{b}_t) \\
\text{Subject to} \begin{cases} \\
\hat{m}_t = m_t - \sum_i m^*_{i,t} + \sum_i q_i b^*_i \\
\hat{b}_{i,t} = b_{i,t} + \frac{m^*_{i,t}}{q_i} - b^*_{i,t} \\
\sum_i m^*_{i,t} \leq m_t \\
m^*_{i,t} \geq 0 \\
b^*_{i,t} \leq b_{i,t} \\
b^*_{i,t} \geq 0
\end{cases}
\]

The Lagrangian:
\[
L = U^i(\hat{m}_t, b^*_{1,t}, \ldots, b^*_{k,t}) + \gamma_{m_t} \left( m_t - \hat{m}_t - \sum_i m^*_{i,t} + \sum_i q_i b^*_{i,t} \right) \\
+ \sum_i \gamma_{b_{i,t}} \left( b_{i,t} - \hat{b}_{i,t} + \frac{m^*_{i,t}}{q_i} - b^*_{i,t} \right) + \epsilon_{m_t} \left( m_t - \sum_i m^*_{i,t} \right) \\
+ \sum_i \epsilon_{b_{i,t}} \left( b_{i,t} - b^*_{i,t} \right) + \sum_i \sigma_{m_{i,t}} m^*_{i,t} + \sum_i \sigma_{b_{i,t}} \hat{b}_{i,t}
\]
First order conditions: (Combine the envelope theorem result of households’ problem in the goods market.)

$$v^j_t - \gamma_{m_t} = 0$$
$$w^i_{t,t} - \gamma_{b_{i,t}} = 0$$

$$- \gamma_{m_t} + \frac{\gamma_{b_{i,t}}}{q_{i,t}} - \epsilon_{m_t} + \sigma_{m_{i,t}} = 0$$
$$q_{i,t} \gamma_{m_t} - \gamma_{b_{i,t}} - \epsilon_{b_{i,t}} + \sigma_{b_{i,t}} = 0$$  \hspace{1cm} (24)

Kuhn-Tucker conditions:

$$\epsilon_{m_t} (m_t - \sum_i m^i_{t,i}) = 0, \epsilon_{m_t} \geq 0; \quad \sigma_{m_{i,t}} m^i_{t,i} = 0, \sigma_{m_{i,t}} \geq 0$$

$$\sum_i \epsilon_{b_{i,t}} (b_{i,t} - b^i_{i,t}) = 0, \epsilon_{b_{i,t}} \geq 0; \quad \sigma_{b_{i,t}} b^i_{i,t} = 0, \sigma_{b_{i,t}} \geq 0$$

Envelope Theorem:

$$\frac{\partial \Omega^j_t}{\partial m_t}(m_t, b_t) = \gamma_{m_t} + \epsilon_{m_t}$$
$$\frac{\partial \Omega^j_t}{\partial b_i}(m_t, b_t) = \gamma_{b_{i,t}} + \epsilon_{b_{i,t}}$$

I denote $\hat{v}^j_t = \gamma_{m_t} + \epsilon_{m_t}$, $\hat{w}^j_i = \gamma_{b_{i,t}} + \epsilon_{b_{i,t}}$

**Claim 3**

1. $\frac{\partial \Omega^j_t}{\partial m_t}(m_t, b_t) = \hat{v}^j_t = \max \left\{ v^j_t, \frac{w^j_t}{q^i_t}, \ldots, \frac{w^j_t}{q^k_t} \right\}$

2. $\frac{\partial \Omega^j_t}{\partial b_i}(m_t, b_t) = \hat{w}^j_i = \max \left\{ q_i v^j_t, w^j_{i,t} \right\}$

I first prove the following claims.

**Claim 4** *There exist $x \in \{ \epsilon_{m_t}, \sigma_{m_{i,t}} \}$ that $x = 0$.*

**Proof.** Suppose $\epsilon_{m_t} > 0$, and $\sigma_{m_{i,t}} > 0$ for all $i$. Then $m_t - \sum_i m^i_{t,i} = 0$ and $m^i_{t,i} = 0$ for all $i$. Since $m_t > 0$, a contradiction is shown. ■

Substitute $v^j_t = \gamma_{m_t}$, and $w^j_{i,t} = \gamma_{b_{i,t}}$ into (24), we have

$$\sigma_{m_{i,t}} + \left( \frac{w^j_{i,t}}{q^i_t} - v^j_t \right) = \epsilon_{m_t}$$  \hspace{1cm} (26)
Claim 5 \( \varepsilon_{m,t} = \max \left\{ \max_i \left( \frac{w_{i,t}}{q_i} - v^j_t \right), 0 \right\} \).

Proof.

1. First of all, suppose \( \max_i \left( \frac{w_{i,t}}{q_i} - v^j_t \right) < 0 \) for all i, then from 26 \( \sigma_{m,i,t} > \varepsilon_{m,t} \) for all i. Since \( \varepsilon_{m,t} \geq 0 \) and at least one of \( \{\varepsilon_{m,t}, \sigma_{m,i,t}\} \) is equal to zero, so \( \varepsilon_{m,t} = 0 \). Moreover, in this case \( \sigma_{m,i,t} > 0 \) for all i, so \( m_{i,t}^+ = 0 \) for all i and \( m_t - \sum_i m_{i,t}^+ = m_t \).

2. Suppose there exists \( \frac{w_{i,t}}{q_i} - v^j_t \geq 0 \) for some i. Since \( \sigma_{m,i,t} > 0 \), so \( \varepsilon_{m,t} = \max_i \left( \frac{w_{i,t}}{q_i} - v^j_t \right) \). Moreover, \( \sigma_{m,i,t} = 0 \) and \( m_{i,t}^+ \geq 0 \) if and only if \( \left( \frac{w_{i,t}}{q_i} - v^j_t \right) = \varepsilon_{m,t} ; \sigma_{m,i,t} > 0 \) and \( m_{i,t}^+ = 0 \) if and only if \( \left( \frac{w_{i,t}}{q_i} - v^j_t \right) < \varepsilon_{m,t} \).

3. Combine 1 and 2, \( \varepsilon_{m,t} = \max \left\{ \max_i \left( \frac{w_{i,t}}{q_i} - v^j_t \right), 0 \right\} = \max \left\{ \left( \frac{w_{i,t}}{q_i} - v^j_t \right), 0 \right\} \).

The claims imply:

\[
\hat{v}^j_t = v^j_t + \varepsilon_{m,t} = \max \left\{ v^j_t, \frac{w^j_{1,t}}{q_1}, \ldots, \frac{w^j_{k,t}}{q_k} \right\}
\]

Symmetric argument shows that

\[
\hat{w}^j_t = \max \left\{ q_i v^j_t, w^j_{i,t} \right\}
\]

7.3 Proof of Lemma 1

Since \( v^c > v^n \), we can split the solutions into three cases:

1. For \( V < v^n < v^c \) : (Region 0)

\[
\hat{v}^c = v^c = 1
\]
\[
\hat{v}^n = v^n
\]
\[
\Rightarrow \hat{v}^n = \frac{1}{1+d} \left[ \alpha + (1-\alpha) \hat{v}^n \right]
\]
\[
\Rightarrow \hat{v}^n = v^n = \frac{\alpha}{\alpha + d}
\]
\[ \frac{\partial \hat{v}^n}{\partial V} = \frac{\partial \hat{v}^c}{\partial V} = \frac{\partial v^n}{\partial V} = \frac{\partial v^c}{\partial V} = 0 \]

Since \( v^n = \frac{\alpha}{\alpha + d} \), we require that \( V < \frac{\alpha}{\alpha + d} \).

2. For \( v^n < V < v^c \) : (Region I)

\[ \hat{v}^c = v^c = 1 \]

\[ \hat{v}^n = \mu V + (1 - \mu) v^n \]

\[ \Rightarrow \hat{v}^n = \mu V + \frac{1 - \mu}{1 + d} [\alpha + (1 - \alpha) \hat{v}^n] \]

\[ \Rightarrow \begin{cases} \hat{v}^n = \frac{\mu(1+d)V+(1-\mu)\alpha}{(1+d)-(1-\mu)(1-\alpha)} \\ v^n = \frac{(1-\mu)\mu V + \alpha}{(1+d)-(1-\mu)(1-\alpha)} \end{cases} \]

\[ \frac{\partial \hat{v}^n}{\partial V} > 0, \frac{\partial \hat{v}^c}{\partial V} > 0, \frac{\partial v^n}{\partial V} = \frac{\partial v^c}{\partial V} = 0 \]

We require that \( V < v^c = 1 \) and

\[ v^n = \frac{(1 - \alpha)\mu V + \alpha}{(1 + d) - (1 - \mu)(1 - \alpha)} < V \]

\[ \Rightarrow V > \frac{\alpha}{\alpha + d} \]

\[ \frac{\partial v^n}{\partial V} = \frac{(1 - \alpha)\mu}{\alpha + d + (1 - \alpha)\mu} < 1, \frac{\partial v^c}{\partial V} = 0 \]

\[ \Rightarrow \frac{\partial (V - v^n)}{\partial V} > 0 \]

3. For \( V > v^c > v^n \) : (Region II)

\[ \hat{v}^c = \mu V + (1 - \mu) \]

\[ \hat{v}^n = \mu V + (1 - \mu) v^n \]

\[ v^n = \frac{1}{1 + d} [\alpha \hat{v}^c + (1 - \alpha) \hat{v}^n] \]
\[
\hat{v}^n(V) = \mu \left[ (1 + d) + (1 - \mu)\alpha \right] V + \alpha(1 - \mu)^2 \frac{\nu^n(V)}{(1 + d) - (1 - \mu)(1 - \alpha)}
\]
\[
\nu^n(V) = \mu V + (1 - \mu) \frac{\nu^n(V)}{(1 + d) - (1 - \mu)(1 - \alpha)}
\]
\[
\hat{v}^c(V) = \mu V + (1 - \mu) \frac{\nu^n(V)}{(1 + d) - (1 - \mu)(1 - \alpha)}
\]
\[
v^c(V) = 1
\]
\[
\frac{\partial \hat{v}^n}{\partial V} > 0, \frac{\partial \hat{v}^c}{\partial V} > 0, \frac{\partial \nu^c}{\partial V} > 0, \frac{\partial \nu^n}{\partial V} = 0
\]
\[
\frac{\partial \nu^n}{\partial V} = \frac{\mu}{\alpha(1 - \mu) + d + \mu} < 1, \frac{\partial \nu^c}{\partial V} = 0
\]
\[
\Rightarrow \frac{\partial (V - \nu^n)}{\partial V} > 0, \frac{\partial (V - \nu^c)}{\partial V} > 0
\]

I have to check \( v^n = \frac{1}{1+\mu} [\alpha \hat{v}^c + (1 - \alpha) \hat{v}^n] < 1 \), or the Transversality conditions will be not violated. Therefore, we require that \( V < \frac{d+\mu}{\mu} \).

### 7.4 Proof of Lemma 2

1. In region I,
\[
q_i = \frac{\lambda_i}{(1+d)-(1-\mu)(1-\alpha)} \left[ (1 - \alpha)\mu(1 + d)V + \alpha(1 + d) \right] \frac{1}{(1 + d) - (1 - \lambda_i)(1 - \alpha\mu)} V - (1 - \lambda_i)\alpha\mu
\]
\[
\Rightarrow \frac{dq_i}{dV} < 0
\]

2. In region II,
\[
q_i = \frac{\lambda_i(1+d)}{(1+d)-(1-\mu)(1-\alpha)} \left[ \mu V + (1 - \mu)\alpha \right] \frac{1}{(\lambda_i + d)V}
\]
\[
\Rightarrow \frac{\partial q_i}{\partial V} < 0
\]
7.5 Proof of Lemma 3

1. Region I:
\[ \dot{\phi}^c = \nu^c; \quad \dot{\phi}^n = \mu(V - \nu^n) + \nu^n; \]
\[ \dot{\omega}^c = \mu(\nu^c - V); \quad \dot{\omega}^n = V. \]
\[ V = \frac{1}{1 + d} \left\{ \frac{\lambda}{q} \{ \alpha \nu^c + (1 - \alpha) [\mu(V - \nu^n) + \nu^n] \} \right\} \]
\[ R = \frac{\lambda}{q} - \lambda = \frac{\lambda}{q} \{ \alpha \nu^c + (1 - \alpha) [\mu(V - \nu^n) + \nu^n] \} \frac{\lambda}{q} - \lambda \]
\[ = \frac{(1 + d)V - \alpha \nu^c + (1 - \alpha) [\mu(V - \nu^n) + \nu^n]}{(1 + d)V - \alpha (V - \nu^n) - (1 - \alpha)(1 - \mu)(V - \nu^n)} \]
\[ \frac{\partial R}{\partial \lambda} = (1 - \mu) \frac{\lambda}{q} \{ \alpha \nu^c + (1 - \alpha) [\mu(V - \nu^n) + \nu^n] \}\frac{\lambda}{q} - \lambda \]
\[ + \left\{ (1 - \lambda) \{ \alpha \nu^c + (1 - \alpha) [\mu(V - \nu^n) + \nu^n] \} \right\} \]
2. Region II: \( \nu^c \geq V \geq \nu^n \)
\[ \dot{\phi}^c = \mu(V - \nu^c) + \nu^c; \quad \dot{\phi}^n = \mu(V - \nu^n) + \nu^n; \]
\[ \dot{\omega}^c = V; \quad \dot{\omega}^n = V. \]
\[ V = \frac{1}{1 + d} \left\{ \frac{\lambda}{q} \{ \alpha \nu^c + (1 - \alpha) [\mu(V - \nu^n) + \nu^n] \} \right\} \]
\[ R = \frac{\lambda}{q} - \lambda = \frac{(1 + d)V - \alpha \nu^c + (1 - \alpha) [\mu(V - \nu^n) + \nu^n]}{(1 + d)V - \alpha (V - \nu^n) - (1 - \alpha)(1 - \mu)(V - \nu^n)} \]
\[ \frac{\partial R}{\partial \lambda} = (1 - \mu) \frac{\lambda}{q} \{ \alpha \nu^c + (1 - \alpha) [\mu(V - \nu^n) + \nu^n] \}\frac{\lambda}{q} - \lambda \]
\[ + \left\{ (1 - \lambda) \{ \alpha \nu^c + (1 - \alpha) [\mu(V - \nu^n) + \nu^n] \} \right\} \]

7.6 Proof of Proposition 1

In region I,
\[ q(\lambda; V) = \frac{\lambda [\alpha + (1 - \alpha) \dot{\phi}^n(V)]}{[(1 + d) - \alpha \mu(V - \nu^n)] V} \]
Where

\[ \hat{v}^n(V) = \frac{\mu(1 + d)V + (1 - \mu)\alpha}{[(1 + d) - (1 - \mu)(1 - \alpha)]} \]

So

\[ R(\lambda; V) = \frac{((1 + d) - (1 - \lambda)(1 - \alpha \mu))V - (1 - \lambda_i)\alpha \mu}{[\alpha + (1 - \alpha)\hat{v}^n(V)]} - \lambda \]

In region II,

\[ q = \frac{\lambda(1 + d)}{(1 + d) - (1 - \mu)(1 - \alpha)} \frac{[\mu V + (1 - \mu)\alpha]}{(\lambda + d)V} \]

\[ R = \frac{(1 + d)}{(1 + d) - (1 - \mu)(1 - \alpha)} \frac{[\mu V + (1 - \mu)\alpha]}{(\lambda + d)V} - \lambda \]

1. For \( \mu = 1 \):
   
   (a) In region I:

   \[ q(\lambda; V) = \frac{\lambda}{[(1 + d) - (1 - \lambda)(1 - \alpha)]} \frac{[\mu V + (1 - \mu)\alpha]}{(\lambda + d)V} \]

   \[ R(\lambda; V) = \frac{-\alpha + (\alpha + d)\hat{v}^n(V)}{\alpha + (1 - \alpha)\hat{v}^n(V)} \]

   \[ \frac{\partial R}{\partial \lambda} (\lambda; V) = 0 \]

   (b) In region II:

   \[ q(\lambda; V) = \frac{\lambda}{(\lambda + d)} \]

   \[ R(\lambda; V) = d \]

   \[ \frac{\partial R}{\partial \lambda} (\lambda; V) = 0 \]

2. For \( \mu < 1 \):

   (a) In Region I:

   \[ R(\lambda; V) = \frac{[(1 + d) - (1 - \lambda)(1 - \alpha \mu)]\frac{\alpha}{\alpha + d} - (1 - \lambda)\alpha \mu}{\left[\frac{\alpha}{\alpha + d} + (1 - \alpha)\frac{\alpha}{\alpha + d}\right]} - \lambda \]

   \[ \frac{\partial R}{\partial \lambda} (\lambda; V) = \frac{(1 - \alpha \mu)\frac{\alpha}{\alpha + d} + \alpha \mu}{\left[\frac{\alpha}{\alpha + d} + (1 - \alpha)\frac{\alpha}{\alpha + d}\right]} - 1 < 0 \]
\[
R(\lambda_i; V) = \frac{[(1 + d) - (1 - \lambda)(1 - \alpha\mu)] - (1 - \lambda_i)\alpha\mu}{[\alpha + (1 - \alpha)\frac{\mu(1 + d) + (1 - \mu)\alpha}{(1 + d) - (1 - \mu)(1 - \alpha)}]} - \lambda
\]

\[
\frac{\partial R}{\partial \lambda}(\lambda; V) = \frac{1}{[\alpha + (1 - \alpha)\frac{\mu(1 + d) + (1 - \mu)\alpha}{(1 + d) - (1 - \mu)(1 - \alpha)}]} - 1 > 0
\]

\[
R = \frac{[(1 + d) - (1 - \lambda_i)(1 - \alpha\mu)] V - (1 - \lambda_i)\alpha\mu}{(1 - \alpha)\mu(1 + d) + (1 + d)\alpha(1 + d)} - \lambda
\]

\[
\frac{\partial R}{\partial \lambda} = \frac{(1 - \alpha\mu) V + \alpha\mu}{(1 + d) - (1 - \mu)(1 - \alpha)} - 1
\]

\[
\frac{\partial^2 R}{\partial \lambda \partial V} > 0
\]

(b) In Region II:

\[
q = \frac{\lambda_i(1 + d)}{(1 + d) - (1 - \mu)(1 - \alpha)} [\mu V + (1 - \mu)\alpha]
\]

\[
R = \frac{(1 + d)(1 - \lambda)? V}{(1 + d) - (1 - \mu)(1 - \alpha)} [\mu V + (1 - \mu)\alpha] - \lambda
\]

\[
\frac{\partial R}{\partial \lambda} = \frac{V}{(1 + d) - (1 - \mu)(1 - \alpha)} [\mu V + (1 - \mu)\alpha] - 1
\]

\[
\frac{\partial^2 R}{\partial \lambda \partial V} > 0
\]