Abstract

This paper exploits a remarkable property of the Hodrick-Prescott filter: the cyclical component of the Hodrick-Prescott filter is equal to the trend component of the Hodrick-Prescott filter when applied to the fourth differences, plus an additional term that is typically small. While it is not hard to show that such a result should approximately hold, we provide an exact version. We first apply our result to show that the cyclical component of the Hodrick-Prescott filter when applied to series that are integrated up to order 3 or more is not weakly dependent. Second, we characterize the behavior of the Hodrick-Prescott filter when applied to deterministic polynomial trends and show that effectively the cyclical component of the filter reduces the order of
the polynomial by 4. Finally, we give a characterization of the Hodrick-Prescott filter when applied to an exponential deterministic trend, and this characterization shows that the filter is effectively incapable of dealing with trend that increase this fast.

1 Introduction

The Hodrick-Prescott (HP) filter is long-standing standard technique in macroeconomics for separating the long run trend in a data series from short run fluctuations. Introduced initially by Whittaker (1923) and popularized in Econometrics by Hodrick and Prescott (1980, 1997), the HP filter is universally used in macroeconomics. The two cited papers by Hodrick and Prescott have thousands of citations; yet, the impact of this work may go beyond that, since the use of the HP filter is universal, and the HP filter is a commonplace tool in macroeconomics.

The HP filter calculates the trend of a series $y_t$, $t = 1, \ldots, T$ by minimizing

$$
\sum_{t=1}^{T} (y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} (\tau_{t+1} - 2\tau_t + \tau_{t-1})^2,
$$

over $\tau = (\tau_1, \ldots, \tau_T)$. The parameter $\lambda$ here is a smoothing parameter that for quarterly data is typically chosen to equal 1600. The minimizer, which we will label $\hat{\tau}_T$, is referred to in the literature as the “trend component,” while $\hat{c}_T = y_t - \hat{\tau}_T$ is referred to as the “cyclical component.” By writing the minimization problem as a vector differentiation problem, it follows that there exists a unique minimizer. The trend component $\hat{\tau}_T$ and the cyclical component $\hat{c}_T$ are both weighted averages of the $y_t$, and in de Jong and Sakarya (2016), an
explicit formula for the weights is found.

Letting $\bar{B}$ and $B$ denote the forward and the backward operators, respectively, the first order conditions of the problem above can be written as

$$(\lambda \bar{B}^2 - 4\lambda \bar{B} + (1 + 6\lambda) - 4\lambda B + \lambda B^2) \hat{\tau}_t = y_t.$$  \tag{2}$$

for $t = 3, 4, \ldots, T - 2$. The expression above can be written as

$$y_t = (\lambda |1 - B|^4 + 1) \hat{\tau}_t.$$  \tag{3}$$

Analyses of the HP filter based on the above first order condition are for example King and Rebelo (1993), Cogley and Nason (1995), Phillips and Jin (2002), Phillips (2010), and McElroy (2008). However, the above first order condition fails to hold for $t = 1, 2$ and $t = T - 1, T$, since the first order conditions for those are

$$(1 + \lambda) - 2\lambda \bar{B} + \lambda \bar{B}^2) \hat{\tau}_1 = y_1,$$  \tag{4}$$

$$(-2\lambda B + (1 + 5\lambda) - 4\lambda \bar{B} + \lambda B^2) \hat{\tau}_2 = y_2,$$  \tag{5}$$

$$(-2\lambda \bar{B} + (1 + 5\lambda) - 4\lambda B + \lambda B^2) \hat{\tau}_{T, T-1} = y_{T-1},$$  \tag{6}$$

$$((1 + \lambda) - 2\lambda B + \lambda B^2) \hat{\tau}_{T, T} = y_T.$$  \tag{7}$$

Therefore, we can interpret the results of aforementioned papers as the results derived from an approximate problem that will likely be valid for values of $t$ away from the begin and end points of the sample. It is obvious that those findings cannot render exact results for the HP filter. Another way to think of this literature is that it gives an informal large $T$
approximation to the HP filter without a formal large $T$ justification. This paper will seek to derive an exact result for the HP filter as the work of (de Jong and Sakarya [2016]). However, the approach taken in this paper towards the analysis of the HP filter is completely different.

The idea behind our result is the following. Using the first order condition

$$y_t = (\lambda|1 - B|^4 + 1)\hat{r}_T(t)$$

and the identity $y_t = \hat{r}_T + \hat{c}_T$, it follows that for $t = 3, \ldots, T - 2$,

$$\dot{c}_T(y_1, \ldots, y_T) = \lambda|1 - B|^4 \hat{r}_T(y_1, \ldots, y_T)$$

$$= \lambda(1 - B)^2(1 - \bar{B})^2\bar{B}^2\hat{r}_T(y_1, \ldots, y_T)$$

$$= \lambda(1 - B)^4\bar{B}^2\hat{r}_T(y_1, \ldots, y_T).$$

Ignoring the fact that $y_{-1}, y_0, y_{T+1}$ and $y_{T+2}$ are undefined, we can now conjecture that in some sense to be made precise, the last expression is approximately equal to

$$\lambda(1 - B)^4\hat{r}_T(y_3, \ldots, y_{T+2})$$

which can be conjectured to approximately equal

$$\lambda\hat{r}_T(\Delta^4y_3, \ldots, \Delta^4y_{T+2}).$$
Therefore, the conjecture presents itself that the cyclical component in a series \( y_t \) is approximately equal to the trend in the fourth difference. In Section 2 of this paper, we will formalize this conjecture. Section 3 explores the consequences of the conjecture; we will show that for series that are integrated up to order 3 or more, the cyclical component does not possess weak dependence properties. In Section 4, we derive the behavior of the HP filter when applied to a polynomial trend, and Section 5 considers the HP filter when applied to an exponential trend.

2 Main result

2.1 A property of the HP filter

In the introduction, an approximate and intuitive explanation is described for the main result. In this section, the result is formally derived for finite \( T \) by respecting the first order conditions for the begin and end points. The main result of the paper is presented in the following theorem.

**Theorem 1.** Let \( \tilde{y}_{T1} = \Delta^2 y_3 \), \( \tilde{y}_{T2} = \Delta^2 y_4 - 2\Delta^2 y_3 \), \( \tilde{y}_{T;T-1} = \Delta^2 y_{T-1} - 2\Delta^2 y_T \), \( \tilde{y}_{T;T} = \Delta^2 y_T \), and for \( t = 3, 4, \ldots, T-2 \), \( \tilde{y}_{Tt} = \Delta^4 y_{t+2} \) for \( t = 3, 4, \ldots, T-2 \). Then

\[
\hat{c}_{Tt}(y_1, y_2, \ldots, y_T) = \lambda \hat{\tau}_{Tt}(\tilde{y}_{T1}, \tilde{y}_{T2}, \ldots, \tilde{y}_{TT}).
\]  

This simple but elegant result provides insights into the structure of the cyclical term and allows us to obtain several results for the HP filter. To the best of our knowledge, this
property of the cyclical term has not been established before. The cyclical component itself is the trend of the fourth difference of the original series, plus an additional term that is typically small. This property sheds light on the behavior of the cyclical component that is obtained from various data generating processes. To be precise, it is now easier to characterize the cyclical component of a polynomial trend or an integrated process. Furthermore, this result shows that the cyclical component contains a trend term, and the filter is incapable of filtering trend of a series if the series increases faster than a certain rate. In order to discover, the implications of Theorem 1, we consider the weak dependence properties of the cyclical component of integrated processes, and characterize the cyclical term of a polynomial trend in the next sections. The cyclical component of an exponential trend is also derived to exemplify the fact that the HP filter is incapable of removing trends of some processes which increase at a fast rate.

3 The HP filter when applied to processes integrated up to order 4

First, we consider the weak dependence properties of the cyclical term which is obtained from processes \( y_t \) that are integrated up to order 4; that is, \( \Delta^q y_t = u_t \), for \( q = 1, 2, 3, \) or 4, where we assume that \( u_t \) has some stationarity or weak dependence properties. We will show that for such processes, the cyclical component has weak dependence properties. To
formulate our result, for \( m \geq 1 \) define the approximator \( \hat{c}_T^m \) as
\[
\hat{c}_T^m = \lambda \sum_{s=3}^{T-2} w_{Ts} \tilde{y}_{Ts} I(|t-s| \leq m).
\] (12)

Note that \( \tilde{y}_t = \Delta^4 y_{t+2} \) for \( s = 3, \ldots, T - 2 \). Since \( \Delta^4 y_{t+2} = \Delta^{4-q} u_{t+2} \) for \( q = 1, 2, 3, 4 \), this approximator depends only on \( u_{t-2-m+q}, \ldots, u_{t+2+m} \). Such approximators are common in the literature on concepts of “fading memory” of time series that ensures that laws of large numbers hold; see for example ...

The next result shows that the cyclical component has an approximability property when the HP filter is applied to a process that is integrated of order 4 or less.

**Theorem 2.** Assume that \( \Delta^q y_t = u_t \), for \( q = 1, 2, 3, \) or 4, where \( \sup_{s \geq 1} \| u_s \|_p < \infty \). Then for any \( \gamma \in (0, 1/2) \), there exists constants \( C_1, C_2, C_3 > 0 \) such that for any \( m \geq 1 \),
\[
\sup_{T \geq 1} \sup_{t \in [\gamma T, (1-\gamma)T]} \| \hat{c}_T^m - w_{T+1 T+1} - w_{T+2 T+2} - w_{T+3 T+3} - w_{T+4 T+4} \|_p \leq C_1 m^{-2} \] (13)

and
\[
\sup_{T \geq 1} \sup_{t \in [\gamma T, (1-\gamma)T]} \| \hat{c}_T - w_{T+1 T+1} - w_{T+2 T+2} - w_{T+3 T+3} - w_{T+4 T+4} \|_p = O(1). \] (14)

The assumption that \( y_t \) is integrated up to order 4 at most implies that, under standard moment assumptions, \( \| \tilde{y}_{T-1,T} \|_p + \| \tilde{y}_{T,T} \|_p = O(T^{3/2}) \). Therefore, under such assumptions,
\[ w_{T11} \tilde{y}_{T1} + w_{T12} \tilde{y}_{T2} + w_{T1,T-1} \tilde{y}_{T,T-1} + w_{T1T} \tilde{y}_{TT} \]  

(15)

will typically be of a small order. On the other hand, it should be noted that \( \hat{c}_{Tt} \) does not approximate to \( \hat{c}_{Tt}^{pi} \) even in the middle of a large sample.

King and Rebelo (1993) has conjectured, based on considering the first order conditions only, that the cyclical component has weak dependence properties for processes integrated up to order 4. The result above is, to the best of our knowledge, the first formal proof of this.

We can now prove the following weak law of large numbers for bounded and continuous functions of the cyclical component:

**Theorem 3.** Assume that \( y_t \) satisfies \( \Delta^q y_t = u_t \) for \( q = 1, 2, 3, \) or 4, and assume that \( u_t \) is strong mixing. In addition, assume that \( |\tilde{y}_{1T}| + |\tilde{y}_{2T}| + |\tilde{y}_{T-1,T}| + |\tilde{y}_{T,T}| = O_p(T^{3/2}) \). Let \( f(\cdot) \) be a function that is bounded and continuous on \( \mathbb{R} \). Then

\[
T^{-1} \sum_{t=1}^{T} (f(\hat{c}_{Tt}) - Ef(\hat{c}_{Tt})) \xrightarrow{p} 0.
\]

(16)

Note that in the above result, the term of Equation (15) play no role asymptotically.
4 The HP filter when applied to deterministic trends

4.1 Deterministic polynomial trends

Theorem 1 also allows us to establish the behavior of the HP filter when applied to deterministic polynomial trends. From Theorem 1, a result for the case of a linear trend \( y_t = a + bt \) immediately follows. After all, for that case, \( \Delta^2 y_t = 0 \), implying that \( \tilde{y}_t = 0 \) for \( t = 1, \ldots, T \), which by Theorem 1 implies that \( \hat{c}_T = 0 \). For higher order polynomials, the result is more complex:

**Theorem 4.** Suppose that \( y_t = t^p \) for \( t = 1, 2, \ldots, T \) and \( p \in \mathbb{N} \). Then,

\[
\hat{c}_T(1,2^p,\ldots,T^p) = t^p - \lambda I(p \geq 4) \sum_{k=0}^{p-4} a_{pk} \hat{c}_T(1,2^k,\ldots,T^k) - \lambda C_{Tp} I(p \geq 2) + \lambda H_{Tp} I(p \geq 4),
\]

(17)

where

\[
a_{pk} = \begin{cases} 
\binom{p}{k} (2p-k+1) - 8 & \text{if } p - k \text{ is even} \\
0 & \text{if } p - k \text{ is odd}.
\end{cases}
\]

(18)

\[
C_{Tp} = \sum_{k=0}^{p-2} c_{pk} \hat{c}_T(1,2^k - 2,0,\ldots,0,(T-3)^k - 2(T-2)^k,(T-2)^k)
\]

(19)
\[ H_{Tp} = \sum_{k=0}^{p-4} a_{pk} \hat{\tau}_{Tt} \left( 1, 2^k, 0, \ldots, 0, (T - 1)^k, T^k \right), \]  

(20)

\[ c_{pk} = \binom{p}{k} (2^{p-k} - 2), \]  

(21)

The result above shows that the cyclical component of a polynomial trend of order \( p \) is

\[ \hat{c}_{Tt}(1, 2^p, \ldots, T^p) = \lambda I(p \geq 4) \sum_{k=0}^{p-4} a_{pk} \hat{\tau}_{Tt}(1, 2^k, \ldots, T^k) + \lambda C_{Tp} I(p \geq 2) - \lambda H_{Tp} I(p \geq 4). \]

For \( p = 2 \) or 3, the cyclical term takes the value of zero in the middle of the sample when the sample size is large. On the other hand, for \( p \geq 4 \), the cyclical term contains a constant, plus some additional terms. This suggests that the HP filter is not capable of filtering out a polynomial trend of order 4 or higher.

Next, we provide an alternative representation for the quadratic trend which highlights the fact that the cyclical term takes values close to zero in the middle of the sample when the sample size is large.

**Theorem 5.** Suppose that \( y_t = t^2 \) for \( t = 1, 2, \ldots, T \). Then, \( \hat{c}_{Tt} = C_1 \left( r^t + r^{T-t} \right) + \)
\( \tilde{C}_1(\tilde{r}^t + \tilde{r}^{T-t}) \) with

\[
\begin{align*}
    r &= \frac{(2i - \sqrt{q}) + \sqrt{q - 4i\sqrt{q}}}{2i} \\
    C_1 &= 2\lambda (\bar{b} + \bar{a}) / (\bar{a} \bar{b} - \bar{a} b),
\end{align*}
\]

where \( q = 1/\lambda, \ |r| < 1 \) for any value of \( q \) and

\[
\begin{align*}
    a &= (1 + \lambda)(r + r^T) - 2\lambda(r^2 + r^{T-1}) + \lambda(r^3 + r^{T-2}) \\
    b &= -2\lambda(r + r^T) + (1 + 5\lambda)(r^2 + r^{T-1}) - 4\lambda(r^3 + r^{T-2}) + \lambda(r^4 + r^{T-3}).
\end{align*}
\]

4.2 Deterministic exponential trends

In this section, we consider exponential deterministic trends in order to exemplify that the HP filter fails to remove the trend of a series when the series increases at a rate that is faster than a certain rate. The following result characterizes an exponential deterministic trend by using the result of Theorem [1]

**Theorem 6.** Let \( y_t = \exp(t) \) for \( t = 1, 2, \ldots, T \). Then

\[
\begin{align*}
    \hat{c}_T (\exp(1), \exp(2), \ldots, \exp(T)) &= C_4\lambda \hat{c}_{T2}(\exp(1), \exp(2), \ldots, \exp(T)) \\
    &= C_4\lambda \hat{c}_{T2} (\exp(1), \exp(2), \ldots, \exp(T)) \\
    &+ C_2\lambda \hat{c}_{T2} ((2 - \exp(-1)), -1, 0, \ldots, 0, -\exp(T - 1), \exp(T - 1)(2 - \exp(1))),
\end{align*}
\]

where \( C_2 = \exp(2)(1 - \exp(-1))^2 \) and \( C_4 = \exp(2)(1 - \exp(-1))^4 \).
The result above suggests that the cyclical component of an exponential deterministic trend contains its HP filter trend multiplied by a constant, plus an additional term which is conjectured to be relatively small, since the rate of convergence for $\hat{\tau}_{Tt}(\exp(1), \exp(2), \ldots, \exp(T))$ is faster than the term in equation (26). Briefly, this result illustrates the incapability of the HP filter in removing the trend of a series which increases at a fast rate.

**Theorem 7.** Let $y_t = \exp(t)$ for $t = 1, 2, \ldots, T$. Then for $k \geq 0$

$$
\lim_{T \to \infty} \frac{\hat{c}_{T,T-k}(\exp(1), \exp(2), \ldots, \exp(T))}{\hat{\tau}_{T,T-k}(\exp(1), \exp(2), \ldots, \exp(T))} = C_4 \lambda - C_2 \lambda \exp(-1) \frac{f_\lambda(k-1) + f_\lambda(k+2) + \xi_\lambda g_\lambda(k+1)g_\lambda(2)}{\sum_{j=0}^{\infty}(f_\lambda(k-j) + f_\lambda(k+j+1) + \xi_\lambda g_\lambda(k+1)g_\lambda(j+1)) \exp(-j)} \\
+ C_2 \lambda(2 \exp(-1) - 1) \frac{f_\lambda(k) + f_\lambda(k+1) + \xi_\lambda g_\lambda(k+1)g_\lambda(1)}{\sum_{j=0}^{\infty}(f_\lambda(k-j) + f_\lambda(k+j+1) + \xi_\lambda g_\lambda(k+1)g_\lambda(j+1)) \exp(-j)}
$$

**References**


Appendix 1: Mathematical proofs

Proof of Theorem 1. First, we rewrite the minimization problem of Equation (1) as a minimization problem over $c_t = y_t - \tau_t$. We then obtain

$$
\sum_{t=1}^{T} c_t^2 + \lambda \sum_{t=2}^{T-1} (c_{t+1} - 2c_t + c_{t-1})^2 - 2\lambda \sum_{t=2}^{T-1} (y_{t+1} - 2y_t + y_{t-1})(c_{t+1} - 2c_t + c_{t-1}) \\
+ \lambda \sum_{t=2}^{T-1} (y_{t+1} - 2y_t + y_{t-1})^2.
$$

(27)
The last term of the expression above is irrelevant to the minimization problem. Applying summation by parts twice gives (Rudin (1976))

\[
\sum_{t=2}^{T-1} (y_{t+1} - 2y_t + y_{t-1})(c_{t+1} - 2c_t + c_{t-1}) = \sum_{t=2}^{T-1} (\Delta^2 y_t - \Delta^2 y_{t+1}) \Delta c_t + \Delta^2 y_T \Delta^2 c_T - \Delta^2 y_3 \Delta c_2
\]

\[
= \sum_{t=3}^{T-2} \Delta^4 y_{t+2} c_t + \Delta^2 y_3 c_1 + (\Delta^2 y_4 - 2\Delta^2 y_3) c_2 + (\Delta^2 y_{T-1} - 2\Delta^2 y_T) c_{T-1} + \Delta^2 y_T c_T,
\]

and therefore, it suffices to minimize

\[
\sum_{t=1}^{T} c_t^2 + \lambda \sum_{t=2}^{T-1} (c_{t+1} - 2c_t + c_{t-1})^2 - 2\lambda \sum_{t=3}^{T-2} \Delta^4 y_{t+2} c_t
\]
\[-2\lambda \Delta^2 y_3 c_1 - 2\lambda (\Delta^2 y_4 - 2\Delta^2 y_3) c_2 - 2\lambda (\Delta^2 y_{T-1} - 2\Delta^2 y_T) c_{T-1} - 2\lambda \Delta^2 y_T c_T
\]

over \((c_1, \ldots, c_T)\). The first order conditions for \(c_t\) for \(t = 1, 2, T-1, T\) are

\[
((1 + \lambda) - 2\lambda \bar{B} + \lambda \bar{B}^2) \hat{c}_{T1} = \lambda \Delta^2 y_3 = \lambda \bar{y}_{T1}, \tag{28}
\]

\[
(-2\lambda \bar{B} + (1 + 5\lambda) - 4\lambda \bar{B} + \lambda \bar{B}^2) \hat{c}_{T2} = \lambda (\Delta^2 y_4 - 2\Delta^2 y_3) = \lambda \bar{y}_{T2}, \tag{29}
\]

\[
(-2\lambda \bar{B} + (1 + 5\lambda) - 4\lambda \bar{B} + \lambda \bar{B}^2) \hat{c}_{T,T-1} = \lambda (\Delta^2 y_{T-1} - 2\Delta^2 y_T) = \lambda \bar{y}_{T,T-1}, \tag{30}
\]

\[
((1 + \lambda) - 2\lambda \bar{B} + \lambda \bar{B}^2) \hat{c}_{TT} = \lambda \Delta^2 y_T = \lambda \bar{y}_{TT}. \tag{31}
\]
respectively. Also, the first order condition for \( c_t \) for \( t = 3, \ldots, T - 2 \) is

\[
(\lambda \bar{B}^2 - 4\lambda \bar{B} + (1 + 6\lambda) - 4\lambda B + \lambda B^2) \hat{c}_T = \lambda \Delta_t^4 y_{t+2} = \lambda \tilde{y}_T.
\] (32)

The analogy between the first order conditions of Equations (28)-(32) and the first conditions of Equations (4)-(2) now reveals that \( \hat{c}_T(y_1, y_2, \ldots, y_T) = \lambda \hat{\tau}_T(y_1, y_2, \ldots, y_T) = \lambda \tilde{\tau}_T(y_1, y_2, \ldots, y_T). \)

**Proof of Theorem 2.** By Theorem 1,

\[
\sup_{t \in [\gamma T, (1 - \gamma)T]} \| \hat{c}_T(y_1, y_2, \ldots, y_T) - w_{T1} \tilde{y}_1 - w_{T2} \tilde{y}_2 - w_{T1,T-1} \tilde{y}_{T,T-1} - w_{T1T} \tilde{y}_{TT} - \hat{c}_T^m \|_p = \sup_{t \in [\gamma T, (1 - \gamma)T]} \| \lambda \hat{\tau}_T(y_1, y_2, \ldots, y_T) - w_{T1} \tilde{y}_1 - w_{T2} \tilde{y}_2 - w_{T1,T-1} \tilde{y}_{T,T-1} - w_{T1T} \tilde{y}_{TT} - \hat{c}_T^m \|_p
\]

\[
\leq \lambda \sup_{t \in [\gamma T, (1 - \gamma)T]} \sum_{s=3}^{T-2} \| w_{T1s} \tilde{y}_{Ts} \|_p I(|t - s| \leq m)
\leq \lambda \sup_{t \in [\gamma T, (1 - \gamma)T]} \sum_{s=3}^{T-2} \| w_{T1s} \|_p I(|t - s| > m).
\]

The above expression equal 0 for \( m \geq T - 1 \). From the discussion in de Jong and Sakarya (2016) following their Theorem 1, it follows that \( w_{T1s} \) can be split into eight parts, as

\[
w_{T1s} = f_T \lambda(t - s) + \sum_{j=2}^{8} w_{T1,t,s}^j \] (33)
where \(|f_T\lambda(0)| \leq 1; for \(m \in \{1, 2, \ldots, T\}\),

\[
|f_T\lambda(m)| \leq Cm^{-3}
\]  \hfill (34)

for some constant \(C > 0\) independent of \(T\); and

\[
\sup_{T \geq 1} \sup_{1 \leq s \leq T} \sup_{\tau \in [\gamma T, (1-\gamma)T]} |T^3 \sum_{j=2}^{8} w_{T,\tau,s}^j| < \infty. \tag{35}
\]

For \(m \geq 1\) and \(T \geq m\), noting that \(\sup_{3 \leq s \leq T-2} \| \bar{y}_{Ts} \|_p \leq 16 \sup_{s \geq 1} \| u_s \|_p\) if \(\Delta^p y_t = u_t\) for \(p = 1, 2, 3,\) or \(4\),

\[
\sup_{t \in [\gamma T, (1-\gamma)T]} \sum_{s=3}^{T-2} |w_{Tts}| \| \bar{y}_{Ts} \|_p I(|t-s| > m)
\]

\[
\leq 16 \sup_{t \in [\gamma T, (1-\gamma)T]} \sum_{s=3}^{T-2} |f_T\lambda(t-s)|I(|t-s| > m) \sup_{s \geq 1} \| u_s \|_p
\]

\[
+ 16 \sup_{t \in [\gamma T, (1-\gamma)T]} \sum_{s=3}^{T-2} \sum_{j=2}^{8} |w_{Tts}^j|I(|t-s| > m) \sup_{s \geq 1} \| u_s \|_p
\]

\[
\leq 32 \sum_{j=m}^{T-2} |f_T\lambda(j)| \sup_{s \geq 1} \| u_s \|_p + 16 \sum_{s \geq 1} \sup_{t \in [\gamma T, (1-\gamma)T]} \sum_{j=2}^{8} |w_{Tts}^j| \sup_{s \geq 1} \| u_s \|_p
\]

\[
\leq 32C \sup_{s \geq 1} \| u_s \|_p \sum_{j=m}^{\infty} j^{-3} + 16T^{-2} \sup_{s \geq 1} \| u_s \|_p \sup_{T \geq 1} \sup_{s \geq 1} \sup_{1 \leq \tau \leq T} |T^3 \sum_{j=2}^{8} w_{T\tau,s}^j| \sup_{T \geq 1} \| u_s \|_p
\]

\[
= O(m^{-2}) \tag{36}
\]
by the results of Equation (34) and Equation (35). This shows the first assertion of the theorem. To show the second assertion, note that similarly to the previous argument,

\[
\sup_{T \geq 1} \sup_{t \in [\gamma T, (1-\gamma)T]} \| \hat{c}_{Tt} - w_{T1}\tilde{y}_{T1} - w_{T2}\tilde{y}_{T2} - w_{T1,T-1}\tilde{y}_{T,T-1} - w_{TT}\tilde{y}_{TT} \|_p \\
\leq \lambda \sup_{T \geq 1} \sup_{t \in [\gamma T, (1-\gamma)T]} \| \sum_{s=3}^{T-2} w_{Ts}\tilde{y}_{Ts} \|_p \\
\leq 32\lambda \sum_{j=0}^{T-2} |f_{T\lambda}(j)| \sup_{s \geq 1} \| u_s \|_p + 16\lambda \sum_{s=3}^{T-2} \sup_{t \in [\gamma T, (1-\gamma)T]} \sum_{j=2}^{8} w_{Ts}^j \sup_{s \geq 1} \| u_s \|_p = O(1)
\]

by a reasoning similar to that of the proof of the first assertion.

Proof of Theorem

This result now follows analogously to the proof of Theorem 6 of de Jong and Sakarya (2016).

Lemma 1. \( \Delta^2 (t+2)^p = \sum_{k=0}^{p-2} c_{p,k} t^k \) for \( t = 1, 2, \ldots, T \) and \( p \geq 2 \).

Proof of Lemma

The Binomial Theorem gives the following equality

\[
(t + a)^p = \sum_{k=0}^{p} \binom{p}{k} t^k a^{p-k}.
\]
By the Binomial Theorem,

\[
\Delta^2(t + 2)^p = (t + 2)^p - 2(t + 1)^p + t^p \\
= \sum_{k=0}^{p} \binom{p}{k} t^k 2^{p-k} - 2 \sum_{k=0}^{p} \binom{p}{k} t^k + t^p \\
= I(p \geq 1) \sum_{k=0}^{p-1} c_{p,k} t^k + (t^p - 2t^p + t^p), \\
= I(p \geq 2) \sum_{k=0}^{p-2} c_{p,k} t^k,
\]

where \(c_{p,k} = \binom{p}{k} (2^{p-k} - 2)\). The last equality follows from the fact that \(c_{p,p-1} = 0\). \(\square\)

**Lemma 2.** \(\Delta^4(t + 2)^p = \sum_{k=0}^{p-4} a_{p,k} t^k\) for \(t = 1, 2, \ldots, T\) and \(p \geq 4\).

**Proof of Lemma 2.** By the Binomial Theorem, we have

\[
\Delta^4(t + 2)^p = (t + 2)^p - 4(t + 1)^p + 6t^p - 4(t - 1)^p + (t - 2)^p \\
= \sum_{k=0}^{p} \binom{p}{k} t^k 2^{p-k} - 4 \sum_{k=0}^{p} \binom{p}{k} t^k + 6t^p - 4 \sum_{k=0}^{p} \binom{p}{k} t^k (-1)^{p-k} + \sum_{k=0}^{p} \binom{p}{k} t^k (-2)^{p-k} \\
= I(p \geq 1) \sum_{k=0}^{p-1} a_{p,k} t^k + (t^p - 4t^p + 6t^p - 4t^p + t^p) \\
= I(p \geq 1) \sum_{k=0}^{p-1} a_{p,k} t^k,
\]

where

\[
a_{p,k} = \binom{p}{k} (2^{p-k} - 4 - 4(-1)^{p-k} + (-2)^{p-k}).
\]
This implies that

\[ a_{p,k} = \begin{cases} \binom{p}{k} (2^{p-k+1} - 8) & \text{if } p - k \text{ is even} \\ 0 & \text{if } p - k \text{ is odd.} \end{cases} \]

Note that \( a_{p,k} = 0 \) for \( k = p - 1 \) and \( k = p - 3 \), since \( p - k \) is odd in both cases. It is also easy to see that \( a_{p,p-2} = 0 \). Thus, we conclude that

\[ \Delta^4(t + 2)^p = I(p \geq 4) \sum_{k=0}^{p-4} a_{p,k} t^k. \]

Proof of Theorem 4

Let \( y_t = t^p \) for \( t = 1, 2, \ldots, T \). Theorem 1 implies that

\[ \hat{c}_{Tt}(1, 2^p, \ldots, T^p) = \lambda \hat{c}_{Tt}(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_T), \]
where

\[ \tilde{y}_1 = 3^p - 2^{p+1} + 1 = I(p \geq 2) \sum_{k=0}^{p-2} c_{p,k}, \]

\[ \tilde{y}_2 = (4^p - 2 \times 3^p + 2^p) - 2(3^p - 2^{p+1} + 1) = I(p \geq 2) \sum_{k=0}^{p-2} c_{p,k} (2^k - 2), \]

\[ \tilde{y}_{T-1} = ((T - 1)^p - 2(T - 2)^p + (T - 3)^p) - 2(T^p - 2(T - 1)^p + (T - 2)^p) \]
\[ = I(p \geq 2) \sum_{k=0}^{p-2} c_{p,k} ((T - 3)^k - 2(T - 2)^k), \]

\[ \tilde{y}_T = T^p - 2(T - 1)^p + (T - 2)^p = I(p \geq 2) \sum_{k=0}^{p-2} c_{p,k} (T - 2)^k, \]

by Lemma 1 and for \( t = 3, 4, \ldots, T - 2, \)

\[ \tilde{y}_t = \Delta^4(t + 2)^p = I(p \geq 4) \sum_{k=0}^{p-4} a_{p,k} t^k, \]

by Lemma 2.

Also, note that

\[ \hat{T}_T(x_1 + y_1, x_2 + y_2, \ldots, x_T + y_T) = \hat{T}_T(x_1, x_2, \ldots, x_T) + \hat{T}_T(y_1, y_2, \ldots, y_T), \tag{37} \]
therefore,

\[ \hat{c}_{Tt}(1, 2^p, \ldots, T^p) \]
\[ = \lambda \hat{r}_{Tt}(0, 0, \tilde{y}_3, \ldots, \tilde{y}_{T-2}, 0, 0) + \lambda \hat{r}_{Tt}(\tilde{y}_1, \tilde{y}_2, 0, \ldots, 0, \tilde{y}_{T-1}, \tilde{y}_T) \]
\[ = \lambda I(p \geq 4) \hat{r}_{Tt}(0, 0, \sum_{k=0}^{p-4} a_{p,k} 3^k, \ldots, \sum_{k=0}^{p-4} a_{p,k} (T - 2)^k, 0, 0) \]
\[ + \lambda I(p \geq 2) \hat{r}_{Tt}(\sum_{k=0}^{p-2} c_{p,k} 2^k, 0, \ldots, 0, \sum_{k=0}^{p-2} c_{p,k} ((T - 3)^k - 2(T - 2)^k), \sum_{k=0}^{p-2} c_{p,k} (T - 2)^k) \]
\[ = \lambda I(p \geq 4) \sum_{k=0}^{p-4} a_{p,k} \hat{r}_{Tt}(0, 0, 3^k, \ldots, (T - 2)^k, 0, 0) \]
\[ + \lambda I(p \geq 2) \sum_{k=0}^{p-2} c_{p,k} \hat{r}_{Tt}(1, 2^k - 2, 0 \ldots, 0, (T - 3)^k - 2(T - 2)^k, (T - 2)^k). \]

By using the identity in equation (37), we write that \( \hat{r}_{Tt}(0, 0, 3^k, \ldots, (T - 2)^k, 0, 0) = \hat{r}_{Tt}(1, 2^k, \ldots, T^k) - \hat{r}_{Tt}(1, 2^k, 0, \ldots, 0, (T - 1)^k, T^k) \), which gives

\[ \hat{c}_{Tt}(1, 2^p, \ldots, T^p) = \lambda I(p \geq 4) \sum_{k=0}^{p-4} a_{p,k} \hat{r}_{Tt}(1, 2^k, \ldots, T^k) \]
\[ + \lambda I(p \geq 2) \sum_{k=0}^{p-2} c_{p,k} \hat{r}_{Tt}(1, 2^k - 2, 0 \ldots, 0, (T - 3)^k - 2(T - 2)^k, (T - 2)^k) \]
\[ - \lambda I(p \geq 4) \sum_{k=0}^{p-4} a_{p,k} \hat{r}_{Tt}(1, 2^k, 0, \ldots, 0, (T - 1)^k, T^k). \]
In order to conclude the proof, we write that

\[
\hat{c}_{T,t}(1, 2^p, \ldots, T^p) = t^p - \hat{c}_{T,t}(1, 2^p, \ldots, T^p)
\]

\[
= t^p - \lambda I(p \geq 4) \sum_{k=0}^{p-4} a_{p,k} \hat{c}_{T,t}(1, 2^k, \ldots, T^k)
\]

\[
- \lambda I(p \geq 2) \sum_{k=0}^{p-2} c_{p,k} \hat{c}_{T,t}(1, 2^k - 2, 0, \ldots, 1, (T - 3)^k - 2(T - 2)^k, (T - 2)^k)
\]

\[
+ \lambda I(p \geq 4) \sum_{k=0}^{p-4} a_{p,k} \hat{c}_{T,t}(1, 2^k, 0, \ldots, 0, (T - 1)^k, T^k).
\]

\[
\square
\]

**Proof of Theorem 5.** For \( t = 3, 4, \ldots, T - 1 \), the first order condition for \( \hat{c}_{T,t} \) given in equation (32) as

\[
\lambda \hat{c}_{T,t+2} - 4\lambda \hat{c}_{T,t+1} + (1 + 6\lambda) \hat{c}_{T,t} - 4\lambda \hat{c}_{T,t-1} + \lambda \hat{c}_{T,t-2} = 0.
\]

Since, the equation above is a fourth order difference equation, it has the following solution

\[
\hat{c}_{T,t} = C_1 r^t + C_2 \bar{r}^t + C_3 r^{-t} + C_4 \bar{r}^{-t},
\]

where \( r \) is the root of the difference equation which is defined in equation (22). The derivation of \( r \) is shown in the proof of Theorem 3 in de Jong and Sakarya (2016). Note that the HP filter always produces trends and cyclical components that are real as long
as the original series is real. Thus,

\[ \hat{c}_{Tt} = \bar{c}_{Tt} \]  

\[ C_1 r^t + C_2 \bar{r}^t + C_3 r^{-t} + C_4 \bar{r}^{-t} = \bar{C}_1 r^t + \bar{C}_2 \bar{r}^t + \bar{C}_3 r^{-t} + \bar{C}_4 \bar{r}^{-t}, \]  

which implies that \( C_2 = \bar{C}_1 \) and \( C_4 = \bar{C}_3 \). Furthermore, Theorem 1 implies that \( \hat{c}_{Tt} = \hat{c}_{T,T-t+1} \) when \( y_t = t^2 \) for \( t = 1, 2, \ldots, T \). Therefore,

\[ C_1 r^t + \bar{C}_1 r^t + C_3 r^{-t} + \bar{C}_3 r^{-t} = C_1 r^{T+1} r^{-t} + \bar{C}_1 \bar{r}^{T+1} \bar{r}^{-t} + C_3 r^{-(T+1)} r^t + \bar{C}_3 \bar{r}^{-(T+1)} \bar{r}^t, \]

since \( \hat{c}_{T,T-t+1} = C_1 r^{T+1} r^{-t} + \bar{C}_1 \bar{r}^{T+1} \bar{r}^{-t} + C_3 r^{-(T+1)} r^t + \bar{C}_3 \bar{r}^{-(T+1)} \bar{r}^t \). The equality above suggests that \( C_3 = C_1 r^{T+1} \). Thus, we write that

\[ \hat{c}_{Tt} = C_1 (r^t + r^{T-t+1}) + \bar{C}_1 (\bar{r}^t + \bar{r}^{T-t+1}). \]

Next, we use the first order conditions for \( t = 1 \) and \( 2 \) to solve for constant \( C_1 \). We obtain

\[ C_1 a + \bar{C}_1 \bar{a} = 2\lambda \]

\[ C_1 b + \bar{C}_1 \bar{b} = -2\lambda, \]

where

\[ a = (1 + \lambda)(r + r^T) - 2\lambda(r^2 + r^{T-1}) + \lambda(r^3 + r^{T-2}) \]

\[ b = -2\lambda(r + r^T) + (1 + 5\lambda)(r^2 + r^{T-1}) - 4\lambda(r^3 + r^{T-2}) + \lambda(r^4 + r^{T-3}). \]  

(42)
The equations above imply that

\[ C_1 = 2\lambda(\bar{b} + \bar{a})/(\bar{a}b - \bar{b}a). \]  

(43)

Lemma 3. \( \Delta^2 \exp(t + 2) = C_2 \exp(t) \) and \( \Delta^4 \exp(t + 2) = C_4 \exp(t) \) where \( C_2 = \exp(2) (1 - \exp(-1))^2 \) and \( C_4 = \exp(2) (1 - \exp(-1))^4 \).

Proof of Lemma 3. Note that

\[
\begin{align*}
\Delta^2 \exp(t + 2) &= \exp(t + 2) - 2\exp(t + 1) + \exp(t) \\
&= \exp(t + 2) (1 - 2\exp(-1) + \exp(-1)) \\
&= \exp(t + 2) (1 - \exp(-1))^2 \\
&= C_2 \exp(t),
\end{align*}
\]

where \( C_2 = \exp(2) (1 - \exp(-1))^2 \). Similarly, we can write that

\[
\begin{align*}
\Delta^4 \exp(t + 2) &= \exp(t + 2) - 4\exp(t + 1) + 6\exp(t) - 4\exp(t - 1) + \exp(t - 2) \\
&= \exp(t + 2) (1 - 4\exp(-1) + 6\exp(-2) - 4\exp(-3) + \exp(-4)) \\
&= \exp(t + 2) (1 - \exp(-1))^4 \\
&= C_4 \exp(t),
\end{align*}
\]

where \( C_4 = \exp(2) (1 - \exp(-1))^4 \).
Proof of Theorem 6. Let \( y_t = \exp(t) \) for \( t = 1, 2, \ldots, T \). By Theorem 1 we write that

\[
\hat{c}_T \left( \exp(1), \exp(2), \ldots, \exp(T) \right) = \lambda \hat{\tau}_T \left( \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_T \right),
\]

where by Lemma 3

\[
\begin{align*}
\tilde{y}_1 &= \exp(3) - 2 \exp(2) + \exp(1) = C_2 \exp(1) \\
\tilde{y}_2 &= (\exp(4) - 2 \exp(3) + \exp(2)) - 2(\exp(3) - 2 \exp(2) + \exp(1)) = C_2(\exp(2) - 2 \exp(1)) \\
\tilde{y}_{T-1} &= (\exp(T - 1) - 2 \exp(T - 2) + \exp(T - 3)) - 2(\exp(T) - 2 \exp(T - 1) + \exp(T - 2)) \\
&= C_2(\exp(T - 3) - 2 \exp(T - 2)) \\
\tilde{y}_T &= \exp(T) - 2 \exp(T - 1) + \exp(T - 2) = C_2 \exp(T - 2),
\end{align*}
\]

and for \( t = 3, 4, \ldots, T - 2 \),

\[
\begin{align*}
\tilde{y}_t &= \exp(t) - 4 \exp(t - 1) + 6 \exp(t - 2) - 4 \exp(t - 3) + \exp(t - 4) \\
&= C_4 \exp(t).
\end{align*}
\]

Since,

\[
\hat{\tau}_T \left( \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_T \right) = \hat{\tau}_T \left( 0, 0, \tilde{y}_3, \tilde{y}_4, \ldots, \tilde{y}_{T-2}, 0, 0 \right) + \hat{\tau}_T \left( \tilde{y}_1, \tilde{y}_2, 0, \ldots, 0, \tilde{y}_{T-1}, \tilde{y}_T \right),
\]

25
we write that

\[ \dot{\hat{c}}_{TT} (\exp(1), \exp(2), \ldots, \exp(T)) \]
\[ = C_4 \lambda \hat{\tau}_{TT} (0, 0, \exp(3), \exp(4), \ldots, \exp(T - 2), 0, 0) \]
\[ + C_2 \lambda \hat{\tau}_{TT} (\exp(1), (\exp(2) - 2 \exp(1)), 0, \ldots, 0, (\exp(T - 3) - 2 \exp(T - 2)), \exp(T - 2)) \]
\[ = C_4 \lambda \hat{\tau}_{TT} (\exp(1), \exp(2), \ldots, \exp(T)) \]
\[ + C_2 \lambda \hat{\tau}_{TT} (\exp(1), (\exp(2) - 2 \exp(1)), 0, \ldots, 0, (\exp(T - 3) - 2 \exp(T - 2)), \exp(T - 2)) \]
\[ - C_4 \lambda \hat{\tau}_{TT} (\exp(1), \exp(2), 0 \ldots, 0, \exp(T - 1), \exp(T)), \]

where the last line follows from

\[ \hat{\tau}_{TT} (0, 0, \exp(3), \exp(4), \ldots, \exp(T - 2), 0, 0) \]
\[ = \hat{\tau}_{TT} (\exp(1), \exp(2), \ldots, \exp(T)) - \hat{\tau}_{TT} (\exp(1), \exp(2), 0 \ldots, 0, \exp(T - 1), \exp(T)). \]

Note that \( C_4 = C_2 (1 - \exp(-1))^2 \), therefore

\[ (C_2 - C_4) \exp(1) = C_2 (1 - (1 - \exp(-1))^2) \exp(1) \]
\[ = C_2 (2 - \exp(-1)), \]

\[ C_2 (\exp(2) - 2 \exp(1)) - C_4 \exp(2) = (C_2 - C_4) \exp(2) - 2C_2 \exp(1) \]
\[ = C_2 (2 \exp(1) - 1) - 2C_2 \exp(1) \]
\[ = -C_2, \]

26
\[
C_2 (\exp(T - 3) - 2 \exp(T - 2)) - C_4 \exp(T - 1)
\]
\[
= C_2 (\exp(T - 3) - 2 \exp(T - 2)) - C_2 (1 - \exp(-1))^2 \exp(T - 1)
\]
\[
= - C_2 \exp(T - 1),
\]

and

\[
\begin{align*}
C_2 \exp(T - 2) - C_4 \exp(T) &= C_2 \exp(T - 2) - C_2 (1 - \exp(-1))^2 \exp(T) \\
&= C_2 \exp(T - 1)(2 - \exp(1)).
\end{align*}
\]

By using the above equalities, we can write that

\[
\begin{align*}
C_2 \lambda \hat{\tau}_{TT} (\exp(1), (\exp(2) - 2 \exp(1)), 0, \ldots, 0, (\exp(T - 3) - 2 \exp(T - 2)), \exp(T - 2)) \\
&- C_4 \lambda \hat{\tau}_{TT} (\exp(1), \exp(2), 0, \ldots, 0, \exp(T - 1), \exp(T)) \\
&= C_2 \lambda \hat{\tau}_{TT} ((2 - \exp(-1)), -1, 0, \ldots, 0, - \exp(T - 1), \exp(T - 1)(2 - \exp(1))).
\end{align*}
\]

Therefore, we can conclude that

\[
\begin{align*}
\hat{c}_{TT} (\exp(1), \exp(2), \ldots, \exp(T)) \\
= C_4 \lambda \hat{\tau}_{TT} (\exp(1), \exp(2), \ldots, \exp(T)) \\
+ C_2 \lambda \hat{\tau}_{TT} ((2 - \exp(-1)), -1, 0, \ldots, 0, - \exp(T - 1), \exp(T - 1)(2 - \exp(1))).
\end{align*}
\]

\[\square\]
Lemma 4. For $k, j \geq 0$,

$$
\lim_{T \to \infty} w_{T,t-k,j} = 0
$$

$$
\lim_{T \to \infty} w_{T,t-k,T-j} = f_\lambda(k - j) + f_\lambda(k + j + 1) + \xi\lambda g_\lambda(k + 1)g_\lambda(j + 1)
$$

Proof of Lemma 4. We use the definition of weights in Theorem 1 of de Jong and Sakarya (2016) and write that

$$
\lim_{T \to \infty} w_{T,t-k,j} = \lim_{T \to \infty} (f_\lambda(T - k - j) + f_\lambda(T)I(T - k + j - 1 = T))
$$

$$
+ \lim_{T \to \infty} (f_\lambda(T - k + j - 1)I(T - k + j - 1 < T) + f_\lambda(T + k - j + 1)I(T - k + j - 1 > T))
$$

$$
+ \lim_{T \to \infty} (\xi_\lambda g_\lambda(T - k)g_\lambda(j) + \phi_\lambda g_\lambda(k + 1)g_\lambda(j))
$$

$$
+ \lim_{T \to \infty} (\phi_\lambda g_\lambda(T - k)g_\lambda(T - j + 1) + \xi_\lambda g_\lambda(k + 1)g_\lambda(T - j + 1))
$$

$$
= 0,
$$

since $\lim_{T \to \infty} f_\lambda(T) = 0$, $\lim_{T \to \infty} g_\lambda(T) = 0$ and $\lim_{T \to \infty} \phi_\lambda = 0$ (by Theorem 2 of de Jong and Sakarya (2016)), $\lim_{T \to \infty} g_\lambda(j) = g_\lambda(j)$, and $|g_\lambda(j)| < \infty$ for all $j \in \mathbb{Z}$. 

28
Similarly, we write that

\[
\lim_{T \to \infty} w_{T, T-k, T-j} = \lim_{T \to \infty} (f_{T\lambda}(k-j) + f_{T\lambda}(T)(2T-k-j-1 = T)) \\
+ \lim_{T \to \infty} (f_{T\lambda}(2T-k-j-1)I(2T-k-j-1 < T) + f_{T\lambda}(k+j+1)(2T-k-j-1 > T)) \\
+ \lim_{T \to \infty} (\xi_{T\lambda}g_{T\lambda}(T-k)g_{T\lambda}(T-j) + \phi_{T\lambda}g_{T\lambda}(k+1)g_{T\lambda}(T-j)) \\
+ \lim_{T \to \infty} (\phi_{T\lambda}g_{T\lambda}(T-k)g_{T\lambda}(j+1) + \xi_{T\lambda}g_{T\lambda}(k+1)g_{T\lambda}(j+1)) \\
= f_{\lambda}(k-j) + f_{\lambda}(k+j+1) + \xi_{\lambda}g_{\lambda}(k+1)g_{\lambda}(j+1),
\]

since \(\lim_{T \to \infty} f_{T\lambda}(m) = f_{\lambda}(m)\) and \(\lim_{T \to \infty} g_{T\lambda}(m) = g_{\lambda}(m)\) for all \(\lambda \geq 0\) and \(m \in \mathbb{Z}\) (by Theorem 3 of de Jong and Sakarya (2016)) and \(\lim_{T \to \infty} \xi_{T\lambda} = \xi_{\lambda}\) (by Theorem 2 of de Jong and Sakarya (2016)). \(\square\)

**Proof of Theorem 7.** By Theorem 6, we have

\[
\lim_{T \to \infty} \frac{\hat{c}_{T, T-k} (\exp(1), \exp(2), \ldots, \exp(T))}{\hat{\tau}_{T, T-k} (\exp(1), \exp(2), \ldots, \exp(T))} = C_{\lambda} + C_{2\lambda} \lim_{T \to \infty} \frac{\hat{c}_{T, T-k} ((2 - \exp(-1)), -1, 0, \ldots, 0, -\exp(T-1), \exp(T-1)(2 - \exp(1)))}{\hat{\tau}_{T, T-k} (\exp(1), \exp(2), \ldots, \exp(T))}.
\]

Note that

\[
\begin{align*}
\lim_{T \to \infty} \frac{\hat{c}_{T, T-k} ((2 - \exp(-1)), -1, 0, \ldots, 0, -\exp(T-1), \exp(T-1)(2 - \exp(1)))}{\hat{\tau}_{T, T-k} (\exp(1), \exp(2), \ldots, \exp(T))} \\
= (2 - \exp(-1)) \lim_{T \to \infty} \frac{w_{T, T-k, 1}}{\sum_{s=1}^{T} w_{T, T-k, s}\exp(s)} - \lim_{T \to \infty} \frac{w_{T, T-k, 2}}{\sum_{s=1}^{T} w_{T, T-k, s}\exp(s)} \\
- \lim_{T \to \infty} \frac{\exp(T-1)w_{T, T-k, T-1}}{\sum_{s=1}^{T} w_{T, T-k, s}\exp(s)} + (2 - \exp(1)) \lim_{T \to \infty} \frac{\exp(T-1)w_{T, T-k, T}}{\sum_{s=1}^{T} w_{T, T-k, s}\exp(s)}.
\end{align*}
\]
The first term of equation (44) can be written as
\[
(2 - \exp(-1)) \lim_{T \to \infty} \frac{\exp(-T)w_{T,T-k,1}}{\sum_{s=1}^{T} \exp(s - T) w_{T,T-k,s}} = (2 - \exp(-1)) \lim_{T \to \infty} \frac{\exp(-T)w_{T,T-k,1}}{\sum_{j=0}^{T-1} \exp(-j) w_{T,T-k,T-j}} = 0,
\]
since \( \lim_{T \to \infty} w_{T,T-k,1} = 0 \), and \( \lim_{T \to \infty} \sum_{j=0}^{T-1} w_{T,T-k,T-j} \exp(-j) = \sum_{j=0}^{\infty} (f_{\lambda}(k-j) + f_{\lambda}(k+j+1) + \xi_{\lambda}g_{\lambda}(k+1)) \exp(-j) \) by Lemma 4.

The second term of equation (44) is zero by the argument similar to the one above.

The third term of equation (44) is equivalent to
\[
\exp(-1) \lim_{T \to \infty} \frac{w_{T,T-k,T-1}}{\sum_{j=0}^{T-1} w_{T,T-k,T-j} \exp(-j)} = \exp(-1) \frac{f_{\lambda}(k-1) + f_{\lambda}(k+2) + \xi_{\lambda}g_{\lambda}(k+1)g_{\lambda}(2)}{\sum_{j=0}^{\infty} (f_{\lambda}(k-j) + f_{\lambda}(k+j+1) + \xi_{\lambda}g_{\lambda}(k+1)g_{\lambda}(j+1)) \exp(-j)},
\]
by Lemma 4.

The last term of equation (44) is equivalent to
\[
(2 \exp(-1) - 1) \lim_{T \to \infty} \frac{w_{T,T-k,T}}{\sum_{j=0}^{T-1} w_{T,T-k,T-j} \exp(-j)} = (2 \exp(-1) - 1) \frac{f_{\lambda}(k) + f_{\lambda}(k+1) + \xi_{\lambda}g_{\lambda}(k+1)g_{\lambda}(1)}{\sum_{j=0}^{\infty} (f_{\lambda}(k-j) + f_{\lambda}(k+j+1) + \xi_{\lambda}g_{\lambda}(k+1)g_{\lambda}(j+1)) \exp(-j)}.
\]