We introduce heterogeneous preferences (heterogeneity in risk aversion and time discount factor) into a two-agent endowment economy with enforcement constraints, aggregate and idiosyncratic uncertainty (Alvarez and Jermann (2001)), and study the corresponding asset pricing and risk sharing implications. We show that the relative time discount factor and the interaction between heterogeneous risk aversion and aggregate uncertainty affect the evolution of the relative Pareto weight of agents over time. The more patient agent tends to obtain a higher Pareto weight, and the less risk averse agent tends to do so in booms and the reverse in recessions. We demonstrate that preference heterogeneity can generate a positive equity premium with only idiosyncratic uncertainty present since the conditional pricing kernel is time-varying depending on which agent is the marginal pricer. We use recursive Lagrangian method to solve a calibrated model and show that preference heterogeneity boosts the mean and volatility of equity premium quantitatively, when the more risk averse and/or the more patient agent cannot trade away most of his income risk with the other agent.

Keywords: Equity Premium, Relative Pareto Weight, Limited Enforcement, Heterogeneous Preferences, Recursive Lagrangian

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1 INTRODUCTION

When the risk-sharing contracts between agents cannot be fully enforced, agents may subject to participation constraints with the outside option of autarky (Kehoe and Levine (1993) and Alvarez and Jermann (2000)). Alvarez and Jermann (2001) show that this contracting friction of limited enforcement generates limited risk sharing and volatile pricing kernel, matching asset pricing moments decently. The two preference parameters, risk aversion and time discount factor, play important roles in determining the amount of risk shared and the properties of pricing kernel. A rise in the agent’s risk aversion will increase the volatility of pricing kernel directly as risk aversion is the curvature parameter of marginal utility. But at the same time, more risk will be shared, and individual consumption volatility decreases, resulting in less volatile pricing kernel. Therefore, the overall effect is ambiguous. The effect of time discount factor is similar as a rise in agent’s time discount factor results in higher pricing kernel directly, but also decreases it through more risk sharing. When two agents have different preferences, how would risk sharing and pricing kernel thus asset prices change in contrast to when they have the same preferences? Will they share more or less risk? Will the pricing kernel be lower or higher, less or more volatile? Will the equity premium be smaller or larger?

In order to answer these questions, we introduce heterogeneity in risk aversion and time discount factor into a two-agent endowment economy with enforcement constraints, aggregate and idiosyncratic uncertainty (Alvarez and Jermann (2001)), and study the corresponding risk sharing and asset pricing implications.

First, we show that the relative time discount factor and the interaction between heterogeneous risk aversion and aggregate uncertainty affect the evolution of the relative Pareto weight (RPW) of agents over time. When neither agents’s enforcement constraint is binding, RPW of the more patient agent goes up, and RPW of the less risk averse agent increases in booms and decreases in recessions. This is absent in Ligon, Thomas, and Worrall (2002), since there is no aggregate uncertainty and agents have the same time preference in their economy. Ligon, Thomas, and Worrall (2002) prove that there exists an interval for RPW to fall into for each state and RPW takes only boundary values of these intervals in the long run if risk sharing is limited. When heterogeneous preferences and aggregate uncertainty are present, I show that RPW takes not only boundary values of those intervals as in Ligon, Thomas, and Worrall (2002), but also certain values inside those intervals. Besides, RPW does not go to zero or infinity, i.e., no agents die or dominate in the long run, since enforcement constraints entitle the agents the option of autarky for all times and agents can consume their non-zero endowment. In addition, we find that full risk sharing is
not possible when agents have heterogeneous preferences.

Then, we demonstrate that preference heterogeneity combined with limited enforcement generates a positive equity premium in a two-state example with only idiosyncratic uncertainty (no aggregate cash flow uncertainty) while agents’ endowment shares are symmetrically distributed. Enforcement constraints induces discount rate shocks as the marginal pricer changes over time depending on which agent is not constrained. When agents have symmetric endowment and the same preferences, the pricing kernel or stochastic discount factor (SDF) is also symmetric across agents. Although the marginal pricing agent is changing, conditional SDF does not vary. Thus there is no time-variation in price-dividend ratios and equity premium is zero. Introducing heterogeneous preferences breaks down the symmetry of SDF across agents. Conditional SDF is different depending on which agent is the marginal pricer, resulting in time-varying price-dividend ratios and positive equity premium. In contrast to the case of same low and same high risk aversion respectively, the amount of risk sharing increases little for the low risk aversion agent, but decreases dramatically for the high risk aversion agent, leading to high consumption volatility for the latter. When the more risk averse agent is unconstrained and becomes the marginal pricer, the stochastic discount factor is large and volatile, resulting in a sizable equity premium. As for the case of heterogeneous time preference, the more patient agent has more chance to be the marginal pricer due to the higher patience level, even though the two agents have symmetric endowment distributions. As a result, SDF is more affected by the more patient agent. When he cannot trade away most of his income risk with the less patient agent, equity premium is high. In addition, when the more patient agent is also more risk averse, least risk is shared and the equity premium is highest.

Last, we use the recursive Lagrangian method of Marcet and Marimon (2016) to solve a calibrated model with both idiosyncratic and aggregate uncertainty. As the intuition of the two-state example carries through, we show that heterogeneous preferences generate asymmetric risk sharing across agents and lead to a higher and more volatile equity premium than homogeneous preferences when agents are subjected to enforcement constraints. Undesirably, the risk-free rate could be too volatile, as the discount rate shocks induced by enforcement constraints become asymmetric with preference heterogeneity present and result in more SDF variation. Besides, heterogeneous time preference shows more promise than heterogeneous risk aversion to better match asset pricing moments. In particular, the former could generate 7.05% mean equity premium and 27.87% equity volatility with moderate risk aversion 3.2 and reasonable heterogeneous time discount factors 0.85

\footnote{We use CRRA utility. Heterogenous risk aversion implies heterogeneous EIS (elasticity of intertemporal substitution). The problem might be solved by Epstein-Zin preference, which separates EIS and relative risk aversion so that agents can have heterogeneous risk aversion but same EIS at the same time. But we do not pursue it because of the associated computational difficulties for the combination of Epstein-Zin and enforcement constraints.}
and 0.75 for two agents\textsuperscript{2}.

Our contribution for asset pricing literature is to show that idiosyncratic income risk and incomplete markets become more relevant for asset prices when agents have heterogeneous preferences. Alvarez and Jermann (2001) and Krueger and Lustig (2010) show that idiosyncratic income risk does not affect (multiplicative) equity premium if its distribution is independent of aggregate risk and aggregate risk is i.i.d. over time. We show that preference heterogeneity renders the discount rate shocks asymmetric across agents induced by enforcement constraints, leading to time-variation in conditional SDF and price-dividend ratios and thus non-zero equity premium even without any aggregate cash flow uncertainty. Quantitatively, heterogeneous time preference improve over homogeneous preferences in matching asset pricing moments when aggregate uncertainty is present\textsuperscript{3}.

As for our contribution to risk sharing literature, we generalize the theoretical result of Ligon, Thomas, and Worrall (2002) on the evolution of the relative Pareto weight (RPW) over time under limited enforcement to include preference heterogeneity and aggregate uncertainty. In particular, we show that relative time discount factor and the interaction between heterogeneous risk aversion and aggregate uncertainty affect the evolution of the RPW. Interestingly, we find that full risk sharing is not possible due to the presence of preference heterogeneity.

Our paper is related to several strands of literature. First, it is related to the literature on heterogeneous preferences and asset prices. For early contribution, see Dumas (1989) and Wang (1996). The two papers, along with Basak and Cuoco (1998), have an undesirable implication that with positive growth the less risk averse agents will dominate the economy in the long run. In order to ensure stationarity, the literature has introduced habit into preferences (Chan and Kogan (2002), Xiouros and Zapatero (2010), and Bhamra and Uppal (2014)) or overlapping generations with agents vanishing each period (Gomes and Michaelides (2008) and Gărleanu and Panageas (2015)) \textsuperscript{4}. In our paper, enforcement constraints naturally emerges as a device to ensure stationary long run distribution as agents always have the option to choose autarky and would not end up with zero consumption. In addition, the papers mentioned studied how heterogeneous preferences affect asset prices with only aggregate risk\textsuperscript{5}. We include both aggregate risk and idiosyncratic risk and show explicitly how the two risks and preference heterogeneity are intertwined through the law of motion for the relative Pareto weight to determine consumption allocations, risk sharing, and asset prices.

\textsuperscript{2}The volatility of risk-free rate is 6.96%.

\textsuperscript{3}Albuquerque, Eichenbaum, Luo, and Rebelo (2015) show that a representative agent model with exogenous time preference shocks accounts for key asset pricing moments. We interpret our time preference shocks as endogenously driven by enforcement constraints as the marginal pricing agent who is not constrained varies over time.

\textsuperscript{4}For other contributions, see Guvenen (2009), Chabakauri (2015), Coen-Pirani (2004), Kogan, Makarov, and Uppal (2007), and Longstaff and Wang (2012).

\textsuperscript{5}Gomes and Michaelides (2008) is an exception.

Our paper is also related to the literature on how incomplete markets and portfolio constraints affect asset prices. For early contributions, see Mankiw (1986) and Constantinides and Duffie (1996). Our main difference is that the market is endogenous incomplete due to endogenous solvency constraints in our model, while the literature usually assumes exogenous incomplete market due to limited securities to trade, trade frictions or exogenous borrowing constraints etc.

Our paper is structured as follows. Section 1 is the introduction. Section 2 outlines the model. Section 3 uses a two-state example to gain the intuition of the model. Section 4 presents calibration and quantitative results. Section 5 concludes.

2 Model

In this section, we outline the model. We first follow promised utility approach to formulate the contracting problem, derive the evolution of the relative Pareto weight, and show its long-run properties. Then we use recursive Lagrangian method of Marcet and Marimon (2016) (see also Kehoe and Perri (2002)) to set up the planner’s problem and present a computation algorithm for solving the consumption allocations. Last, we decentralize the economy following Alvarez and Jermann (2000) and pin down the asset prices with the solved allocations.

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6 There is no aggregate uncertainty in her model. As shown in equation (2.7), heterogeneous risk aversion affects the law of motion for the relative Pareto weight (RPW) only when aggregate uncertainty is present. And her focus is risk sharing, while we study asset prices as well. Besides, she does not derive explicitly how RPW evolves over time.

2.1 Environment

There are two agents endowed with random stream of income $e_{it}$, $i = 1, 2$. Aggregate endowment is $e_t = e_{1t} + e_{2t}$ and grows over time $g_{t+1} = e_{t+1}/e_t$. Agents’ income share is $\hat{e}_{it} = e_{it}/e_t$. We assume $z = (g, \hat{e})$ jointly follows a finite-state Markov process. We denote $z^t = (z_0, z_1, z_2, \ldots, z_t)$ as the history up to $t$. Transition probabilities from $t - 1$ to $t$ are denoted as $\pi(z^t | z^{t-1})$.

Agents have CRRA utility, but may differ in the relative risk aversion coefficient and the time discount factor. Utility function is

$$u^i(c_{it}) = \frac{c_{it}^{1-\gamma_i}}{1-\gamma_i},$$

and life-time utility is defined as

$$U^i_t(c_i) \equiv E_t \sum_{j=0}^{\infty} \beta^j u^i(c_{it+j}),$$

where $c_{it}$ is agent $i$’s consumption at time $t$, $\gamma_i$ is the relative risk aversion coefficient, and $\beta_i$ is the time discount factor. Contracts are not fully enforceable and agents have autarky as outside option. So agents’ consumption choices should satisfy the participation constraints,

$$U^i_t(c_i) \geq U^i_t(e_i), \quad i = 1, 2, \quad t = 0, 1, 2, \ldots,$$

where $U^i_t(e_i)$ is the utility value of autarky starting from time $t$. Note that for notational convenience we write out the states $z^t$ only when it is necessary to avoid confusion.

2.2 Promised Utility Formulation

We follow promised utility approach to formulate the contracting problem. Pareto frontiers should satisfy

$$V(w, z) = \max_{c_i, w'(z')} \left\{ u(c_1) + \beta_1 \sum_{z' \in Z} \pi(z'|z) V(w'(z'), z') \right\}$$

$$c_1 + c_2 \leq e,$$

$$\lambda: \quad u(c_2) + \beta_2 \sum_{z' \in Z} \pi(z'|z) w'(z') \geq w,$$

$$\beta_2 \pi(z'|z) \eta^2(z'): \quad w'(z') \geq U^2(z'), \quad z' \in Z,$$

$$\beta_1 \pi(z'|z) \eta^1(z'): \quad V(w'(z'), z') \geq U^1(z'), \quad z' \in Z,$$
where $V$ is the life-time utility of agent 1, $w$ is promised life-time utility to agent 2, $\lambda$ is the Lagrangian multiplier on the promise-keeping constraint, and $\eta^i(z')$ is the state-dependent Lagrangian multiplier on agent $i$'s participation constraints.

The first order conditions are

$$\frac{u_1'(c_1)}{u_2'(c_2)} = \lambda,$$

$$-V'(w'(z'), z') = \lambda + \eta^1(z') \beta_1,$$

together with the envelop condition

$$-V'(w, z) = \lambda.$$

There is aggregate growth in the economy. To obtain a stationary economy, we normalize some variables as follows,

$$\hat{c}_{it} \equiv \frac{c_{it}}{e_t}, \quad \hat{e}_{it} \equiv \frac{e_{it}}{e_t},$$

$$\hat{w} \equiv \frac{w}{e^{1-\gamma_2}}, \quad \hat{V}(\hat{w}, z) \equiv \frac{V(w, z)}{e^{1-\gamma_1}},$$

$$\hat{U}^i(z) \equiv \frac{U^i(z)}{e^{1-\gamma_i}},$$

$$\hat{\pi}_1(z'|z) \equiv \frac{\pi(z'|z)g(z')^{1-\gamma_1}}{\sum_{z'} \pi(z'|z)g(z')^{1-\gamma_1}},$$

$$\hat{\beta}_i(z) \equiv \beta_i \sum_{z'} \pi(z'|z)g(z')^{1-\gamma_i}.$$  

The original recursive problem can then be rewritten as

$$\hat{V}(\hat{w}, z) = \max_{\hat{c}_1, \hat{c}_2} \{ u(\hat{c}_1) + \hat{\beta}_1 \sum_{z' \in Z} \hat{\pi}_1(z'|z)\hat{V}(\hat{w}'(z'), z') \}$$

$$\hat{c}_1 + \hat{c}_2 \leq 1,$$

$$\hat{\lambda} : \quad u(\hat{c}_2) + \hat{\beta}_2 \sum_{z' \in Z} \hat{\pi}_2(z'|z)\hat{w}'(z') \geq \hat{w},$$

$$\hat{\beta}_2 \hat{\pi}_2(z'|z)\hat{\eta}^2(z') : \quad \hat{w}'(z') \geq \hat{U}^2(z'), \quad z' \in Z,$$

$$\hat{\beta}_1 \hat{\pi}_1(z'|z)\hat{\eta}^1(z') : \quad \hat{V}(\hat{w}'(z'), z') \geq \hat{U}^1(z'), \quad z' \in Z,$$

where $\hat{\lambda}$ and $\hat{\eta}^i(z')$ are Lagrangian multipliers on the constraints after normalization.
The first order conditions are

\[
\frac{u'_1(c_1)}{u'_2(c_2)} = \hat{\lambda},
\]

\[-\hat{V}'(\hat{w}'(z'), z') = \frac{\hat{\lambda} + \hat{\eta}^2(z') \hat{\beta}_2 \hat{\pi}_2}{1 + \hat{\eta}^1(z') \hat{\beta}_1 \hat{\pi}_1} = \frac{\hat{\lambda} + \hat{\eta}^2(z') \hat{\beta}_2 g(z')^{\gamma_1 - \gamma_2}}{1 + \hat{\eta}^1(z') \hat{\beta}_1}.
\]

together with the envelop condition

\[-\hat{V}'(\hat{w}, z) = \hat{\lambda}.
\]

It is easy to see that

\[\hat{\lambda} = \frac{\lambda}{e^{\gamma_2 - \gamma_1}}.
\]

\(\hat{\lambda}\) equals marginal utility of consumption share, which is also the (normalized or preference-adjusted) relative Pareto weight (RPW, of agent 2 with respect to agent 1) in the planner’s problem as discussed in subsection 2.4\(^8\). Combining the last two equations, we obtain

\[
\hat{\lambda}' = \frac{\hat{\lambda} + \hat{\eta}^2(z') \hat{\beta}_2}{1 + \hat{\eta}^1(z') \hat{\beta}_1} g(z')^{\gamma_1 - \gamma_2}. \tag{2.1}
\]

The above equation describes how idiosyncratic risk, aggregate risk and preference heterogeneity are intertwined to influence RPW and thus consumption allocations, risk sharing, and asset prices. When agent \(i\) gets a high idiosyncratic income shock tomorrow, his participation constraint binds, \(\hat{\eta}^i > 0\), and his RPW rises. When both agents are not constrained, \(\hat{\eta}^1 = \hat{\eta}^2 = 0\), the RPW of the more patient agent will increase due to the term \(\beta_2/\beta_1\), and the RPW of the less risk averse agent will rise in booms and decline in recessions due to the term \(g(z')^{\gamma_1 - \gamma_2}\).

2.3 Risk Sharing

We characterize the evolution of \(\hat{\lambda}\) over time and its long-run properties.

**Proposition 1.** Suppose agents could have heterogenous preferences. A constrained-efficient contract can be characterized as follows: There exists \(S\) state dependent intervals \([\bar{\lambda}_s, \tilde{\lambda}_s]\), \(s = 1, 2, ..., S\)

\(^8\)For convenience, we call \(\hat{\lambda}\) as the relative Pareto weight hereafter.
such that given $\lambda_t$ and next period occurring state $s$, $\lambda_{t+1}$ updates as:

$$
\lambda_{t+1} = \begin{cases} 
\lambda_s, & \text{if } \lambda_t g_s^{\gamma_1 - \gamma_2 \frac{\beta_2}{\beta_1}} < \lambda_s \\
\lambda_t g_s^{\gamma_1 - \gamma_2 \frac{\beta_2}{\beta_1}}, & \text{if } \lambda_t g_s^{\gamma_1 - \gamma_2 \frac{\beta_2}{\beta_1}} \in [\lambda_s, \bar{\lambda}_s] \\
\bar{\lambda}_s, & \text{if } \lambda_t g_s^{\gamma_1 - \gamma_2 \frac{\beta_2}{\beta_1}} > \bar{\lambda}_s.
\end{cases}
$$

**Proof.** The proof follows Ligon, Thomas, and Worrall (2002). First, there exists $S$ state dependent intervals $[V_s, \bar{V}_s]$ and $[\bar{w}_s, \bar{\bar{w}}_s]$ for agent 1 and 2’s possible lifetime utility values, following Thomas and Worrall (1988). Obviously, $V_s = \bar{U}_s^1$ and $w_s = \bar{U}_s^2$. Second, let $\lambda_s = -\bar{V}'(w_s)$ and $\bar{\lambda}_s = -\bar{V}'(\bar{w}_s)$. Then considering the three cases

- If $\lambda_t g_s^{\gamma_1 - \gamma_2 \frac{\beta_2}{\beta_1}} < \lambda_s$, $\lambda_{t+1} \in [\lambda_s, \bar{\lambda}_s] > \lambda_t g_s^{\gamma_1 - \gamma_2 \frac{\beta_2}{\beta_1}}$. Equation 2.1 then implies $\bar{\eta}_s^2 > 0$. Thus $\bar{w}_s = w_s$. Hence $\lambda_{t+1} = -\bar{V}'(w_s) = \lambda_s$.

- If $\lambda_t g_s^{\gamma_1 - \gamma_2 \frac{\beta_2}{\beta_1}} > \bar{\lambda}_s$, $\lambda_{t+1} \in [\lambda_s, \bar{\lambda}_s] \subset \lambda_t g_s^{\gamma_1 - \gamma_2 \frac{\beta_2}{\beta_1}}$. Equation 2.1 then implies $\bar{\eta}_s^1 > 0$. Thus $\bar{V}_s = V_s$ and $\bar{w}_s = \bar{w}_s$. Hence $\lambda_{t+1} = -\bar{V}'(\bar{w}_s) = \bar{\lambda}_s$.

- If $\lambda_t g_s^{\gamma_1 - \gamma_2 \frac{\beta_2}{\beta_1}} \in [\lambda_s, \bar{\lambda}_s]$, it must have $\bar{\eta}_s^1 = \bar{\eta}_s^2 = 0$ and hence $\lambda_{t+1} = \lambda_t g_s^{\gamma_1 - \gamma_2 \frac{\beta_2}{\beta_1}}$. Suppose the contrary that $\bar{\eta}_s^2 > 0$ and $\bar{\eta}_s^1 = 0$. Then we have $\bar{w}_s = w_s$ and $\lambda_{t+1} = \lambda_s$. But equation (2.1) implies $\lambda_{t+1} > \lambda_t g_s^{\gamma_1 - \gamma_2 \frac{\beta_2}{\beta_1}}$. Contradiction. The symmetric argument holds for $\bar{\eta}_s^2 = 0$ and $\bar{\eta}_s^1 > 0$. And it is not possible that $\bar{\eta}_s^2 > 0$ and $\bar{\eta}_s^1 > 0$.

\end{proof}

**Corollary 1.** Suppose agents have the same preferences, i.e., $\beta_1 = \beta_2, \gamma_1 = \gamma_2$. Given $\lambda_t$ and next period occurring state $s$, $\lambda_{t+1}$ updates as:

$$
\lambda_{t+1} = \begin{cases} 
\lambda_s, & \text{if } \lambda_t < \lambda_s \\
\lambda_t, & \text{if } \lambda_t \in [\lambda_s, \bar{\lambda}_s] \\
\bar{\lambda}_s, & \text{if } \lambda_t > \bar{\lambda}_s.
\end{cases}
$$

This is the same result as in Ligon, Thomas, and Worrall (2002). When there is no enforcement constraints, the first best is achieved and $\lambda$ is constant over time, $\lambda_{t+1} = \lambda_t$ for all $t$. With participation constraints present, next period $\lambda$ keeps unchanged if possible, and changes the minimum to be inside the interval of the possible values of (preference-adjusted) relative Pareto weight if this period $\lambda$ is outside the interval. When heterogeneous preferences are present, the extra term $g_s^{\gamma_1 - \gamma_2 \frac{\beta_2}{\beta_1}}$ shows up in the updating rule of $\lambda$. 


Proposition 2. Assume agents have heterogeneous preferences.

(i) Both agents survive in the long run, i.e., \( \lim_{T \to \infty} \hat{\lambda}_T \neq 0 \) and \( \lim_{T \to \infty} \hat{\lambda}_T \neq \infty \).

(ii) There exists no constrained efficient contract featuring full risk sharing.

(iii) When the constrained efficient contract features limited risk sharing, the long-run ergodic set of the relative Pareto weight is several certain boundary \( \hat{\lambda}_s \)'s plus sets of other points inside \( \hat{\lambda} \) intervals.

Proof. (i) From Proposition 1, \( \lim_{T \to \infty} \hat{\lambda}_T \in [\min_s \lambda_s, \max_s \hat{\lambda}_s] \). Thus, \( \lim_{T \to \infty} \hat{\lambda}_T \neq 0 \) and \( \lim_{T \to \infty} \hat{\lambda}_T \neq \infty \).

(ii) From Corollary 1, if there is an overlapping interval for all \( [\lambda_s, \hat{\lambda}_s]^S \), constrained efficient contract achieves full risk sharing under homogenous preferences. From Proposition 1, even if an overlapping interval exists, there is some positive probability that \( \lim_{T \to \infty} \hat{\lambda}_T \) will be outside the interval under heterogeneous preferences, due to the term \( g^\gamma_1 - \gamma_2 s \beta_2 \beta_1 \). Hence there exists no optimal full risk-sharing contract.

(iii) From Corollary 1, \( \lim_{T \to \infty} \hat{\lambda}_T \) will only take values from \( \{\lambda_s, \hat{\lambda}_s\}^S \) when agents have homogeneous preferences. From Proposition 1, \( \lim_{T \to \infty} \hat{\lambda}_T \) will take values not only from \( \{\lambda_s, \hat{\lambda}_s\}^S \) but also inside \( [\lambda_s, \hat{\lambda}_s]^S \) due to the term \( g^\gamma_1 - \gamma_2 s \beta_2 \beta_1 \), when agents have heterogeneous preferences. \( \square \)

2.4 Recursive Lagrangian Formulation

There is a difficulty for computation associated with the promised utility approach. That is, to find the maximum promised utility \( \bar{w}_s \) for each state \( s \), which is endogenous determined. Thus we resort to the relative managable recursive Lagrangian approach–introducing the relative Pareto weight (RPW) as the co-state variable–to compute the model solutions. The section of promised utility approach is kept to facilitate proofs for the evolution of the RPW.

Given initial Pareto weight \( \phi_i \), the planner’s problem is

\[
\max_{\{c_{1t}, c_{2t}\}} \left[ E_0 \sum_{t=0}^{\infty} \sum_{i=1,2} \beta_t^i \phi_i u^i(c_{it}) \right] \]

\( U^i_t(c_i) \geq U^i_t(e_i), \quad i = 1, 2, \quad t = 0, 1, 2, ... \)

\( c_{1t} + c_{2t} = e_t, \quad t = 0, 1, 2, ... \)

We follow Marcet and Marimon (2016) (see also Kehoe and Perri (2002)) and use recursive Lagrangian formulation to solve the problem. Let the Lagrange multiplier on the enforcement con-
The Lagrangian is
\[
L = \sum_{i=1,2} E_0 \sum_{t=0}^{\infty} \beta_i^t \left[ \phi_i u^i(c_{it}) + \mu_{it} \left[ E_t \sum_{j=0}^{\infty} \beta_i^j u^i(c_{it+j}) - U_t^i(e_i) \right] \right].
\]

By Abel’s summation, \( L \) can be simplified as
\[
L = \sum_{i=1,2} E_0 \sum_{t=0}^{\infty} \beta_i^t \left[ H_{it} u^i(c_{it}) - \mu_{it} U_t^i(e_i) \right]
\]
where \( H_{it} = H_{it-1} + \mu_{it} \)
\( H_{i,-1} = \phi_i \),

where \( H_{it} \) is the time-varying Pareto weight for agent \( i \) at \( t \) and it equals the sum of initial Pareto weight \( \phi_i \) and all historical multipliers up to \( t \), \( \{\mu_{is}\}_{s=0}^{\infty} \). First order conditions wrt. consumption imply
\[
\frac{u_1^i(c_{1t})}{u_2^i(c_{2t})} = \frac{H_{2t}}{H_{1t}} \left( \frac{\beta_2}{\beta_1} \right)^t,
\]
where \( u^i(c_{it}) \) denotes the marginal utility of consumption for agent \( i \). Redefine the multipliers,
\[
\lambda_t \equiv \frac{H_{2t}}{H_{1t}} \left( \frac{\beta_2}{\beta_1} \right)^t, \quad \nu_{it} \equiv \frac{\mu_{it}}{H_{it}},
\]
where \( \lambda_t \) is the relative Pareto weight (RPW) of agent 2 to agent 1. The law of motion for it can then be derived as
\[
\lambda_t = \frac{1 - \nu_{1t}}{1 - \nu_{2t}} \lambda_{t-1} \frac{\beta_2}{\beta_1}.
\]
When agent \( i \)’s enforcement constraint binds, \( \nu_{it} > 0 \), his RPW goes up. When both agents are not constrained, \( \nu_{it} = 0 \; i = 1,2 \), the more patient (higher \( \beta \)) agent has an increase in his RPW.

**Definition 1.** Given initial Pareto weights \( \phi_i \), constrained efficient allocations for the growth economy are \( \{c_1, c_2, \nu_1, \nu_2, \lambda\} \) which satisfy the following (2.2), (2.3), complementary slackness condition (2.4), and resource constraint (2.5),
\[
\frac{u_1^i(c_{1t})}{u_2^i(c_{2t})} = \lambda_t, \quad \lambda_t = \frac{1 - \nu_{1t}}{1 - \nu_{2t}} \lambda_{t-1} \frac{\beta_2}{\beta_1}, \quad \lambda_{-1} = \frac{\phi_2}{\phi_1},
\]
\[
\nu_{it} \geq 0, \quad \nu_{it} \left[ U_t^i(c_i) - U_t^i(e_i) \right] = 0, \quad (2.4)
\]
To obtain a stationary economy, we normalize some variables as follows,

\[ \hat{c}_{it} \equiv \frac{c_{it}}{e_t}, \quad \hat{\epsilon}_{it} \equiv \frac{\epsilon_{it}}{e_t}, \]

\[ \hat{\lambda}_t \equiv \frac{\lambda_t}{e_t^{\gamma_2-\gamma_1}}, \quad \hat{\lambda}_{-1} \equiv \frac{\lambda_{-1}}{e_{-1}}, \quad (e_{-1} \equiv 1) \]

\[ \hat{\pi}_i(z_{t+1}|z_t) \equiv \frac{\pi_i(z_{t+1}|z_t)g_{t+1}^{1-\gamma_i}}{\sum_{z_{t+1}} \pi_i(z_{t+1}|z_t)g_{t+1}^{1-\gamma_i}}, \]

\[ \hat{\beta}_i(z_t) \equiv \beta_i \sum_{z_{t+1}} \pi_i(z_{t+1}|z_t)g_{t+1}^{1-\gamma_i}. \]

**Definition 2.** Given initial Pareto weights \( \phi_i \), constrained efficient allocations for the stationary economy are \( \{\hat{c}_1, \hat{c}_2, \nu_1, \nu_2, \hat{\lambda}\} \) which satisfy the following \( (2.6), (2.7) \), complementary slackness condition \( (2.8) \), and resource constraint \( (2.9) \),

\[ \frac{u_1^i(\hat{c}_{1t})}{u_2^i(\hat{c}_{2t})} = \hat{\lambda}_t, \] (2.6)

\[ \hat{\lambda}_t = \frac{1 - \nu_1 \hat{\lambda}_{-1} g_{t+1}^{\gamma_1-\gamma_2} \beta_2}{\nu_2 \hat{\beta}_1}, \] (2.7)

\[ \hat{\lambda}_{-1} = \frac{\phi_2}{\phi_1}, \]

\[ \nu_{it} \geq 0, \quad \nu_{it} \left[ \hat{U}_i^i(\hat{c}_i) - \hat{U}_i^i(\hat{\epsilon}_i) \right] = 0, \] (2.8)

\[ \hat{c}_{1t} + \hat{c}_{2t} = 1. \] (2.9)

It is easy to derive the equivalence between the growth economy and stationary economy with redefined variables. And we see the consistence between the promised utility formulation and recursive Lagrangian formulation. In particular, \( \hat{\lambda} \), the normalized relative Pareto weight (RPW) in \( (2.7) \), is exactly the marginal utility ratio of consumption share in \( (2.1) \). One difference is that \( \hat{\lambda} \) is implied by envelop condition in promised utility formulation, while it is a state variable in the recursive Lagrangian formulation.

### 2.5 Computation

Denote state variables \( x = \{\hat{\lambda}, z\} \), where \( z \) is the joint Markov process of idiosyncratic uncertainty and aggregate uncertainty, and the added co-state variable \( \hat{\lambda} \) is the relative Pareto weight of agent 2 wrt. agent 1. Denote the set of policy functions and value functions as
\[ F(x) = \{ \hat{c}_i(x), \hat{\lambda}'(x), \nu_i(x), W_i(x) \}, i = 1, 2, \]

where
\[ W_i(x) = u(\hat{c}_i(x)) + \beta_i(z) \sum_{z'} \pi_i(z'|z) W_i(x'). \]

The computation algorithm follows the several steps,

1. Set up a grid \( X \) over the state space.
2. Set the full risk sharing solution as the initial guess \( F^0(x) \).
3. For each \( x \in X \), guess neither enforcement constraint binds.
   3a. If satisfied, set the new policies and value functions \( F^1(x) \) to be \( F^0(x) \)
   3b. If agent 1 or 2’s constraint is not satisfied, impose the binding constraint and recalculate the solution as \( F^1(x) \).
4. Iterate until \( |F^n(x) - F^{n-1}(x)| < \epsilon \).

We use linear interpolation for states not on the grid points. See appendix for more details.

2.6 Decentralization and Asset Prices

We follow the Alvarez and Jermann (2000) to decentralize the economy.\(^9\)

Definition 3. An competitive equilibrium with solvency constraints \( \{ B_i \} \) that are not too tight for initial conditions \( \{ a_{i,0} \} \) has quantities \( \{ a_i \} \) and prices \( \{ q \} \) s.t.

1. For each \( i \), given \( \{ a_{i,0} \} \) and prices \( \{ q \} \), \( \{ c_i, a'_i \} \) solves

\[
J_i (a_i(z^t)) = \max \left\{ u^i \left( c_i(z^t) \right) + \beta_i E [J_i \left( a_i(z^{t+1}) \right)] \right\}
\]

\[
c_i(z^t) + \sum_{z^{t+1}|z^t} a_i(z^{t+1}) q(z^{t+1}|z^t) = e_i(z^t) + a_i(z^t),
\]

\[
a_i(z^{t+1}) \geq B_i(z^{t+1}), \tag{2.10}
\]

where \( a_i \) is agent \( i \)'s Arrow security holdings, \( q \) is the price of Arrow security, \( B_i \) is agent \( i \)'s endogenous determined borrowing constraint, and \( J_i \) is agent \( i \)'s value function.

2. Good market and asset markets clear,
\[
\sum_i c_i(z^t) = e(z^t)
\]

\(^9\)Preference heterogeneity do not affect the proofs for the welfare theorems.
\[
\sum_i a_i (z^{t+1}) = 0
\]

3. Solvency constraint is not-too-tight,

\[ J_i \left( B_i(z^{t+1}) \right) = U^i_{t+1}(c_i). \]

The Euler equation can be derived as

\[ q_t(z^{t+1}|z^t) = \beta_i \pi(z^{t+1}|z^t) \frac{u^i_k(c_{i,t+1}(z^{t+1}))}{u^i_k(c_{i,t}(z^t))} + \zeta_{i,t+1}(z^{t+1}) \frac{u^i_k(c_{i,t}(z^t))}{u^i_k(c_{i,t}(z^t))}, \]

where \( \zeta_{i,t+1}(z^{t+1}) \) is the lagrangian multiplier on the solvency constraint (2.10). Since the two agents cannot have binding constraints at the same time, it follows that \( \zeta_{1,t+1} = 0 \) or \( \zeta_{2,t+1} = 0 \) or both. Thus

\[ q_t(z^{t+1}|z^t) = \pi(z^{t+1}|z^t) \max_{i=1,2} \left( \beta_i \frac{u^i_k(c_{i,t+1}(z^{t+1}))}{u^i_k(c_{i,t}(z^t))} \right). \] (2.11)

The agent whose solvency constraint is not binding has the highest intertemporal marginal rate of substitution (IMRS) and price the Arrow security at that state.

The asset pricing equation is

\[ E_t[M_{t+1} R_{t+1}] = 1, \]

where the stochastic discount factor (SDF) \( M_{t+1} \) is

\[ M_{t+1} = \max_{i=1,2} \left( \beta_i \frac{u^i_k(c_{i,t+1})}{u^i_k(c_{i,t})} \right) = \max_{i=1,2} \left( \beta_i \left( \frac{c_{i,t+1}}{c_{i,t}} \right)^{-\gamma} \left( \frac{e_{t+1}}{e_t} \right) \right). \] (2.12)

The risk-free rate is

\[ R_{f,t+1} = 1/E_t[M_{t+1}] \]

and the return of aggregate consumption claim is

\[ R_{s,t+1} = \frac{P_{t+1} + e_{t+1}}{P_t} = \frac{1 + \frac{P_{t+1}}{e_{t+1}}}{e_t} \frac{e_{t+1}}{P_t e_t} e_t. \]

where

\[ \frac{P_t}{e_t} = E_t \left[ M_{t+1} \left( 1 + \frac{P_{t+1}}{e_{t+1}} \right) \frac{e_{t+1}}{e_t} \right]. \]

\( P_t \) is the price of aggregate consumption claim. And the equity premium is defined as

\[ R_{e,t+1} = R_{s,t+1} - R_{f,t+1}. \]
We use the solution of consumption allocations from the planner’s problem to pin down the SDF and then the price-consumption ratio and asset returns.

3 Two-state Example

We introduce preference heterogeneity into the same two-state example as in Alvarez and Jermann (2001)\(^{10}\). We show that it induces discount rate shocks, which combined with the limited risk sharing generated from enforcement constraints generates a positive and volatile equity premium with the absence of aggregate uncertainty\(^{11}\).

Figure 1 compares the long-run consumption allocation under homogeneous risk aversion with that under heterogeneous risk aversion where agent 2 has a higher risk aversion (2.7) than agent 1 (1.5). Agents have the same time discount factor, which is fixed at \(\beta = 0.65\). With the same risk aversion, the two agents have symmetric consumption shares because transition probability and endowment shares are symmetric. The upper solid line shows the consumption share when the agent has a higher income share, while the lower dashed line for lower income share. As risk aversion increases, agents change from autarky to limited risk sharing to full risk sharing. The squares (for agent 1) and diamonds (for agent 2) present a particular case where agents have different risk aversion parameters, i.e., \(\gamma_1 = 1.5\) and \(\gamma_2 = 2.7\). The more risk averse agent 2 pays a premium to less risk averse agent 1 for insurance so that the former consumes less than the latter in both states (\(c_{1L} > c_{2L}\) and \(c_{1H} > c_{2H}\)). Besides, agent 1’s consumption allocation deviates little from the autarky if they have had the same low risk aversion of 1.5, but much less risk is shared for agent 2 than when they have the same high risk aversion of 2.7.

Figure 2 compares the long-run consumption allocation under homogeneous time discount factor with that under heterogeneous time discount factor where agent 1 has a higher patience level (0.7) than agent 1 (0.5). Agents have the same time risk aversion coefficient, which is fixed at \(\gamma = 3.0\). Similarly, the two agents have symmetric consumption shares when they have the same patience level \(\beta\). The upper solid (lower dashed) line shows the consumption share when the agent has a higher (lower) income share. As \(\beta\) rises, agents change from autarky to limited risk sharing to full risk sharing. The squares (for agent 2) and diamonds (for agent 1) present a particular heterogeneous \(\beta\) case \(\beta_1 = 0.7\) and \(\beta_2 = 0.5\). The more patient agent 1 pays a premium to less patient agent 2 for insurance so that the former consumes less than the latter in both states

\(^{10}\)The transition probability and endowment shares are

\[\Pi = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}, \quad \tilde{c}_1(s_1) = \begin{bmatrix} 0.641 \\ 0.359 \end{bmatrix}, \quad \tilde{c}_2(s_1) = \begin{bmatrix} 0.359 \\ 0.641 \end{bmatrix}.\]

\(^{11}\)Without aggregate uncertainty, equity premium is essentially term premium of a perpetual consol bond.
\( (c_{1L} < c_{2L} \text{ and } c_{1H} < c_{2H}) \).

When agents have the same preference, there will be no risk premium because agents are symmetric and there is no aggregate uncertainty. Although the agent who prices the asset is time-varying, the conditional SDF and thus price-(aggregate) consumption ratio do not change due to the symmetry assumption. But when agents have heterogeneous preferences, positive risk premium results instead. Preference heterogeneity breaks the symmetry of SDF and induces conditional variation in SDF and thus in price-consumption ratio. Therefore, preference heterogeneity generates asymmetric discount rate shocks and leads to positive equity premium even without aggregate cash flow uncertainty.

Table 1 dissects how heterogeneous preferences produces positive equity premium. Panel A contrasts the case of heterogeneous \( \gamma \) with homogeneous \( \gamma \) one while keeping \( \beta \) fixed at 0.65. When agents have the same \( \gamma \) (1.5 or 2.7), there is no variation in conditional SDF due to the symmetric transition probability matrix and symmetric endowment shares. Thus the stock return equals the risk-free rate \( (R_s = R_f) \) state by state and there is no equity premium. But when agents have different \( \gamma \), the symmetry in SDF breaks down. Especially for the transition from \( s_t = s_2 \) to \( s_{t+1} = s_1 \) which is priced by the more risk averse agent 2, SDF is much higher because he has high consumption volatility and higher risk aversion. This boosts the mean and volatility of SDF. Despite no cash flow uncertainty, conditional variation in SDF generates variation in price-consumption ratio, resulting in positive equity premium. Panel B contrasts the case of heterogeneous \( \beta \) with homogeneous \( \beta \) while keeping \( \gamma \) fixed at 3.0. The results are similar as in Panel A. When agents have the same \( \beta \) (0.7 or 0.5), there is no variation in conditional SDF. Thus \( R_s = R_f \) state by state and there is no equity premium. But when agents have different \( \beta \), the symmetry in SDF breaks down. Especially for the transition from \( s_t = s_1 \) to \( s_{t+1} = s_2 \) which is priced by the more patient agent 1, SDF is much higher due to high consumption volatility and higher patience. Besides, agent 1 is also the marginal pricer for the transition from \( s_t = s_2 \) to \( s_{t+1} = s_2 \) as he has a larger IMRS (larger \( \beta \) and no consumption change for both agents). Despite of no cash flow uncertainty, conditional variation in SDF generates variation in price-consumption ratio and positive equity premium as a result.

4 **Quantitative Results**

4.1 **Calibration**

We assume that there are four states and agents are symmetric in their endowment processes. We follow the high beta annual calibration of Alvarez and Jermann (2001) with ten moments to pin
down 10 parameters, including 2 aggregate growth rates, 2 idiosyncratic income shares, 2 aggregate transition probabilities and 4 idiosyncratic transition probabilities. See Appendix 6.2 for details. For preference parameters $\beta$ and $\gamma$, we pick values around the numbers used in Alvarez and Jermann (2001) ($\beta = 0.78$ and $\gamma = 3.5$). We simulate the model 1000 times and 5000 periods with the first 500 periods burned out to obtain the long run distribution.

4.2 Heterogeneous Risk Aversion

Figure 3 shows how the degree of risk sharing measured by consumption volatility and SDF change when agents have heterogeneous risk aversion $\gamma$ but the same time preference $\beta$. We compare the case where both agents have the same risk aversion of 2.5 or 4 with the case where one agent has risk aversion $\gamma_1 = 2.5$ while the other $\gamma_2 = 4$ for a range of $\beta$. When $\beta$ is small, risk sharing incentive is low. Agents stay in autarky and consumption volatility is high. As $\beta$ increases, the degree of risk sharing rises. When $\beta$ becomes large enough, agents achieve full risk sharing. The degree of risk sharing for the heterogeneous risk aversion case lies between the case of both low and both high risk aversion for most $\beta$. In particular, the amount of risk sharing does not increase much for the less risk averse agent 1, but decreases a fair amount for the more risk averse agent 2, compared with the case of same low and same high risk aversion respectively. Besides, agent 2’s consumption profile is more volatile than agent 1’s. Note that when $\beta > 0.96$, heterogeneous risk aversion generates less risk sharing for both agents than homogeneous ones. Even when $\beta$ is large enough, full risk sharing cannot be achieved as proved by Proposition 2.

The magnitude of $\beta$ affects SDF through two opposite channels. On one hand, higher $\beta$ leads to higher and more volatile SDF directly for given individual consumption growth. On the other hand, higher $\beta$ induces more risk sharing and thus lower consumption volatility, reducing the size and volatility of SDF. When $\beta$ is small, agents stay in autarky and only the first effect is present. Therefore, the mean and vol of SDF increase with $\beta$. As $\beta$ becomes larger, the second effect dominates and the mean and volatility of SDF decline with $\beta$. For a range of moderate $\beta$, heterogeneous risk aversion generates higher and more volatile SDF than both the case of same low and same high risk aversion. Especially, when $\beta$ lies in between 0.78 and 0.85, heterogeneous agents share limited risk, and the more risk averse agent with volatile consumption pushes up the mean and volatility of SDF at the states where he is not constrained and is the marginal pricer.

Figure 4 shows how the mean and volatility of asset returns change when agents have heterogeneous risk aversion but the same time preference. The two effects of $\beta$ on SDF is present inversely on the risk-free rate. The mean of $R_f$ ($E(R_f)$) first declines and then increases with $\beta$. But the volatility of $R_f$ ($\sigma(R_f)$) declines all the way down. This is because the direct effect of $\beta$ always
dominates the indirect effect on $\sigma(R_f)$. When $\beta$ lies in between 0.78 and 0.85, limited risk is shared and heterogeneous risk aversion produces lower $E(R_f)$ than homogeneous risk aversion. For most range of $\beta$, heterogeneity in risk aversion generates higher equity premium and higher equity volatility, because the more risk averse agent 2 feels very unsafe to hold stocks in recessions when his income volatility is higher and he cannot trade away most of his income risk with the less risk averse agent 1. Besides, $\sigma(R_f)$ is much higher when agents have different risk aversion coefficients. This results in over-volatile risk-free rate as discussed in Section 4.4.

4.3 Heterogeneous Time Preference

Figure 5 shows how the degree of risk sharing and SDF change when agents have heterogeneous time preference but the same risk aversion. We compare the case where both agents have the same time preference $\beta$ of 0.75 or 0.85 with the case where agent 1 has $\beta$ of 0.85 while agent 2 0.75 for each $\gamma$. When $\gamma$ is small, risk sharing incentive is low. Agents stay in autarky and consumption volatility is high. As $\gamma$ increases, the degree of risk sharing rises. When $\gamma$ becomes large enough, agents achieve full risk sharing. The degree of risk sharing for the heterogeneous $\beta$ case lies between the case of both low and both high $\beta$ for $\gamma \in [3, 4.2]$, where the amount of risk sharing does not increase much for the less patient agent 2, but decreases a fair amount for the more patient agent 1, in contrast to the case of same low and same high $\beta$ respectively. Besides, agent 2’s consumption profile is more volatile than agent 1’s. Note that when $\gamma > 4.7$, heterogeneous time discount factor generates less risk sharing for both agents than homogeneous ones do. Even when $\gamma$ is large enough, full risk sharing cannot be achieved as proved by Proposition 2.

The magnitude of $\gamma$ affects SDF through two opposite channels. On one hand, higher $\gamma$ leads to higher and more volatile SDF directly for given individual consumption growth. On the other hand, higher $\gamma$ induces more risk sharing and thus smaller individual consumption growth, reducing the size and volatility of SDF. When $\gamma$ is small, agents stay in autarky and only the first effect is present. Therefore, the mean and vol of SDF increase with $\gamma$. As $\gamma$ becomes larger, the second effect dominates and the mean and volatility of SDF decline with $\gamma$. For most $\gamma$, heterogeneous $\beta$ generates a higher mean of SDF, because the more patient agent will price more states including all transitions with no state change $s_{t+1} = s_t$ and thus no consumption change. Yet, the volatility of SDF for heterogeneous $\beta$ is smaller than for same low $\beta$ when $\gamma \in [2.9, 4.7]$, because agents have higher consumption volatility in the latter case.

Despite of the low SDF volatility, heterogeneous $\beta$ produces a high and volatile equity premium, as show in Figure 6. There is a spike for mean equity premium $E(R_e)$ and equity return volatility $\sigma(R_e)$ at $\gamma$ where agents change from autarky to little risk sharing. This is because when the
more patient agent 1 receives a higher income share (thus not constrained) and prices the assets, he requires a high compensation for bearing risk as little of his income risk can be traded away with the less patient agent 2. The pattern of the mean risk-free rate \( E(R_f) \) corresponds inversely with the mean of SDF \( E(M) \). Heterogeneous \( \beta \) generates variation in conditional SDF, leading to excessively volatile risk-free rate when \( \gamma \) is large. As demonstrated by the two-state example, preference heterogeneity in \( \gamma \) or \( \beta \) renders discount rate shocks asymmetric, generates conditional SDF variation, and results in volatile asset returns.

### 4.4 Asset Pricing Moments

Table 2 presents the simulation moments for asset prices and consumption allocation. Panel A and B show the effect of heterogeneous risk aversion and heterogeneous time preference respectively. For Panel A, \( \beta \) is chosen to match the mean of risk-free rate \( E(R_f) \) as closely as possible after \( \gamma \) is picked around the value 3.5 from Alvarez and Jermann (2001). For Panel B, \( \gamma \) is chosen to match \( E(R_f) \) as closely as possible after \( \beta \) is picked around the value 0.78 from Alvarez and Jermann (2001). Panel A shows that heterogeneity in \( \gamma \) \((\gamma_1 = 2.5, \gamma_2 = 4.0)\) increases the mean equity premium and equity volatility, 4.11% and 12.70%, in comparison with homogenous low risk aversion \((\gamma_1 = \gamma_2 = 2.5)\), 1.59% and 6.64%. But heterogeneous risk aversion does not necessarily improve over homogenous high risk aversion \((\gamma_1 = \gamma_2 = 4.0)\), where equity premium is slightly higher (4.52%) and equity volatility is slightly lower (10.74%) in the latter. Besides, risk-free rate is too volatile (10.93%). Panel B shows that heterogeneity in \( \beta \) \((\beta_1 = 0.85, \beta_2 = 0.75)\) does not necessarily increase SDF volatility but produce a much higher equity premium (7.05%) and more volatile equity returns (27.87%) than homogenous \( \beta \) \((3.99\% \text{ and } 9.94\% \text{ for } \beta_1 = \beta_2 = 0.75 \text{ and } 2.09\% \text{ and } 7.42\% \text{ for } \beta_1 = \beta_2 = 0.85)\). Moreover, risk-free rate volatility rises little (6.96% vs. 6.14% and 4.08%) although it is already more volatile than in the data (4.01%). The original Alvarez and Jermann (2001) matches the volatility of risk-free rate 5.67% from Mehra and Prescott (1985). In light that we follow their calibration, the heterogeneous \( \beta \) case shows much promise for better matching the mean and volatility of equity premium. That the risk-free rate is too volatile is because the main mechanism of the model originates from the discount rate shocks induced by enforcement constraints, which become asymmetric with the presence of preference heterogeneity. We mention the potential remedy for this in footnote 1.

As proved in Alvarez and Jermann (2001) and Krueger and Lustig (2010), when the distribution of idiosyncratic shocks is independent of aggregate shocks and aggregate shocks are i.i.d. (in short as independent risk), the consumption share \( \hat{c} \) does not depend on aggregate uncertainty. Hence term premium is zero and multiplicative equity premium is the same as in representative agent
economy. Table 3 illustrates that heterogeneous preferences generate positive term premium and higher equity premium than homogeneous preferences when risk is independent\textsuperscript{12}. Panel A shows the results for heterogeneous risk aversion when $\beta$ is fixed at 0.5. When agents have the same $\gamma = 2.5$ or 3.5, term premium is zero and equity premium is small (less than half percent). But when agents have different risk aversion ($\gamma_1 = 2.5$ and $\gamma_2 = 3.5$), term premium becomes 0.72% and multiplicative equity premium rises to 2.44%. But all returns become quite volatile. Panel B shows the results for heterogenous time discount factor when $\gamma$ is fixed at 3.0. Similarly, when agents have the same $\beta = 0.45$ or 0.55, term premium is zero and equity premium is small (less than half percent). But when agents have heterogeneous time preference ($\beta_1 = 0.55$ and $\beta_2 = 0.45$), small difference of $\beta(0.1)$ causes a 4.37% term premium and 7.72% multiplicative equity premium. Besides, equity and perpetual bond are much more volatile than risk-free rate. In contrast to the case of heterogeneous $\gamma$, heterogenous $\beta$ can generate a higher and more volatile equity premium without inducing too much volatility in risk-free rate.

In the benchmark model, all agent’s income is labor income. Agents are not endowed with any assets at the beginning. We relax this assumption and let the agents be endowed with both labor income and half Lucas tree initially. Lucas tree bears fruits as dividend income, which is a constant fraction of total income, $\omega = D_t / e_t$\textsuperscript{13}. The rest $1 - \omega$ fraction of $e_t$ is labor income and agents’ share of labor income is subject to idiosyncratic shocks as in the benchmark model. In default, the Lucas tree will be seized and agents consume only labor income in autarky. Table 4 presents how asset prices change as we vary $\omega$ from zero to 10%\textsuperscript{14}. Asset prices are very sensitive to the magnitude of $\omega$. As it increases, autarky becomes less attractive and agents share more risk. The volatility of consumption share and SDF decrease. Risk-free rate becomes much higher, while equity premium declines a lot, from 3.45% to 0.63%, 4.11% to 0.72%, and 7.05% to 2.59% for homogeneous preferences, heterogenous risk aversion, and heterogenous time discount factor, respectively. In order for limited enforcement to matter more for equity premium, we could alleviate the punishment for defaulting, such as allowing agents to trade a risk-free bond as in Krueger and Perri (2006) or allowing agents to come back to financial markets in several years after defaulting. These less severe punishment mechanisms instead of permanent exclusion from financial markets can reduce optimal risk sharing and increase equity premium\textsuperscript{15}.

\textsuperscript{12}The risk aversion and time discount factor parameters are chosen for qualitative illustration, not for matching moments quantitatively.
\textsuperscript{13}Other parameter values are not changed. So the only difference is that $\omega = 0$ in the benchmark model while it is positive here.
\textsuperscript{14}Chien, Cole, and Lustig (2012) calibrate the fraction of collateralizable income $\omega$ to be 10%.
\textsuperscript{15}Another way to boost equity premium is to view equity as a levered aggregate consumption claim.
5 Conclusion

We introduce heterogeneous preferences (heterogeneity in risk aversion and time discount factor) into a two-agent endowment economy with enforcement constraints, aggregate and idiosyncratic uncertainty (Alvarez and Jermann (2001)), and study the corresponding asset pricing and risk sharing implications. We find that the relative time discount factor and the interaction between heterogeneous risk aversion and aggregate uncertainty affect the evolution of the relative Pareto weight of agents. The more patient agent tends to obtain a higher Pareto weight, and the less risk averse agent tends to do so in booms and the reverse in recessions. We demonstrate that preference heterogeneity can generate a positive equity premium with only idiosyncratic uncertainty present since the conditional pricing kernel is time-varying depending on which agent is the marginal pricer. With the calibrated model, we show that preference heterogeneity boosts the mean and volatility of equity premium quantitatively, when the more risk averse and/or the more patient agent cannot trade away most of his income risk with the other agent. Especially, heterogeneous time preference shows much promise for better matching key asset pricing moments.

References


Table 1: SDF and Returns Across States: Heterogeneous v.s. Homogeneous Preferences

![Table](image)

Note: This table shows the asset prices in the two-state example for each transition from state at $t$ to state at $t+1$. Panel A fixes $\beta = 0.65$ and compares the case of homogeneous $\gamma$ with heterogeneous $\gamma$. Panel B fixes $\gamma = 3.00$ and compares the case of homogeneous $\beta$ with heterogeneous $\beta$. $\gamma_i$ is agent $i$’s risk aversion parameter and $\beta_i$ is agent $i$’s time discount factor. SDF stands for stochastic discount factor, $R_s$ is (gross) aggregate market return, $R_f$ is (gross) risk-free rate, and $R_e$ is equity premium.
Table 2: Moments for Asset Pricing and Risk Sharing

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<th>(\sigma(R_s))</th>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\gamma = 3.75), (\beta_1 = \beta_2 = 0.75)</td>
<td>0.87</td>
<td>3.99</td>
<td>6.14</td>
<td>9.94</td>
<td>0.40</td>
<td>0.99</td>
<td>1.29</td>
<td>0.273</td>
<td>0.273</td>
</tr>
<tr>
<td>(\beta_1 = \beta_2) vs. (\beta_1 = 0.85, \beta_2 = 0.85)</td>
<td>1.17</td>
<td>2.09</td>
<td>4.08</td>
<td>7.42</td>
<td>0.28</td>
<td>0.99</td>
<td>0.80</td>
<td>0.275</td>
<td>0.275</td>
</tr>
<tr>
<td>(\beta_1 \neq \beta_2)</td>
<td>1.12</td>
<td>7.05</td>
<td>6.96</td>
<td>27.87</td>
<td>0.25</td>
<td>0.99</td>
<td>0.88</td>
<td>0.273</td>
<td>0.280</td>
</tr>
</tbody>
</table>

Note: This table presents the unconditional moments from the model simulations. The moments are averaged across 1000 simulations, each with 5000 periods and the first 500 periods burned. \(E(R_f) (\sigma(R_f))\) is the mean (volatility) of risk-free rate, \(E(R_s - R_f)\) is mean of equity premium, and \(\sigma(R_s)\) is equity return volatility. Sharpe stands for Sharpe ratio, \(E(M) (\sigma(M))\) is the mean (volatility) of SDF, and \(\sigma(ln(\hat{c}_i))\) is the volatility of agent \(i\)'s consumption share. U.S. data sample moments of market excess return, market return volatility, and Sharpe ratio are from Bansal and Yaron (2004). And real risk-free rate sample moments are from Chien and Lustig (2010) using a long sample 1928-2007. AJ denotes the calibration and results from Alvarez and Jermann (2001). Panel A and B show the effects of heterogeneous \(\gamma\) and heterogeneous \(\beta\), respectively. The preference parameters are picked around the values from Alvarez and Jermann (2001). For heterogeneous \(\gamma\) (\(\beta\)) case, risk aversion coefficients are first picked, and then time discount factor is chosen to match \(E(R_f)\) as closely as possible.
Table 3: Moments under Independent Risk

<table>
<thead>
<tr>
<th></th>
<th>$E(R_f)$</th>
<th>$E(R_b - R_f)$</th>
<th>$E(R_s - R_f)$</th>
<th>$\sigma(R_f)$</th>
<th>$\sigma(R_b)$</th>
<th>$\sigma(R_s)$</th>
<th>Sharpe</th>
<th>$\sigma(ln\hat{c}_1)$</th>
<th>$\sigma(ln\hat{c}_2)$</th>
<th>$E(R_s)/E(R_f) - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma_1 = \gamma_2$</td>
<td>15.81</td>
<td>0.00</td>
<td>0.36</td>
<td>0.00</td>
<td>0.00</td>
<td>4.07</td>
<td>0.09</td>
<td>0.296</td>
<td>0.296</td>
<td>0.31</td>
</tr>
<tr>
<td>vs. $\gamma = 3.5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma_1 \neq \gamma_2$</td>
<td>24.87</td>
<td>0.00</td>
<td>0.54</td>
<td>0.00</td>
<td>0.00</td>
<td>4.40</td>
<td>0.12</td>
<td>0.196</td>
<td>0.196</td>
<td>0.43</td>
</tr>
<tr>
<td>fix $\beta = 0.5$</td>
<td>$\gamma_1 = 2.5$, $\gamma_2 = 3.5$</td>
<td>5.96</td>
<td>0.72</td>
<td>1.26</td>
<td>17.03</td>
<td>15.12</td>
<td>16.68</td>
<td>0.08</td>
<td>0.276</td>
<td>0.279</td>
</tr>
<tr>
<td><strong>Panel B</strong></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta = 0.45$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_1 = \beta_2$</td>
<td>8.19</td>
<td>0.00</td>
<td>0.40</td>
<td>0.00</td>
<td>0.00</td>
<td>3.81</td>
<td>0.10</td>
<td>0.296</td>
<td>0.296</td>
<td>0.37</td>
</tr>
<tr>
<td>vs. $\beta = 0.55$</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_1 \neq \beta_2$</td>
<td>15.01</td>
<td>0.00</td>
<td>0.43</td>
<td>0.00</td>
<td>0.00</td>
<td>4.05</td>
<td>0.11</td>
<td>0.222</td>
<td>0.222</td>
<td>0.37</td>
</tr>
<tr>
<td>fix $\gamma = 3.0$</td>
<td>$\beta_1 = 0.55$, $\beta_2 = 0.45$</td>
<td>6.42</td>
<td>4.37</td>
<td>6.77</td>
<td>9.22</td>
<td>25.82</td>
<td>33.02</td>
<td>0.21</td>
<td>0.273</td>
<td>0.277</td>
</tr>
</tbody>
</table>

Note: This table shows the unconditional moments under independent risk, where aggregate risk is i.i.d. and the distribution of the idiosyncratic risk is independent of aggregate risk ($M_1=0$, $M_{2,8,9,10}=1$ in the calibration Table 5). The moments are averaged across 1000 simulations, each with 5000 periods and the first 500 periods burned. $E(R_f)$ ($\sigma(R_f)$) is the mean (volatility) of risk-free rate, $E(R_s - R_f)$ ($E(R_b - R_f)$) is the mean of equity premium (the term premium of a perpetual bond), and $\sigma(R_s)$ ($\sigma(R_b)$) is the volatility of equity return (perpetual bond return). Sharpe stands for Sharpe ratio, $E(M)$ ($\sigma(M)$) is the mean (volatility) of SDF, and $\sigma(ln\hat{c}_i)$ is the volatility of agent $i$’s consumption share. $E(R_s)/E(R_f) - 1$ is the multiplicative equity premium. Panel A fixes $\beta = 0.5$ and contrasts homogeneous risk aversion of $\gamma = 2.5$ and $\gamma = 3.5$ with heterogeneous risk aversion of $\gamma_1 = 2.5$ and $\gamma_2 = 3.5$. Panel B fixes $\gamma = 3.0$ and contrasts homogeneous time discount factor of $\beta = 0.45$ and $\beta = 0.55$ with heterogeneous time discount factor of $\beta_1 = 0.55$ and $\beta_2 = 0.45$. 


### Table 4: Moments under Positive Assets

<table>
<thead>
<tr>
<th></th>
<th>$\omega$</th>
<th>$E(R_f)$</th>
<th>$E(R_s - R_f)$</th>
<th>$\sigma(R_f)$</th>
<th>$\sigma(R_s)$</th>
<th>Sharpe</th>
<th>$\sigma(ln\tilde{c}_1)$</th>
<th>$\sigma(ln\tilde{c}_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$</td>
<td>0.00</td>
<td>0.77</td>
<td>3.45</td>
<td>5.56</td>
<td>9.31</td>
<td>0.37</td>
<td>0.28</td>
<td>0.28</td>
</tr>
<tr>
<td>$\beta = 0.78$</td>
<td>0.01</td>
<td>13.93</td>
<td>2.17</td>
<td>4.70</td>
<td>8.18</td>
<td>0.27</td>
<td>0.21</td>
<td>0.21</td>
</tr>
<tr>
<td>$\gamma = 3.5$</td>
<td>0.05</td>
<td>27.54</td>
<td>1.06</td>
<td>3.46</td>
<td>6.92</td>
<td>0.15</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>$\beta = 0$</td>
<td>0.10</td>
<td>34.28</td>
<td>0.63</td>
<td>2.72</td>
<td>6.28</td>
<td>0.10</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td><strong>Panel B:</strong></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma_1 \neq \gamma_2$</td>
<td>0.00</td>
<td>0.82</td>
<td>4.11</td>
<td>10.93</td>
<td>12.70</td>
<td>0.32</td>
<td>0.27</td>
<td>0.28</td>
</tr>
<tr>
<td>$\beta = 0.81$</td>
<td>0.01</td>
<td>11.01</td>
<td>2.41</td>
<td>6.83</td>
<td>9.03</td>
<td>0.27</td>
<td>0.21</td>
<td>0.23</td>
</tr>
<tr>
<td>$\gamma_1 = 2.5$, $\gamma_2 = 4$</td>
<td>0.05</td>
<td>22.37</td>
<td>0.98</td>
<td>3.64</td>
<td>6.75</td>
<td>0.15</td>
<td>0.12</td>
<td>0.14</td>
</tr>
<tr>
<td>$\beta_1 = 0.85$, $\beta_2 = 0.75$</td>
<td>0.10</td>
<td>27.85</td>
<td>0.72</td>
<td>3.02</td>
<td>6.47</td>
<td>0.11</td>
<td>0.06</td>
<td>0.08</td>
</tr>
<tr>
<td><strong>Panel C:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_1 \neq \beta_2$</td>
<td>0.00</td>
<td>1.12</td>
<td>7.05</td>
<td>6.96</td>
<td>27.87</td>
<td>0.25</td>
<td>0.27</td>
<td>0.28</td>
</tr>
<tr>
<td>$\gamma = 3.2$</td>
<td>0.01</td>
<td>8.43</td>
<td>4.65</td>
<td>8.45</td>
<td>12.32</td>
<td>0.38</td>
<td>0.23</td>
<td>0.25</td>
</tr>
<tr>
<td>$\beta_1 = 0.85$, $\beta_2 = 0.75$</td>
<td>0.05</td>
<td>17.59</td>
<td>3.35</td>
<td>8.40</td>
<td>11.38</td>
<td>0.29</td>
<td>0.17</td>
<td>0.21</td>
</tr>
</tbody>
</table>

Note: This table shows the unconditional moments under positive collateral income. The moments are averaged across 1000 simulations, each with 5000 periods and the first 500 periods burned. $\omega$ is the share of dividend income as of total income, which is seizable in default. $E(R_f)$ ($\sigma(R_f)$) is the mean (volatility) of risk-free rate, $E(R_s - R_f)$ is mean of equity premium, and $\sigma(R_s)$ is equity return volatility. Sharpe stands for Sharpe ratio, and $\sigma(ln\tilde{c}_i)$ is the volatility of agent $i$'s consumption share. Panel A shows the results under homogeneous preferences $\beta = 0.78$ and $\gamma = 3.5$ as in Alvarez and Jermann (2001). Panel B shows the results under heterogeneous risk aversion $\beta = 0.81$, $\gamma_1 = 2.5$ and $\gamma_2 = 4$. Panel C shows the results under heterogeneous time discount factor $\gamma = 3.2$, $\beta_1 = 0.85$, and $\beta_2 = 0.75$. 
Figure 1: Consumption Share in the Two-state Example: Homogeneous $\gamma$ vs. Heterogeneous $\gamma$

Note: This figure shows the long-run stationary consumption shares of agent 1 and 2 in the two-state example. The horizontal ax denotes risk aversion and the vertical ax consumption shares. The agents’ time discount factors are the same, and fixed at $\beta = 0.65$. The upper solid (lower dashed) line depicts how the agent’s consumption share changes with risk aversion when he receives a high (low) endowment realization under homogeneous risk aversion ($\gamma_1 = \gamma_2$). The squares and diamonds present a particular case where agents have different risk aversion coefficients, i.e., $\gamma_1 = 1.5$ and $\gamma_2 = 2.7$. The squares denotes the consumption share of the less risk averse agent 1 ($\gamma_1 = 1.5$) when he receives a high (upper square) or low (lower square) endowment realization. The similar diamonds are for the more risk averse agent 2 ($\gamma_2 = 2.7$). $c_{1L} = 0.377$, $c_{1H} = 0.628$, $c_{2L} = 0.372$, $c_{2H} = 0.623$. 

\begin{align*}
  c_{H}, \gamma_1 = \gamma_2 &= c_{L}, \gamma_1 = \gamma_2 \\
  c_{1L}, \gamma_1 < \gamma_2 &= c_{1H}, \gamma_1 < \gamma_2 \\
  c_{2L}, \gamma_1 < \gamma_2 &= c_{2H}, \gamma_1 < \gamma_2
\end{align*}
Figure 2: Consumption Share in the Two-state Example: Homogeneous $\beta$ vs. Heterogeneous $\beta$

Note: This figure shows the long-run stationary consumption shares of agent 1 and 2 in the two-state example. The horizontal ax denotes time discount factor and the vertical ax consumption shares. The agents’ risk aversion coefficients are the same, and fixed at $\gamma = 3.0$. The upper solid (lower dashed) line depicts how the agent’s consumption share changes with time discount factor when he receives a high (low) endowment realization under homogeneous time discount factor ($\beta_1 = \beta_2$). The squares and diamonds present a particular case where agents have different time discount factors, i.e., $\beta_1 = 0.7$ and $\beta_2 = 0.5$. The squares denotes the consumption share of the less patient agent 2 ($\beta_2 = 0.5$) when he receives a high (upper square) or low (lower square) endowment realization. The similar diamonds are for the more patient agent 1 ($\beta_1 = 0.7$). $c_{1L} = 0.422$, $c_{1H} = 0.531$, $c_{2L} = 0.469$, $c_{2H} = 0.578$. 
Note: This figure shows how agents’ consumption shares and the resulting SDF (stochastic discount factor) change with time discount factor $\beta$. Agents have the same $\beta$ but may have different risk aversion coefficients $\gamma$. The upper part of the figure presents the standard deviation of consumption shares, which measures the degree of risk sharing. The diamond-marked line indicates that both agents have the same risk aversion $\gamma = 2.5$, while the square-marked line same $\gamma = 4.0$. The rest two lines presents the case of heterogeneous risk aversion, $\gamma_1 = 2.5$ (the dashed line for agent 1) and $\gamma_2 = 4.0$ (the solid line for agent 2). The more risk averse agent 2 has less volatile consumption than the less risk averse agent 1 when there is non-zero risk sharing. The lower part of the figure presents the mean and standard deviation of SDF. The diamond-marked (square-marked) line stands for homogeneous risk aversion $\gamma = 2.5$ ($\gamma = 4.0$) while the solid line stands for heterogeneous risk aversion $\gamma_1 = 2.5$ and $\gamma_2 = 4.0$. 

Figure 3: Risk Sharing: Heterogeneous vs. Homogeneous Risk Aversion
Figure 4: Asset Prices: Heterogeneous vs. Homogeneous Risk Aversion

Note: This figure shows how the unconditional moments of the risk-free rate and equity returns change with time discount factor $\beta$. Agents have the same $\beta$ but may have different risk aversion coefficient $\gamma$. The diamond-marked (square-marked) line stands for homogeneous risk aversion $\gamma_1 = 2.5$ ($\gamma_2 = 4.0$), while the solid line stands for heterogeneous risk aversion $\gamma_1 = 2.5$ and $\gamma_2 = 4.0$. 
Figure 5: Risk Sharing: Heterogeneous vs. Homogeneous Time Discount Factor

Note: This figure shows how agents’ consumption shares and the resulting SDF (stochastic discount factor) change with risk aversion $\gamma$. Agents have the same $\gamma$ but may have different time discount factor $\beta$. The upper part of the figure presents the standard deviation of consumption shares, which measures the degree of risk sharing. The diamond-marked line indicates that both agents have the same time discount factor $\beta = 0.75$, while the square-marked line same $\beta = 0.85$. The rest two lines present the case of heterogeneous time discount factor, $\beta_1 = 0.85$ (the dashed line for agent 1) and $\beta_2 = 0.75$ (the solid line for agent 2). The more patient agent 1 has less volatile consumption than the less patient agent 2 when there is non-zero risk sharing. The lower part of the figure presents the mean and standard deviation of SDF. The diamond-marked (square-marked) line stands for homogeneous time discount factor $\beta = 0.75$ ($\beta = 0.85$) while the solid line stands for heterogeneous time discount factor $\beta_1 = 0.85$ and $\beta_2 = 0.75$. 

Figure 6: Asset Prices: Heterogeneous vs. Homogeneous Time Discount Factor

Note: This figure shows how the unconditional moments of the risk-free rate and equity returns change with risk aversion \( \gamma \). Agents have the same \( \gamma \) but may have different time discount factor \( \beta \). The diamond-marked (square-marked) line stands for homogeneous time discount factor \( \beta = 0.75 \) (\( \beta = 0.85 \)), while the solid line stands for heterogeneous time discount factor \( \beta_1 = 0.75 \) and \( \beta_2 = 0.85 \).
6 Appendix

6.1 Computation

Let state variables be \( x = (\lambda, z) \). The policy and value functions are \( \hat{c}_i(x), \lambda'(x), \nu_i(x), W_i(x) \), where the value function

\[
W_i(x) = u(\hat{c}_i(x)) + \beta_i(z) \sum_{z'} \pi_i(z'|z) W_i(x').
\] (6.1)

The computation algorithm follows the steps,

1. Set up a grid \( X \) over the state space.
2. Set initial guess to be the solution to the planner’s problem without enforcement constraints.
   \( \nu^0_i(x) = 0, z^0_i(x) = z, \hat{c}^0_i(x), W^0_i(x) \) satisfy (2.6), (2.9) and (6.1).
3. Consider three possible binding patterns of enforcement constraints
   - Neither constraint binds
   - Agent 1’s constraint binds
   - Agent 2’s constraint binds

3.1 For each \( x \in X \), compute allocations assuming neither constraint binds.
3.2 Then check
   \[
u^1_i(x) = \nu^0_i(x), \lambda^1(x) = \lambda^0(x), \hat{c}^1_i(x) = \hat{c}^0_i(x), W^1_i(x) = W^0_i(x).
\]
   (6.2)
   3.2.1 If (6.2) satisfied for \( i = 1, 2 \), then set new policies
   \[
   u^1(\hat{c}^1_i(x)) + \beta(z) \sum_{z'} \pi(z'|z) W^1_i(x') \geq U^i(\hat{e}_i(z)) \quad \text{for } i = 1, 2 \]
   3.2.2 If (6.2) satisfied for \( i = 2 \) but not \( i = 1 \), then set \( \nu^2_1(x) = 0 \), solve \( \nu^1_i(x), \lambda^1_i(x), \hat{c}^1_i(x), \hat{c}^2_i(x) \) from (2.6), (2.7), (2.9) and
   \[
   u^1(\hat{c}^1_i(x)) + \beta(z) \sum_{z'} \pi(z'|z) W^1_i(x') = U^1(\hat{e}_1(z)).
\] (6.3)
   Set \( W^1_i(x) \) as LHS (left-hand-side) of (6.3) and \( W^2_1(x) \) as LHS of (6.2).

3.2.3 If (6.2) satisfied for \( i = 1 \) but not \( i = 2 \), then set \( \nu^1_i(x) = 0 \), solve \( \nu^2_i(x), \lambda^1_i(x), \hat{c}^1_i(x), \hat{c}^2_i(x) \) from (2.6), (2.7), (2.9) and
   \[u^2(\hat{c}^2_i(x)) + \beta(z) \sum_{z'} \pi(z'|z) W^2_i(z') = U^2(\hat{e}_2(z)).\] (6.4)
Set $W_1^i(x)$ as LHS of (6.2) and $W_2^j(x)$ as LHS of (6.4).

4.1 If the difference between $(\nu_0^i(x), \lambda_0^i(x), \hat{c}_0^i(x), W_0^i(x))$ and $(\nu_1^i(x), \lambda_1^i(x), \hat{c}_1^i(x), W_1^i(x))$ is small enough for each $x \in X$, then stop.

4.2 If not, then set initial guess equal to the new set of policy, multiplier, and value functions. Keep iterating until the value functions and policy functions converge.

6.2 Calibration

The endowment process is

<table>
<thead>
<tr>
<th>State</th>
<th>$g$</th>
<th>$\hat{e}_2$</th>
<th>$\hat{e}_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$g_L$</td>
<td>$\frac{1}{2} - \theta$</td>
<td>$\frac{1}{2} + \theta$</td>
</tr>
<tr>
<td>2</td>
<td>$g_H$</td>
<td>$\frac{1}{2} - \eta$</td>
<td>$\frac{1}{2} + \eta$</td>
</tr>
<tr>
<td>3</td>
<td>$g_L$</td>
<td>$\frac{1}{2} + \theta$</td>
<td>$\frac{1}{2} - \theta$</td>
</tr>
<tr>
<td>4</td>
<td>$g_H$</td>
<td>$\frac{1}{2} + \eta$</td>
<td>$\frac{1}{2} - \eta$</td>
</tr>
</tbody>
</table>

The transition matrix is

$$
\begin{bmatrix}
 p_{LL}\pi_{LL} & p_{LH}(1 - \pi_{LL}) & (1 - p_{LL})\pi_{LL} & (1 - p_{LH})(1 - \pi_{LL}) \\
p_{HL}(1 - \pi_{HH}) & p_{HH}\pi_{HH} & (1 - p_{HL})(1 - \pi_{HH}) & (1 - p_{HH})\pi_{HH} \\
(1 - p_{LL})\pi_{LL} & (1 - p_{LH})(1 - \pi_{LL}) & p_{LL}\pi_{LL} & p_{LH}(1 - \pi_{LL}) \\
(1 - p_{HL})(1 - \pi_{HH}) & (1 - p_{HH})\pi_{HH} & p_{HL}(1 - \pi_{HH}) & p_{HH}\pi_{HH}
\end{bmatrix}
$$

Aggregate growth rate follows a Markov process and the idiosyncratic income shares follows a Markov process conditional on the aggregate transition. $\pi_{LL}$ ($\pi_{HH}$) denotes the aggregate transition probability from recession (boom) to recession (boom). $p_{ij}$ denotes the idiosyncratic transition probability of agents having the same relative status (higher or lower than the other agent) of income share conditional on the aggregate state transition from $i$ to $j$.

There are ten parameters to be calibrated besides preference parameters:

$$ g_L, g_H, \theta, \eta, \pi_{LL}, \pi_{HH}, p_{LL}, p_{LH}, p_{HL}, p_{HH}. $$

The calibrated parameters following Alvarez and Jermann (2001) are in Table 5.

The resulting endowment process is

<table>
<thead>
<tr>
<th>State</th>
<th>$g$</th>
<th>$\hat{e}_2$</th>
<th>$\hat{e}_1$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9602</td>
<td>0.3562</td>
<td>0.6438</td>
</tr>
<tr>
<td>2</td>
<td>1.0402</td>
<td>0.3562</td>
<td>0.6438</td>
</tr>
<tr>
<td>3</td>
<td>0.9602</td>
<td>0.6438</td>
<td>0.3562</td>
</tr>
<tr>
<td>4</td>
<td>1.0402</td>
<td>0.6438</td>
<td>0.3562</td>
</tr>
</tbody>
</table>
And the resulting transition matrix is

\[
\begin{bmatrix}
0.1414 & 0.8200 & 0.0309 & 0.0077 \\
0.2637 & 0.6820 & 0.0486 & 0.0057 \\
0.0309 & 0.0077 & 0.1414 & 0.8200 \\
0.0486 & 0.0057 & 0.2637 & 0.6820 \\
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aggregate growth rate in recessions (L)</td>
<td>( g_L )</td>
<td>0.9602</td>
<td>Mehra-Prescott (1985): M1: ( \rho(g) = -0.14 )</td>
</tr>
<tr>
<td>Aggregate growth rate in booms (H)</td>
<td>( g_H )</td>
<td>1.0402</td>
<td>M2: ( E(g) = 1.83% )</td>
</tr>
<tr>
<td>Transition probability from L to L</td>
<td>( \pi_{LL} )</td>
<td>0.1723</td>
<td>M3: ( \text{Std}(g) = 3.57% )</td>
</tr>
<tr>
<td>Transition probability from H to H</td>
<td>( \pi_{HH} )</td>
<td>0.6877</td>
<td>NBER 1889-1991: M4: ( \text{Pr}(\text{boom})/\text{Pr}(\text{recession}) = 2.65 )</td>
</tr>
<tr>
<td>High income share in recessions (1/2 + ( \theta ))</td>
<td>0.6438</td>
<td></td>
<td>Alvarez-Jermann (2001):</td>
</tr>
<tr>
<td>High income share in booms (1/2 + ( \eta ))</td>
<td>0.6438</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Idiosyncratic transition probability of agent 2 (high-high) income shares conditional on LL</td>
<td>( p_{LL} )</td>
<td>0.8206</td>
<td>M5: ( \text{Std}(\hat{e}_i(z)) = 0.296 )</td>
</tr>
<tr>
<td>Idiosyncratic transition probability of agent 2 (high-high) income shares conditional on HH</td>
<td>( p_{HH} )</td>
<td>0.9907</td>
<td>M6: ( \rho(\ln(\hat{e}_i(z))) = 0.9 )</td>
</tr>
<tr>
<td>Idiosyncratic transition probability of agent 2 (high-high) income shares conditional on LH</td>
<td>( p_{LH} )</td>
<td>0.8444</td>
<td>M7: ( \sum_{i=1,2} [\hat{e}<em>i(L) - \frac{1}{2}]^2 = \sum</em>{i=1,2} [\hat{e}_i(H) - \frac{1}{2}]^2 )</td>
</tr>
<tr>
<td>Idiosyncratic transition probability of agent 2 (high-high) income shares conditional on HL</td>
<td>( p_{HL} )</td>
<td>0.9917</td>
<td>M8: ( \text{Std}(\ln(\hat{e}<em>i(z</em>{t+1}))</td>
</tr>
<tr>
<td>Idiosyncratic transition probability of agent 2 (high-high) income shares conditional on HH</td>
<td>( p'_{LH} )</td>
<td>( \sigma_{L} \sigma_{H} )</td>
<td>M9: ( \text{Std}(\ln(\hat{e}<em>i(z</em>{t+1}))</td>
</tr>
<tr>
<td>Idiosyncratic transition probability of agent 2 (high-high) income shares conditional on LH</td>
<td>( p'_{HH} )</td>
<td>( \sigma_{L} \sigma_{H} )</td>
<td>M10: ( \text{Std}(\ln(\hat{e}<em>i(z</em>{t+1}))</td>
</tr>
</tbody>
</table>

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