Constructing Counterfactual Wage Distribution Using
A General Equilibrium Labor Market Model with
Heterogeneity and Unobservable Efficiency

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Abstract
Welfare analysis of wage inequality requires constructing counterfactual wage distributions. I propose a method based on a fully nonparametric general equilibrium labor market model in which heterogenous workers and firms trade effective labor. Effective labor depends on two factors, observable hours and unobservable efficiency. Contrary to previous partial equilibrium approaches, counterfactual interventions in my model affect the behaviors of both workers and firms, and hence the market equilibrium. I show nonparametric identification of the structural functions of the model, in particular the unobservable efficiency function. The identified structural functions are used to generate counterfactual wage samples through a simulation method I prescribe. As a preliminary step towards analyzing identification and estimation of counterfactual wages, I introduce the operators that map the structural functions to parameters of interests. My model works under a wide range of counterfactual interventions, predicting an individual wage for each worker in the sample, based on which one could obtain the distribution.

1 Introduction
Constructing counterfactual distributions constitutes an indispensable step in empirical analysis (e.g. ?). Importance of counterfactual distributions is particularly prominent in welfare analysis of wage inequality. For example, to estimate welfare loss from gender discrimination, one needs to decompose differences in wage distributions between men and women into a discrimination effect and a composition effect (see ?; and reference therein). Or, to understand the reasons of increasing wage inequality, one needs to quantify the contributions of various causes to the differences in wage distributions over time (e.g. ?).

In all these contexts, the notion of wage distributions involves the entire employed population in the market. Any counterfactual intervention that potentially affects the entire employed population could render the market to a new equilibrium. Relationships among observable quantities estimated using a partial equilibrium model generally fail
to hold in other equilibria. Therefore, economists need a general equilibrium model to analyze counterfactual wage distributions.

Efficiency of work, as well as hours, determines how much workers get paid. Efficiency also affects workers’ choice of hours. Therefore, structural labor market models used to predict wages should take efficiency into account. An obstacle to this endeavor, however, is that efficiency is unobservable to economists, and difficult to quantify even for workers and firms themselves.

This paper is the first to construct counterfactual wage distributions using a general equilibrium labor market model with heterogeneity and unobservable efficiency. In contrast to other attempts, my approach explicitly characterizes the labor market equilibrium in which both workers and firms make optimal decisions. Compared with other equilibrium models with heterogeneous agents on both sides of the market, my model incorporates unobservable quality (efficiency in labor market). Moreover, my model allows for market level heterogeneity.

One of the contributions of my paper is to provide nonparametric identification of the structural functions of the model. In particular, efficiency function is nonparametrically identified up to a scale normalization. My identification strategy proceeds in four steps. Firstly, I nonparametrically identify the reduced form relationships among observed quantities, such as wage income, hours of work and worker’s observable characteristics, in each market. Secondly, the efficiency function is nonparametrically identified up to a scale normalization using multiple market variations. Thirdly, the reduced form wage scheme function is nonparametrically identified using the wage equation in each market. And finally, the marginal utility function and the marginal revenue function are nonparametrically identified using optimization conditions.

Another contribution is that I prescribe a simulation method of generating individual counterfactual wage for each worker in the sample, using the identified structural functions. Counterfactual wages are invariant to the scale normalization of the efficiency function. One could then compare the counterfactual wage sample \( \{\tilde{I}_m^{n} \}_{m=1}^{M} \) with the observed sample \( \{I_1^{n}, \ldots, I_N^{n}\}_{n=1}^{N} \) for a variety of analysis. Here are just some examples of summary statistics one might be interested in:

- **C.D.F.:** \( \hat{F}_{I_m}(t) = \frac{1}{N^m} \sum_{i=1}^{N^m} I_i^m \leq t \)
- **Quantiles:** \( \hat{Q}_{I_m}(p) = I_{\lfloor p N^m \rfloor} \)
- **Gini coefficient:** \( \hat{G}_{I_m} = 1 - \frac{2}{n-1} \left( n - \frac{\sum_{i=1}^{N^m} i I_i^m}{\sum_{i=1}^{N^m} I_i^m} \right) \)
- **Theil index:** \( \hat{T}_{I_m} = \frac{1}{N^m} \sum_{i=1}^{N^m} \left( \frac{I_i^m}{\tilde{I}_m^m} \right) \ln \left( \frac{I_i^m}{\tilde{I}_m^m} \right) \) where \( \tilde{I}_m^m = \frac{1}{N^m} \sum_{i=1}^{N^m} I_i^m \)
where \( \{I_i^m\}_{i=1}^{N^m} \) are (ascending) order statistics in market \( m \in \{1, \ldots, M\} \). The counterfactual versions of these statistics are obtained by replacing \( I_i^m \)'s in the above formulae with \( \tilde{I}_i^m \)'s.

Even though tailored to analyze labor markets, the essence of my model applies to other counterfactual distribution problems in markets where the unobservable quality of the differentiated products plays a substantial role.

In the existing literature, there are two established approaches to constructing counterfactual wage distributions in response to changes in distributions of individual characteristics. The “reweighting” approach reweights the actual wage distribution using Bayes’ rule. On the other hand, the “direct” approach first estimates the marginal distribution of individual characteristics and the conditional wage distribution given individual characteristics separately, then combines them together under different counterfactual interventions. ?, ? took the re-weighting approach. ?, ?, ? took the direct approach.

Both approaches made strong assumptions about how changes in individual characteristics affect changes in wage distributions. These assumptions sometimes were explicitly stated, sometimes not. To summarize them, here I introduce a simple model general enough to cover most cases presented in previous literature. Suppose for individual \( i \) in group \( D_i \), her wage income \( I_i \) has the following reduced form relationship with her observable characteristics \( x_i \) and unobservable characteristic \( a_i \)

\[
I_i = g(x_i, a_i, D_i)
\]

where \( g \) is an unknown function, and \( D_i \) is a group index. For example, \( D_i = 1 \) for treatment group and \( D_i = 0 \) for control group; or \( D_i = 79 \) for individuals in year 1979 and \( D_i = 88 \) for individuals in year 1988. Both approaches in the literature make the following two assumptions:

1. \( D_i \perp\!
\perp a_i | x_i \) a.s.;
2. Changes in the distribution of \( x_i \) don’t affect the unknown function \( g \).

The first assumption is the usual unconfoundedness assumption in the treatment effect literature. The second assumption excludes the general equilibrium effects, which will result in different functional forms for \( g \) in counterfactual equilibria. While partial equilibrium analysis may apply to small scale experiment contexts, constructing counterfactual wage distributions for the whole employed population entails a general equilibrium framework. This paper shows identification of the structural functions in a general equilibrium labor market model, and constructs counterfactual wage distributions based on the structural functions.

Another strand of literature closely related to this paper is the one on hedonic equilibrium models. In classic hedonic equilibrium models (e.g. ?, ?), differentiated products are only described by their observable (to economists) characteristics. ? investigated semi-parametric identification and estimation of the structural functions in a hedonic equilibrium model with unobservable characteristics. ? developed identification and estimation
of the structural functions in a nonseparable hedonic equilibrium model with some shape restrictions on the structural functions.

This paper advances the literature on hedonic equilibrium models in two aspects: modeling unobservable quality and constructing counterfactual outcomes. With regard to the first aspect, existing literature assumes that price depends only on observable quantity of the good or service being transferred. But in labor markets, wage depends not only on observable hours (quantity), but also on unobservable efficiency (quality). I explicitly model and identify the unobservable quality in a hedonic equilibrium labor market model. As for the second aspect, the main purpose of existing literature is to achieve identification of the structural functions. However, counterfactual analysis requires one step further: solving for counterfactual equilibria using the identified structural functions. I define the operators that map the structural functions to the equilibrium outcomes in order to facilitate counterfactual analysis.

Last but not least, my model has implications on labor supply and wage determination. In existing labor economics literature (e.g. ?; ?; and reference therein), labor supply is analyzed in partial equilibrium framework, where wage scheme is either exogenously given or derived under special parametric specifications. The wage scheme in my model is endogenously determined in the equilibrium of a nonparametric, nonseparable model, and the supply side of my model is consistent with standard utility-based labor supply model (see ? for example). Moreover, existing labor supply and wage determination literature mainly concerns quantity of labor, but rarely quality. My model takes both into account.

The rest of this paper is organized as follows. Section 2 describes the model. Section 3 discusses the identification of the structural functions of the model. Section 4 gives the method of simulating counterfactual wage samples using the identified structural functions. Section 5 formally defines the operators which map the structural functions to the equilibrium outcomes under counterfactual interventions. Section 6 points out some important questions requiring further research. A few complimentary results are collected in Appendices.

2 Model

I consider a large number of competitive labor markets, indexed by \( m \in \mathcal{M} \). In every market there are a large number of workers and a large number of single-worker firms. Let the \( d_x \times 1 \) vector \( x \) be a worker’s observable characteristics, the scalar \( a \) be the worker’s unobservable characteristic. Suppose \( f_{x,a}^m \) is the joint density of \( (x,a) \) on \( \mathcal{X} \times \mathcal{A} \) in market \( m \). And let the scalar \( h \) be the worker’s observable hours of work, and the scalar \( e \) be the worker’s unobservable efficiency. Moreover, let the \( d_{m^*} \times 1 \) vector \( m^* \) be market-level characteristics concerning the workers.

What matters ultimately to the firms is the amount of effective labor \( z \) delivered by the workers. The amount of effective labor is determined by two factors, the hours \( h \) (quantity) and the efficiency \( e \) (quality), through their product. Given the amount of
effective labor $z$, the combinations of $h$ and $e$ do not make a difference from the firms’ perspective. Workers’ efficiency $e$ is inherently determined by their observable characteristics $x$, unobservable characteristics $a$, and possibly the market characteristics $m^*$. On the other hand, the hours $h$ is chosen by the workers to maximize their contemporary utility. I assume that the workers derive disutility only from hours.

**Assumption 1.** Suppose the unobservable amount of effective labor $z$ is determined as the product of efficiency $e = e(x, a, m^*)$ and hours $h$, that is, $z = h \times e$.

In Assumption 1, $e(x, a, m^*)$ denotes the efficiency of a worker with $(x, a)$ characteristics in a market with $m^*$ characteristics. Market level characteristics may affect efficiency of an individual worker through, for example, peer pressure, spill-over of knowledge, or complementarities among the workers. The variables $m^*$ in the specification of the efficiency function capture these effects.

Let $U(h, x, a, m^*)$ be the disutility of working $h$ hours for a worker with $(x, a)$ characteristics in a market with $m^*$ characteristics. Assume that the functions $U: \mathcal{H} \times \mathcal{X} \times \mathcal{A} \times \mathcal{M}^* \rightarrow \mathbb{R}$ and $e: \mathcal{X} \times \mathcal{A} \times \mathcal{M}^* \rightarrow \mathbb{R}$ do not vary among workers or among different markets, so they are structural functions.

Since what matters ultimately to the firms is the amount of effective labor $z$. One can define an alternative disutility function $V: \mathcal{Z} \times \mathcal{X} \times \mathcal{A} \times \mathcal{M}^* \rightarrow \mathbb{R}$ as

$$V(z, x, a, m^*) = U\left(\frac{z}{e(x, a, m^*)}, x, a, m^*\right)$$

(2.1)

which is a compound of the disutility function $U$ and the efficiency function $e$.

Let the $d_y \times 1$ vector $y$ be a firm’s observable characteristics, the scalar $b$ be the firm’s unobservable characteristic. And let the $d_{m^d} \times 1$ vector $m^d$ be market-level characteristics concerning the firms. Suppose $f_{y,b}^m$ is the joint density of $(y, b)$ on $Y \times B$ in market $m$. Moreover, let $R(z, y, b, m^d)$ be the net revenue of receiving $z$ unit of effective labor for a firm with $(y, b)$ characteristics in a market with $m^d$ characteristics. Market level characteristics may affect revenue of an individual firm through, among others, industry agglomeration effects, or local demand for the firm’s products. The variables $m^d$ in the specification of the revenue function capture these effects.

My primary goal is to conduct counterfactual analysis of wages, which requires as a preliminary step identifying the disutility function $V$ (or equivalently $U$ and $e$), the revenue function $R: \mathcal{Z} \times \mathcal{Y} \times \mathcal{B} \times \mathcal{M}^d \rightarrow \mathbb{R}$, and the joint densities $f_{x,a}^m \in \mathcal{L}_2(\mathbb{R}^{d_x+1})$ and $f_{y,b}^m \in \mathcal{L}_2(\mathbb{R}^{d_y+1})$. Because the market is competitive, the identification of the supply side functions and the demand side functions can be separated, provided that the wage scheme function can be identified from one side of the market. In what follows, I will focus on the supply side structural functions $U$, $e$ and the joint density $f_{x,a}^m$. The demand side structural function $R$ and the joint density $f_{y,b}^m$ employs similar arguments and is simpler.

**Assumption 2.** Suppose $x \indep a|m^*$, $y \indep b|m^d$, $f_{x,a}^m > 0$ and $f_{y,b}^m > 0$ for all $m \in \mathcal{M}$.  

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Assumption 2 implies that $f_{x,a}^m = f_x^m \times f_a^m$ and $f_{y,b}^m = f_y^m \times f_b^m$. Since $f_x^m$ and $f_y^m$ are trivially identified from data, and only $f_a^m$ and $f_b^m$ need to be identified. Assumption 2 is inarguably a strong assumption. However, it is imposed for the mere purpose of exposition. All of the identification results to be showed in this paper will follow by almost the same argument if one allows $x$ to be correlated with $a$ (or $y$ to be correlated with $b$) and make use of a vector of instrumental variables $w^s$ (or $w^d$). Then endogeneity of $x$ (or $y$) can be dealt with, for example, using the control variable approach in $\gamma$. A brief account of applying their approach to my model is given in Appendix B, but I will not elaborate on this digression in the main text.

**Assumption 3.** Suppose firms’ revenue function $R(z,y,b,m^d)$, workers’ disutility function $U(h,x,a,m^s)$, and efficiency function $e(x,a,m^s)$ are all twice continuously differentiable on their respective supports. Also suppose that $e(x,a,m^s)$ is uniformly bounded below away from zero.

**Assumption 4.** Suppose $U_h > 0$, $U_a < 0$, $U_{ha} > 0$ and $U_{hh} > 0$ for all $(h,x,a,m^s) \in H \times X \times A \times M^s$, and suppose $R_z > 0$, $R_b > 0$, $R_{zb} > 0$ and $R_{zz} < 0$ for all $(z,y,b,m^d) \in Z \times Y \times B \times M^d$.

**Assumption 5.** Suppose $e_a > 0$, that is, the efficiency function is strictly increasing in the unobservable characteristic of the worker, for all $(x,a,m^s) \in X \times A \times M^s$.

Assumption 1 to Assumption 4 imply that a unique hedonic equilibrium exists, the detailed arguments being provided in Appendix A. Assumption 5 guarantees that the effective labor supply function $s^m$ to be defined in the following is strictly increasing in workers’ unobservable characteristic $a$. This monotonicity is a sufficient but not necessary condition for my identification strategy.

Given the efficiency function $e$ and the wage scheme function $P^m$ in the market $m$, a worker with characteristics $(x,a)$ in the market $m$ chooses hours $h$ to maximize her contemporary net utility

$$\max_h P^m(h(e(x,a,m^s)) - U(h,x,a,m^s)) \tag{2.2}$$

The $P^m$ function has a superscript $m$ since it is an equilibrium outcome depending on market primitives $(f_x^m, f_y^m, U, e, R)$ in market $m$. The worker’s first order condition is

$$P_x^m(h(e(x,a,m^s))e(x,a,m^s) - U_h(h,x,a,m^s)) = 0 \tag{2.3}$$

The worker’s utility maximization problem can be equivalently expressed as

$$\max_z P^m(z) - V(z,x,a,m^s) \tag{2.4}$$

And it implies that the worker’s first order condition is

$$P_z^m(z) - V_z(z,x,a,m^s) = 0$$
which is equivalent to (2.3) in the light of (2.1). Assumption 3 and Assumption 4 imply that the second order condition

\[ R_{zz} \cdot e^2 - U_{hh} < 0 \]

holds for all \((h, x, a, m^*) \in H \times X \times A \times M^*\) and all \((z, y, b, m^d) \in Z \times Y \times B \times M^d\).

By the properties of the equilibrium elaborated in Appendix A, this ensures that each worker has a unique interior optimum to the maximization problem (2.4). Moreover, by the Implicit Function Theorem (7), there exists an effective labor supply function \(z = s^m(x, a, m^*)\) in each market \(m\) such that

\[ P_m^m(s^m(x, a, m^*)) - V_z(s^m(x, a, m^*), x, a, m^*) = 0 \]

And \(s^m(x, a, m^*)\) is strictly increasing in worker’s unobservable characteristic \(a\). Therefore the unique interior optimum to the maximization problem (2.2) constitutes an hours function \(h = h^m(x, a, m^*) = s^m(x, a, m^*)/e(x, a, m^*)\) that satisfies (2.3). However, the hours function \(h^m\) is not necessarily strictly increasing in \(a\).

The wage income function \(I^m\) in each market \(m\) is determined by workers’ effective labor supply function \(s^m\) and the wage scheme function \(P^m\) through

\[ I^m(x, a, m^*) = P^m(s^m(x, a, m^*)) \]  

(2.5)

By Assumption 3, 4 and the first order condition (2.3), we have that the wage scheme function \(P^m\) is strictly increasing in its argument. Therefore, the wage income function \(I^m\) is also strictly increasing in workers’ unobservable characteristic \(a\). This monotonicity is important to the identification of the wage income function \(I^m\), as I will explain in section 3.

A remark fits here. Since the market-level variables \(m^*\) take the same values for all workers in the same market, variations with respect to \(m^*\) of the functions \(h^m, s^m\) and other market specific reduced form functions cannot be identified. However, identification of the structural functions does not require this knowledge. Another way of thinking of this issue is that \(m^*\) serve as a market identifier just like the super-script \(m\) on those reduced form functions, and I do not need to include them as arguments of the functions.

### 3 Identifying Structural Functions in Two Types of Markets

Distributions of workers’ unobservable characteristic \(a\) are likely to vary among different markets. For example, it is hard to believe that unobservable characteristic \(a\) among workers in Manhattan, New York has the same distribution as that among workers in Manhattan, Kansas. This market level heterogeneity cannot be separated from variations in market equilibria without any further structure, due to the inherent unobservability
of $a$. Some market level observable characteristics $m^s$, however, may be correlated with this market level heterogeneity. For example, big metropolitan areas should have more dispersed distributions of unobservable characteristic $a$ among their workers than small towns. Such difference can be controlled by population sizes of the markets. After controlling relevant market level observable characteristics $m^s$, one may assume that the residual unobservable characteristic (still call it $a$) of workers follows the same distribution in all markets.

If $m^s$ are discrete variables with finite support, then there are only finite types of markets. In this case, it suffices to show the identification of the structural functions in two types of market. Identification can be generalized to any finite types of markets without any but notational complexity. Therefore, I focus on the identification of the structural functions in the model with two types of markets in this section.

If $m^s$ are continuous variables, then one has two ways of proceeding with the identification. One way assumes that the number of underlying types of markets is finite, and the types are reflected in the observable characteristics $m^s$ with some noise. Based on this assumption, one could conduct a preliminary cluster analysis of the markets using $m^s$. Then the identification strategy discussed in this section applies. Another way admits that $m^s$ are inherently continuous. In that case, it requires some additional continuity conditions (and probably more) to achieve the identification of the structural functions. I will not discuss the second way in this paper.

I suppose there are large numbers of markets of both types. Let $M^{(0)}$ denote the market index set for one type of markets, and $M^{(1)}$ the other, such that $M = M^{(0)} \cup M^{(1)}$ and $M^{(0)} \cap M^{(1)} = \emptyset$. All market level variables for the same type of markets take the same values. Then one can drop the market level variables $m^s$ and $m^d$ altogether, and simply use a binary variable $T^m$ to denote the market type. In this case, the efficiency function $e(x, a, m^s)$ becomes two separate functions, $e^0(x, a)$ and $e^1(x, a)$. I will consider their identification separately. Similarly, the marginal disutility function $U_h(h, x, a, m^s)$ becomes two functions $U^0_h(h, x, a)$ and $U^1_h(h, x, a)$, one for each type of markets. On the demand side, $R^0(z, y, b)$ and $R^1(z, y, b)$ are considered separately.

Suppose a multiple market worker-firm matched data set is available. The data includes variables $\{(x^m_i, h^m_i, T^m_i, y^m_i, r^m_i)\}_{i=1}^{N^m}, T^m\}_{m=1}^{M}$, where $i$ is the worker-firm pair identifier and $m$ is the market identifier. $r$ is firms’ revenue, and $T^m = 0$ or $1$ indicates the market type.

Note that the workers and the firms need to be matched with each other in the data set in order to identify the firms’ revenue function $R$. Because the amount of effective labor $z$ is not directly observable but common to the matched worker and firm, I use the information on the supply side to recover $z$ first, then use it to identify the demand side structural functions. In standard hedonic equilibrium models, on the contrary, $z$ is observable, so one can match the sellers with the buyers through $z$ even if they are observed in different data sets.
3.1 Identifying Supply Side Structural Functions in Markets $m \in \mathcal{M}^{(0)}$

The supply side structural functions $(e^0, U^0_h)$ and $(e^1, U^1_h)$ play symmetric roles in my model. Identifying any of the pairs up to its own normalization (to be introduced in Assumption 7 and Definition 1) is the same exercise as identifying the other. It suffices to first show the identification of $(e^0, U^0_h)$, and then to align the normalizations of the two pairs.

The identification strategy of $(e^0, U^0_h)$ consists of four steps: (1) identification of wage income function $I^m(x, a)$ and hours function $h^m(x, a)$ in each market; (2) exploit the multidimensionality of the arguments of these functions to identify the efficiency function $e^0(x, a)$; (3) use the wage equation to identify the wage scheme function $P^m(z)$; and (4) use workers’s first order condition to identify the marginal disutility function $U^0_h(h, x, a)$. These results are given in the following lemmas and theorems.

I start with the identification of the wage income function $I^m(x, a)$ in each market. As was made clear in ?, non-additive structural functions cannot be identified separately from the distribution of unobserved heterogeneity without any further restriction, I therefore make the following identifying assumption.

**Assumption 6.** Suppose the workers’ unobservable characteristic $a$ follows the uniform distribution $U[0,1]$ in all markets.

The uniform distribution assumption for unobservable characteristic $a$ may seem restrictive at the first glance. Yet another equivalent interpretation is that $a$, thus distributed, is not worker’s unobservable characteristic itself, but the quantile of the unobservable characteristic. Under this interpretation, Assumption 6, together with the form of the disutility function $U$, only requires that the unobservable characteristic has the same distribution across all markets, after controlling market-level characteristics. And one does not have to know the distribution of the unobservable characteristic to achieve the identification.

It is easy to see that the overall level of the efficiency function $e^0$ is not identifiable because efficiency is unobservable, so I make the following normalization assumption.

**Assumption 7.** Suppose for a known fixed vector $(\bar{x}, \bar{a}) \in X \times A$, we have that $e^0(\bar{x}, \bar{a}) = 1$.

**Lemma 1.** Under Assumptions 1 to 6, the wage income function $I^m(x, a)$ is strictly increasing in (the quantile of) worker’s unobservable characteristic $a$. Then $I^m(x, a)$ is nonparametrically identified within each market $m$.

**Proof.** By the wage equation (2.5), $I^m(x, a)$ is strictly increasing in $a$ if $P^m$ is strictly increasing in $z$ and $s^m$ is strictly increasing in $a$. Given workers’ first order condition (2.3), Assumptions 3 and 4 guarantee the strict monotonicity of $P^m$. On the other hand, by the expression of $\partial s^m(x, a)/\partial a$ to be shown in Appendix A, it is positive under Assumptions 4 and 5 and the setup of the model.
Given the strict monotonicity of $I^m(x, a)$ and Assumption 6, the identification of $I^m(x, a)$ follows the same argument as in ?. In particular, by monotonicity, Assumption 2 and 6, we have

$$F_{I^m_{|x}}(I^m(x, a)) = F_{a} a^m = a$$

Then

$$I^m(x, a) = F_{I^m_{|x}}^{-1}(a)$$

Once one identifies the wage income function $I^m$, she can invert it with respect to $a$ to obtain $a^m_i = (I^m)^{-1}(x^m_i, I^m)$, for $i = 1, \ldots, N^m$ and $m = 1, \ldots, M$. With these individual quantiles, it is easy to see the identification of the hours function $h^m$.

**Lemma 2.** Under the conditions for Lemma 1, the hours function $h^m(x, a)$ is nonparametrically identified within each market $m$.

**Proof.** Since both $x^m_i$ and $a^m_i$ are known now, identification of $h^m(x, a)$ follows straightforwardly.

A few remarks are due here. Firstly, $I^m$ and $h^m$ vary for every market, due to the cross market variations in $f^m_x$ and $f^m_y$. $I^m$ and $h^m$ are identified within each market, and the superscript $m$ absorbs all market-level variables. Secondly, only identification of $I^m$ requires strict monotonicity with respect to $a$, but not that of $h^m$. In fact, when $a$ rises, $h$ will increase if the substitute effect outweighs the income effect, and $h$ will decrease if the other way around. The relative magnitudes of the substitute effect and the income effect is an empirical question, the answer to which depends on individual characteristics $(x, a)$, market level characteristics $m^*$, and the equilibrium in the market. Moreover, the non-monotonicity of hours with respect to wage income, both within and across markets, has been well documented in empirical works (?; ?; and reference therein).

The last two lemmas identify reduced form functions in each market. What follows is a key step in my identification strategy, where the structural efficiency function $e^0$ is identified. The identifiable set for $e^0$ is the union of iso-income sets going through $(\bar{x}, \bar{a})$ in all markets $m \in \mathcal{M}^{(0)}$.

$$S^{(0)}(\bar{x}, \bar{a}) = \{(x, a) \in \mathcal{X} \times \mathcal{A}: \text{there exists a market } m \in \mathcal{M}^{(0)} \text{ such that in equilibrium } I^m(x, a) = I^m(\bar{x}, \bar{a})\}$$

And the iso-income set going through $(\bar{x}, \bar{a})$ in market $m$ is

$$S^m(\bar{x}, \bar{a}) = \{(x, a) \in \mathcal{X} \times \mathcal{A}: \text{in equilibrium of market } m, I^m(x, a) = I^m(\bar{x}, \bar{a})\}$$
The result of the theorem follows (3.1) and (3.2). Hence we have

\[ S^{(0)}(\bar{x}, \bar{a}) = \bigcup_{m \in \mathcal{M}^{(0)}} S^m(\bar{x}, \bar{a}) \]  

(3.1)

**Theorem 1.** Under the conditions for Lemma 1 and Assumption 7, the efficiency function \( e^0(x, a) \) for markets \( m \in \mathcal{M}^{(0)} \) is nonparametrically identified on \( S^{(0)}(\bar{x}, \bar{a}) \).

**Proof.** In every market \( m \in \mathcal{M}^{(0)} \), for the fixed vector \((\bar{x}, \bar{a})\) defined in Assumption 7, let \( \bar{I}^m = I^m(\bar{x}, \bar{a}) \) and \( \bar{h}^m = h^m(\bar{x}, \bar{a}) \). Then for all \((x, a) \in S^m(\bar{x}, \bar{a})\), \( I^m(x, a) = I^m \). Within the same market, this implies that those workers provide the same amount of effective labor as the worker \((\bar{x}, \bar{a})\), that is \( h^m(x, a) \times e^0(x, a) = \bar{h}^m \times 1 = \bar{h}^m \). Therefore, the values of the efficiency function \( e^0 \) for workers \((x, a) \in S^m(\bar{x}, \bar{a})\) is identified by

\[ e^0(x, a) = \frac{\bar{h}^m}{h^m(x, a)} \text{ for } (x, a) \in S^m(\bar{x}, \bar{a}) \]  

(3.2)

The result of the theorem follows (3.1) and (3.2).

The idea of identifying \( e^0(x, a) \) on \( S^m(\bar{x}, \bar{a}) \) is illustrated in Figure 3.1. It shows the special case where \( x \) is a scalar. The colorful surface is the efficiency function \( e^0 \) over a support \( \mathcal{X} \times \mathcal{A} = [1, 10] \times [0, 1] \). The red solid dot in the \( x-a \) plane is the normalization point \((\bar{x}, \bar{a})\). The red solid line through \((\bar{x}, \bar{a})\) represents the iso-income set \( S^m(\bar{x}, \bar{a}) \) in a market \( m \in \mathcal{M}^{(0)} \). In this market, the value of \( e^0(x, a) \), more precisely, the ratio \( e^0(x, a) / e^0(\bar{x}, \bar{a}) \) is identified for \((x, a) \in S^m(\bar{x}, \bar{a})\). Such values of \( e^0(x, a) \) are represented by the red solid curve on the surface of the efficiency function \( e^0 \). In another market \( m' \in \mathcal{M}^{(0)} \), the iso-income set \( S^{m'}(\bar{x}, \bar{a}) \) will be represented by another curve in the \( x-a \) plane, resulting in another curve on the \( e^0 \) surface. Theorem 1 says that the identifiable set of \( e^0 \) under normalization at \((\bar{x}, \bar{a})\) is the union of the identifiable sets \( S^m(\bar{x}, \bar{a}) \) in every market \( m \in \mathcal{M}^{(0)} \).

As a by-product, the identification of the wage scheme function \( P^m \) is given in the following corollary.

**Corollary 1.** Under the conditions for Theorem 1, wage scheme function \( P^m(z) \) for markets \( m \in \mathcal{M}^{(0)} \) are nonparametrically identified on \( Z_s^{(0)}(\bar{x}, \bar{a}) \).

**Proof.** The wage equation (2.5), Lemma 1, 2, and Theorem 1 imply the result.

In the above corollary, the identifiable set \( Z_s^{(0)}(\bar{x}, \bar{a}) \) for the wage scheme function \( P^m \) is defined as

\[ Z_s^{(0)}(\bar{x}, \bar{a}) = \{ z \in Z; \text{ there exists a market } m \in \mathcal{M}^{(0)} \text{ and some } (x, a) \in S^m(\bar{x}, \bar{a}) \text{ such that in equilibrium } z = h^m(x, a)e^0(x, a) \} \]

The next important result is the identification of the marginal disutility function \( U_h \). Before stating the theorem, I need to define some more notations. Define the identifiable
Theorem 2. Under the conditions for Theorem 1, the marginal disutility function $U_0^0(h, x, a)$ is nonparametrically identified on $H_S^{(0)}(\bar{x}, \bar{a})$.

Proof. By workers’ first order condition in any market $m \in \mathcal{M}^{(0)}$

$$P_z^m(h^m(x, a) \cdot e^0(x, a)) \cdot e^0(x, a) = U_0^0(h^m(x, a), x, a)$$

where $(h^m(x, a), x, a) \in H_S^{(0)}(\bar{x}, \bar{a})$. 

3.1.1 Extended/Over-Identification of the Efficiency Function in Markets $m \in \mathcal{M}^{(0)}$

I have shown that the efficiency function $e^0$ is identified on $S^{(0)}(\bar{x}, \bar{a})$. However, cross market variations within the same market type provide more information about structural functions than exact identification. In this sub-subsection, I will show that the efficiency function $e^0$ is over-identified on $S^{(0)}(\bar{x}, \bar{a})$. Moreover, using the same idea underlying Theorem 1, I can extend the identification of $e^0$ to $\bar{S}^{(0)}(\bar{x}, \bar{a})$, which includes $S^{(0)}(\bar{x}, \bar{a})$ as
a subset.

The intuition is as follows (see Figure 3.2). Consider two markets \( m = 1 \) and \( m = 2 \) of the same type \( \mathcal{M}^{(0)} \). In market \( m = 1 \) (Step 1 of Figure 3.2), given the normalization point \((\tilde{x}, \tilde{a})\), the ratio of the efficiency functions \( e^0(x, a)/e^0(\tilde{x}, \tilde{a}) = e^0(x, a) \) is identified for \((x, a) \in S^1(\tilde{x}, \tilde{a}) \) (by Theorem 1). The red solid line represents \( S^1(\tilde{x}, \tilde{a}) \) in Figure 3.2. If one were to pick a different normalization point \((\tilde{x}, \tilde{a}) \notin S^1(\tilde{x}, \tilde{a}) \), and normalize \( e^0(\tilde{x}, \tilde{a}) = \tilde{e}^0 \), then one would have obtained a different identifiable set \( S^1(\tilde{x}, \tilde{a}) \), using the same idea underlying Theorem 1. That is, the ratio of the efficiency functions \( e^0(x, a)/e^0(\tilde{x}, \tilde{a}) = e^0(x, a)/\tilde{e}^0 \) is also identified for \((x, a) \in S^1(\tilde{x}, \tilde{a}) \). Note that \( S^1(\tilde{x}, \tilde{a}) \cap S^1(\tilde{x}, \tilde{a}) = \emptyset \), since they are different iso-income sets. The red dashed line through \((\tilde{x}, \tilde{a})\) represents \( S^1(\tilde{x}, \tilde{a}) \). By varying the normalization point \((\tilde{x}, \tilde{a})\) one will obtain a family of disjoint identifiable sets \( S^1(\tilde{x}, \tilde{a}) \), each being a different iso-income set. In Figure 3.2, this family of disjoint identifiable sets is represented by the parallel red dashed lines. Within each of them, the ratio of the efficiency functions at each point relative to that at the normalization point is identified. On the other hand, across such disjoint identifiable sets, \( \tilde{e}^0 \), the relative level of the efficiency function at different normalization points are not identified from the information in a single market \( m = 1 \).

In market \( m = 2 \) (Step 2 of Figure 3.2), one can similarly obtain a family of disjoint identifiable sets \( S^2(x, a) \) by varying the normalization point \((x, a)\). They are represented by blue solid and dashed lines. As long as the distributions \((f_x^1, f_y^1)\) and \((f_x^2, f_y^2)\) are not the same almost surely, the two markets will exhibit different equilibria. Graphically, as a result, red lines and blue lines cross. This implies that the sets \( S^1(\tilde{x}, \tilde{a}) \cap S^2(\tilde{x}, \tilde{a}) \neq \emptyset \) for some \((\tilde{x}, \tilde{a}) \notin S^1(\tilde{x}, \tilde{a}) \) (there exist some intersecting identifiable sets from the two markets). This helps us to pin down the level of the efficiency functions for \( S^1(\tilde{x}, \tilde{a}) \) relative to \( S^1(\tilde{x}, \tilde{a}) \).

Let \((\tilde{x}, \tilde{a}) \in S^1(\tilde{x}, \tilde{a}) \cap S^2(\tilde{x}, \tilde{a}) \) (shown in Step 2 of Figure 3.2). I have shown that both \( e^0(\tilde{x}, \tilde{a})/e^0(\tilde{x}, \tilde{a}) = e^0(\tilde{x}, \tilde{a}) \) and \( e^0(\tilde{x}, \tilde{a})/e^0(\tilde{x}, \tilde{a}) = e^0(\tilde{x}, \tilde{a})/\tilde{e}^0 \) are identified. So the value of \( e^0 \) is also identified.

\[
e^0 = e^0(\tilde{x}, \tilde{a}) = e^0(\tilde{x}, \tilde{a})/e^0(\tilde{x}, \tilde{a}) = h^2(\tilde{x}, \tilde{a})/h^1(\tilde{x}, \tilde{a})
\]

Hence the function \( e^0 \) for all \((x, a) \in S^1(\tilde{x}, \tilde{a}) \) are identified up to the initial normalization at \((\tilde{x}, \tilde{a})\).

Collect all points \((\tilde{x}, \tilde{a})\) such that \( S^1(\tilde{x}, \tilde{a}) \cap S^2(\tilde{x}, \tilde{a}) \neq \emptyset \), and denote it as \( S^{1,2}(\tilde{x}, \tilde{a}) \) (represented by the red solid lines, thick or thin, in Step 3 of Figure 3.2).

\[
S^{1,2}(\tilde{x}, \tilde{a}) = \{ (x, a) \in X \times A : S^1(x, a) \cap S^2(x, a) \neq \emptyset \}
\]

Then the above argument implies that \( e^0 \) is identified for all \((x, a) \in S^{1,2}(\tilde{x}, \tilde{a}) \). Note that \( S^{1,2}(\tilde{x}, \tilde{a}) \supseteq S^1(\tilde{x}, \tilde{a}) \).
Similarly, $e^0$ is identified for $(x,a)$ in the set

$$S^{2,1}(x,a) = \{(x,a) \in X \times A: S^2(x,a) \cap S^1(x,a) \neq \emptyset\}$$

which is represented by blue solid lines in Step 4 of Figure 3.2. Again, $S^{2,1}(x,a) \supseteq S^2(x,a)$.  

One can carry on this process and inductively define larger and larger identifiable sets for $e^0$.  

$$S_{n,k}^{m,n,m,n,\cdots}(x,a) = \{(x,a) \in X \times A: S^n(x,a) \cap S^m(x,a) \cap \cdots \neq \emptyset\}$$

for $m,n \in M(0)$ and $k \in \mathbb{N}^+$, and the largest possible identifiable set for $e^0$ is

$$\bar{S}^{(0)}(x,a) = \lim_{k \to \infty} \left( \bigcup_{m,n \in M(0)} S_{n,k}^{m,n,m,n,\cdots}(x,a) \right)$$

These sets are represented by more and more solid lines, red and blue, in Step 5 and 6 of Figure 3.2. Note that the largest possible identifiable set $\bar{S}^{(0)}(x,a)$ includes as a subset the identifiable set $S^{(0)}(x,a)$ given in Theorem 1. So the following Proposition extends the identification result in Theorem 1 to a larger support.  

**Proposition 1.** Under the conditions for Theorem 1, the efficiency function $e^0(x,a)$ is nonparametrically identified on $\bar{S}^{(0)}(x,a)$ using multiple market information.  

Figure 3.2 also makes it clear that the two markets provide two separate sources of identification for the efficiency function. In fact, the information from two markets is more than enough to exactly identify the efficiency function $e^0$, not to mention the information from more markets. Define the set

$$\tilde{S}^{(0)}(x,a) = \bigcup_{k,k' \in \mathbb{N}^+, m,n,m',n' \in M(0)} \left( S_{n,k}^{m,n,m,n,\cdots}(x,a) \cap S_{n,k'}^{m',n',m',n',\cdots}(x,a) \right)$$

Then we have the following over-identification result.  

**Proposition 2.** Under the conditions for Theorem 1 and assume $(f_x^m, f_y^m) \neq (f_x^{m'}, f_y^{m'})$ with a positive Lebesgue measure for at least two markets $m, m' \in M(0)$, then the efficiency function $e^0(x,a)$ is nonparametrically over-identified on $\tilde{S}^{(0)}(x,a)$.  

In the reasoning above, the largest possible identifiable set $\bar{S}^{(0)}(x,a)$, and the over-identified set $\tilde{S}^{(0)}(x,a)$ for the efficiency function $e^0$ may both depend on the initial normalization point $(\bar{x}, \bar{a})$. However, relevant applications would be like the case shown in Figure 3.2, where $\tilde{S}^{(0)}(x,a) = \bar{S}^{(0)}(x,a) = X \times A$. That is, the efficiency function
Figure 3.2: Over-Identification of $e^0(x, a)$ from Two Markets Illustrated
$e^0$ is over-identified on the whole support using information from merely two markets, and the largest possible identifiable set $S^{(0)}(\bar{x}, \bar{a})$ actually does not depend on the initial normalization point $(\bar{x}, \bar{a})$. This extends the identification result in Theorem 1 in a surprising way. To see the difference, note that the identifiable set $S^1(\bar{x}, \bar{a}) \cup S^2(\bar{x}, \bar{a})$ given by Theorem 1 using two markets was only two curves (illustrated by the red and blue solid curves in Step 2 of Figure 3.2), and $e^0$ was only exactly identified on $S^1(\bar{x}, \bar{a}) \cup S^2(\bar{x}, \bar{a})$.

Even without unobservable efficiency, $U_h$ is not identifiable from a single market without being imposed any shape restriction (\cite{3.1}). In spite of the over-identification of $e^0$, one still needs large number of markets $m \in \mathcal{M}^{(0)}$ to identify $U_h$.

### 3.2 Identification of Supply Side Structural Functions in Markets $m \in \mathcal{M}^{(1)}$

Again, identifying $(e^1, U^1_h)$ up to its own normalization is the same exercise as what I have shown in the previous subsection. The key question here is to align the two normalizations. Since the normalizations give the unobservable efficiency functions an arbitrary scale, the key question is to pin down the relative scales of the efficiency function between the two types of markets.

Similar to the last subsection, I define the identifiable set for the efficiency function in markets of type $T_m = 1$,

$$S^{(1)}(\bar{x}, \bar{a}) = \{(x, a) \in \mathcal{X} \times \mathcal{A}: \text{there exists a market } m \in \mathcal{M}^{(1)} \text{ such that in equilibrium } I^m(x, a) = I^m(\bar{x}, \bar{a})\}$$

Then we have

$$S^{(1)}(\bar{x}, \bar{a}) = \cup_{m \in \mathcal{M}^{(1)}} S^m(\bar{x}, \bar{a})$$

Moreover, define the identifiable set for the wage scheme function

$$Z^{*,(1)}(\bar{x}, \bar{a}) = \{z \in \mathcal{Z}: \text{there exists a market } m \in \mathcal{M}^{(1)} \text{ and some } (x, a) \in S^m(\bar{x}, \bar{a}) \text{ such that in equilibrium } z = h^m(x, a)e^1(x, a)\}$$

and the identifiable set for the marginal disutility function

$$\mathcal{H}S^{(1)}(\bar{x}, \bar{a}) = \{(h, x, a) \in \mathcal{H} \times \mathcal{X} \times \mathcal{A}: \text{there exists a market } m \in \mathcal{M}^{(1)} \text{ such that in equilibrium } I^m(x, a) = I^m(\bar{x}, \bar{a}) \text{ and } h = h^m(x, a)\}$$

**Definition 1.** For the same known fixed vector $(\bar{x}, \bar{a})$ as in Assumption 7, we define $e^1(\bar{x}, \bar{a}) = e^*$.  

By exactly the same argument as in the previous subsection, one can identify the structural functions $e^1, U^1_h$, and the wage scheme function $P^m(z)$ for all markets $m$ of type $T_m = 1$. These results are collected in the following lemma.
Lemma 3. Under the conditions for Lemma 1 and Definition 1, the efficiency function $e^1(x,a)$ is nonparametrically identified on $S^{(1)}(\bar{x}, \bar{a})$, the wage scheme function $P_m^*(z)$ is nonparametrically identified on $Z^{(1)}(\bar{x}, \bar{a})$, and the marginal disutility function $U_h^1(h,x,a)$ is nonparametrically identified on $HS^{(1)}(\bar{x}, \bar{a})$. These identification results are all up to the scale normalization in Definition 1.

It is clear that the identification of the structural functions and the reduced form functions in both types of markets depend on their respective normalizations (Assumption 7 and Definition 1). In other words, the absolute scales of these functions (value of $e^*$) are not identifiable. This result is much less harmful than it seems, since the identification of counterfactual wage income (hence its distribution) is invariant with respect to such normalizations.\(^1\) The following additional assumptions allow one to identify the value of $e^*$, but are not essential to constructing counterfactual wages. In what remains in this paper, however, I will maintain them to ease exposition.

Assumption 8. Suppose the mapping from $e^*$ to $U_h^1$ defined on $HS^{(1)}(\bar{x}, \bar{a})$ is bijective, given other primitives of the model.

Assumption 9. Suppose the common support $HS^{(0)}(\bar{x}, \bar{a}) \cap HS^{(1)}(\bar{x}, \bar{a})$ of the marginal disutility functions is non-degenerated. And suppose $U_h^0(h,x,a) = U_h^1(h,x,a) = U_h(h,x,a)$ a.e. for $(h,x,a) \in HS^{(0)}(\bar{x}, \bar{a}) \cap HS^{(1)}(\bar{x}, \bar{a})$. Therefore, $U_h^0(h,x,a) = U_h^1(h,x,a) = U_h(h,x,a)$ a.e. for $(h,x,a) \in HS^{(0)}(\bar{x}, \bar{a}) \cap HS^{(1)}(\bar{x}, \bar{a})$.

Lemma 4. Under the conditions for Theorem 2 and Lemma 3, Assumptions 8 and 9, the value of $e^*$ is uniquely identified. As a result, the marginal disutility function $U_h(h,x,a)$ is nonparametrically identified on $HS^{(0)}(\bar{x}, \bar{a}) \cup HS^{(1)}(\bar{x}, \bar{a})$.

Proof. With Assumptions 8 and 9, a proof by contradiction argument leads to the result. \(\Box\)

3.3 Identifying Demand Side Structural Functions

Identifying demand side structural function $R_0^d(z,y,b)$ (or $R_1^d(z,y,b)$) and accompanying reduced form effective labor demand function $d^m(y,b)$ makes little difference from ?’s method. The only tweak stems from the fact that $z$ is not directly observable but is identified using previously identified supply side functions.

Lemma 5. Under the conditions for Lemma 3, $z_i^m = h^m(x_i^m,a_i^m) \times e^m(x_i^m,a_i^m)$ is identified.

Define the identifiable set for the effective labor demand function

$$D^{(T)}(\bar{x}, \bar{a}) = \{(y,b) \in \mathcal{Y} \times \mathcal{B}: \text{there exists a market } m \in \mathcal{M}^{(T)} \text{ and } (x,a) \in S^{(T)}(\bar{x}, \bar{a}) \text{ such that } d^m(y,b) = h^m(x,a) \times e^m(x,a)\}$$

\(^1\)Easy to show, and is not provided in this paper.
and the identifiable set for the marginal revenue function

$\mathcal{H} \mathcal{D}^{(T)}(\tilde{x}, \tilde{a}) = \{(z, y, b) \in \mathbb{Z} \times \mathbb{Y} \times \mathbb{B} : \text{there exists a market } m \in \mathcal{M}^{(T)}$
and $(x, a) \in \mathcal{S}^{(T)}(\tilde{x}, \tilde{a}) \text{ such that } z = d^m(y, b)$
and $z = h^m(x, a) \times e^{T_m(x,a)} \}$

for $T = 0, 1$.

**Assumption 10.** Suppose the firms’ unobservable characteristic $b$ follows the uniform distribution $U[0, 1]$ in all markets.

**Lemma 6. (Theorem 4.1)** Under the conditions for Lemma 3 and Assumption 10, the marginal revenue function $R^T_z(z, y, b)$ is nonparametrically identified for all $(z, y, b) \in \mathcal{H} \mathcal{D}^{(T)}(\tilde{x}, \tilde{a})$. The effective labor demand function $d^m(y, b)$ is nonparametrically identified for all $(y, b) \in \mathcal{D}^{(T)}(\tilde{x}, \tilde{a})$. Under the same conditions, the revenue function $R^T_z(z, y, b)$ is nonparametrically identified using revenue data for all $(z, y, b) \in \mathcal{H} \mathcal{D}^{(T)}(\tilde{x}, \tilde{a})$. In all these statements, $T = 0, 1$.

### 4 Constructing Counterfactual Wage Samples through Simulation

The next step in counterfactual analysis is to construct counterfactual wage income samples $\{\tilde{I}_m^1, \ldots, \tilde{I}_m^{N_m}\}_{m=1}^M$ using the structural functions identified in section 3 and the equilibrium conditions given in Appendix A. As is exposed in Appendix A, effective labor supply, demand and wage scheme functions are determined in the equilibrium, and the equilibrium wage scheme function is generically the solution to a differential equation without explicit expression (Echelon, Hickman and Nesheim, 2004). Therefore, the counterfactual wage samples should be constructed through simulation for each market $m$ in the sample.

One counterfactual scenario is of particular interest, so that it was investigated by ?, ?, and ?, among many other authors. They looked at the wage distribution that would prevail if the observable characteristics of the workers were to change to $\{(\tilde{x}_m^1, \ldots, \tilde{x}_N^m)\}_{m=1}^M$. I take this scenario as example.

In the following steps for simulating the counterfactual wage income samples, all structural functions $U_h, e$ and $R_z$ are assumed to be known. In practice, these structural functions are replaced by their estimates.

1. For each worker in market $m$, obtain the efficiency $\tilde{e}_i^m = e^{T_m}(\tilde{x}_i^m, a_i^m)$.

2. Suppose the first derivative of the wage scheme function $P^m_z$ were known, then solve workers’ first order condition $P^m_z(h \tilde{e}_i^m) \tilde{z}_i^m = U_h^{T_m}(h, \tilde{x}_i^m, a_i^m)$ to obtain $\tilde{h}_i^m$ for each worker; this further generates an effective labor supply $\tilde{z}_i^{x,m} = \tilde{h}_i^{m} \tilde{e}_i^m$ for each worker.
3. Under the same first derivative of the wage scheme function \( P^m \), solve firms’ first order condition \( R^m_z (z, y^m_i, b^m_i) = P^m (z) \) to obtain an effective labor demand \( z^{d,m}_i \) for each firm.

4. Choose the first derivative of the wage scheme function \( \tilde{P}^m_\tilde{z} \) from the set
\[
\{ P_\tilde{z}: \mathbb{R}^+ \to \mathbb{R}^+ \}
\]
to minimize some divergence measure between \( \{ \tilde{z}^{s,m}_1, \ldots, \tilde{z}^{s,m}_N \} \) and \( \{ \tilde{z}^{d,m}_1, \ldots, \tilde{z}^{d,m}_N \} \). Repeat steps 2 and 3 until one finds the optimal \( \tilde{P}^m_\tilde{z} \).

5. Then counterfactual effective labor supply \( \{ \tilde{z}^{s,m}_i \}_{i=1}^N \) are obtained in step 2 under the optimal \( \tilde{P}^m_\tilde{z} \). And finally the counterfactual wage income sample is given by
\[
\tilde{I}^m_i = \int_0^{\tilde{z}^m_i} \tilde{P}^m_\tilde{z} (z) dz + C
\]
for each worker in the market, where \( C \) is a constant.

6. Repeat steps 1-5 to obtain the counterfactual wage income sample \( \{ \tilde{I}^m_1, \ldots, \tilde{I}^m_N \} \) for each market \( m \).

Other counterfactual interventions may change values of \( y \)'s, \( a \)'s or \( b \)'s to another values \( \tilde{y} \)'s, \( \tilde{a} \)'s or \( \tilde{b} \)'s. Or they may change the structural functions to \( \tilde{U}_h, \tilde{e}_z \). Simulation still follows the above steps, with corresponding values or structural functions replaced by their “tilde” versions.

With the counterfactual wage income samples, one can then use the formulae in section 1 to obtain the counterfactual wage summary statistics of interest.

5 Operators That Define Counterfactual Wage Distribution

The simulated counterfactual wage income sample will resemble the true counterfactual wage distribution sufficiently well if: (1) a unique wage scheme function \( \tilde{P}^m \) can be recovered from the equilibrium condition, and (2) the unique \( \tilde{P}^m \) does not change abruptly with estimation errors in the estimates of the structural functions. The first requirement pertains to the identification of the counterfactual equilibrium outcomes given the structural functions. The second requirement concerns whether the equilibrium condition is well-posed with regard to \( \tilde{P}^m \). To investigate the performance of the simulation method proposed in section 4, I need to formally characterize the properties of the operators which take the structural functions as inputs, and produce equilibrium reduced form functions and quantities as outputs.

For each market \( m \), the primitive structural functions are
\[
\begin{align*}
&f^m_{x,a} \in \mathcal{L}_2(\mathbb{R}^{d_x+1}), \\
&f^m_{y,b} \in \mathcal{L}_2(\mathbb{R}^{d_y+1}), \\
&U^m_T \in C^2(\mathcal{H} \times \mathcal{X} \times \mathcal{A}), \\
&R^m_T \in C^2(\mathcal{Z} \times \mathcal{Y} \times \mathcal{B}), \\
&e^m_T \in C^2(\mathcal{X} \times \mathcal{A}).
\end{align*}
\]
These primitives generate equilibrium outcomes through the equilibrium conditions.

Firstly, define a functional \( \Phi \) on an open neighborhood around some \( (x^*, a^*, z^*) \) \( \in \mathcal{X} \times \mathcal{A} \times \mathcal{Z} \) and the true structural functions \( (U^m_T, e^m_T, P^m) \), where \( z^* \) is the op-
timal amount of effective labor supply for the worker \((x^*, a^*)\) under \((U_{T^m}, e_{T^m}, P_{T^m})\), and \(P_{T^m}\) is the equilibrium wage scheme function in the market \((f_{x,a}^m, f_{y,b}^m, U_{T^m}^*, R_{T^m}^*, e_{T^m}^*)\). \(\Phi\) characterizes worker \((x, a)\)’s first order condition (2.3) in market \(m\),

\[
\Phi(x, a, z, U_{T^m}, e_{T^m}, P_{T^m}) = P_{T^m}(z) \cdot e_{T^m}(x, a) - U_{T^m}^\prime \left( \frac{z}{e_{T^m}(x, a)}, x, a \right)
\]

By the Implicit Function Theorem, there exists a functional \(\gamma_1: (x, z, U_{T^m}, e_{T^m}, P_{T^m}^*) \rightarrow a \in A\) in the neighborhood around \((x^*, z^*, U_{T^m}^*, e_{T^m}^*, P_{T^m}^*)\) such that \(a = \gamma_1(x, z, U_{T^m}, e_{T^m}, P_{T^m})\) satisfies

\[
\Phi(x, \gamma_1(x, z, U_{T^m}, e_{T^m}, P_{T^m}^*), z, U_{T^m}, e_{T^m}, P_{T^m}^*) = 0
\]

Moreover, \(\gamma_1\) is continuously differentiable and

\[
\frac{\partial \gamma_1(x, z, U_{T^m}, e_{T^m}, P_{T^m}^*)}{\partial z} = \Phi_a^{-1}(x, \gamma_1(x, z, U_{T^m}, e_{T^m}, P_{T^m}^*), z, U_{T^m}, e_{T^m}, P_{T^m}^*) \Phi_z(x, \gamma_1(x, z, U_{T^m}, e_{T^m}, P_{T^m}^*), z, U_{T^m}, e_{T^m}, P_{T^m}^*), \Phi(z, \gamma_1(x, z, U_{T^m}, e_{T^m}, P_{T^m}^*), z, U_{T^m}, e_{T^m}, P_{T^m}^*)
\]

Similarly, define a functional \(\Psi\) on an open neighborhood around some \((y^*, b^*, z^*, R_{T^m}, P_{T^m}^*)\), which characterizes firm \((y, b)\)’s first order condition,

\[
\Psi(y, b, z, R_{T^m}, P_{T^m}^*) = R_{T^m}^*(z, y, b) - P_{T^m}^*(z)
\]

By the Implicit Function Theorem, there exists a functional \(\gamma_2: (y, z, R_{T^m}, P_{T^m}^*) \rightarrow b \in B\) in the neighborhood around \((y^*, z^*, R_{T^m}^*, P_{T^m}^*)\) such that \(b = \gamma_2(y, z, R_{T^m}, P_{T^m})\) satisfies

\[
\Psi(y, \gamma_2(y, z, R_{T^m}, P_{T^m}^*), z, R_{T^m}, P_{T^m}^*) = 0
\]

Moreover, \(\gamma_2\) is continuously differentiable and

\[
\frac{\partial \gamma_2(y, z, R_{T^m}, P_{T^m}^*)}{\partial z} = \Psi_b^{-1}(y, \gamma_2(y, z, R_{T^m}, P_{T^m}^*), z, R_{T^m}, P_{T^m}^*) \Psi_z(y, \gamma_2(y, z, R_{T^m}, P_{T^m}^*), z, R_{T^m}, P_{T^m}^*) \Psi(z, \gamma_2(y, z, R_{T^m}, P_{T^m}^*), z, R_{T^m}, P_{T^m}^*)
\]

The proof of existence and properties of \(\gamma_1\) and \(\gamma_2\) using the Implicit Function Theorem is provided in the Appendix C.

Secondly, define a functional \(\Xi\) around some open neighborhood around \((f_{x,a}^m, f_{y,b}^m, U_{T^m}^*, R_{T^m}^*, e_{T^m}^*, P_{T^m}^*)\) for every \(z \in Z\) that characterizes the equilibrium condition (A.1) (to be formally defined in subsection A),

\[
\Xi(f_{x,a}^m, f_{y,b}^m, U_{T^m}, e_{T^m}, R_{T^m}, P_{T^m}^*)(z)
\]

\[
= \int_{X} f_{x,a}^m \left( x, \gamma_1(x, z, U_{T^m}, e_{T^m}, P_{T^m}^*) \right) \frac{\partial \gamma_1(x, z, U_{T^m}, e_{T^m}, P_{T^m}^*)}{\partial z} dx
\]

\[
- \int_{Y} f_{y,b}^m \left( y, \gamma_2(y, z, R_{T^m}, P_{T^m}^*) \right) \frac{\partial \gamma_2(y, z, R_{T^m}, P_{T^m}^*)}{\partial z} dy
\]
Equilibrium requires that \( \sup_{z \in Z} |\Xi(f_{x,a}^m, f_{y,b}^m, U^{Tm}, e^{Tm}, R^{Tm}, P^m)(z)| = 0 \) for any given \( (f_{x,a}^m, f_{y,b}^m, U^{Tm}, e^{Tm}, R^{Tm}) \) in the domain of \( \Xi \), implying \( \|\Xi\| = 0 \). Under proper conditions, there exists an operator \( \kappa_3: (f_{x,a}^m, f_{y,b}^m, U^{Tm}, e^{Tm}, R^{Tm}) \rightarrow P^m_z \) such that \( P^m_z(\cdot) = \kappa_3(f_{x,a}^m, f_{y,b}^m, U^{Tm}, e^{Tm}, R^{Tm})(\cdot) \) solves the equilibrium differential equation

\[
\sup_{z \in Z} |\Xi(f_{x,a}^m, f_{y,b}^m, U^{Tm}, e^{Tm}, R^{Tm}, \kappa_3(f_{x,a}^m, f_{y,b}^m, U^{Tm}, e^{Tm}, R^{Tm}))(z)| = 0
\]

(5.5)

for any given \( (f_{x,a}^m, f_{y,b}^m, U^{Tm}, e^{Tm}, R^{Tm}) \) in the domain of \( \Xi \).

The system (5.1) to (5.5) characterizes the equilibrium, their solution \( \gamma_1, \gamma_2 \) and \( \kappa_3 \) determining the wage income of each worker in every market under counterfactual interventions.

Now it is straightforward to define the functional that gives the wage income for worker \((x, a)\) in market \(m\) as

\[
\Upsilon(x,a,f_{x,a}^m, f_{y,b}^m, U^{Tm}, e^{Tm}, R^{Tm}) = \int_0^\infty \kappa_1(x,a,U^{Tm}, e^{Tm}, \kappa_3(f_{x,a}^m, f_{y,b}^m, U^{Tm}, e^{Tm}, R^{Tm})) \kappa_3(f_{x,a}^m, f_{y,b}^m, U^{Tm}, e^{Tm}, R^{Tm})(z) \, dz + C
\]

where \( C \) is a constant. Suppose that \( \Upsilon \) is invertible with respect to \( a \), giving rise to \( \Upsilon^{-1}: (x, \iota, f_{x,a}^m, f_{y,b}^m, U^{Tm}, e^{Tm}, R^{Tm}) \rightarrow a \in [0, 1] \), where \( \iota = \Upsilon(x, a, f_{x,a}^m, f_{y,b}^m, U^{Tm}, e^{Tm}, R^{Tm}) \). We can then define the operator that gives the density function of wage as

\[
\Theta(f_{x,a}^m, f_{y,b}^m, U^{Tm}, e^{Tm}, R^{Tm})(\iota) = \int_{x} f_{x,a}^m \left( x, \Upsilon^{-1}(x, \iota, f_{x,a}^m, f_{y,b}^m, U^{Tm}, e^{Tm}, R^{Tm}) \right) \frac{\partial \Upsilon^{-1}(x, \iota, f_{x,a}^m, f_{y,b}^m, U^{Tm}, e^{Tm}, R^{Tm})}{\partial \iota} \, dx
\]

Let \( \Delta \) denote the functional which maps from the wage density function to any of the summary statistics in section 1, then those summary statistics are given by

\[
\Delta(\Theta(f_{x,a}^m, f_{y,b}^m, U^{Tm}, e^{Tm}, R^{Tm}))
\]

In particular, the inequality measures should not be affected by the constant \( C \).

6 Discussion

A number of questions entail further research: the estimation procedure of the structural functions, the identification and estimation of the counterfactual wage.

Using series approximation, we may follow the identification steps to estimate the structural functions. But it remains to show the asymptotics of such procedure.

Identification of the counterfactual equilibrium outcomes hinges on the uniqueness of
the solution \((\kappa_1, \kappa_2, \kappa_3)\) of the equilibrium system (5.1) to (5.5).

Estimation of counterfactual wages takes three major steps: (1) estimate structural functions \((\hat{U}^T m, \hat{e}^T m, \hat{R}^T m)\) for \(T = 0, 1\); (2) plug estimated structural functions into the equilibrium system (5.1) to (5.5) to solve for the equilibrium reduced form functions \((\hat{s}^m, \hat{d}^m, \hat{P}^m)\); and (3) plug the solution into the functionals \((\Upsilon, \Theta, \Delta)\) to obtain the counterfactual wage, wage density and summary statistics. Step (2) is the key one, since it involves solving the wage scheme function \(\hat{P}^m\) from a non-linear equation involving integrals and differentials, where the operators are estimated. It remains to investigate the properties of such estimators.

A few other issues are worth more investigating, but out of the scope of the current paper.

First order conditions characterize workers’ and firms’ optimal choices, only on the premise of continuous hours of working. In reality, utility maximizing workers may choose from a relatively small number of hours levels, rather than being able to vary hours continuously (see ?; ? for example). If hours take discrete values, then my first-order-condition-based identification strategy need to be revised. It is likely that one only gets partial identification of structural functions.

Workers may serve in different capacities; what matters to the firms might be a combination of effective labors along several skill dimensions. In housing markets, buyers and sellers evaluate a house based on the amenity provided by its location, bedrooms, garage, to name a few. In both markets, among others, \(z\) is a vector, not a scalar. ? examined the single market identification of the structural functions in hedonic equilibrium models with multiple dimensional \(z\) and without unobservable quality. To conduct counterfactual analysis, however, it is necessary to check through the identification of the counterfactual equilibrium outcomes in such models.

My model is static. However, it might be fruitful to generalize my model to a dynamic one, where workers make a sequence of labor supply choices throughout their careers, and workers’ efficiency increases in a learning-by-doing fashion. Given the identified structural functions in the dynamic model, we may answer a wider range of questions if we could construct the whole trajectories of wage distributions under various counterfactual interventions.
A Properties of Equilibrium

Under Assumptions 1 and 3, applying the Implicit Function Theorem \((\text{?})\) to workers’ first order condition (2.3) gives rise to

\[
\frac{\partial z}{\partial a} = \frac{\partial s^m(x, a, m^s)}{\partial a} = \frac{eU_{ha} - P^m_z e e_a - U_{hh} h^m e_a}{P^m_z e^2 - U_{hh}}
\]

where for the suppressed arguments of the functions, we have \(h = h^m(x, a, m^s), z = s^m(x, a, m^s) = h^m(x, a, m^s) \times e(x, a, m^s)\). Note that under the usual restrictions on the disutility and the revenue functions in Assumption 3, Assumption 5 is sufficient for \(s^m\) to be strictly increasing in \(a\), but is not necessary. Strict monotonicity of \(s^m\) is a sufficient condition to achieve identification of the wage income function \(I^m\) (see Lemma 1). This implies that the partial derivative of the inverse effective labor supply function \((s^m)^{-1}(x, z, m^s)\) with respect to \(a\) is

\[
\frac{\partial (s^m)^{-1}(x, z, m^s)}{\partial z} = P^m_z e^2 - U_{hh}
\]

By similar argument, one gets the partial derivative of the inverse effective labor demand function \(b = (d^m)^{-1}(y, z, m^d)\) is

\[
\frac{\partial (d^m)^{-1}(y, z, m^d)}{\partial z} = \frac{R_{zb}}{P^m_z - R_{zz}}
\]

where \(z = d^m(y, b, m^d)\). The equilibrium is defined as that the density of the supplied and demanded effective labor \(z\) equals almost surely in every market. That is

\[
\int_X f^m_{x,a} (x, (s^m)^{-1}(x, z, m^s)) \frac{\partial (s^m)^{-1}(x, z, m^s)}{\partial z} dx = \int_Y f^m_{y,b} (y, (d^m)^{-1}(y, z, m^d)) \frac{\partial (d^m)^{-1}(y, z, m^d)}{\partial z} dy
\]

(A.1)

for all \(z \in Z\) and all \(m \in M\). This implies that the pricing function \(P^m\) is a solution of the differential equation (A.1). In particular, the curvature of the pricing function \(P^m\) can be regarded as a “weighted average” of the curvatures of the disutility and the revenue functions.

\[
P^m_{zz}(z) = \int_Y \frac{f^m_{x,b}}{R_{zb}} R_{zz} dy + \int_X \frac{1}{(U_{ha} - P^m_z e e_a - U_{hh} h^m e_a)} U_{hh} dx
\]

(A.2)

where the arguments of the functions are suppressed as well. Assumption 4 combined with (A.2) implies that both workers’ and firms’ second order conditions hold, hence guarantees their unique interior solutions.
showed that the classic hedonic equilibrium model is equivalent to a stable matching problem. The same argument applies to my model as well, since the efficiency is observable to both workers and firms. They also provided sufficient conditions for existence, uniqueness, and purity (assertive matching) of the equilibrium. My Assumptions 3 to 5 guarantee that the Spence-Mirrlees single-crossing condition, a sufficient condition for these properties of equilibrium is satisfied.

B Identifying Structural Functions with Endogenous $x$

In the main text, I assumed that $x \perp a|m^s$ for all $m \in \mathcal{M}$, for the sheer purpose of preventing unnecessary digression from blurring the key idea of identification. In this section, I explain with a simple example how one could tackle the endogeneity of $x$ using the control function approach. For simplicity, suppose $x$ is workers’ education attainment and is a scalar. Suppose $a$ represents the quantile of workers’ ability. Then $x$ is liable to correlate with $a$. However, I assume that a worker decides her education attainment to maximize her expected net utility from the labor market, based on cost of education $c(\xi, w^s_m)$ and a noisy signal $\eta^m_m$ about her ability $a^m_m$ before start working. That is,

$$x^m_i = \arg \max_{\xi} \left\{ \mathbb{E} \left[ P^m(Z) - U \left( \frac{Z}{e(\xi, a^m_i, m^s_i)}, \xi, a^m_i \right) \big| w^s_m, \eta^m_m \right] - c(\xi, w^s_m, \eta^m_m) \right\} \quad (B.1)$$

where $w^s_m$ is the worker’s education cost shifter, $c(\xi, w^s_m)$ is the cost function of education level $\xi$. The maximization problem (B.1) leads to a solution in the form of $x^m_i = g(w^s_m, \eta^m_m)$. This solution forms a triangular system with the wage income function $I^m_i = I^m_i(x^m_i, a^m_i)$.

Assumption 2’. Suppose $w^s_m \perp (a^m_i, \eta^m_m)|m^s$ for all $m \in \mathcal{M}$.

showed that under Assumption 2’, $x^m_i$ and $a^m_i$ are independent conditional on $v^m_i = F_{x^m_i|v^m_i}(x^m_i, w^s_m)$.

As a result, one can obtain identification of the wage income function $I^m_i(x, a)$.

**Lemma 1’.** Under Assumptions 1, 2’, and 3 to 6, the wage income function $I^m_i(x, a)$ is nonparametrically identified within each market $m$.

**Proof.** Given the strict monotonicity of $I^m_i(x, a)$, Assumption 2’ and 6, and ? Theorem 1, we have

$$\mathbb{E}_{v^m_i} [F_{L^{-1}(x^m_i, a^m_i)}|x^m_i, v^m_i] (I^m_i(x^m_i, a^m_i)|x^m_i, v^m_i) \triangleq L(I^m_i(x^m_i, a^m_i), x^m_i)$$

$$\Rightarrow I^m_i(x^m_i, a^m_i) = L^{-1}(x^m_i, a^m_i)$$

where $L^{-1}$ is the inverse function of $L$ with respect to its first argument. \hfill \Box

---

\textsuperscript{2} We can drop the conditional variables $m^s$ here since they take the same values for all the workers in the same market. Some additional regularity conditions are needed. See ? for details.
The remaining identification results in section 3 hold without modification in assumptions or proofs.

C Proofs for Section 5

Assumption 11. Suppose the structural functions $U^T$ is fourth continuously differentiable, and its derivatives up to the fourth order are uniformly bounded on a subset of its support. Suppose $e^T$ is second continuously differentiable, and its derivatives up to the second order are uniformly bounded on a subset of its support. Suppose $R^T$ is fourth continuously differentiable, and its derivatives up to the fourth order are uniformly bounded on a subset of its support.

Let $\|U^T\|$ denote the maximum of the suprema of $U^T$ and its derivatives up to the fourth order over its support; $\|e^T\|$ up to the second order; $\|R^T\|$ up to the fourth order; and $\|P^m\|$ up to the second order.

Condition 1. $P^m$ is fourth continuously differentiable, and its derivatives up to the fourth order are uniformly bounded on a subset of its support.

Note that Condition 1 is about an endogenous function. Ideally, restrictions on the structural functions that guarantee this condition should be found.3

Under Assumption 11 and Condition 1, the Implicit Function Theorem applies to $\Phi$ and $\Psi$. As a result, there exist functionals $\gamma_1$ and $\gamma_2$ such that (5.1) and (5.3) hold. Since $\Phi$ and $\Psi$ are continuous mappings, $\gamma_1$ and $\gamma_2$ are continuous, too.4

In particular,

$$
\frac{\partial \gamma_1}{\partial z}(x, z, U^T, e^T, P^m) = \Phi_a^{-1} \left( x, \gamma_1(x, z, U^T, e^T, P^m), z, U^T, e^T, P^m \right) \Phi_z \left( x, \gamma_1(x, z, U^T, e^T, P^m), z, U^T, e^T, P^m \right)
$$

where

$$
\Phi_a(x, a, z, U^T, e^T, P^m) = P^m(z) e^T_a(x, a) - U^T_a \left( \frac{z}{e^T_a(x, a)}, x, a \right) \left( \frac{z e^T_a(x, a)}{e^T_a(x, a)^2} \right)
$$

and

$$
\Phi_z(x, a, z, U^T, e^T, P^m) = P^m(z) e^T_a(x, a) - U^T_{hz} \left( \frac{z}{e^T_a(x, a)}, x, a \right) \frac{1}{e^T_a(x, a)}
$$

3This problem will be addressed in the discussion of the solution to (5.5).
4(5.2) and (5.4) require $\gamma_1$ and $\gamma_2$ to be differentiable. For them to be $C^1$ mapping, I need to show that $\Phi$ and $\Psi$ are $C^1$ mappings. Assumption 11 guarantees that $(\Phi_x, \Phi_a, \Phi_z)$ and $(\Psi_y, \Psi_b, \Psi_z)$ are continuous/bounded. But additional assumptions and conditions need to be found to ensure $(\Phi_U, \Phi_e, \Phi_P)$ and $(\Psi_R, \Psi_P)$ are continuous/bounded.
And

$$\frac{\partial \gamma_2(x, z, U^{T_m}, e^{T_m}, P_{z}^{m})}{\partial z} = \Psi_{b}^{-1} \left( y, \gamma_2(y, z, R^{T_m}, P_{z}^{m}), z, R^{T_m}, P_{z}^{m} \right)$$

$$\Psi_{z} \left( y, \gamma_2(y, z, R^{T_m}, P_{z}^{m}), z, R^{T_m}, P_{z}^{m} \right)$$

where\(^5\)

$$\Psi_{b}(y, b, z, R^{T_m}, P_{z}^{m}) = R_{z_b}^{T_m} (z, y, b)$$

and

$$\Psi_{z}(y, b, z, R^{T_m}, P_{z}^{m}) = R_{z}^{T_m} (z, y, b) - P_{z}^{m}(z)$$

Under proper assumptions,\(^6\) the Fréchet differential \(D_{P_{z}}\Xi\) is given by

$$D_{P_{z}}\Xi = \int_{X} \frac{\partial f_{x, a}}{\partial a} \left( x_{1}(x, z, U^{T_m}, e^{T_m}, P_{z}^{m}) \right) \frac{\partial \gamma_1(x, z, U^{T_m}, e^{T_m}, P_{z}^{m})}{\partial z} D_{P_m} \gamma_1(x, z, U^{T_m}, e^{T_m}, P_{z}^{m}) dx$$

$$+ \int_{X} f_{x, a} \left( x_{1}(x, z, U^{T_m}, e^{T_m}, P_{z}^{m}) \right) D_{P_m} \gamma_1(x, z, U^{T_m}, e^{T_m}, P_{z}^{m}) dx$$

$$- \int_{Y} \frac{\partial f_{y, b}}{\partial a} \left( y_{1}(y, z, R^{T_m}, P_{z}^{m}) \right) \frac{\partial \gamma_2(y, z, R^{T_m}, P_{z}^{m})}{\partial z} D_{P_m} \gamma_2(y, z, R^{T_m}, P_{z}^{m}) dy$$

$$- \int_{Y} f_{y, b} \left( y_{1}(y, z, R^{T_m}, P_{z}^{m}) \right) D_{P_m} \left( \frac{\partial \gamma_2(y, z, R^{T_m}, P_{z}^{m})}{\partial z} \right) dy$$

where

$$D_{P_{z}} \gamma_1(x, z, U^{T_m}, e^{T_m}, P_{z}^{m}) = \frac{e^{T_m}(x, a) P_{z}^{m}(\cdot)}{P_{z}^{m}(z) e^{T_m}(x, a) - U^{T_m}(z, x, a) + U^{T_m}(x, a) \frac{z}{e^{T_m}(x, a)}}$$

$$D_{P_{z}} \gamma_2(y, z, R^{T_m}, P_{z}^{m}) = - \frac{P_{z}^{m}(\cdot)}{R_{z_b}^{T_m} (x, y, b)}$$

and in the last two equations, \(D\) denotes the differential operator.

---

\(^5\)The following two expressions need to be checked later.

\(^6\)To be determined later.