A Jump and Smile Ride: Continuous and Jump Variance Risk Premia in Option Pricing

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Abstract

Stochastic and time-varying volatility models typically fail to correctly price out-of-the-money put options at short maturity. We extend Realized Volatility option pricing models by adding a jump component estimated from high-frequency data, and the associated risk premium. The inclusion of jumps provides a rapidly moving volatility factor, which improves on the fitting properties under the physical measure. The change of measure is performed adopting a Stochastic Discount Factor (SDF) with three risk premia: equity, and two variance risk premia associated to the continuous and discontinuous components.

Employing an SDF with multiple premia further improve the flexibility under risk neutral

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dynamics while preserving analytical tractability. It also provides new way of separately estimate variance risk premia by coherently combining high-frequency returns and option data in a multi-factor pricing model. The empirical analysis illustrates the influence of the jump factor on the pricing performances of Standard and Poor’s 500 Index options.

1 Introduction

Stochastic and time-varying volatility models, such as Heston (1993), Duan (1995), and Heston and Nandi (2000), are able to qualitatively reproduce the smile (i.e. excess kurtosis) and the smirk (i.e. negative skewness) observed in short term equity options. However, they fail to address these features quantitatively. As a result, they severely under-price out-of-the-money put options. To cope with this problem, a variety of models have been developed to include jumps in returns (see Merton (1976), Bates (1996), Bates (2000), Kou (2002), Pan (2002), Broadie and Detemple (2004), Huang and Wu (2004), Bates (2006) and Cai and Kou (2011) in continuous-time, and Maheu and McCurdy (2004), Duan et al. (2006), and Christoffersen et al. (2010) in discrete-time) and jumps in volatility (see, e.g., Eraker et al. (2003), Eraker (2004), and Broadie et al. (2007)). Christoffersen et al. (2008) employ a modified version of the two-factor component GARCH in Engle and Lee (1999) for options pricing in discrete-time, while Bates (2000) proposes a two-factor jump-diffusion model to fit the implicit distribution in futures options. Similarly, the family of RV option pricing models recently proposed by Corsi et al. (2013) (i.e., ARG, HARG and HARGL) have difficulties in generating realistic level and dynamics of the steepness of the implied volatility at short maturity, although, the general shape and dynamics of the smile is much closer to the empirical one compared to the standard GARCH option pricing models. Therefore, the HARGL implies some degree of under-pricing for deep out-of-the-money (DOTM) put options. This is a common feature of stochastic volatility option pricing models without jumps, since they cannot completely capture the probability mass in the right tail of the volatility density.

In this paper we extend the class of Realized Volatility option pricing models by adding
a jump component in volatility and its associated risk premium. The inclusion of jumps in
the variance dynamics provides a rapidly moving volatility factor, which will improve on the
fitting properties under the physical measure, \( \mathbb{P} \), and on the pricing performance under the risk-
neutral measure, \( \mathbb{Q} \). Consequently our change of measure employs a SDF with three different
risk premia: one for equity, and two variance risk premia related to the continuous and jump
components. The proposed multiple risk premia SDF allows to improve the flexibility of the
option pricing model under the risk neutral dynamics while preserving analytical tractability.
In addition, it provides a new methodology of separate estimation of the continuous and jump
variance risk premia which coherently combines information from both high-frequency returns
and option data.

More specifically, we develop a model where the log-return dynamics is completely de-
termined by specifying the RV dynamics chosen among the family of the HAR-RV processes.
These processes, introduced by Corsi (2009), successfully describe the impact that past realized
variances aggregated on different time scales (daily, weekly and monthly) have on the current
level of realized variance. Recently, Corsi et al. (2013) have studied the application of the
HARG-RV model to option pricing in discrete time introducing the HARGL extension which
accounts for the leverage through a daily binary component. More recently, Majewski et al.
(2015) have widened the HARG-RV class and included a heterogeneous parabolic structure for
leverage, defining the LHARG-RV model.

In this work, we extend the LHARG-RV model to account for a possibility of extreme move-
ments in the evolution of volatility. The newly proposed model is labelled as JLHARG-RV or
Heterogeneous AutoRegressive Gamma model for Realized Volatility with Leverage and Jumps.
JLHARG-RV assumes that the dynamics for realized variance is given by the sum of two inde-
pendent random variables which account for the continuous and the discontinuous components
of the volatility. We model the former as an autoregressive gamma process (see Gourieroux and
Jasiak (2006)) whose conditional mean is assumed to be a linear function of the past realized
variances and leverage terms aggregated over different time scales (daily, weekly, and monthly).
The latter is described as a compound Poisson process where the jump size is sampled from a gamma distribution. For this model we first show how to compute analytically the moment generating function (MGF) of the log-returns, under the physical measure.

In order to obtain an analytical option pricing formula, we derive the MGF under the risk neutral measure. The change of measure is performed adopting the same approach in Christoffersen et al. (2008), Gagliardini et al. (2011), Corsi et al. (2013), based on a discrete-time exponential affine SDF which allows to incorporate risk premia for the continuous and discontinuous components of the volatility, in addition to the equity risk premium. We stress the importance of having risk premia for both the volatility factors in order to compensate for two new sources of risk, in addition to the traditional premium related to shocks in the log-return. In particular, including a premium for the jump component represents an important novel contribution of this work which helps to better understand the negative skew effect implied by out-of-the-money (OTM) option prices quoted on the market. Due to the analytical tractability of exponential-affine forms, we are able to derive the risk-neutral MGF and show that the risk-neutral model still belongs to the JLHARG model class. In particular, we prove the existence of a one-to-one mapping among the parameters describing the physical and risk-neutral dynamics of the JLHARG model.

An additional advantage of JLHARG is related to the model estimation. This is due to the observability of RV, directly built from the high-frequency time series of log-returns. We compute the RV time series from tick-by-tick returns for the Standard and Poor’s 500 (S&P500) Futures, from January 1, 1990 to December 31, 2007. In order to separate the two dynamics of volatility, we exploit the Threshold Bipower Variation methodology introduced in Corsi et al. (2010) which allows to detect the jumps in the RV. Having the time-series for the continuous and discontinuous volatility factors, we estimate the parameters of the JLHARG processes employing the Maximum Likelihood Estimator (MLE) on both sets of historical data.

We empirically assess the option pricing performance of the model. Being an extension of HARG class of models, we choose it as a natural benchmark of our newly proposed JLHARG.
We perform our analysis on OTM Plain Vanilla options written on S&P500 Index whose valuation is given each Wednesday from January 1, 1996 to December 31, 2004. We calibrate the premia on the whole implied volatility surfaces and we compute the option prices using the efficient COS method introduced by Fang and Oosterlee (2008). The results clearly illustrates the important contribution of the jump factor in the pricing performance of S&P500 Index OTM options and the economic significance of the jump risk premia.

2 The model

2.1 Real-World dynamics

We consider a risky asset with the following log-return dynamics

\[ y_t = r + \lambda RV_t + \sqrt{RV_t} \epsilon_t, \]  

(2.1)

where \( r \) is the risk-free rate, \( \lambda \) is the market price of risk, \( \epsilon_t \) are \( i.i.d. \) standard normal innovations, and \( RV_t \) is realized variance at day \( t \). The aggregate daily dynamics (2.1) is formally equivalent to that employed to price options in Corsi et al. (2013), Christoffersen et al. (2014), and Majewski et al. (2015). As a major difference, in this paper we distinguish two separate components of realized variance: a continuous component \( RV_t^c \) and a jump component \( RV_t^j \) (details on the RV measure employed in the implementation of the model are given in Section 3).

Our approach is motivated by the empirical analyses of Andersen et al. (2001), who find that the distributions of daily equity returns standardized by the corresponding RV is approximately Gaussian and Andersen et al. (2010) who investigate the deviation from normality ascribed to a jump component in the price process. The latter results indicate that the discontinuous component has a minor impact on the distributional properties, since the jump-adjusted standardized
series are not systematically closer to the Gaussian than the \( y_t / \sqrt{RV_t} \) standardized returns.\(^2\) This is especially true for time series generated from futures contracts on the S&P500 Index, which are recognized in Andersen et al. (2010) to suffer from minimal microstructure distortion and low liquidity effects. As can be seen from the density plots of Figure 1, we observe the same feature for the S&P500 Futures in our sampling period. The two-sample Kolmogorov-Smirnov test between the RV standardized and jump-adjusted series indicates that the two distributions cannot be distinguished. If any, by judging on the value of the kurtosis of 3.64 for the jump-adjusted distribution and 3.06 for the RV standardized, we conclude that the latter is closer to a normal distribution than the former one. Consequently, we employ the log-return dynamics

\[
y_t = r + \lambda (RV^c_t + RV^j_t) + \sqrt{RV^c_t + RV^j_t} \epsilon_t
\]

(2.2)

as a reasonable approximation of the process observed at daily frequency.

Given the information at time \( t, \mathcal{F}_t \), a new realization of the RV components is obtained by sampling at time \( t + 1 \) from two conditionally independent distributions. The continuous part of RV depends on past realizations of \( RV^c \) and of a leverage component \( \ell_t \), which corresponds to a quadratic function of the total realized variance

\[
\ell_t = (\epsilon_t - \gamma \sqrt{RV^c_t + RV^j_t})^2.
\]

Then, introducing the notation \( RV^c_{t+1} | \mathcal{F}_t \sim \tilde{\gamma}(\delta, \Theta(RV^c_t, L_t), \theta) \),

(2.3)

\(^2\) Perhaps surprisingly, the results indicate that neither of the jump-adjusted standardized series are systematically closer to Gaussian than the non-adjusted realized volatility standardized returns. [\ldots] One reason is that jumps largely self-standardize: a large jump tends to inflate the (absolute) value of both the return (numerator) and the realized volatility (denominator) of standardized returns, so the impact is muted.” - Andersen et al. (2010)
where $\delta$ is the shape parameter, and $\theta$ is the scale. The non-centrality is given by

$$
\Theta(RV^c_t, L_t) = d + \beta_d RV^c_t + \beta_w RV^c_t + \beta_m RV^c_t + \alpha_d \ell_t(d) + \alpha_w \ell_t(w) + \alpha_m \ell_t(m),
$$

(2.4)

where $d \in \mathbb{R}$, $\beta_i \in \mathbb{R}^+$, $\alpha_i \in \mathbb{R}^+$ are constant, and the quantities

$$
\begin{align*}
RV^c_t(d) &= RV^c_t, \\
RV^c_t(w) &= \frac{1}{4} \sum_{i=1}^4 RV^c_{t-i}, \\
RV^c_t(m) &= \frac{1}{17} \sum_{i=5}^{21} RV^c_{t-i},
\end{align*}
$$

represent the heterogeneous components corresponding to the short-term or daily ($d$), medium-term or weekly ($w$) and long-term or monthly ($m$) realized variance and leverage terms, respectively on the left and right columns above.

The jump component of the realized variance is instead modelled as a compound Poisson process with intensity $\tilde{\Theta}$ and sizes sampled from a gamma distribution with shape $\tilde{\delta}$ and scale $\tilde{\theta}$

$$
RV^i_{t+1} | F_t \sim \sum_{i=1}^{n_{t+1}} Y_i \quad \text{with} \quad n_{t+1} \sim \mathcal{P}(\tilde{\Theta}) \quad \text{and} \quad Y_i \text{i.i.d.} \sim \gamma(\tilde{\delta}, \tilde{\theta}).
$$

(2.5)

Equations (2.2)-(2.5) completely characterise the log-return dynamics as an Autoregressive Gamma model in Realized Volatility with Heterogeneous Leverage and Jumps, and we acronym it JLHARG-RV model. The crucial advantage of the JLHARG model is that it satisfies the affine property. The importance of affine processes in finance - due to their analytical tractability - has been acknowledged in many studies (see Duffie et al. (2000); Darolles et al. (2006); Majewski et al. (2015) among others). We prove the following

**Proposition 1.** Under $\mathbb{P}$, the MGF of the log-return $y_{t,T} = \sum_{t=t+1}^{T} y_k$ for JLHARG model has the following form

$$
\phi^P(t, T, z) = \mathbb{E}^P[e^{zy_{t,T} | F_t}] = \exp \left( a_t + \sum_{i=1}^{p} b_{t,i} RV^c_{t+1-i} + \sum_{i=1}^{q} c_{t,i} \ell_{t+1-i} \right),
$$

7
where \(a_t, b_{t,i}\) and \(c_{t,i}\) are given by recursive relations.

Proof: See Appendix A.

2.2 Risk-neutralization

To preserve analytical tractability of the model under the martingale measure we risk-neutralize it employing an SDF within the family of exponential affine factors, whose high flexibility allows to incorporate multiple factor-dependent risk premia. This approach has been extensively used in literature.\(^3\) We propose an SDF of the following form

\[
M_{s,s+1} = \frac{e^{-\nu_c \text{RV}_c^s - \nu_j \text{RV}_j^s - \nu_y y_{s+1}}}{\mathbb{E}^F [e^{-\nu_c \text{RV}_c^s - \nu_j \text{RV}_j^s - \nu_y y_{s+1}} | \mathcal{F}_s]},
\]

which represents the Esscher transform from the physical log-return density to the risk neutral one (see Gerber and Shiu (1994) and Bühlmann et al. (1996)). The main advantage of the SDF (2.6) is to clearly identify the sources of risk and explicitly compensate them with separated risk premia. Specifically, this form allows to have both the continuous (\(\nu_c\)) and discontinuous (\(\nu_j\)) variance risk premia, in addition to the standard equity premium (\(\nu_y\)). The equity premium has to satisfy the following no-arbitrage condition.

**Proposition 2.** The JLHARG model defined by equations (2.2) – (2.5) with SDF given by (2.6) satisfies the no-arbitrage condition if and only if

\[\nu_y = \lambda + \frac{1}{2}.\]

Proof: Appendix B.

\(^3\)For example in Gagliardini et al. (2011), Corsi et al. (2013), Christoffersen et al. (2013) and Majewski et al. (2015).
Moreover, we are able to provide a one-to-one mapping of the parameters under \( \mathbb{P} \) to those under the \( \mathbb{Q} \) measure, ensuring that the risk-neutral log-return dynamics is still governed by a JLHARG process.

**Proposition 3.** Under risk-neutral measure \( \mathbb{Q} \) the realized variance follows a JLHARG process with parameters

\[
\begin{align*}
\beta_d^* &= \frac{\beta_d}{1 - \theta_y}, \\
\beta_w^* &= \frac{\beta_w}{1 - \theta_y}, \\
\beta_m^* &= \frac{\beta_m}{1 - \theta_y}, \\
\alpha_d^* &= \frac{\alpha_d}{1 - \theta_y}, \\
\alpha_w^* &= \frac{\alpha_w}{1 - \theta_y}, \\
\alpha_m^* &= \frac{\alpha_m}{1 - \theta_y}, \\
\theta^* &= \frac{\theta}{1 - \theta_y}, \\
\delta^* &= \delta, \\
\gamma^* &= \gamma + \frac{1}{2}, \\
d^* &= \frac{d}{1 - \theta_y}, \\
\tilde{\Theta}^* &= \frac{\tilde{\Theta}}{(1 - \tilde{\theta}_y)}, \\
\tilde{\delta}^* &= \tilde{\delta}, \\
\tilde{\theta}^* &= \frac{\tilde{\theta}}{1 - \tilde{\theta}_y},
\end{align*}
\]  
(2.7)

where \( y^c = -\lambda^2/2 - \nu_c + \frac{1}{5} \) and \( y^j = -\lambda^2/2 - \nu_j + \frac{1}{5} \).

Proof: Appendix C.

Knowing the dynamics of the process under \( \mathbb{Q} \), the moment generating function under the risk-neutral measure is a straightforward consequence of Proposition 1.

**Corollary 1.** Under \( \mathbb{Q} \) the MGF of the JLHARG model is formally the same as in Proposition 1 with equity risk premium \( \lambda^* = -0.5 \), and \( d^*, \delta^*, \theta^*, \tilde{\Theta}^*, \tilde{\delta}^*, \tilde{\theta}^*, \gamma^*, \alpha_l^*, \beta_l^* \) for \( l = d, w, m \) as in (2.7).

We point out that the risk premia in the vector \( (\nu_c, \nu_j) \) are the only parameters that need to be calibrated on option data. Then, all the parameters governing the dynamics of the process under \( \mathbb{Q} \) can be explicitly computed from the values estimated under \( \mathbb{P} \) starting from the values of \( (\nu_c, \nu_j) \) through the relations given by (2.7).
2.3 Particular cases

The JLHARG-RV family nests a variety of RV models as special cases. The first instance is the JHARG model which preserves the heterogeneous autoregressive structure for RV but lacks the leverage term. This model can be seen as a natural extension of the HARG model, by Corsi et al. (2013), accounting for a discontinuous component. The second model is the JLHARG model with Parabolic Leverage (P-JLHARG) that we obtain setting \( d = 0 \) in (2.4). The third one is a JLHARG with zero-mean leverage term (ZM-JLHARG) inspired by the Component GARCH model of Christoffersen et al. (2008). At variance with the latter paper, we follow the approach of Majewski et al. (2015) including time heterogeneity in leverage through the following relations

\[
\begin{align*}
\bar{\ell}_t^{(d)} &= \epsilon_t^2 - 1 - 2\epsilon_t\gamma\sqrt{RV_t^c + RV_t^j}, \\
\bar{\ell}_t^{(w)} &= \frac{1}{4} \sum_{i=1}^{4} \left( \epsilon_{t-i}^2 - 1 - 2\epsilon_{t-i}\gamma\sqrt{RV_{t-i}^c + RV_{t-i}^j} \right), \\
\bar{\ell}_t^{(m)} &= \frac{1}{17} \sum_{i=5}^{21} \left( \epsilon_{t-i}^2 - 1 - 2\epsilon_{t-i}\gamma\sqrt{RV_{t-i}^c + RV_{t-i}^j} \right).
\end{align*}
\]

The linear \( \Theta(RV_t^c, L_t) \) in this case reads

\[
\beta_d RV_t^c + \beta_w RV_t^e + \beta_m RV_t^e + \alpha_d \bar{\ell}_t^{(d)} + \alpha_w \bar{\ell}_t^{(w)} + \alpha_m \bar{\ell}_t^{(m)},
\]

which can be reduced to the form (2.4) setting \( d = -(\alpha_d + \alpha_w + \alpha_m) \), \( \beta_l = \beta_l - \alpha_l\gamma^2 \) for \( l = d, w, m \). The larger flexibility of the leverage term \( \bar{\ell}_t \) allows the model to better describe the skewness and kurtosis of the empirical data.
3 Model estimation and statistical properties

The estimation of the parameter under $\mathbb{P}$ is greatly simplified by the direct observability of RV which avoids the use of filtering procedure of the latent volatility process. In this paper, the RV time series is obtained from tick-by-tick data for the S&P500 Futures, from January 1, 1990 to December 31, 2007. Our estimation procedure for the continuous and jump component is the following:

(i) we estimate the total quadratic variation of the log-prices using the Two-Scale estimator introduced by Zhang et al. (2005);

(ii) we identify the discontinuous component using the Threshold Bipower variation method by Corsi et al. (2010) which detects the spikes in RV time series and separates it from the continuous component.

The RV, so far defined, is built from open-to-close data, thus neglecting the overnight contribution. We adjust our RV estimator by rescaling the time series so to match the unconditional mean of the squared daily returns (close-to-close).

Having the two time series for the RV components, we can estimate the parameters under $\mathbb{P}$ of the JLHARG-RV processes using the Maximum Likelihood Estimator. According to the model specified in equation (2.3) and (2.5), the log-likelihood functions for the continuous and
jump RV components, respectively $l^c_{t,T}$ and $l^j_{t,T}$, are given by the following series-expansions

$$l^c_{t,T} (\delta, \theta, d, \beta, \beta, \alpha, \mu, \alpha, \alpha, \gamma) = - \sum_{t=1}^{T} \left( \frac{RV^c_t}{\theta} + \Theta (RV^c_{t-1}, L_{t-1}) \right)$$

$$+ \sum_{t=1}^{T} \log \left( \sum_{k=1}^{\infty} \frac{(RV^c_t)^{\delta+k-1}}{(\delta + k) \theta^{\delta+k}} \frac{\Theta (RV^c_{t-1}, L_{t-1})^k}{k!} \right),$$

$$l^j_{t,T} (\tilde{\delta}, \tilde{\theta}, \tilde{\Theta}) = - \sum_{t=1}^{T} \left( \frac{RV^j_t}{\tilde{\theta}} + \tilde{\Theta} \right) + \sum_{t=1}^{T} \log \left( \sum_{k=1}^{\infty} \frac{(RV^j_t)^{k+1}}{\theta \Gamma (k \delta)} \frac{\tilde{\Theta}^k}{k!} \right).$$

Both log-likelihoods have a term involving an infinite series. To overcome this issue we operate a truncation of the infinite sum to the $90th$ order as suggested in Corsi et al. (2013). Finally, we determine the market price of risk $\lambda$ in equation (2.2) regressing the centered and normalized log-return on the realized volatility. This regression is performed by rewriting the equation (2.2) as

$$\frac{y_{t+1} - r}{\sqrt{RV^c_{t+1} + RV^j_{t+1}}} = \lambda \sqrt{RV^c_{t+1} + RV^j_{t+1}} + \epsilon_{t+1}.$$

We choose the FED Fund rate as proxy for the risk-free rate $r$.

In Table 1 the values of the estimated parameters under $P$ are reported. We show the results for four different models JHARG, JHARGL, P-JLHARG and ZM-JLHARG together with their standard deviations and the values of the log-likelihood function. We also include estimation results for a version of the HARGL model, presented by Corsi et al. (2013) accounting for a discontinuous component in RV, termed JHARGL. Our results confirm that the impact of past RV on the current one decreases with the increase of the aggregation horizon. The same evidence has been documented by Corsi (2009), Corsi and Renò (2012) and Majewski et al. (2015).
In view of an option pricing application, an important role for reproducing the shape of the implied volatility surface is played by the skewness and kurtosis term structures generated by the underlying dynamics. Adding a heterogeneous leverage considerably improves the skewness and the excess kurtosis of the log-return probability distribution. In this paper, we not only preserve the heterogeneity of the leverage, but we also add a discontinuous component which captures extreme price movements. With this choice, our JLHARG class of models is able to reproduce a stronger leverage effect. In Figure 2 we show the skewness and the excess kurtosis from a simulation of the P-JLHARG model with parameters from Table 1 at different aggregation time – from one 1 day to 250 days – under both $\mathbb{P}$ and $\mathbb{Q}$ measures. The model is able to reproduce significant negative values of skewness and positive excess kurtosis under the physical measure. When moving to the $\mathbb{Q}$ measure, the effect, induced by the presence of the variance risk premia $\nu_c$ and $\nu_j$, is strengthened.

4 Option valuation

4.1 Data set description

Our data set consists of Plain Vanilla OTM options on S&P500 Index for each Wednesday from January 1, 1996 to December 31, 2004. We first apply a standard filter removing options with maturity less than 10 days or more than 365 days, implied volatility larger than 70% and prices less than 0.05$ (see Barone-Adesi et al. (2008), Corsi et al. (2013) and Majewski et al. (2015)). Using $K/S_t$ as definition of moneyness, we filter out DOTM options with moneyness larger than 1.3 for call options and less than 0.7 for put options. This choice yields a total number of 46066 observations. For our purposes, put options are identified as DOTM if their moneyness is between $0.7 \leq m \leq 0.9$ and OTM if $0.9 < m \leq 0.98$. On the other hand, call options are said to be DOTM if $1.1 < m \leq 1.3$ and OTM if $1.02 < m \leq 1.1$. Options are called at-the-money (ATM) if $0.98 < m \leq 1.02$. As far as the time to maturity $\tau$ is concerned, we identify options as short maturity ($\tau \leq 50$ days), short-medium maturity ($50 < \tau \leq 90$ days),
long-medium maturity \((90 < \tau \leq 160 \text{ days})\), and long maturity \((\tau > 160 \text{ days})\).

### 4.2 Model calibration and pricing method

In order to derive the risk-neutral dynamics, the values for risk premium parameters \((\nu_c, \nu_j, \nu_y)\) need to be identified. According to Proposition 2, in our framework \(\nu_y\) is fixed by the no-arbitrage condition, while the \((\nu_c, \nu_j)\) remains a vector of free parameters that needs to be calibrated from option prices.

For the calibration procedure, we adopt a method based on the unconditional minimization of the distance between the market implied and the model implied volatility surface. For this reason, we divide our dataset in different intervals of moneyness and maturity – as described in Section 4.1 – obtaining a \(5 \times 4\) moneyness-maturity grid. Then, for each subset, we compute the unconditional mean of the market implied volatilities. In this way, as shown in Table 2, we obtain a 20-point discrete representation of the implied volatility surface. Finally, we compute the same discrete grid for the model implied volatility and we identify the optimal values of \((\nu_c, \nu_j)\) which minimize the distance between the two grids, i.e.

\[
\arg \min_{(\nu_c, \nu_j)} \{ f_{\text{obj}}(\nu_c, \nu_j) \}.
\]

The objective function \(f_{\text{obj}}(\nu_c, \nu_j)\) is defined as

\[
f_{\text{obj}}(\nu_c, \nu_j) = \sqrt{\sum_{i=1}^{5} \sum_{j=1}^{4} (\text{IV}_{ij}^{\text{mod}}(\nu_c, \nu_j) - \text{IV}_{ij}^{\text{mkt}})^2},
\]

and represents the quadratic distance between the model implied volatility surface and the market one, whose elements are \(\text{IV}_{ij}^{\text{mod}}(\nu_c, \nu_j)\) and \(\text{IV}_{ij}^{\text{mkt}}\), respectively. In order to compute the option prices – and associated implied volatilities – we employ a numerical scheme introduced by Fang and Oosterlee (2008), termed the COS method. This method, based on Fourier-cosine
expansions, efficiently evaluates the price of Plain Vanilla options from the characteristic function of log-returns.

In Table 3 we report the calibrated variance risk premia for JLHARG models. The risk premia for the HARG and HARGL models of Corsi et al. (2013) and for the LHARG model of Majewski et al. (2015) are also obtained with the same calibration scheme. This allows us to have a fair comparison among the different models in the pricing performance analysis of the next sections. We observe that the presence of a jump premium reduces in absolute term the values of the premium for the continuous component. This fact indicates that the discontinuous component has a double effect on the dynamics. On one hand it induces more skew in the distribution of returns. This is coherently captured by the negative jump premium. On the other hand, since it represents an additional source of randomness with no autoregressive structure, it has the global effect to remove some persistence from the continuous component of the realized volatility process. Furthermore, the jump premium decreases in absolute term when a more refined form of the leverage is adopted. The most negative jump premium is obtained for the JHARG model. It becomes less negative for the JHARGL model, which has a quite elementary binary leverage, and reduces further for the JLHARG with heterogeneous parabolic leverage. Finally, it reaches the smallest value for the ZM-JLHARG where the heterogeneous leverage is centred. This is due to the fact that large negative innovations in the price rise the future variance through the leverage term, and so reduce the premium of the jump component and increase the one of the continuous part.

4.3 Pricing results

We can summarize the option pricing procedure in four steps: (i) estimation of the parameters under the physical measure $\mathbb{P}$; (ii) unconditional calibration of the parameter vector $(\nu_c, \nu_j)$; (iii) mapping of parameter values from $\mathbb{P}$ to $\mathbb{Q}$ using expressions (2.7); (iv) numerical computation
of option prices through COS method using the MGF recursive formulas in (A.10).

In order to disentangle the contribution to the pricing performance of the different features of the newly proposed model, we compare our results with those from HARG and HARGL models in Corsi et al. (2013), and LHARG models in Majewski et al. (2015). Since HARGL model involves a functional form of the leverage which can not be treated in analytical way in our MGF framework, the option prices are obtained using a Monte Carlo simulation method.

As customary in the literature (Renault (1997), Corsi et al. (2013), Majewski et al. (2015)), we employ the Root Mean Square Error (RMSE) on the percentage IV as performance measure, i.e.

\[
RMSE_{IV} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} \left( \frac{IV_{mod}^i - IV_{mkt}^i}{N} \right)^2},
\]

where \( N \) is the number of options, and \( IV_{mod}^i \) and \( IV_{mkt}^i \) are the model and the market implied volatility, respectively.

In Table 4 we report the global comparison of the option pricing performances between the two classes of competitor models. We build ratios between the RMSE of each model with jumps and its natural competitor without jumps. The table shows that for all considered pairs of models, the RMSE of the models accounting for jumps is always smaller than the corresponding ones without jumps. In particular, the performances are improved by about 4% - 5% in the range of moneyness \( 0.9 < m < 1.1 \) and 4% - 7% in the range \( 0.8 < m < 1.2 \). This first result confirms that including a discontinuous component in the description of the log-returns dynamics has the genuine effect to improve the option pricing.

Table ?? shows in detail how the pricing performance is distributed across different moneyness and maturity as described in Section 4.1. We observe that the largest performance improvements are obtained for short-maturity DOTM and OTM put options, for which the
models with jumps outperform the corresponding model without jumps by percentages ranging from 5% to 20%. This is consistent with what is observed in option markets. Short-maturity OTM put options are systematically over-priced due to market agents’ concern of the occurrence of extreme negative events which could generate large losses on a short-maturity time horizon. For longer maturities (50 < τ ≤ 90, 90 < τ ≤ 160 and 160 < τ), the difference of performance in pricing OTM options becomes thinner since the impact of large discontinuous movements is mitigated by the strong mean-reversion behavior of the variance observed over long time horizons. This is in agreement with the less pronounced smile, or smirk, observed in market implied volatilities profiles of long maturity OTM options.

In Table 6 and 7, we report the pricing results limiting the analysis to those days in which jumps could have had a greater impact on implied volatilities. Thus, we restrict the analysis to those Wednesday preceded by a Tuesday on which a jump occurred. We observe that the global improvements in performances induced by the presence of the discontinuous component are much more pronounced with a 8%-11% RMSE reduction in the range of moneyness 0.9 < m < 1.1, and 5% - 9% in the range 0.8 < m < 1.2. The detailed RMSE ratios computed for each couple of competitor models for different moneyness and maturities confirm the overall improvements. This is a desired feature for the class of JLHARG models since it captures the effect of jumps to induce higher expected skewness and volatility.

To sum up, when compared to available realized volatility models not accounting for a discontinuous component, the proposed JLHARG models manage to better reproduce the implied volatilities for OTM options, especially at short time to maturity.

4.4 U-shaped density log-ratio

This final section establishes if the SDF (2.6), combined with the JLHARG-RV dynamics, implies a U-shaped profile for the log-ratio between the risk-neutral and physical probability
densities. This interesting property of a multi-dimensional pricing kernel has been recently investigated in Christoffersen et al. (2013) for a class of GARCH models. The authors show that a premium for the variance explains a number of puzzles concerning the level and movement of implied option variance compared with observed time series variance. Among others, they consider – and solve – the puzzle pointed out by Bates (1996) and Broadie et al. (2007) that the physical and risk-neutral volatility smiles differ. The key feature of their modeling approach is that the projection of the pricing kernel onto the stock price return alone is U-shaped. The strong option smile associated to this non-monotonic relation can be quantified looking at the natural logarithm of the ratio of the risk-neutral and physical conditional densities implied by the model.

We check the non-monotonicity of the log-ratio of the densities implied by the JLHARG class of models. In the GARCH case, the analytical derivation of this property is facilitated by the fact that the variance term at time $t + 1$ is an $\mathcal{F}_t$-measurable quantity. Thus, it can be directly projected onto returns at time $t$. This is clearly not the case for JLHARG models. As it can be seen from (2.3) and (2.5), in the multi-risk premia kernel (2.6) the quantities $RV_{t+1}^c$ and $RV_{t+1}^j$ are not $\mathcal{F}_t$-measurable, so the analytical projection-onto-returns method by Christoffersen et al. (2013) can not be directly replicated. We proceed as follows. We simulate a large sample of log-returns according to the P-JLHARG dynamics under the physical and risk-neutral measures with variance risk premia fixed by the model calibration. Specifically, we draw 280000 log-returns for four different time-scales – 1 day, 1 month, 3 months, and 6 months – we build the histograms for each time series (Figure 3), and we compute their log-ratios. By this method we indirectly enlighten the properties of the pricing kernel in the way it modifies the log-return probability density from the physical to the risk-neutral measure. The blue solid line in Figure 4 clearly shows the emergence of a non-monotonic relationship, and confirms the remarkable stability of such U-shaped relation across different time horizons – as already noticed in Figure 1 of Christoffersen et al. (2013). In Figure 4, we also show the results for
the density log-ratio in the particular case where both variance risk premia are set equal to zero (green dotted line). We observe that the relation – apart from the noisy behavior in the tail regions – becomes approximately linear. In fact, in absence of variance risk premia, the effect of the change of measure on the parameter of the RV process is immaterial \(^4\). As a result the risk-neutral dynamics of the realized volatility is very close to the physical dynamics. A monotonic behavior is typical for the Black-Scholes model – adopting a pricing kernel function only of log-returns – which has been proved to be inconsistent with the empirical evidence of a U-shaped density log-ratio. These results underline the importance of having a variance dependent SDF with non-zero variance risk-premia in order to reconcile the time series return distribution with that implied by option prices.

5 Conclusions

In this paper, we present a class of heterogeneous autoregressive models accounting for a discontinuous component in Realized Volatility. We demonstrate how to analytically characterize the moment generating function of the log-return process under physical and risk-neutral measure. For the change of measure, we adopt a flexible exponential affine pricing kernel which allows to clearly identify all the sources of financial risk, and separately compensate for them, introducing separate premia for the equity, continuous, and discontinuous variance components. Then, we detail an application to option pricing, and show the improvements of the novel class of models in capturing the pronounced volatility smile of short-maturity out-of-the-money options.

\(^4\)This can be seen from formula (2.7) noticing that when \(\nu_c = 0\) and \(\nu_j = 0\), \(y^c\) and \(y^j\) are of order one while \(\theta \sim 10^{-5}\) so all the denominators are approximately equal to one.
References


A MGF computations under $\mathbb{P}$ measure

For the ease of computation, the expression (2.4) is rewritten as

$$\Theta(RV^c_t, L_t) = d + \sum_{i=1}^{22} \beta_i RV^c_{t+1-i} + \sum_{i=1}^{22} \alpha_i \left( \epsilon_{t+1-i} - \gamma \sqrt{RV^c_{t+1-i} + RV^j_{t+1-i}} \right)^2,$$

with

$$\beta_i = \begin{cases} \beta_d & \text{for } i = 1 \\ \beta_w/4 & \text{for } 2 \leq i \leq 5 \\ \beta_w/17 & \text{for } 6 \leq i \leq 22 \end{cases}$$

and

$$\alpha_i = \begin{cases} \alpha_d & \text{for } i = 1 \\ \alpha_w/4 & \text{for } 2 \leq i \leq 5 \\ \alpha_w/17 & \text{for } 6 \leq i \leq 22 \end{cases} \quad \text{(A.1)}$$

We start showing that JLHARG processes satisfy the affine relation

$$\mathbb{E} \left[ e^{z\gamma_{s+1} + b RV_{s+1} + c \ell_{s+1}} | \mathcal{F}_s \right] = e^{\alpha(z,b,c) + \sum_{i=1}^{n} B_i(z,b,c) RV_{s+1-i} + \sum_{i=1}^{p} C_j(z,b,c) \ell_{s+1-j}}, \quad \text{(A.2)}$$

for some functions $\mathcal{A} : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, $\mathcal{B}_i : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$, $\mathcal{C}_j : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, where $RV_t = (RV^c_t, RV^j_t)$, $b \in \mathbb{R}^2$, $c \in \mathbb{R}$, and $\cdot$ is the scalar product in $\mathbb{R}^2$. To derive the explicit form of the functions $\mathcal{A}, \mathcal{B}_i, \mathcal{C}_j$ which allows to characterise the MGF we show that

$$\mathbb{E}^P \left[ e^{z\gamma_{t-1} + b RV_t + c \ell_{t-1}} | \mathcal{F}_{t-1} \right]$$

$$= \mathbb{E}^P \left[ e^{z \gamma_t + b RV_t c \ell_t} | \mathcal{F}_{t-1} \right]$$

$$= \mathbb{E}^P \left[ e^{z (\gamma_t + \gamma_{RV^c_t} + \gamma_{RV^j_t}) + \gamma_{RV^c_t} + \gamma_{RV^j_t}} | \mathcal{F}_{t-1} \right]$$

$$= \mathbb{E}^P \left[ e^{z (\gamma_t + \gamma_{RV^c_t} + \gamma_{RV^j_t}) + \gamma_{RV^c_t} + \gamma_{RV^j_t} b RV^c_t + b RV^j_t - \frac{1}{2} \ln(1-2\epsilon) + \frac{1 + \epsilon^2 - 2\epsilon \gamma_t}{1 - 2\epsilon}\gamma_{RV^c_t} + \frac{1 + \epsilon^2 - 2\epsilon \gamma_t}{1 - 2\epsilon}\gamma_{RV^j_t}} | \mathcal{F}_{t-1} \right]$$

$$= \mathbb{E}^P \left[ e^{z (\gamma_t + \gamma_{RV^c_t} + \gamma_{RV^j_t}) + \gamma_{RV^c_t} + \gamma_{RV^j_t} b RV^c_t + b RV^j_t - \frac{1}{2} \ln(1-2\epsilon) + \frac{1 + \epsilon^2 - 2\epsilon \gamma_t}{1 - 2\epsilon}\gamma_{RV^c_t} + \frac{1 + \epsilon^2 - 2\epsilon \gamma_t}{1 - 2\epsilon}\gamma_{RV^j_t}} | \mathcal{F}_{t-1} \right]$$

$$= e^{z \gamma_t - \frac{1}{2} \ln(1-2\epsilon)} \mathbb{E}^P \left[ e^{\left( \gamma_t + \gamma_{RV^c_t} + \gamma_{RV^j_t} \right) RV^c_t} | \mathcal{F}_{t-1} \right]$$

$$\times \mathbb{E}^P \left[ e^{\left( \gamma_t + \gamma_{RV^c_t} + \gamma_{RV^j_t} \right) RV^j_t} | \mathcal{F}_{t-1} \right].$$
In the third line we have used the result that if $Z \sim \mathcal{N}(0,1)$ then

$$E \left[ \exp \left( x (Z + y)^2 \right) \right] = \exp \left( - \frac{1}{2} \ln (1 - 2x) + \frac{xy^2}{1 - 2x} \right).$$

For a noncentred gamma random variable, from Gourieroux and Jasiak (2006) we know that

$$E^P \left[ e^{xRV_i} | \mathcal{F}_{t-1} \right] = \exp \left( - \delta \mathcal{W}(x_1, \theta) + \mathcal{V}(x_1, \theta) \left( d + \sum_{i=1}^p \beta_i RV_{s-i}^c + \sum_{j=1}^q \alpha_j \ell_{s-j} \right) \right),$$

where

$$\mathcal{V}(x, \theta) = \frac{\theta x}{1 - \theta x}, \quad \mathcal{W}(x, \theta) = \ln(1 - x\theta),$$

and

$$x(z, b, c) = z\lambda + b + \frac{z^2 + \gamma^2 c - 2c\gamma z}{1 - 2c}, \quad x_1 = x(z, b_1, c). \tag{A.4}$$

For the computation of the last expectation in the final line of (A.3), we use the property that if $Z_t$ is a compound Poisson process with rate $\omega$ and i.i.d. jump sizes $D_i$, then

$$E \left[ e^{xZ_t} | \mathcal{F}_{t-1} \right] = \exp \left( \omega \left( M_D(x) - 1 \right) \right), \tag{A.5}$$

where $M_D(x)$ is the MGF of the jump size random variable $D_i$. Since sizes are distributed according to a gamma distribution, we have

$$M_D(x) = \frac{1}{\left( 1 - x\tilde{\theta} \right)^\tilde{\delta}}, \tag{A.6}$$

From expressions (A.5) and (A.6) we obtain

$$E^P \left[ e^{xRV_i} | \mathcal{F}_{t-1} \right] = \exp \left( \tilde{\Theta} J \left( x_2, \tilde{\theta}, \tilde{\delta} \right) \right),$$

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where
\[ J(x, \tilde{\theta}, \tilde{\delta}) = \frac{1 - (1 - \tilde{\theta} x)^\delta}{(1 - \tilde{\theta} x)^\delta} \] and \( x_2 = x(z, b_2, c) \).

Gathering all the previous results, we finally conclude
\[
\mathbb{E}^P \left[ e^{z y_t + b \cdot \mathbf{RV}_t + c \ell_t} \big| \mathcal{F}_{t-1} \right] = \\
\exp \left[ z r - \frac{1}{2} \ln(1 - 2c) + \mathcal{V}(x_1, \theta) \left( d + \sum_{i=1}^{p} \beta_i \mathbf{RV}_{t-i}^c + \sum_{j=1}^{q} \alpha_j \ell_{t-j} \right) \\
- \delta \mathcal{W}(x_1, \theta) + \tilde{\Theta} J(x_2, \tilde{\theta}, \tilde{\delta}) \right],
\]

where we have introduced two functions \( x_1 = x(z, b_1, c) \) and \( x_2 = x(z, b_2, c) \), while the expression for \( x \) is given by (A.4). The direct comparison of the last expression with (A.2) allows to derive the following explicit expressions
\[
\mathcal{A}(z, b, c) = z r - \frac{1}{2} \ln(1 - 2c) - \delta \mathcal{W}(x_1, \theta) + d \mathcal{V}(x_1, \theta) + \tilde{\Theta} J(x_2, \tilde{\theta}, \tilde{\delta}), \\
\mathcal{B}_i(z, b_1, c) = \mathcal{V}(x_1, \theta) \beta_i, \\
\mathcal{C}_j(z, b_1, c) = \mathcal{V}(x_1, \theta) \alpha_j.
\]

As shown in Majewski et al. (2015), once we have above expressions we obtain
\[
\phi^P(t, T, z) = \mathbb{E}^P \left[ e^{z y_{t,T}} \big| \mathcal{F}_t \right] = \exp \left( a_t + \sum_{i=1}^{p} b_{t,i} \mathbf{RV}_{t-i+1}^c + \sum_{i=1}^{q} c_{t,i} \ell_{t+1-i} \right)
\]
where

\[ a_s = a_{s+1} + zr - \frac{1}{2} \log(1 - 2c_{s+1,1}) + dV(x_{s+1}^c, \theta) - \delta W(x_{s+1}^c, \theta) + \tilde{\Theta} J(x_{s+1}^j, \tilde{\theta}) \]

\[ b_{s,i} = \begin{cases} 
- b_{s+1,i} + V(x_{s+1}^c, \theta) \beta_i & \text{for } 1 \leq i \leq p - 1 \\
V(x_{s+1}^c, \theta) \beta_i & \text{for } i = p 
\end{cases} \]

\[ c_{s,i} = \begin{cases} 
- c_{s+1,i} + V(x_{s+1}^c, \theta) \alpha_i & \text{for } 1 \leq i \leq q - 1 \\
V(x_{s+1}^c, \theta) \alpha_i & \text{for } i = q 
\end{cases} \]

with

\[ x_{s+1}^c = z\lambda + b_{s+1,1} + \frac{1}{2} z^2 + \gamma^2 c_{s+1,1} - 2c_{s+1,1} \gamma z}{1 - 2c_{s+1,1}} \]

\[ x_{s+1}^j = z\lambda + \frac{1}{2} z^2 + \gamma^2 c_{s+1,1} - 2c_{s+1,1} \gamma z}{1 - 2c_{s+1,1}} \]

The functions \( V, W \) and \( J \) are defined as

\[ V(x, \theta) = \frac{\theta x}{1 - \theta x}, \quad W(x, \theta) = \ln(1 - x\theta), \quad J(x, \tilde{\theta}, \tilde{\delta}) = \frac{1 - (1 - \tilde{\theta} x)^\delta}{(1 - \theta x)^\delta}, \]

and the terminal conditions read \( a_T = b_{T,i} = c_{T,j} = 0 \) for \( i = 1, 2, ..., p \) and \( j = 1, 2, ..., q \).

\section*{B No-arbitrage condition}

The no-arbitrage conditions are

\[ \mathbb{E}^p [M_{s+1} | \mathcal{F}_s] = 1 \text{ for } s \in \mathbb{N}, \]

\[ \mathbb{E}^p [M_{s+1} e^{j_{s+1}} | \mathcal{F}_s] = e^r \text{ for } s \in \mathbb{N}. \]
The first relation is satisfied by definition of $M_{s,s+1}$. From a general result in Majewski et al. (2015), condition (B.1) is satisfied if, and only if

$$
\mathcal{A}(1 - \nu_y, -\nu, 0) = r + \mathcal{A}(-\nu_y, -\nu, 0),
$$

$$
\mathcal{B}_i(1 - \nu_y, -\nu, 0) = \mathcal{B}_i(-\nu_y, -\nu, 0),
$$

$$
\mathcal{C}_j(1 - \nu_y, -\nu, 0) = \mathcal{C}_j(-\nu_y, -\nu, 0),
$$

with $\nu = (\nu_c, \nu_j)$. From the last two relations, using the explicit expressions of $\mathcal{B}_i$ and $\mathcal{C}_j$ given in (A.8) and (A.9) we obtain

$$
\mathcal{V}(x(1 - \nu_y, -\nu_c, 0), \theta) = \mathcal{V}(x(-\nu_y, -\nu_c, 0), \theta),
$$

which is equivalent to

$$
x(1 - \nu_y, -\nu_c, 0) = x(-\nu_y, -\nu_c, 0).
$$

Simple computations show that the latter equation fixes the value of the equity premium

$$
\nu_y = \lambda + \frac{1}{2}.
$$

It is worth noticing that the result holding for the equity premium does not constrain the value of the variance risk premia. From the condition on $\mathcal{A}$ it follows that

$$
d\mathcal{V}(x(1 - \nu_y, -\nu_c, 0), \theta) - \delta\mathcal{W}(x(1 - \nu_y, -\nu_c, 0), \theta) + \hat{\Theta}\mathcal{J}(x(1 - \nu_y, -\nu_j, 0), \tilde{\theta})
$$

$$
= d\mathcal{V}(x(-\nu_y, -\nu_c, 0), \theta) - \delta\mathcal{W}(x(-\nu_y, -\nu_c, 0), \theta) + \hat{\Theta}\mathcal{J}(x(-\nu_y, -\nu_j, 0), \tilde{\theta}),
$$

which – in light of the relation (B.2) – reduces to

$$
\hat{\Theta}\mathcal{J}(x(1 - \nu_y, -\nu_j, 0), \tilde{\theta}) = \hat{\Theta}\mathcal{J}(x(-\nu_y, -\nu_j, 0), \tilde{\theta}).
$$

(B.3)
Equation (B.3) is identically satisfied if

\[ x(1 - \nu_y, -\nu_j, 0) = x(-\nu_y, -\nu_j, 0), \]

which holds for any value of \( \nu_j \). In conclusion, the no-arbitrage conditions fix the level of the equity risk premium, while both the continuous and discontinuous variance risk premia remain free parameters to be calibrated on option data.

C Risk-neutral dynamics

JLHARG models imply a risk-neutral MGF for log-returns whose exponential affine terms can be re-parametrized in order to obtain an expression formally equivalent to the physical MGF. Firstly we observe that the risk-neutral MGF can be expressed with a recursive set of expressions, involving a combination of the functions \( A, B_i, C_j \). Then, recalling the results given in Majewski et al. (2015), the MGF for JLHARG model under measure \( Q \) has the following form

\[
\phi^Q_{\nu_c \nu_y \nu_j}(t, T, z) = E^Q\left[e^{zy_t,T} \mid \mathcal{F}_t\right] = \exp\left(a^*_t + \sum_{i=1}^{p} b^*_t RV_{t+1-i}^c + \sum_{i=1}^{q} c^*_t \ell_{t+1-i}\right),
\]

where

\[
\begin{align*}
a^*_s &= a^*_{s+1} + zr - \frac{1}{2} \log(1 - 2c^*_{s+1,1}) + dV(x^*_{s+1}, \theta) - dV(y^*_{s+1}, \theta) \\
&\quad - \delta W(x^*_{s+1}, \theta) + \delta W(y^*_{s+1}, \theta) + \hat{\Theta} J(x^*_{s+1}, \hat{\theta}) - \hat{\Theta} J(y^*_{s+1}, \hat{\theta}) \\
b^*_{s,i} &= \begin{cases} b^*_{s+1,i} + \left(V(x^*_{s+1}, \theta) - V(y^*_{s+1}, \theta)\right) \beta_i & \text{for } 1 \leq i \leq p - 1 \\
\left(V(x^*_{s+1}, \theta) - V(y^*_{s+1}, \theta)\right) \beta_i & \text{for } i = p \\
\end{cases} \\
c^*_{s,j} &= \begin{cases} c^*_{s+1,j} + \left(V(x^*_{s+1}, \theta) - V(y^*_{s+1}, \theta)\right) \alpha_j & \text{for } 1 \leq j \leq q - 1 \\
\left(V(x^*_{s+1}, \theta) - V(y^*_{s+1}, \theta)\right) \alpha_j & \text{for } j = q \\
\end{cases}
\end{align*}
\] (C.1)
where

\[ x_{s+1}^{c*} = (z - \nu_y)\lambda + b_{s+1,1}^* - \nu_c + \frac{1}{2}(z - \nu_y)^2 + \gamma^2 c_{s+1,1}^* - 2c_{s+1,1}^* \gamma(z - \nu_y)}{1 - 2c_{s+1,1}^*} \]

\[ x_{s+1}^{j*} = (z - \nu_y)\lambda - \nu_j + \frac{1}{2}(z - \nu_y)^2 + \gamma^2 c_{s+1,1}^* - 2c_{s+1,1}^* \gamma(z - \nu_y)}{1 - 2c_{s+1,1}^*} \]

\[ y_{s+1}^{l*} = -\nu_y\lambda - \nu_l + \frac{1}{2}\nu_y^2, \]

with \( l = c, j \) and the terminal conditions are \( a_{T, i}^* = b_{T, i}^* = c_{T, j}^* = 0 \) for \( i = 1, 2, ..., p \) and \( j = 1, 2, ..., q \).

The first passage consists in comparing expression (C.1) with (A.10). We have to find a set of new parameters for which the recursive expressions for \( a_t^*, b_t^*, c_t^* \) under \( \mathbb{Q} \) correspond to the expressions under \( \mathbb{P} \). We start defining

\[ x_{s+1}^{c**, i} = z\lambda + b_{s+1,1}^* + \frac{1}{2}z^2 + (\gamma^*)^2 c_{s+1,1}^* - 2c_{s+1,1}^* \gamma^* z}{1 - 2c_{s+1,1}^*} \]

\[ x_{s+1}^{j**, i} = z\lambda + \frac{1}{2}z^2 + (\gamma^*)^2 c_{s+1,1}^* - 2c_{s+1,1}^* \gamma^* z}{1 - 2c_{s+1,1}^*} \]

Then, the following relations have to hold

\[ \delta \left( \mathcal{W}(x_{s+1}^{c*}, \theta) - \mathcal{W}(y_{s+1}^{c*}, \theta) \right) = \delta^* \mathcal{W}(x_{s+1}^{c**}, \theta^*) \] (C.2)

\[ \beta_i \left( \mathcal{V}(x_{s+1}^{c*}, \theta) - \mathcal{V}(y_{s+1}^{c*}, \theta) \right) = \beta_i^* \mathcal{V}(x_{s+1}^{c**}, \theta^*) \] (C.3)

\[ \alpha_j \left( \mathcal{V}(x_{s+1}^{c*}, \theta) - \mathcal{V}(y_{s+1}^{c*}, \theta) \right) = \alpha_j^* \mathcal{V}(x_{s+1}^{c**}, \theta^*) \] (C.4)

\[ \tilde{\Theta} \left( \mathcal{J}(x_{s+1}^{j*}, \theta) - \mathcal{J}(y_{s+1}^{j*}, \theta) \right) = \tilde{\Theta}^* \mathcal{J}(x_{s+1}^{j**}, \theta^*) \] (C.5)

with \( y_{s+1}^{c*} = -\lambda^2/2 - \nu_c + \frac{1}{8} \) and \( y_{s+1}^{j*} = -\lambda^2/2 - \nu_j + \frac{1}{8} \).

Equation (C.2) can be explicitly written as

\[ \delta \log \left[ 1 - \frac{\theta}{1 - \theta y_{s+1}^{c*}} (x_{s+1}^{c*} - y_{s+1}^{c*}) \right] = \delta^* \log \left( 1 - \theta^* x_{s+1}^{c**} \right), \]
which implies the following three sufficient conditions

\[ \delta^* = \delta \]
\[ \theta^* = \frac{\theta}{1 - \theta y^*} \]
\[ x_{s+1}^{c**} = x_{s+1}^{c*} - y^*. \]  \hspace{1cm} (C.6)

It can be easily verified that the last condition (C.6) is satisfied by substituting

\[ \lambda^* = -\frac{1}{2}; \]
\[ \gamma^* = \gamma + \lambda + \frac{1}{2}. \]

The equation (C.3) can be equivalently expressed in the form

\[ \frac{\beta_i}{1 - \theta y^*} \frac{\theta}{1 - \theta y^*} \frac{x_{s+1}^{c*} - y^*}{1 - \theta/(1 - \theta y^*) (x_{s+1}^{c*} - y^*)} = \beta_i^* \frac{\theta^* x_{s+1}^{c**}}{1 - \theta^* x_{s+1}^{c**}} \]

which gives another sufficient condition for the mapping

\[ \beta_i^* = \frac{\beta_i}{1 - \theta y^*}. \]

An analogous consideration about the third condition (C.4) allows to obtain the condition on \( \alpha_i^* \),

\[ \alpha_i^* = \frac{\alpha_i}{1 - \theta y^*}. \]

Relation (A.1) gives us the expressions for \( \beta_d^*, \beta_u^*, \beta_m^*, \alpha_d^*, \alpha_u^* \) and \( \alpha_m^* \). Finally, equation (C.5) provides the last sufficient condition

\[ \frac{\tilde{\Theta}^{\delta*}}{1 - \tilde{\Theta}^{\delta*}} \frac{1 - \left(\left(1 - \tilde{\theta} x_{s+1}^{j*}\right) / \left(1 - \tilde{\theta} y^j\right)\right)^{\delta*}}{1 - \left(\left(1 - \tilde{\theta} x_{s+1}^{j*}\right) / \left(1 - \tilde{\theta} y^j\right)\right)^{\delta*}} = \tilde{\Theta}^{\delta*} \frac{1 - \tilde{\theta} x_{s+1}^{j*}}{1 - \tilde{\theta} x_{s+1}^{j*}}. \]
which is satisfied if

\[
\tilde{\delta}^* = \tilde{\delta},
\]

\[
\tilde{\Theta}^* = \frac{\tilde{\Theta}}{(1 - \tilde{\theta}y^{j^*})^{\tilde{\delta}}},
\]

\[
\tilde{\theta}^* = \frac{\tilde{\theta}}{1 - \tilde{\theta}y^{j^*}},
\]

\[
x^{j^*}_{s+1} = x^{j^*}_{s+1} - y^{j^*}.
\]

(C.7)

As it can be seen the last condition (C.7) is redundant when compared to the condition (C.6).
Table 1: Maximum likelihood estimates, robust standard errors, and log-likelihood values. The historical data for the JHARG, JHARGL, P-JLHARG and ZM-JLHARG models are given by the daily RV computed on tick-by-tick data for the S&P500 Futures. For all three models, the estimation period ranges from the period 1990-2007.
<table>
<thead>
<tr>
<th>Moneyness</th>
<th>0.7 ≤ m ≤ 0.9</th>
<th>0.9 &lt; m ≤ 0.98</th>
<th>0.98 &lt; m ≤ 1.02</th>
<th>1.02 &lt; m ≤ 1.1</th>
<th>1.1 &lt; m ≤ 1.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Implied Volatility</td>
<td>0.3564</td>
<td>0.2353</td>
<td>0.1958</td>
<td>0.1767</td>
<td>0.2317</td>
</tr>
<tr>
<td>50 &lt; τ ≤ 90</td>
<td>0.3056</td>
<td>0.2269</td>
<td>0.2023</td>
<td>0.1790</td>
<td>0.1946</td>
</tr>
<tr>
<td>90 &lt; τ ≤ 160</td>
<td>0.2866</td>
<td>0.2232</td>
<td>0.2059</td>
<td>0.1849</td>
<td>0.1836</td>
</tr>
<tr>
<td>160 &lt; τ</td>
<td>0.2662</td>
<td>0.2230</td>
<td>0.2108</td>
<td>0.1923</td>
<td>0.1842</td>
</tr>
</tbody>
</table>

Table 2: Mean market implied volatilities of S&P500 Index options on each Wednesday from January 1, 1996 to December 31, 2004 (46066 observations) sorted by moneyness and maturity. Moneyness is defined as \( m = \frac{K}{S_t} \), where \( K \) and \( S_t \) are the strike and the underlying price, respectively. Maturity is measured in calendar days.

<table>
<thead>
<tr>
<th>Model</th>
<th>( \nu_c )</th>
<th>( \nu_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>HARG</td>
<td>-5117</td>
<td>-</td>
</tr>
<tr>
<td>HARGL</td>
<td>-5632</td>
<td>-</td>
</tr>
<tr>
<td>P-LHARG</td>
<td>-5973</td>
<td>-</td>
</tr>
<tr>
<td>ZM-LHARG</td>
<td>-6592</td>
<td>-</td>
</tr>
<tr>
<td>JHARG</td>
<td>-2735</td>
<td>-9819</td>
</tr>
<tr>
<td>JHARGL</td>
<td>-4456</td>
<td>-3946</td>
</tr>
<tr>
<td>P-JLHARG</td>
<td>-4722</td>
<td>-3802</td>
</tr>
<tr>
<td>ZM-JLHARG</td>
<td>-4442</td>
<td>-1033</td>
</tr>
</tbody>
</table>

Table 3: Continuous and discontinuous variance risk premia for different models calibrated on market implied volatilities.
### Implied Volatility RMSE

<table>
<thead>
<tr>
<th>Model</th>
<th>Moneyness</th>
<th>0.9 &lt; m &lt; 1.1</th>
<th>0.8 &lt; m &lt; 1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>HARG</td>
<td></td>
<td>5.957</td>
<td>8.039</td>
</tr>
<tr>
<td>HARGL</td>
<td></td>
<td>5.846</td>
<td>9.275</td>
</tr>
<tr>
<td>LHARG</td>
<td></td>
<td>5.732</td>
<td>7.429</td>
</tr>
<tr>
<td>ZM-LHARG</td>
<td></td>
<td>5.690</td>
<td>7.186</td>
</tr>
<tr>
<td>JHARG/HARG</td>
<td></td>
<td>0.958</td>
<td>0.933</td>
</tr>
<tr>
<td>JHARGL/HARGL</td>
<td></td>
<td>0.946</td>
<td>0.958</td>
</tr>
<tr>
<td>P-JLHARG/P-LHARG</td>
<td></td>
<td>0.947</td>
<td>0.951</td>
</tr>
<tr>
<td>ZM-JLHARG/ZM-LHARG</td>
<td></td>
<td>0.959</td>
<td>0.956</td>
</tr>
</tbody>
</table>

Table 4: Global option pricing performance on S&P500 options from January 1, 1996 to December 31, 2004, computed with the RV measure estimated from 1990 to 2007. We use the maximum likelihood parameter estimates from Table 1. First four rows: percentage $RMSE_{IV}$ for the HARG, HARGL, LHARG and ZM-LHARG models (benchmarks). Subsequent rows: $RMSE_{IV}$ ratios for different models with jumps relative to their natural competitors.
<table>
<thead>
<tr>
<th>Moneyness</th>
<th>( \tau \leq 50 )</th>
<th>( 50 &lt; \tau \leq 90 )</th>
<th>( 90 &lt; \tau \leq 160 )</th>
<th>( 160 &lt; \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A</td>
<td>JHARG/HARG Implied Volatility RMSE</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 0.7 \leq m \leq 0.9 )</td>
<td>0.805</td>
<td>0.921</td>
<td>0.977</td>
<td>1.065</td>
</tr>
<tr>
<td>( 0.9 &lt; m \leq 0.98 )</td>
<td>0.799</td>
<td>0.873</td>
<td>0.939</td>
<td>0.978</td>
</tr>
<tr>
<td>( 0.98 &lt; m \leq 1.02 )</td>
<td>1.022</td>
<td>1.005</td>
<td>0.995</td>
<td>0.949</td>
</tr>
<tr>
<td>( 1.02 &lt; m \leq 1.1 )</td>
<td>1.123</td>
<td>1.108</td>
<td>1.051</td>
<td>0.926</td>
</tr>
<tr>
<td>( 1.1 &lt; m \leq 1.3 )</td>
<td>0.872</td>
<td>1.027</td>
<td>0.983</td>
<td>0.881</td>
</tr>
<tr>
<td>Panel B</td>
<td>JHARGL/HARGL Implied Volatility RMSE</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 0.7 \leq m \leq 0.9 )</td>
<td>0.953</td>
<td>0.971</td>
<td>1.037</td>
<td>1.110</td>
</tr>
<tr>
<td>( 0.9 &lt; m \leq 0.98 )</td>
<td>0.898</td>
<td>0.938</td>
<td>0.960</td>
<td>0.963</td>
</tr>
<tr>
<td>( 0.98 &lt; m \leq 1.02 )</td>
<td>0.963</td>
<td>0.961</td>
<td>0.956</td>
<td>0.915</td>
</tr>
<tr>
<td>( 1.02 &lt; m \leq 1.1 )</td>
<td>0.993</td>
<td>1.004</td>
<td>0.976</td>
<td>0.888</td>
</tr>
<tr>
<td>( 1.1 &lt; m \leq 1.3 )</td>
<td>0.955</td>
<td>0.956</td>
<td>0.928</td>
<td>0.855</td>
</tr>
<tr>
<td>Panel C</td>
<td>P-JLHARG/P-LHARG Implied Volatility RMSE</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 0.7 \leq m \leq 0.9 )</td>
<td>0.943</td>
<td>0.951</td>
<td>0.992</td>
<td>1.051</td>
</tr>
<tr>
<td>( 0.9 &lt; m \leq 0.98 )</td>
<td>0.878</td>
<td>0.909</td>
<td>0.954</td>
<td>0.973</td>
</tr>
<tr>
<td>( 0.98 &lt; m \leq 1.02 )</td>
<td>0.960</td>
<td>0.965</td>
<td>0.988</td>
<td>0.970</td>
</tr>
<tr>
<td>( 1.02 &lt; m \leq 1.1 )</td>
<td>0.989</td>
<td>1.033</td>
<td>1.045</td>
<td>0.969</td>
</tr>
<tr>
<td>( 1.1 &lt; m \leq 1.3 )</td>
<td>0.893</td>
<td>0.943</td>
<td>0.985</td>
<td>0.947</td>
</tr>
<tr>
<td>Panel D</td>
<td>ZM-JLHARG/ZM-LHARG Implied Volatility RMSE</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 0.7 \leq m \leq 0.9 )</td>
<td>0.950</td>
<td>0.944</td>
<td>0.972</td>
<td>1.017</td>
</tr>
<tr>
<td>( 0.9 &lt; m \leq 0.98 )</td>
<td>0.912</td>
<td>0.932</td>
<td>0.964</td>
<td>0.976</td>
</tr>
<tr>
<td>( 0.98 &lt; m \leq 1.02 )</td>
<td>0.970</td>
<td>0.970</td>
<td>0.990</td>
<td>0.980</td>
</tr>
<tr>
<td>( 1.02 &lt; m \leq 1.1 )</td>
<td>0.983</td>
<td>1.008</td>
<td>1.025</td>
<td>0.979</td>
</tr>
<tr>
<td>( 1.1 &lt; m \leq 1.3 )</td>
<td>0.924</td>
<td>0.941</td>
<td>0.970</td>
<td>0.959</td>
</tr>
</tbody>
</table>

Table 5: Option pricing performance in implied volatility. \( RMSE_{IV} \) ratios for different models with jumps relative to their natural competitors sorted by moneyness and maturity.
Table 6: Global option pricing performance on S&P500 options traded on particular Wednesday, from January 1, 1996 to December 31, 2004, preceded by a Tuesday on which a jump occurred, computed with the RV measure estimated from 1990 to 2007. First four rows: percentage $RMSE_{IV}$ for the HARG, HARGL, LHARG and ZM-LHARG models (benchmarks). Subsequent rows: $RMSE_{IV}$ ratios for different models with jumps relative to their natural competitors.
<table>
<thead>
<tr>
<th>Moneyness</th>
<th>τ ≤ 50</th>
<th>50 &lt; τ ≤ 90</th>
<th>90 &lt; τ ≤ 160</th>
<th>160 &lt; τ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.7 ≤ m ≤ 0.9</td>
<td>0.899</td>
<td>0.914</td>
<td>0.969</td>
<td>1.068</td>
</tr>
<tr>
<td>0.9 &lt; m ≤ 0.98</td>
<td>0.771</td>
<td>0.820</td>
<td>0.894</td>
<td>0.980</td>
</tr>
<tr>
<td>0.98 &lt; m ≤ 1.02</td>
<td>0.914</td>
<td>0.904</td>
<td>0.904</td>
<td>0.937</td>
</tr>
<tr>
<td>1.02 &lt; m ≤ 1.1</td>
<td>1.045</td>
<td>1.053</td>
<td>0.991</td>
<td>0.892</td>
</tr>
<tr>
<td>1.1 &lt; m ≤ 1.3</td>
<td>0.848</td>
<td>1.010</td>
<td>0.957</td>
<td>0.850</td>
</tr>
<tr>
<td>Panel B</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.7 ≤ m ≤ 0.9</td>
<td>0.955</td>
<td>0.975</td>
<td>1.028</td>
<td>1.127</td>
</tr>
<tr>
<td>0.9 &lt; m ≤ 0.98</td>
<td>0.888</td>
<td>0.918</td>
<td>0.969</td>
<td>0.988</td>
</tr>
<tr>
<td>0.98 &lt; m ≤ 1.02</td>
<td>0.919</td>
<td>0.918</td>
<td>0.932</td>
<td>0.923</td>
</tr>
<tr>
<td>1.02 &lt; m ≤ 1.1</td>
<td>0.959</td>
<td>0.966</td>
<td>0.955</td>
<td>0.858</td>
</tr>
<tr>
<td>1.1 &lt; m ≤ 1.3</td>
<td>0.946</td>
<td>0.944</td>
<td>0.913</td>
<td>0.820</td>
</tr>
<tr>
<td>Panel C</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.7 ≤ m ≤ 0.9</td>
<td>0.945</td>
<td>0.945</td>
<td>0.983</td>
<td>1.053</td>
</tr>
<tr>
<td>0.9 &lt; m ≤ 0.98</td>
<td>0.864</td>
<td>0.869</td>
<td>0.914</td>
<td>0.955</td>
</tr>
<tr>
<td>0.98 &lt; m ≤ 1.02</td>
<td>0.906</td>
<td>0.889</td>
<td>0.907</td>
<td>0.934</td>
</tr>
<tr>
<td>1.02 &lt; m ≤ 1.1</td>
<td>0.942</td>
<td>0.970</td>
<td>0.979</td>
<td>0.926</td>
</tr>
<tr>
<td>1.1 &lt; m ≤ 1.3</td>
<td>0.899</td>
<td>0.921</td>
<td>0.953</td>
<td>0.916</td>
</tr>
<tr>
<td>Panel D</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.7 ≤ m ≤ 0.9</td>
<td>0.951</td>
<td>0.938</td>
<td>0.964</td>
<td>1.016</td>
</tr>
<tr>
<td>0.9 &lt; m ≤ 0.98</td>
<td>0.900</td>
<td>0.897</td>
<td>0.926</td>
<td>0.957</td>
</tr>
<tr>
<td>0.98 &lt; m ≤ 1.02</td>
<td>0.934</td>
<td>0.916</td>
<td>0.927</td>
<td>0.948</td>
</tr>
<tr>
<td>1.02 &lt; m ≤ 1.1</td>
<td>0.955</td>
<td>0.959</td>
<td>0.966</td>
<td>0.942</td>
</tr>
<tr>
<td>1.1 &lt; m ≤ 1.3</td>
<td>0.931</td>
<td>0.927</td>
<td>0.945</td>
<td>0.931</td>
</tr>
</tbody>
</table>

Table 7: Option pricing performance in implied volatility for particular Wednesday, from January 1, 1996 to December 31, 2004, preceded by a Tuesday on which a jump occurred. $RMSE_{IV}$ ratios for different models with jumps relative to their natural competitors sorted by moneyness and maturity.
Figure 1: Standardized log-return distribution. Comparison of the S&P500 Futures log-return distribution under different rescaling measures: Standard normal distribution (red line), jump-adjusted standardized log-return by RV^c (green line) and standardized log-return by total RV (blue line).
Figure 2: Skewness and excess kurtosis of the JLHARG process under physical measure $\mathbb{P}$ (left column) and risk-neutral measure $\mathbb{Q}$ (right column).
Figure 3: Histograms of log-returns over different time horizons. The physical density is obtained by simulating a P-LHARG model with parameters reported in Table 1, while the risk-neutral density corresponds to parameter values rescaled according to (2.7). On the y-axis we report log-counts per bin.
Figure 4: Log-ratio of the risk-neutral and physical log-return conditional densities over different time horizons. The physical and risk-neutral densities has been computed proceeding as in Figure 3. The blue solid line represents the log-ratio with variance risk premia obtained from calibration, while the green dotted line refers to the case with both premia set equal to zero.