Consider a society with a finite number of sectors (social issues or commodities). In a partial equilibrium mechanism a sector authority aims to elicit agents’ preference rankings for outcomes at hand, presuming separability of preferences, while such presumption is false in general and such isolated rankings are artifacts. Therefore, its participants are required to behave as if they had separable preferences. This paper studies what can be implemented if we take such misspecification as a given constraint. Specifically, in our implementation model there are several sector authorities, agents are constrained to submit their rankings to each sector authority separately and, moreover, sector authorities cannot communicate with each other. When a social choice rule (SCR) can be Nash implemented by a product set of partial equilibrium mechanisms, we say that it can be implemented in partial equilibrium. We identify necessary conditions for SCRs to be implemented in partial equilibrium and show that they are also sufficient under mild auxiliary conditions. Thus, the implementation in partial equilibrium of SCRs is examined in several environments, mainly in auction and matching environments.

JEL classification: C72; D71.

Keywords: Nash implementation, non-separable preferences, partial equilibrium.

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1 Introduction

To understand how to address a single social issue, economists have used the methodology of partial equilibrium mechanism design. This methodology isolates items to be allocated as well as people's preferences for those items from the rest of the world, under a *ceteris paribus* (all else equal) assumption. Because of such isolation, it has provided exact mechanisms and algorithms on how to govern individual behavior so as to achieve desirable objectives and has proved capable of handling a wide variety of issues, not only economic but political and legal. The prominently successful cases are auction and matching, such as auction houses, labor market for medical interns and school admissions.

The *ceteris paribus* (all else equal) assumption, however, cannot be true in general, since people's preferences are generally non-separable. For example, which school one would like to be admitted depends on where she lives and, moreover, where she would like to live depends on which school she could be admitted to. When the central authority of one social issue assumes that each of its participants has a single preference ranking for the social issue at hand and requires participants to report their rankings, it is forcing its participants to behave *as if* their preferences were separable, while such rankings are artifacts.

However, if we change something in the school admission program, it will have a general equilibrium effect, such as changes in the housing market and how people choose where to live, etc. Likewise, if we change something in the house auction rule, it will have a general equilibrium effect on how people consume other goods related to the auctioned house and, moreover, will affect bidders' willingness to pay, and so on. In this paper, we ask the following questions: What do we lose by ignoring such general equilibrium effects? If we take the misspecification described above as a given institutional constraint, how does it restrict the set of viable social arrangements?

Furthermore, even if people's preferences are separable and every individual has a single conditional preference for each social issue, that methodology hinders our understanding of how to assign priorities to individuals. However, priorities are eventually established in reality. It is likely, for example, that one individual should be prioritized
in the school admission and another should be prioritized in the housing allocation, whereas the actual decisions go in the opposite directions. This leads to inefficiency at a general equilibrium level.

This paper answers the above questions by studying what social choice rules (SCR) can be Nash implemented in a society in which there is a finite number of social issues or sectors, agents are constrained to submit their preferences to sector authorities separately and sector authorities are unable to communicate with each other, for example, due to misspecification by the designer or due to technical/institutional constraints. Simply put, our Nash implementation problem consists of designing partial equilibrium mechanisms, one for each sector, with the property that the Nash equilibrium outcomes of their product set coincides with the recommendations of a given SCR. If this can be done, we say that a SCR can be implemented in partial equilibrium. Therefore, we answer the above questions by using a positive economic approach.

We show that SCRs that can be implemented in partial equilibrium satisfy (Maskin) monotonicity, a decomposability condition and a decomposable (Maskin) monotonicity condition. In addition, if a SCR satisfies the above properties and some other mild auxiliary conditions, reminiscent of Maskin’s Theorem (Maskin, 1999), and if there are at least three agents, then the SCR can be implemented in partial equilibrium.

The ideas are actually quite intuitive: (i) under separability, a SCR that can be implemented in partial equilibrium must induce one-dimensional SCRs, one for each sector, each of which depends only on conditional preferences; (ii) such sector-specific SCRs must satisfy the standard invariance condition due to Maskin, and that condition is also sufficient for constructing an implementing partial equilibrium mechanism under mild additional conditions; and finally (iii) given a list of implementing partial equilibrium mechanisms, its extension to general preferences is obtained by simply letting each agent submit preferences to each sector authority separately, as if she had separable preferences.

We also provide a characterization of what can be implemented when we start with a list of sector-specific SCRs defined over conditional preferences. We provide a natural extension of sector-specific SCRs, based on the as if idea, and show that it is the smallest SCR that can be implemented in partial equilibrium.
Our analysis shows that the positive nature of partial equilibrium mechanisms to require agents to behave as if they had separable preferences imposes constraints not only on what kinds of outcomes a sector authority can achieve, but also, and most importantly, on what kinds of outcomes the society as a whole can achieve.

Section 2 provides motivating examples while section 3 presents the theoretical framework and outlines the basic model, with necessary conditions presented in section 4. Section 5 presents our characterization result, with its implications discussed in section 6. Section 7 concludes.

2 Motivating examples

To illustrate our points we discuss two prominent cases of partial equilibrium mechanism design: matching and auction.

2.1 Matching

In a matching problem, each involved agent is required to submit a preference ranking over mates or items. Moreover, the assignment authority solves that problem in isolation from other matching problems in which agents can also be involved as participants. For example, in a school choice program, parents of a student submit a strict ranking over schools to an education authority, which decides which school each student will attend, independently from other authorities’ decisions and after having taken into consideration the assignment priorities of schools. However, parental preferences over schools are typically not independent of the decision made by other assignment authorities such as the housing authority. Examples below illustrate the problems arising when agents’ preferences are assumed to be separable, while they are not, and assignment authorities’ decisions are not coordinated.

The economy consists of two indivisible and non-homogeneous types of items, type 1 and type 2, and two agents, agent A and agent B. An agent is indexed with the subscripted letter i and a type is indexed with the superscripted letter s. Each agent starts with some initial bundle of items. Let $e_i^s$ denote item of type s owned by agent i. The set of items of type s is denoted by $X^s = \{e_A^s, e_B^s\}$. 
We imagine that a new distribution of items of type $s$, or sector $s$ allocation, is proposed by sector $s$ assignment authority. A sector $s$ allocation $x^s = (x^s_A, x^s_B)$ is a list of items of type $s$ that is consistent with the initial endowments of sector $s$. An allocation is a list of bundles of items $x = ((x^1_A, x^1_B), (x^2_A, x^2_B))$ that is consistent with the initial endowments.

Our interpretation is that sector 1 authority proposes school seat allocations to agents, where each agent already owns one school seat, and that sector 2 authority proposes house allocations to agents, where each agent already owns one house. Basically, this is a model of barter exchange in which only items of the same type can be traded.

Suppose that preferences of agent $i$ are represented by an ordering $R_i$ defined on the set of bundles $X^1 \times X^2$. As noted in the previous section, in the ‘usual’ case, preferences for items of type 1 will depend upon the consumption of items of type 2. The case in which such dependence does not occur is that in which preferences are separable.

To be precise, suppose sector 2 authority has assigned the item $x^2_i$ to agent $i$. We define the conditional ordering, $R^1_i (x^2_i)$, on $X^1$ of agent $i$ induced by the ordering $R_i$ by:

$$\text{if } e^2_A R^1_i (x^2_i) e^2_B \text{ then } (e^1_A, x^2_i) R_i (e^1_A, x^2_B).$$

Likewise, one can define the conditional ordering, $R^2_i (x^1_i)$, on $X^2$ given that sector 1 authority has assigned item $x^1_i$ to agent $i$.

Agent $i$’s ordering $R_i$ is separable in $X^1$ if the conditional orderings on $X^1$ are identical, that is, $R^1_i (e^2_B) = R^1_i (e^2_A)$. We say that the ordering $R_i$ is separable if that ordering is separable in $X^1$ as well as in $X^2$. We subsequently indicate the conditional ordering on $X^s$ induced by an ordering $R_i$ separable in $X^s$ by $R^s_i$.

Let $(R_A, R_B)$ be a profile of agents’ orderings that represent agents’ preferences. We say that an allocation $x$ is Pareto dominated for $(R_A, R_B)$ if some other allocation $y$ would make at least one agent better off without hurting the other agent, that is, $(y^1_i, y^2_i) R_i (x^1_i, x^2_i)$ for all $i$, with at least one of the preferences being strict. An allocation is Pareto efficient for $(R_A, R_B)$ if it is not Pareto dominated by any other allocation.
An allocation \( x \) is individually rational for \((R_A, R_B)\) if it leaves each agent \( i \) as well off as her endowment, that is, \((x_1^i, x_2^i) \geq (e_1^i, e_2^i)\) for all \( i \). An allocation is a core allocation for \((R_A, R_B)\) if it is Pareto efficient and individually rational.

Let us suppose that the objective of sector authorities is to propose sector \( s \) allocations that are sector-wise core allocations. To be precise, suppose that sector 2 authority has proposed \((x_1^B, x_2^B)\) as a sector 2 allocation. We say that sector 1 allocation \( x_1 = (x_1^A, x_1^B) \) is a sector 1 core allocation for the profile of conditional orderings \((R_1^A(x_2^A), R_1^B(x_2^B))\) if \( x_1 \) is individually rational and Pareto efficient for \((R_1^A(x_2^A), R_1^B(x_2^B))\). Then, an allocation \( x = ((x_1^A, x_2^A), (x_1^B, x_2^B)) \) is a sector-wise core allocation if \( x_1 = (x_1^A, x_1^B) \) is a sector 1 core allocation for the profile of conditional orderings \((R_1^A(x_2^A), R_1^B(x_2^B))\) and \( x_2 = (x_2^A, x_2^B) \) is a sector 2 core allocation for the profile of conditional orderings \((R_2^A(x_1^A), R_2^B(x_1^B))\). The idea behind this definition is that each agent is supposed to submit her preferences for items of type \( s \) to a sector \( s \) authority as if her preferences were separable.

Examples 1 and 2 below show that when sector authorities make non-coordinated decisions and each sector \( s \) authority considers it optimal to assign a sector \( s \) core allocation to agents, then sector-wise core allocations are not necessarily Pareto efficient according to agents’ preferences. This is regardless of whether agents’ preferences are separable.

**Example 1.** Suppose that agents \( A \) and \( B \)’s separable strict orderings on \( X^1 \times X^2 \) are as follows:

For \( A \):

\[
(e_B^1, e_A^2) R_A(e_B^1, e_B^2) R_A(e_A^1, e_A^2) R_A(e_A^1, e_B^2)
\]

For \( B \):

\[
(e_B^1, e_A^2) R_B(e_A^1, e_A^2) R_B(e_B^1, e_B^2) R_B(e_A^1, e_B^2),
\]

where we say agent \( i \) “strictly prefers \( x \) to \( y \) according to \( R_i \)” if “\( x R_i y \)”. One can check that the conditional strict orderings of agent \( i \) are:

For items of type 1:

\[
R_i^1 e_A^1 e_B^1,
\]

For items of type 2:

\[
R_i^2 e_A^2 e_B^1.
\]
The unique core allocation for the profile \((R_A, R_B)\) is the allocation \(x\) given by

\[(x_A^1, x_A^2) = (e^1_B, e^2_B) \text{ and } (x_B^1, x_B^2) = (e^1_A, e^2_A).\]

Clearly, no single agent will want to block \(x\) because every agent strictly prefers \((x_i^1, x_i^2)\) to her initial endowment \((e_i^1, e_i^2)\) according to \(R_i\). Moreover, the allocation \(x\) is Pareto efficient.

Sector 1 allocation \(y^1\) given by

\[(y_A^1, y_B^1) = (e^1_A, e^1_B)\]

is the unique sector 1 core allocation for the profile of conditional orderings \((R^1_A, R^1_B)\). This is so because the move from \(e^1_B\) to \(e^1_A\) is a bad deal for agent \(B\). Moreover, neither agent would block \(y^1\) because each agent keeps her initial item of type 1.

Likewise, one can check that sector 2 allocation \(y^2\) given by:

\[(y_A^2, y_B^2) = (e^2_A, e^2_B)\]

is the unique sector 2 core allocation for \((R^2_A, R^2_B)\). Thus, a sector-wise core allocation is the allocation \(y\) given by

\[((y_A^1, y_A^2), (y_B^1, y_B^2))\].

However, \(y\) is not a Pareto efficient allocation for \((R_A, R_B)\). This is so because the move from \((y_i^1, y_i^2)\) to \((x_i^1, x_i^2)\) is a good deal for both agents. In short, if sector authorities proposed the sector-wise core allocation \(y\) and agents could freely barter exchange items, we would not expect barter exchange to lead to the allocation \(y\).

**Example 2.** Suppose that agents \(A\) and \(B\)’s non-separable strict orderings on \(X^1 \times X^2\) are as follows:

for \(A\) : \((e^1_B, e^2_B)R_A(e^1_B, e^2_B)R_A(e^1_A, e^2_A)R_A(e^1_A, e^2_B)\)

for \(B\) : \((e^1_A, e^2_A)R_B(e^1_B, e^2_A)R_B(e^1_B, e^2_B)R_B(e^1_A, e^2_B)\).
One can check that the conditional strict orderings of agent $A$ are

for items of type 1 : $e_1^A R_A^1 e_A^1$
for items of type 2, given $e_A^1$ : $e_A^2 R_A^2 (e_A^1) e_B^2$
for items of type 2, given $e_B^1$ : $e_B^2 R_A^2 (e_B^1) e_A^2$,

and the conditional strict orderings of agent $B$ are

for items of type 1, given $e_A^2$ : $e_A^1 R_B^1 (e_A^2) e_B^1$
for items of type 1, given $e_B^2$ : $e_B^1 R_B^1 (e_B^2) e_A^1$

As in the preceding example, the unique core allocation for the profile $(R_A, R_B)$ is the allocation $x$ given by

$$(x_A^1, x_A^2) = (e_A^1, e_A^2) \text{ and } (x_B^1, x_B^2) = (e_A^1, e_A^2).$$

Indeed, each agent $i$ likes the bundle $(x_i^1, x_i^2)$ as much as she likes her endowment $(e_i^1, e_i^2)$. Moreover, each agent receives her top ranked bundle. Since $x$ is Pareto efficient and since $x$ would not be blocked by either agent, $x$ is a core allocation for $(R_A, R_B)$.

Sector 1 no-trade allocation

$$y^1 = (y_A^1, y_B^1) = (e_A^1, e_B^1)$$

is a sector 1 core allocation for the profile of conditional orderings $(R_A, R_B^1 (e_B^2))$ provided that there is no trade in sector 2. Indeed, given that agent $i$ consumes her own endowment $e_i^1$, neither agent would block $y^1$ provided that there is no trade in sector 2. Moreover, provided that there is no trade in sector 2, the move from $y_B^1 = e_B^1$ to $e_A^1$ is a bad trade for $B$ according to her conditional ordering $R_B^1 (e_B^2)$. Then, $y^1$ is Pareto efficient for $(R_A, R_B^1 (e_B^2))$. Reasoning such as the one just used
shows that sector 2 no-trade allocation \( y^2 \) given by
\[
(y^2_A, y^2_B) = (e^2_A, e^2_B)
\]
is a sector 2 core allocation for the profile of conditional orderings \( (R^2_A(e^1_A), R^2_B) \) provided that there is no trade in sector 1. Thus, a sector-wise core allocation is the no-trade allocation \( y \) given by
\[
((y^1_A, y^2_A), (y^1_B, y^2_B)).
\]
However, \( y \) is not a Pareto efficient allocation for \( (R_A, R_B) \). This is so because \( x \) will make both agents better off than they are at the no-trade allocation \( y \). In other words, if sector authorities proposed the allocation \( y \) and agents could barter exchange items, we would not expect barter exchange to lead the economy to the allocation \( y \). We conclude by noting that the \( x \) allocation is also a sector-wise core allocation.\(^1\)

A common feature of the examples above is that the only Pareto efficient allocation for the profile of agents’ orderings was the core allocation \( x \). Moreover, one can easily check that the \( x \) allocation is still the only core allocation even though we treat each agent \( i \)’s endowment \( (e^1_i, e^2_i) \) as a single commodity. One then may wonder whether the Pareto inefficiency of sector-wise core allocations can be circumvented by considering core allocations computed as if each agent’s endowment were a single commodity. The answer is no. We prove this fact by means of the following example where agent \( i \) still views her endowment \( e^1_i \) of type 1 and her endowment \( e^2_i \) of type 2 as separated items.

**Example 3.** Suppose that agents \( A \) and \( B \)’s non-separable strict orderings on \( X^1 \times X^2 \) are as follows:

for \( A \) : \( (e^1_A, e^2_A)R_A(e^1_B, e^2_B)R_A(e^1_A, e^2_A) \)
for \( B \) : \( (e^1_A, e^2_B)R_B(e^1_B, e^2_A)R_B(e^1_A, e^2_B) \).

\(^1\) The sector 1 allocation \( x^1 = (x^1_A, x^1_B) \) recommended by \( x \) is a sector 1 core allocation for \( (R^1_A, R^1_B(e^2_A)) \) and the sector 2 allocation \( x^2 = (x^2_A, x^2_B) \) is a sector 2 core allocation for \( (R^2_A(e^1_B), R^2_B) \).
One can check that the conditional strict orderings of agent \(i\) are

- for items of type 1, given \(e^2_A\): \(e^1_B R^1_i (e^2_A) e^1_A\)
- for items of type 1, given \(e^2_B\): \(e^1_A R^1_i (e^2_B) e^1_B\)
- for items of type 2, given \(e^1_A\): \(e^2_B R^2_i (e^1_A) e^2_A\)
- for items of type 2, given \(e^1_B\): \(e^2_A R^2_i (e^1_B) e^2_B\).

The unique Pareto efficient allocation for the profile \((R_A, R_B)\) is the allocation \(z\) given by

\[(z^1_A, z^2_A) = (e^1_B, e^2_A)\] and \[(z^1_B, z^2_B) = (e^1_A, e^2_B)\].

The reason is that each agent receives her top ranked bundle according to \(R_i\).

Consider the allocation \(x\) given, as above, by

\[(x^1_A, x^2_A) = (e^1_B, e^2_B)\] and \[(x^1_B, x^2_B) = (e^1_A, e^2_A)\].

Let us treat each agent’s endowment as a single commodity. Clearly, no agent will want to block \(x\) because each agent \(i\) likes \((x^1_i, x^2_i)\) as much as she likes \((e^1_i, e^2_i)\) according to \(R_i\). Moreover, \(x\) is Pareto efficient for \((R_A, R_B)\). This is so because the move from \(x\) to the no-trade allocation is a bad deal for each agent \(i\) according to \(R_i\).

Therefore, since \(x\) is individually rational and Pareto efficient for \((R_A, R_B)\) provided that each agent’s endowment is treated as a single commodity, \(x\) is in the “core” of this economy. However, both agents would be better off with \(z\) than they would be under the “core” allocation \(x\), since \((z^1_A, z^2_A) R_A (x^1_A, x^2_A)\) and \((z^1_B, z^2_B) R_B (x^1_B, x^2_B)\). In short, if authorities proposed the allocation \(x\) and agents could act on their own, they would exchange items so as to arrive at the allocation \(z\).

Consider the allocation \(y\) given by

\[(y^1_A, y^2_A) = (e^1_A, e^2_A)\] and \[(y^1_B, y^2_B) = (e^1_B, e^2_B)\].

Reasoning like that used in the preceding example shows that sector 1 allocation \(y^1 = (y^1_A, y^1_B)\) is a sector 1 core allocation for \((R^1_A (y^2_A), R^1_B (y^2_B))\) and that sector 2
allocation $y^2 = (y^2_A, y^2_B)$ is a sector 2 core allocation for $(R^2_A(y^1_A), R^2_B(y^1_B))$. Therefore, the allocation $y$ is a sector-wise core allocation for this economy. However, $y$ is not a Pareto efficient allocation for $(R_A, R_B)$. This is so because $z$ will make both agents better off than they are at the allocation $y$. In short, if agents could freely barter exchange items, they would rearrange them so as to arrive at the allocation $z$. We conclude by noting that the $z$ allocation is a sector-wise core allocation.

2.2 Auction

In auctions and, more generally, in social decision problems with income transfers, a social decision is a pair $(d, t)$, where $d$ denotes a pure social decision and $t$ denotes a vector of income transfers across agents, which may either add up to zero or to a non-positive number depending on the situation. The task of the central designer is to elicit agents’ true preferences for “pure” social decisions, such as preferences for scales of a public project or for an exclusive licence and to specify income transfers across agents as a function of the elicited preferences. With few exceptions, the analysis of a social decision problem is typically performed by isolating that problem from others. Needless to say, such simplification can be purchased only at the cost of realism. However, what is left unclear is the extent to which that simplification limits the practical relevance of the analysis. We clarify that such simplification restricts the relevance of the analysis to problems where the values of pure social decisions are small relative to agents’ total wealths. In other words, it is limited to pure social decisions for which income effects are minor.

To this end, consider two non-identical social decision problems with income transfers. Let us refer to each of them as the social decision problem of sector $s = 1, 2$. Given that the noun “income” echoes the existence of some kind of mechanism that operates in the rest of the economy but sector authorities do not know what type of mechanism(s) is at work, we subsequently assume that income transfers are made by means of some physical good, which we name commodity money. Let $I$ denote the set of agents.
Let \( D^s \) denote the set of pure social decisions in sector \( s = 1, 2 \). Let

\[
T = \left\{ t \in [-\bar{t}, \infty)^n : \sum_{i \in I} t_i \leq 0 \right\}
\]

denote the set of closed transfers, where the real number \( \bar{t} > 0 \) denotes some predetermined upper-bound for payments. Let \( e_i \) denote the initial endowment of commodity money of agent \( i \in I \), which is assumed to be \( e_i \geq 2\bar{t} \). Therefore, let \( X^s = D^s \times T \) denote the set of outcomes of sector \( s = 1, 2 \) where its elements are denoted by \( x^s = (d^s, t^s) \).

Suppose that agent \( i \)'s ordering \( R_i \) of outcomes in \( X^1 \times X^2 \) can be represented by a utility function \( u_i : X^1 \times X^2 \to \mathbb{R}_+ \) of the form

\[
u_i (x^1, x^2) = U_i (d^1, d^2, t^1_i + t^2_i + e_i),
\]

where \( U_i : D^1 \times D^2 \times \mathbb{R}_+ \to \mathbb{R}_+ \) is strictly increasing in its third argument, that is, for all \( d^1 \in D^1 \), all \( d^2 \in D^2 \) and all \( a, b \geq 0 \),

\[ a > b \implies U_i (d^1, d^2, a) > U_i (d^1, d^2, b). \]

The source of the limited practical relevance of that analysis can be identified in the standard assumption that agent \( i \)'s ordering \( R_i \) represents separable preferences of agent \( i \in I \). The reason is that if agent \( i \)'s preferences are separable and represented by an ordering \( R_i \), which has a utility representation of the form given in (2), and if agent \( i \)'s willingness to pay/accept is well defined,\(^2\) then her preferences have a quasi-linear representation in the commodity money. One way to show it consists in proving that each conditional ordering \( R^s_i \) induced by \( R_i \) has a quasi-linear representation in the commodity money. We show it below by focusing on the conditional ordering \( R^1_i \) of sector 1, given that the arguments for the other conditional ordering \( R^2_i \) are entirely symmetric.

One can easily see that the fact that \( U_i \) is strictly increasing in its third argument as-

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2. In the sense that for any two decisions \( d^s \) and \( \bar{d}^s \) of sector \( s \) there exists a decision \( (d^{sc}, t^{sc}) \) of the other sector \( s_C \) and two income transfers, say \( t^s \) and \( \bar{t}^s \), of sectors \( s \) such that agent \( i \) finds \( (d^s, d^{sc}, t^s_i + t^{sc}_i + e_i) \) and \( (\bar{d}^s, d^{sc}, \bar{t}^s_i + t^{sc}_i + e_i) \) equally good according to \( U_i \).
sures that more commodity money is better than less according to agent $i$’s conditional ordering $R_i^1$. Furthermore, the assumption that agent $i$’s willingness to pay/accept is well defined assures that no matter how much better the pure social decision $\hat{d}^1$ is than $d^1$, according to her conditional ordering $R_i^1$, some amount of commodity money compensates her for getting $d^1$ instead of $\hat{d}^1$. Therefore, to see that the conditional ordering $R_i^1$ induced by $R_i$ has a quasi-linear utility representation in the commodity money, we are left to show that $R_i^1$ exhibits no income effects. In other words, the conditional ordering $R_i^1$ needs to satisfy the property that the trade-offs between pure social decisions and commodity money do not change with equal increases in the commodity money. Formally: For all $d^1$ and $\hat{d}^1$ in $D^1$ and all income transfers $t^1$, $\hat{t}^1$, $\hat{t}^2$ and $\hat{t}^1$ in $T$ such that

$$q = \hat{t}^1 - t^1 = \hat{t}^1 - \hat{t}^1;$$

it holds that

$$(d^1, t^1_i + q)R_i^1(\hat{d}^1, \hat{t}^1_i + q) \iff (d^1, t^1_i)R_i^1(\hat{d}^1, \hat{t}^1_i).$$

Then, consider any two pure social decisions of sector 1, say $d^1$ and $\hat{d}^1$, and any four income transfers in $T$, say $t^1$, $\hat{t}^1$, $\hat{t}^2$ and $\hat{t}^1$, such that (3) holds. The separability requirement implies that for any two outcomes $(d^2, t^2)$ and $(\hat{d}^2, \hat{t}^2)$ of sector 2 such that agent $i$’s income transfer is $q$ at $t^2$ and zero at $\hat{t}^2$, it holds that

$$(d^1, t^1_i)R_i^1(\hat{d}^1, \hat{t}^1_i) \iff U_i(d^1, d^2, t^1_i + t^2_i + e_i) \geq U_i(\hat{d}^1, d^2, \hat{t}^1_i + t^2_i + e_i) \iff U_i(d^1, d^2, (t^1_i + q) + t^2_i + e_i) \geq U_i(\hat{d}^1, d^2, (\hat{t}^1_i + q) + t^2_i + e_i) \iff (d^1, t^1_i + q)R_i^1(\hat{d}^1, \hat{t}^1_i + q).$$

Thus, the conditional ordering $R_i^1$ satisfies the property of no income effect, and so we conclude that agent $i$’s separable ordering $R_i$ has a quasi-linear utility representation in the commodity money.

Therefore, the exercise to isolate a particular sector from the rest of the economy implicitly relies on the assumption that income effects on that sector are minor, meaning
that where income effects are large those isolations fail and economic design recommendations based on them are misleading (see also Vives 1987; Hayashi, 2013).

3 The basic framework

We consider a finite set of agents indexed by \( i \in I = \{1, \cdots, n\} \) and a finite set of elementary sectors indexed by \( s \in S = \{1, \cdots, \ell\} \). The set of outcomes of sector \( s \) available to agents is represented by \( X^s \), with \( x^s \) as a typical element. \( X^s \) is called sector \( s \) outcome space. We assume that the set of outcomes available to agents is the product space

\[
X = \prod_{s \in S} X^s.
\]

To economize on notation, for any sector \( s \), write \( s_C \) for the complement of \( s \) in \( S \). Thus, \((x^s, x^{s_C})\) is an outcome of \( X \), where it is understood that \( x^{s_C} \) is an element of the product space \( X^{s_C} = \prod_{\bar{s} \in S \setminus s} X^\bar{s} \). Unless stated otherwise, the same notational convention will be followed for any profile of items.

In the usual fashion, agent \( i \)'s preferences over \( X \) are given by a complete and transitive binary relation, subsequently an ordering, \( R_i \) on \( X \). The corresponding strict and indifference relations are denoted by \( P (R_i) \) and \( I (R_i) \), respectively.

The condition of separability of preferences that must hold if the isolation of sector \( s \) decision problem from others is legitimate can be formulated as follows. For each \( x^{s_C} \), we define the \( s \) conditional ordering, \( R_i^s (x^{s_C}) \), on \( X^s \) by

\[
\text{for all } y^s, z^s \in X^s : y^s R_i^s (x^{s_C}) z^s \iff (y^s, x^{s_C}) R_i (z^s, x^{s_C}).
\]

We say that the ordering \( R_i \) is separable in \( X^s \) (over \( X^{s_C} \)) if and only if

\[
\text{for all } x^{s_C}, y^{s_C} \in X^{s_C} : R_i^s (x^{s_C}) = R_i^s (y^{s_C}).
\]

In other words, \( R_i \) is separable in \( X^s \) if the agent \( i \)'s preferences over outcomes of \( X^s \) are independent of outcomes chosen from \( X^{s_C} \). Again, to save writing, for any separable ordering \( R_i \) in \( X^s \), write \( R_i^s \) for the \( s \) conditional ordering. The ordering \( R_i \) is separable
provided that for each sector \( s \) the ordering \( R_i \) is separable in \( X^s \).

We assume that the central designer does not know agent \( i \)'s true preferences. Thus, write \( \mathcal{R}(X) \) for the set of orderings on \( X \), \( \mathcal{R}^{sep}(X) \) for the set of separable orderings on \( X \), \( \mathcal{R}_i \) for the domain of (allowable) orderings on \( X \) for agent \( i \) and \( \mathcal{R}_i^{sep} \) for the domain of (allowable) separable orderings on \( X \) for agent \( i \).

We assume, however, that there is complete information among the agents in \( I \). This implies that the central designer knows \( \mathcal{R}_i \) and \( \mathcal{R}_i^{sep} \). Moreover, given that any separable ordering \( R_i \) induces \( \ell \) agent \( i \)'s conditional orderings, one for each \( s \), the central designer also knows \( \mathcal{D}_s^i \), which is the set of the \( s \) conditional orderings on \( X^s \) induced by agent \( i \)'s domain \( \mathcal{R}_i^{sep} \). In summary, the assumption of complete information implies that the central designer knows the domain of preferences for the set \( I \), which is the product set of \( \mathcal{R}_i \)’s, that is,

\[
\mathcal{R}_I = \prod_{i \in I} \mathcal{R}_i,
\]

with \( R \) as a typical profile, and knows the domains \( \mathcal{R}_i^{sep} \) and \( \mathcal{D}_s^i \), which are respectively the product set of \( \mathcal{R}_i^{sep} \)’s and of \( \mathcal{D}_s^i \)’s. A typical element of \( \mathcal{D}_s^i \) is denoted by \( R_s^i \).

The goal of the central designer is to implement a SCR \( \varphi : \mathcal{R}_I \to X \) where \( \varphi(R) \) is non-empty for any \( R \in \mathcal{R}_I \). We shall refer to \( x \in \varphi(R) \) as a \( \varphi \)-optimal outcome at \( R \). The central designer delegates the choice to agents according to a partial equilibrium mechanism which forces agents to behave as if they had separable orderings. Formally, for any sector \( s \), the central designer delegates the choice to agents according to a partial equilibrium mechanism \( \Gamma^s = ((M_i^s)_{i \in N}, h^s) \), where \( M_i^s \) is the strategy space of agent \( i \) in sector \( s \) and \( h^s : M^s \to X^s \), the outcome function, assigns to every strategy profile

\[
m^s \in M^s = \prod_{i \in I} M_i^s
\]

a unique outcome in \( X^s \). A product set of partial equilibrium mechanisms \( \Gamma = ((M_i)_{i \in N}, h) \) is a mechanism, where \( M_i \) is the strategy space of agent \( i \) defined by

\[
M_i = \prod_{s \in S} M_i^s
\]
and \( h : M \to X \), the outcome function, assigns to every strategy profile

\[
m \in M = \prod_{i \in I} M_i
\]

a unique outcome in \( X \) such that

\[
h(m) = (h^s(m^s))_{s \in S}.
\]

A product set of partial equilibrium mechanisms \( \Gamma \) and a profile \( R \in \mathcal{R}_I \) induce a strategic game \((\Gamma, R)\). A strategy profile \( m \in M \) is a Nash equilibrium (in pure strategies) of \((\Gamma, R)\) if for all \( i \in I \), it holds that

\[
\text{for all } \tilde{m}_i \in M_i : h(m) R_i h(\tilde{m}_i, m_{-i}),
\]

where, as usual, \( m_{-i} \) is the strategy profile of all agents except \( i \) such that \((m_i, m_{-i}) = m\). Write \( NE(\Gamma, R) \) for the set of Nash equilibrium profiles of \((\Gamma, R)\). Likewise, any mechanism \( \Gamma^s \) together with the profile \( R^s \in \mathcal{D}^*_I \) defines a strategic game in \( s \). A strategy profile \( m^s \in M^s \) is a Nash equilibrium (in pure strategies) of \((\Gamma^s, R^s)\) if for all \( i \in I \), it holds that

\[
\text{for all } \tilde{m}^s_i \in M^s_i : h(m^s) R^s_i h(\tilde{m}^s_i, m^s_{-i}).
\]

The following definition is then our formulation of the central designer’s (Nash-)implementation problem. A SCR \( \varphi \) is (Nash-)implementable in partial equilibrium if there exists a product set of partial equilibrium mechanisms \( \Gamma \) such that for all \( R \in \mathcal{R}_I \), it holds that

\[
h(NE(\Gamma, R)) = \varphi(R).
\]

The lemma below shows that the separability property implies that the set of Nash equilibrium strategy profiles has a product structure.

**Lemma 1.** Let \( \Gamma \) be a product set of partial equilibrium mechanisms. For all \( R \in \mathcal{R}^{sep}_I \),

\[
NE(\Gamma, R) = \prod_{s \in S} NE(\Gamma^s, R^s),
\]
where for all $i \in I$ and all $s \in S$, $R^s_i$ is the $s$ conditional ordering induced by $R_i$.

**Proof.** Let $\Gamma$ be a product set of partial equilibrium mechanisms. Take any $R \in R^sep_\Gamma$.

For any $i \in I$ and any $s \in S$, write $R^s_i$ for the $s$ conditional ordering induced by $R_i$.

Consider any $m \in NE(\Gamma, R)$. Thus, it follows that

$$h(m) R_i h(\bar{m}_i, m_{-i}) \text{ for all } \bar{m}_i \in M_i.$$

Fix any $s \in S$ and any $i \in I$. Since $R_i \in R^sep_i$, it holds that

$$h^s (m^s) R^s_i h^s (\bar{m}_i^s, m_{-i}^s) \text{ for all } \bar{m}_i^s \in M_i^s.$$

Since it holds for any $i \in I$, we have that $m^s \in NE(\Gamma^s, R^s)$. Finally, given that the choice of $s$ was arbitrary, we have that $m \in \prod_{s \in S} NE(\Gamma^s, R^s)$.

Consider any $m \in \prod_{s \in S} NE(\Gamma^s, R^s)$. Thus,

$$h^s (m^s) R^s_i h^s (\bar{m}_i^s, m_{-i}^s) \text{ for all } \bar{m}_i^s \in M_i^s.$$

Assume, to the contrary, that $m \notin NE(\Gamma, R)$. Then, for at least one $i_o \in I$ and one $\bar{m}_{i_o} \in M_{i_o}$, it holds that $h(\bar{m}_{i_o}, m_{-i_o}) P(R_{i_o}) h(m)$.

Since for sector 1, it holds that

$$h^1 (m^1) R^1_{i_o} h^1 (\bar{m}^1_{i_o}, m_{-i_o}),$$

it follows from $R_{i_o} \in R^sep_{i_o}$ that

$$h(m) R_{i_o} \left(h^1 (\bar{m}^1_{i_o}, m_{-i_o}), (h^s (m^s))_{s \in S \setminus \{1\}}\right).$$

Reasoning like that used in the preceding lines shows that for any $s \in S \setminus \{1, \ell\}$, it holds that

$$\left(h^p (\bar{m}^p_{i_o}, m^p_{-i_o})\right)_{p=1, \ldots, s-1}, (h^q (m^q))_{q=s, \ldots, \ell} R_{i_o} \left(h^p (\bar{m}^p_{i_o}, m^p_{-i_o})\right)_{p=1, \ldots, s}, (h^q (m^q))_{q=s+1, \ldots, \ell}.\)
Likewise, for sector \( \ell \), it holds that

\[
\left( (h^p(m_{\ell}^p, m_{-\ell}^p))_{p=1, \ldots, \ell-1}, h^\ell(m^\ell) \right)_{R_{\ell}^p} h(m_{\ell}, m_{-\ell}).
\]

Since \( R_{\ell} \) is transitive, it follows that

\[
h(m) R_{\ell} h(m_{\ell}, m_{-\ell}),
\]

in violation of \( h(m_{\ell}, m_{-\ell}) \neq R_{\ell} h(m) \). Thus, \( m \in NE(\Gamma, h) \). □

4 Necessary conditions

In this section, we discuss conditions that are necessary for the implementation in partial equilibrium. We end the section by showing that no acceptable Pareto optimal SCR defined on the domain of separable orderings can be implemented in partial equilibrium.

A condition that is central to the implementation of SCRs in Nash equilibrium is monotonicity (in the Maskin sense). This condition says that if an outcome \( x \) is \( \varphi \)-optimal at the profile \( R \) and this \( x \) does not strictly fall in preference for anyone when the profile is changed to \( R' \), then \( x \) must remain a \( \varphi \)-optimal outcome at \( R' \). Let us formalize that condition as follows. For any ordering \( R_i \) and outcome \( x \), the weak lower contour set of \( R_i \) at \( x \) is defined by

\[
L(x; R_i) = \{ x' \in X \mid x R_i x' \}.
\]

Therefore:

**Definition 1.** The SCR \( \varphi : \mathcal{R}_I \rightarrow X \) is (Maskin) monotonic provided that for all \( x \in X \) and all \( R, R' \in \mathcal{R}_I \), if \( x \in \varphi(R) \) and \( L(x, R_i) \subseteq L(x, R'_i) \) for all \( i \in I \), then \( x \in \varphi(R') \).

**Theorem 1.** The SCR \( \varphi : \mathcal{R}_I \rightarrow X \) is monotonic if \( \varphi \) is implementable in partial equilibrium.

**Proof.** Suppose the SCR \( \varphi : \mathcal{R}_I \rightarrow X \) is implementable in partial equilibrium. Then, there exists a product set of partial equilibrium mechanisms \( \Gamma \) such that for all \( R \in \mathcal{R}_I \), it holds \( \varphi(R) = h(NE(\Gamma, R)) \). For some profile \( R \in \mathcal{R}_I \) consider \( x \in \varphi(R) \).
Then, there exists $m \in NE(\Gamma, R)$ such that $h(m) = x$ and such that for all $i \in I$, it holds that

(4) $h(M_i, m_{-i}) \subseteq L(x, R_i)$,

where $h(M_i, m_{-i})$ is the set of outcomes that agent $i$ can generate by varying her own strategy choice, keeping her opponents’ actions fixed at $m_{-i}$.

Consider the profile $R' \in \mathcal{R}_I$ such that for all $i \in I$, it holds that $L(h(m), R_i) \subseteq L(h(m), R'_i)$. It follows from (4) that for all $i \in I$, it holds that

$$h(M_i, m_{-i}) \subseteq L(h(m), R'_i).$$

Therefore, $m \in NE(\Gamma, R')$. From the definition of implementability in partial equilibrium, we conclude that $x \in \varphi(R')$. Thus, $\varphi$ is monotonic. ■

The relevance of implementation theory comes from the fact that it provides a theoretical construct within which to study the way in which a society shall trade off agent preferences to achieve its goals. Unless the SCR is dictatorial, this involves a compromise. In light of Lemma 1, the second condition identifies a property of how a SCR must handle the compromise across sectors where agents’ preferences are separable.

**Definition 2.** The SCR $\varphi : \mathcal{R}_I \rightarrow X$ is decomposable provided that for all $s \in S$, there exists a (non-empty) correspondence $\varphi^s : \mathcal{D}_I^s \rightarrow X^s$ with the following property: for all $R \in \mathcal{R}_I^{sep}$, $\varphi(R) = \prod_{s \in S} \varphi^s(R^s)$, where for all $i \in I$ and all $s \in S$, $R^s_i$ is the $s$ conditional ordering induced by $R_i$.

This says that if a SCR is decomposable, then the $s$th dimension of the SCR depends only on the profiles of conditional orderings of the $s$th sector. Differently put, the SCR can be decomposed into the product of one-dimensional SCRs. Furthermore, it implies that the social objectives that a society or its representatives want to achieve can be decomposed in ‘small’ social objectives, one for each sector. Therefore, to analyze the way in which the society should trade off agent preferences for the $s$th sector to achieve its goal we can ignore consumption trade-offs across sectors and focus only on the profiles of conditional orderings of $s$th sector.
**Theorem 2.** The SCR $\varphi : \mathcal{R}_I \to X$ is decomposable if $\varphi$ is implementable in partial equilibrium.

**Proof.** Suppose $\varphi : \mathcal{R}_I \to X$ is implementable in partial equilibrium. Then, there exists a product set of partial equilibrium mechanisms $\Gamma$ such that for all $R \in \mathcal{R}_I$, it holds $\varphi(R) = h(NE(\Gamma, R))$. Furthermore, Lemma 1 implies

$$h(NE(\Gamma, R)) = \prod_{s \in S} h^s(NE(\Gamma^s, R^s))$$

for all $R \in \mathcal{R}_I^{sep}$, and so, by the fact that $\Gamma$ implements $\varphi$ in partial equilibrium, it holds that

$$\varphi(R) = \prod_{s \in S} h^s(NE(\Gamma^s, R^s)) \text{ for all } R \in \mathcal{R}_I^{sep}.$$

For all $s \in S$, define $\varphi^s : \mathcal{D}_I^s \to X^s$ by $\varphi^s(\hat{R}^s) = h^s(NE(\Gamma^s, \hat{R}^s))$ for any $\hat{R}^s \in \mathcal{D}_I^s$. We conclude from the definition of $\varphi^s$ and (5) that $\varphi(R) = \prod_{s \in S} \varphi^s(R^s)$. Thus, $\varphi$ is decomposable. ■

In the literature of strategy-proof social choice functions it has been shown that decomposability is implied by strategy-proofness where agents have separable preferences (as per Barberà et al., 1991; Le Breton and Sen, 1999). A natural question, then, is whether decomposability is implied by Nash implementation. The answer is no. We prove this by means of the following example.

**Example 4.** There are two types of agents, say type $A$ and type $B$, two sectors, say sector 1 and sector 2, and two distinct items per sector, say $x^s$ and $y^s$. Consider a profile $R$ where the separable strict orderings of types are

- for type $A$ : $(x^1, x^2)R_A(x^1, y^2)R_A(y^1, x^2)R_A(y^1, y^2)$
- for type $B$ : $(y^1, y^2)R_B(x^1, y^2)R_B(y^1, x^2)R_B(x^1, x^2)$.

---

3. A SCR $\varphi : \mathcal{R}_I \to X$ is Nash-implementable if there exists a mechanism $\gamma \equiv (M, h)$ such that for all $R \in \mathcal{R}_I$, $\varphi(R) = h(NE(\gamma, R))$. 

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Furthermore, consider a profile $\bar{R}$ where the separable strict orderings of types are

for type $A$ : $(x^1, x^2) \bar{R}_A(y^1, x^2) \bar{R}_A(x^1, y^2) \bar{R}_A(y^1, y^2)$
for type $B$ : $(y^1, y^2) \bar{R}_B(y^1, x^2) \bar{R}_B(x^1, y^2) \bar{R}_B(x^1, x^2)$.

One can check that $R$ and $\bar{R}$ induce the following conditional strict orderings:

for type $A$, sector 1 : $x^1 R^1_A y^1$
for type $A$, sector 2 : $x^2 R^2_A y^2$
for type $B$, sector 1 : $y^1 R^1_B x^1$
for type $B$, sector 2 : $y^2 R^2_B x^2$.

Suppose that there are three agents, where agents 1 and 2 are of type $A$ and agent 3 is of type $B$. Furthermore, suppose that the profiles $R$ and $\bar{R}$ are the only allowable profiles of separable orderings.

Consider the SCR $\varphi : \{R, \bar{R}\} \rightarrow X$ such that

\begin{equation}
\varphi(R) = \{(x^1, y^2), (x^1, x^2)\} \neq \varphi(\bar{R}) = \{(y^1, x^2), (x^1, x^2)\}.
\end{equation}

This SCR is Maskin monotonic and satisfies the condition of no veto-power.\footnote{No veto-power says that if an outcome $x$ is at the top of the preferences of all but possibly one of the agents, then $x$ should be selected by the SCR $\varphi$.} Therefore, the SCR $\varphi$ is Nash-implementable, according to Maskin’s Theorem (Maskin, 1999).

Suppose that the SCR $\varphi$ is decomposable. By construction, one has that the set of conditional orderings of sector 1 and sector 2 induced by $R$ and $\bar{R}$ are

for type $A$ : $D^1_A = \{R^1_A\}$ and $D^2_A = \{R^2_A\}$
for type $B$ : $D^1_B = \{R^1_B\}$ and $D^2_B = \{R^2_B\}$.
Decomposability implies that

$$
\varphi(R) = \varphi^1(R_A^1, R_A^2, R_B^1) \times \varphi^2(R_A^2, R_A^3, R_B^2) = \varphi(\bar{R}),
$$

in violation of (6). Thus, the SCR $\varphi$ is not decomposable.

An equivalent statement of (Maskin) monotonicity stated above follows the reasoning that if $x$ is $\varphi$-optimal at $R$ but not $\varphi$-optimal at $R'$, then the outcome $x$ must have fallen strictly in someone’s ordering at the profile $R'$ in order to break the Nash equilibrium via some deviation. Therefore, there must exist some preference reversal if an equilibrium strategy profile at $R$ is to be broken at $R'$. When the new profile $R'$ satisfies the requirement of separability and the SCR $\varphi$ is implementable in partial equilibrium, then the $s$th sector of the SCR depends only on the profiles of conditional orderings of the $s$th sector. Therefore, a variant of monotonicity follows the reasoning that if $x$ is $\varphi$-optimal at $R$ but not $\varphi$-optimal at $R'$ and if $R'$ is a profile of separable orderings, then the outcome $x$ must have fallen strictly in someone’s conditional ordering. Simply put, if an equilibrium strategy profile at $R$ is to be broken at $R'$, then the preference reversal must happen in one of the sectors.

To introduce this variant of monotonicity, for any ordering $R_i$, outcome $x$ and sector $s$, let the weak lower contour set of $R_i$ and sector $s$ at $x$ be defined by $L^s(x, R_i) = \{(y^s, x^{\neq c}) \in X \mid xR_i (y^s, x^{\neq c})\}$. Then:

**Definition 3.** The SCR $\varphi : \mathcal{R}_I \to X$ is decomposable (Maskin) monotonic provided that for all $x \in X$, all $R \in \mathcal{R}_I$ and all $R' \in \mathcal{R}_I^{\text{sep}}$, if $x \in \varphi(R)$ and for all $i \in I : L^s(x, R_i) \subseteq L^s(x, R'_i)$ for all $s \in S$, then $x \in \varphi(R')$.

**Theorem 3.** The SCR $\varphi : \mathcal{R}_I \to X$ is decomposable monotonic if $\varphi$ is implementable in partial equilibrium.

**Proof.** Suppose the SCR $\varphi : \mathcal{R}_I \to X$ is implementable in partial equilibrium. Then, there exists a product set of partial equilibrium mechanisms $\Gamma$ such that for all $R \in \mathcal{R}_I$, it holds $\varphi(R) = h(NE(\Gamma, R))$. For some profile $R \in \mathcal{R}_I$ consider $x \in \varphi(R)$. Then, there exists a Nash equilibrium strategy profile $m \in NE(\Gamma, R)$ such that $h(m) =$
where

\[ (h^s(m^s))_{s \in S} = x. \]

Moreover, for all \( i \in I \), it holds that the set of obtainable outcomes, that is, \( h(M_i, m_{-i}) = \{ h(m'_i, m_{-i}) \in X | m'_i \in M_i \} \), is contained in \( L(x, R_i) \).

Consider any sector \( s \) and any agent \( i \). Let

\[
(h^s(M_i^s, m^s_{-i}), (h^s(m^s))_{s \in S_{SC}}) = \{ (h^s(\bar{m}_i^s, m^s_{-i}), (h^s(m^s))_{s \in S_{SC}}) | \bar{m}_i^s \in M_i^s \}
\]

be the set of outcomes that agent \( i \) can generate by varying his own strategy of sector \( s \), keeping his own strategy choices and those of other agents for sector \( \bar{s} \) different from \( s \) fixed at \( m^\bar{s} \) and keeping the strategy choices of other agents of sector \( s \) fixed at \( m^s_{-i} \). It follows from \( h(M_i, m_{-i}) \subseteq L(h(m), R_i) \) that \( (h^s(M_i^s, m^s_{-i}), (h^s(m^s))_{s \in S_{SC}}) \subseteq L^s(h(m), R_i) \). Since agent \( i \) and sector \( s \) were arbitrary, it follows that

\[
(7) \quad \text{for all } i \in I \text{ and all } s \in S : (h^s(M_i^s, m^s_{-i}), (h^s(m^s))_{s \in S_{SC}}) \subseteq L^s(h(m), R_i).
\]

Consider the profile \( \bar{R} \in \mathcal{R}^{sep}_i \) such that

\[
\text{for all } i \in I \text{ and all } s \in S : L^s(h(m), R_i) \subseteq L^s(h(m), \bar{R}_i).
\]

Then, from (7), it follows that

\[
(8) \quad \text{for all } i \in I \text{ and all } s \in S : (h^s(M_i^s, m^s_{-i}), (h^s(m^s))_{s \in S_{SC}}) \subseteq L^s(h(m), \bar{R}_i).
\]

Given that \( \bar{R} \) is a profile of separable orderings, let \( \bar{R}_i^s \) be the \( s \) conditional ordering induced by \( \bar{R}_i \). Thus, from (8), we have that

\[
\text{for all } i \in I \text{ and all } s \in S : h^s(M_i^s, m^s_{-i}) \subseteq L(h^s(m^s), \bar{R}_i^s),
\]

where \( L(h^s(m^s), \bar{R}_i^s) \) is the weak lower contour set of \( \bar{R}_i^s \) at \( h^s(m^s) \). Then, for all \( s \in S \), \( m^s \) is a Nash equilibrium strategy profile of \( (\Gamma^s, \bar{R}^s) \), and so, by Lemma 1, it follows that \( m \in NE(\Gamma, \bar{R}) \). From the definition of implementability in partial equilibrium, we conclude that \( x \in \varphi(\bar{R}) \). Thus, \( \varphi \) is decomposable monotonic.

Important properties of SCRs are as follows.
Definition 4. The SCR $\varphi : \mathcal{R}_I \rightarrow X$ is Pareto optimal provided that for all $R \in \mathcal{R}_I$ and all $x \in \varphi (R)$, there is no $x' \in X$ such that $x'R_ix$ for all $i \in I$ and $x'P(R_j)x$ for some $j \in I$.

Definition 5. The SCR $\varphi : \mathcal{R}_I \rightarrow X$ is dictatorial provided that there exists an agent $i \in I$ such that for all $R \in \mathcal{R}_I$ and all $x \in X$,

$$x \in \varphi (R) \iff xR_ix' \text{ for all } x' \in X.$$ 

A social choice function (SCF) is a single-valued SCR. A SCF is strategy-proof if each agent does herself no good by misrepresenting her own ordering.\(^5\) Furthermore, a SCF is nonimposed if the set of outcomes is included in its range.

A classic result due to Gibbard (1973) and Satterthwaite (1975) shows that a nonimposed, strategy-proof SCF defined on the domain of all possible linear orderings is dictatorial, provided that the unstructured finite set of outcomes contains at least three outcomes.\(^6\) Using a framework similar to ours, Le Breton and Sen (1999) identify domain richness conditions that are sufficient for a nonimposed, strategy-proof SCF to be decomposable into one-dimensional strategy-proof SCFs. Therefore, where the finite set of outcomes is a product set and each sector set contains at least three outcomes, the decomposability theorem of Le Breton and Sen implies that a nonimposed, strategy-proof SCF defined on the domain of all possible separable linear orderings can be decomposed into one-dimensional dictatorial SCFs.

Given Le Breton and Sen’s negative result, one is forced to relax some of their assumptions in the hope of finding more encouraging results. A requirement weaker than strategy-proofness is that of requiring truth-telling when the other agents are also telling the truth, that is, that of Nash equilibrium. We show below that the prospects for implementing in partial equilibrium a Pareto optimal SCR on an unrestricted domain

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\(^5\) The SCF $\varphi : \mathcal{R}_{I^{sep}} \rightarrow X$ is strategy-proof if for all $i \in I$, all $R \in \mathcal{R}_{I^{sep}}$, and all $R'_i \in \mathcal{R}_{I^{sep}}(X)$ such that $(R'_i, R_{-i}) \in \mathcal{R}_{I^{sep}}, \varphi (R) R_i \varphi (R'_i, R_{-i}).$

\(^6\) A linear order $R$ on $X$ is a complete, transitive and antisymmetric (binary) relation. Barberà and Peleg (1990) show that the result of Gibbard (1973) and Satterthwaite (1975) holds true if one drops the assumption of universal domain of preferences and agents’ preferences are required to be continuous. The result of Gibbard-Satterthwaite is also basically robust to the consideration of SCRs, as per Barberà et al. (2001).
of separable orderings are quite bleak as well. The reason is that a decomposable, Pareto optimal SCR defined on $R_{I}^{sep}(X)$ is dictatorial, provided that each sector set contains at least two outcomes. This negative result is similar in spirit to a classic result due to Hurwicz and Schmeidler (1978) and Maskin (1999), which states that if a two-agent Pareto optimal SCR defined on the domain of all possible linear orderings is Nash implementable, then it is dictatorial. Furthermore, it is similar to a result due to Barberà et al. (1991), according to which there is no Pareto optimal, strategy-proof and non-dictatorial voting scheme defined on the domain of separable linear orderings.

**Theorem 4.** Suppose $n \geq 2$ and that $\ell \geq 2$. For any $s \in S$, let $|X^{s}| \geq 2$. The SCR $\varphi : R_{I}^{sep}(X) \rightarrow X$ is dictatorial if $\varphi$ is Pareto optimal and decomposable.

**Proof.** Let $n \geq 2$ and $\ell \geq 2$. For any $s \in S$, let $|X^{s}| \geq 2$. Suppose the SCR $\varphi : R_{I}^{sep}(X) \rightarrow X$ is not dictatorial and that it is decomposable. We show that $\varphi$ is not Pareto optimal. Fix any two distinct agents $i_{0}, j_{0} \in I$ and any two distinct sectors $s(i_{0}), s(j_{0}) \in S$. Suppose agent $i_{0}$ is a dictator in sector $s(i_{0})$, whereas agent $j_{0}$ is a dictator in sector $s(j_{0})$. Fix any $x^{s(i_{0})}, y^{s(i_{0})} \in X^{s(i_{0})}$, with $x^{s(i_{0})} \neq y^{s(i_{0})}$, and any $x^{s(j_{0})}, y^{s(j_{0})} \in X^{s(j_{0})}$, with $x^{s(j_{0})} \neq y^{s(j_{0})}$. Let $\bar{S} = \{ s(i_{0}), s(j_{0}) \}$, and write $\bar{S}_{C}$ for the complement of $\bar{S}$ in $S$.

Consider any profile $R \in R_{I}^{sep}(X)$ such that the conditional orderings of agents $i_{0}$ and $j_{0}$ are, respectively, for sector $s(i_{0})$:

$$x^{s(i_{0})} I \left( R_{i_{0}}^{s(i_{0})} \right) y^{s(i_{0})} P \left( R_{i_{0}}^{s(i_{0})} \right) \ldots P \left( R_{i_{0}}^{s(i_{0})} \right) \ldots$$

and

$$y^{s(i_{0})} P \left( R_{j_{0}}^{s(i_{0})} \right) x^{s(i_{0})} P \left( R_{j_{0}}^{s(i_{0})} \right) \ldots P \left( R_{j_{0}}^{s(i_{0})} \right) \ldots,$$

and for sector $s(j_{0})$:

$$x^{s(j_{0})} P \left( R_{i_{0}}^{s(j_{0})} \right) y^{s(j_{0})} P \left( R_{i_{0}}^{s(j_{0})} \right) \ldots P \left( R_{i_{0}}^{s(j_{0})} \right) \ldots$$

and

$$x^{s(j_{0})} I \left( R_{j_{0}}^{s(j_{0})} \right) y^{s(j_{0})} P \left( R_{j_{0}}^{s(j_{0})} \right) \ldots P \left( R_{j_{0}}^{s(j_{0})} \right) \ldots.$$
Note that by the separability requirement it holds that

for all \( z^{(j_o)C} \in X^{(j_o)C} : (x^{(j_o)}, z^{(j_o)C}) P (R_{i_o}) (y^{(j_o)}, z^{(j_o)C}) \)

and

for all \( z^{(i_o)C} \in X^{(i_o)C} : (y^{(i_o)}, z^{(i_o)C}) P (R_{j_o}) (x^{(i_o)}, z^{(i_o)C}) \).

Furthermore, suppose that \( R \) is such that for all \( i \in I \setminus \{i_o, j_o\} \), it holds that

for all \( x^{\tilde{S}C} \in X^{\tilde{S}C} : (y^{(i_o)}, x^{(j_o)}, x^{\tilde{S}C}) R_i (x^{(i_o)}, y^{(j_o)}, x^{\tilde{S}C}) \).

Since \( \varphi \) is decomposable and since, moreover, \( i_o \) is a dictator in sector \( s \ (i_o) \) and \( j_o \) is a dictator in sector \( s \ (j_o) \), we have that \( \varphi^{(i_o)} (R^{(i_o)}) = \{x^{(i_o)}, y^{(i_o)}\} \) and \( \varphi^{(j_o)} (R^{(j_o)}) = \{x^{(j_o)}, y^{(j_o)}\} \). It follows that the SCR \( \varphi \) is not a Pareto optimal SCR for \( R \) given that for all \( i \in I \) and all \( x^{\tilde{S}C} \in \prod_{s \in S_C} \varphi (R^s) \), it holds that

\[
\left( y^{(i_o)}, x^{(j_o)}, x^{\tilde{S}C} \right) R_i \left( x^{(i_o)}, y^{(j_o)}, x^{\tilde{S}C} \right),
\]

and, moreover,

\[
\left( y^{(i_o)}, x^{(j_o)}, x^{\tilde{S}C} \right) P (R_{i_o}) \left( x^{(i_o)}, y^{(j_o)}, x^{\tilde{S}C} \right)
\]

and

\[
\left( y^{(i_o)}, x^{(j_o)}, x^{\tilde{S}C} \right) P (R_{j_o}) \left( x^{(i_o)}, y^{(j_o)}, x^{\tilde{S}C} \right).
\]

\[\blacksquare\]

5 Sufficient conditions

In implementation theory, it is Maskin’s Theorem (Maskin, 1999) that shows that where the central designer faces at least three agents, a SCR is implementable in (pure-strategies) Nash equilibrium if it is monotonic and it satisfies the auxiliary condition of no veto-power.\(^7\)

\(^7\) Moore and Repullo (1990), Dutta and Sen (1991), Sjöström (1991) and Lombardi and Yoshihara (2013) refined Maskin’s Theorem by providing necessary and sufficient conditions for a SCR to be im-
In the abstract Arrovian domain, the condition of no veto-power says that if an outcome is at the top of the preferences of all agents but possibly one, then it should be chosen irrespective of the preferences of the remaining agent: that agent cannot veto it. The condition of no veto-power implies two conditions. First, it implies the condition of unanimity, which states that if an outcome is at the top of the preferences of all agents, then that outcome should be selected by the SCR. Thus, as a part of sufficiency, we require a variant of unanimity, which states that if all agents agree on which outcome is best for sector $s$, then this outcome should be chosen by the $s$th dimension of a decomposable SCR.

**Definition 6.** A decomposable SCR $\varphi : \mathcal{R}_I \rightarrow X$ satisfies unanimity provided that for all $R \in \mathcal{R}_I^{sep}$, all $s \in S$ and all $x^s \in X^s$, if $X^s \subseteq L(x^s, R_i^s)$ for all $i \in I$, then $x^s \in \varphi^s (R^s)$.

Second, the condition of no veto-power implies the condition of weak no veto-power, which states that if an outcome $x$ is $\varphi$-optimal at one profile $\tilde{R}$ and if the profile change from $\tilde{R}$ to $R$ in a way that under the new profile an outcome $y$ that was no better than $x$ at $\tilde{R}_i$ for some agent $i$ is weakly preferred to all outcomes in the weak lower contour set of $\tilde{R}_i$ at $x$ according to the ordering $R_i$ and this $y$ is maximal for all other agents in the set $X$, then $y$ should be a $\varphi$-optimal outcome at $R$. As a part of sufficiency, we require the following adaptation of weak no veto-power to our implementation model.

**Definition 7.** A decomposable SCR $\varphi : \mathcal{R}_I \rightarrow X$ satisfies weak no veto-power provided that for all $R \in \mathcal{R}_I^{sep}$, all $s \in S$ and all $x^s \in X^s$, if for some $\tilde{R}^s \in \mathcal{D}_I^s$ it holds that $x^s \in \varphi^s (\tilde{R}^s)$, $y^s \in L(x^s, \tilde{R}_i^s) \subseteq L(y^s, R_i^s)$ for some $i \in I$ and $X^s \subseteq L(y^s, R_i^s)$ for all $j \in I \setminus \{i\}$, then $y^s \in \varphi^s (R^s)$.

The main result of the section is also established with the aid of two domain conditions, Property A and Property B.

Property A requires the following. For any arbitrary collection of admissible conditional orderings, one for each sector, there exists an admissible separable ordering on implementable in (pure strategies) Nash equilibrium. For an introduction to the theory of implementation see Jackson (2001) and Maskin and Sjöström (2002).
such that the induced conditional orderings over every sector coincides with the ones in the arbitrary collection. Formally:

**Definition 8.** The domain $\mathcal{R}_i \subseteq \mathcal{R}(X)$ satisfies Property A if, for all $(R_i^1, \ldots, R_i^d) \in \prod_{s \in S} \mathcal{D}_s$, there exists a separable ordering $\bar{R}_i \in \mathcal{R}_i$ such that the $s$-conditional ordering $\bar{R}_i^s$ induced by $\bar{R}_i$ coincides with $R_i^s$, that is, $\bar{R}_i^s = R_i^s$ for all $s \in S$.

Property B is central to implementation in partial equilibrium. It guarantees that one can always behave as if her preference was separable. Specifically, Property B says the following. For any ordering $R_i$ and outcome $x$, there exists an admissible separable ordering $R'_i$ on $X$, such that the preferences change from $R'_i$ to $R_i$ in a (Maskin) monotonic way around $x$ (that is, whenever $xR'_ix'$, one has that $xR_ix'$) and, moreover, for any sector $s$ and any outcome of the set $X$ that differs from $x$ only for the the values of sector $s$, the outcome $x$ must not strictly fall in preference for agent $i$ when her preference changes from $R_i$ to $R'_i$. Formally:

**Definition 9.** The domain $\mathcal{R}_i \subseteq \mathcal{R}(X)$ satisfies Property B if, for all $R_i \in \mathcal{R}_i$ and all $x \in X$, there exists a separable ordering $\bar{R}_i \in \mathcal{R}_i$ such that

\[
\text{(9)} \quad \text{for all } s \in S : L^s(x, R_i) \subseteq L^s(x, \bar{R}_i),
\]

\[
\text{(10)} \quad \text{and } L(x, \bar{R}_i) \subseteq L(x, R_i).
\]

Note that a separable ordering $\bar{R}_i$ that satisfies (9) and (10) also satisfies the following property: $L^s(x, R_i) = L^s(x, \bar{R}_i)$ for all $s \in S$.

**Definition 10.** The domain $\mathcal{R}_I$ satisfies Property A and Property B provided that for each $i \in I$, the domain $\mathcal{R}_i$ satisfies Property A as well as Property B.

Note that Property A imposes no restrictions of domains of interest. A discussion of the implications of Property B for the domain $\mathcal{R}_I$ is provided in section 6.

We now present our characterization result.
Theorem 5. Let \( \mathcal{R}_I \) satisfy Property A and Property B. If \( n \geq 3 \) and \( \varphi : \mathcal{R}_I \to X \) is a SCR satisfying decomposability, monotonicity, decomposable monotonicity, weak no veto-power and unanimity, then it is implementable in partial equilibrium.

Proof. Suppose \( \mathcal{R}_I \) satisfies Property A and Property B and that \( n \geq 3 \). Suppose the SCR \( \varphi : \mathcal{R}_I \to X \) is decomposable, monotonic, decomposable monotonic and it satisfies unanimity and weak no veto-power. We first construct a canonical mechanism \( \Gamma^s \) for each \( s \in S \). Then, we show that the constructed product set of partial equilibrium mechanisms \( \Gamma = (\Gamma^s)_{s \in S} \) implements the SCR \( \varphi \) in partial equilibrium.

For all \( i \in I \), all \( R^s_i \in \mathcal{D}_i^s \) and all \( x^s \in X^s \), the weak lower contour set of \( R^s_i \) at \( x^s \) is defined by

\[
L(x^s, R^s_i) = \{ y^s \in X^s | x^s R^s_i y^s \}.
\]

For each sector \( s \in S \) and each agent \( i \in I \), define the strategy space of sector \( s \) by

\[
M^s_i = \mathcal{D}^s_i \times X^s \times \mathbb{Z}_+,
\]

where \( \mathcal{D}^s_i \) is the domain of profiles of conditional orderings of sector \( s \) induced by \( \mathcal{R}_I^{sep} \) and \( \mathbb{Z}_+ \) is the set of nonnegative integers. Denote the generic member of \( M^s_i \) as \( m^s_i = ((R^s)^i, (x^s)^i, z^i) \). Then, as usual, agent \( i \) chooses as a strategy a triple consisting of a profile of the \( s \) conditional orderings \( (R^s)^i \) in \( \mathcal{D}^s_i \), an outcome \( (x^s)^i \) from \( X^s \) and an integer \( z^i \) in \( \mathbb{Z}_+ \). Let \( M^s = \prod_{i \in I} M^s_i \) and define \( h^s : M^s \to X^s \) as follows: For any \( m^s \in M^s \),

Rule 1: If \( m^s_i = (R^s, x^s, z^i) \) for all \( i \in N \) and \( x^s \in \varphi^s (R^s) \), then \( h^s (m^s) = x^s \).

Rule 2: If for some \( i \in N \), \( m^s_j = (R^s, x^s, z^j) \) for all \( j \in N \setminus \{i\} \), \( x^s \in \varphi^s (R^s) \) and \( ((R^s)^i, (x^s)^i) \neq (R^s, x^s) \), then

\[
h^s (m^s) = \begin{cases} (x^s)^i & \text{if } (x^s)^i \in L(x^s, R^s_i) \\ x^s & \text{otherwise.} \end{cases}
\]

Rule 3: Otherwise, \( h^s (m) = (x^s)^{i^*} \) where \( i^* = \max \{ i \in N | z^i = \max \{ z^1, \ldots, z^n \} \} \).

We show that \( \Gamma = (\Gamma^s)_{s \in S} \) implements the SCR \( \varphi \) in partial equilibrium. Therefore, in the subsequent discussion we consider an arbitrary \( R \in \mathcal{R}_I \).
We first show that $\varphi (R) \subseteq h(NE(\Gamma, R))$. Take an arbitrary $x \in \varphi (R)$, where $x = (x^s)_{s \in S}$. Given that $\mathcal{R}_I$ satisfies Property B, there exists a profile $\tilde{R} \in \mathcal{R}_I^{sep}$ such that for each agent $i \in I$, it holds that

\begin{equation}
(11) \quad \text{for all } s \in S : L^s(x, R_i) \subseteq L^s(x, \tilde{R}_i)
\end{equation}

and that

\begin{equation}
(12) \quad L(x, \tilde{R}_i) \subseteq L(x, R_i).
\end{equation}

Recall that we also write $(x^s, x^{sc})$ for $x \in X$ and that $L^s(x, R'_i) = \{(y^s, x^{sc}) \in X | xR'_i(y^s, x^{sc})\}$ for any $R'_i \in \mathcal{R}_i$.

Since $x \in \varphi (R)$ and since, moreover, the SCR $\varphi$ is decomposable monotonic and $\tilde{R} \in \mathcal{R}_I^{sep}$, it follows from (11) that $x \in \varphi \left( \tilde{R} \right)$. Furthermore, the decomposability of $\varphi$ implies that $\varphi \left( \tilde{R} \right) = \prod_{s \in S} \varphi^s(\tilde{R}_s)$ given that $\tilde{R} \in \mathcal{R}_I^{sep}$, where $\tilde{R}^i_s$ denotes the conditional ordering induced by $\tilde{R}_i$ and where $\tilde{R}^s = (\tilde{R}_1^s, \cdots, \tilde{R}_n^s)$.

Fix an arbitrary $s \in S$. Let $m^s_i = \left( \tilde{R}^s, x^s, z^i \right)$ for all $i \in I$. Then, the strategy profile $m^s$ falls into Rule 1 and, therefore, $h^s(m^s) = x^s$. By the definition of the outcome function $h^s$, any agent $i$ who deviates from $m^s$ gets an outcome in $h^s(M^s_i, m_{-i}) \subseteq L \left( x^s, \tilde{R}^s_i \right)$, which is no better for her than $x^s$. Therefore, $m^s$ is a Nash equilibrium of $\left( \Gamma^s, \tilde{R}^s \right)$.

Since the choice of $s$ was arbitrary and $\tilde{R} \in \mathcal{R}_I^{sep}$, it follows from Lemma 1 that the strategy profile $m = (m_1, \cdots, m_n)$, where $m_i = (m^s_i)_{s \in S}$ for all $i \in I$, is a Nash equilibrium of $\left( \Gamma, \tilde{R} \right)$. Therefore, it holds that

for all $i \in I : h \left( M_i, m_{-i} \right) \subseteq L \left( x, \tilde{R}_i \right),

where $M_i = \prod_{s \in S} M^s_i$ and $h \left( M_i, m_{-i} \right) = \{ h \left( m'_i, m_{-i} \right) \in X | m'_i \in M_i \}$. By simply invoking (12) we have that

for all $i \in I : h \left( M_i, m_{-i} \right) \subseteq L \left( x, R_i \right).

Therefore, $m$ is a Nash equilibrium of $(\Gamma, R)$ such that $h \left( m \right) = x$. We conclude from
this that there is a Nash equilibrium of \((\Gamma, R)\) corresponding to each \(x \in \varphi (R)\).

To prove that \(h (NE (\Gamma, R)) \subseteq \varphi (R)\), consider any strategy profile \(m \in NE (\Gamma, R)\). Thus, for all \(i \in I\) and all \(s \in S\), it holds that

\[
(13) \text{ for all } \tilde{m}^i_s \in M^i_s : h(m) R_i (h^s (\tilde{m}^i_s, m^s_{-i}), (h^s (m^s))_{s \in s_C}).
\]

Given that \(\mathcal{R}_I\) satisfies Property B, there exists a profile \(\tilde{R} \in \mathcal{R}^{sep}_I\) such that for all \(i \in I\), it holds that

\[
(14) \text{ for all } s \in S: L^s (h(m), R_i) \subseteq L^s (h(m), \tilde{R}_i)
\]

and, moreover,

\[
(15) \quad L (h(m), \tilde{R}_i) \subseteq L (h(m), R_i).
\]

Then, from (13) and (14) and from the fact that \(\tilde{R} \in \mathcal{R}^{sep}_I\), it holds that for all \(i \in I\) and all \(s \in S\)

\[
h^s (m^s) \tilde{R}_i h^s (\tilde{m}^i_s, m^s_{-i}) \text{ for all } \tilde{m}^i_s \in M^i_s,
\]

and so

\[
\text{for all } s \in S: m^s \in NE \left(\Gamma^s, \tilde{R}^s\right).
\]

Hence, Lemma 1 implies that \(m \in NE (\Gamma, \tilde{R})\).

To prove that \(h (m) \in \varphi (R)\) we first need to show that \(h (m) \in \varphi (\tilde{R})\). To this end, for the given \(m \in NE (\Gamma, \tilde{R})\), define \(L^1 (m) = \{m^s | m^s \text{ falls into Rule 1}\}\), \(L^2 (m) = \{m^s | m^s \text{ falls into Rule 2}\}\) and \(L^3 (m) = \{m^s | m^s \text{ falls into Rule 3}\}\). We proceed according to whether or not \(L^1 (m)\) is an empty set.

**Case 1:** \(L^1 (m)\) is empty.

Consider any \(m^s \in L^2 (m)\), so that for some \(i \in I\), \(\left((\tilde{R}^s)^j, (x^s)^j\right) = (\tilde{R}^s, x^s)\) for all \(j \in I \setminus \{i\}\), \(x^s \in \varphi^s (\tilde{R}^s)\) and \(\left((\tilde{R}^s)^i, (x^s)^i\right) \neq (\tilde{R}^s, x^s)\). Each agent \(j \neq i\) can deviate from \(m^s_j\) and induce any outcome \(y^s \in X^s\) by choosing \(z^j\) high enough so as to win the integer game (that is, \(z^j > z^p\) for all \(p \in I \setminus \{j\}\)). The fact that \(m^s\) is a Nash equilibrium of \(\left(\Gamma^s, \tilde{R}^s\right)\) implies that \(X^s \subseteq L (h^s (m^s), \tilde{R}^s_j)\). Consider agent
Therefore, for all $i$ agent $i$ could deviate from $m_i^s$ and induce any outcome $w^s$ in $L \left( x^s, \tilde{R}^s_i \right) \setminus \{ x^s \}$ she wishes by changing $m_i^s$ into $\tilde{m}_i^s = (\tilde{R}^s, w^s, z^s)$. To attain the outcome $x^s$ agent $i$ can change $m_i^s$ into $\tilde{m}_i^s = (\tilde{R}^s, x^s, z^s)$ so as to induce Rule 1. Then, we have that $h^s(m^s) \in L \left( x^s, \tilde{R}^s_i \right) \subseteq h^s \left( M_i^s, m_{-i}^s \right)$. Since $m^s$ is a Nash equilibrium of $\left( \Gamma^s, \tilde{R}^s \right)$, it follows that $L \left( x^s, \tilde{R}^s_i \right) \subseteq h^s \left( m^s, \tilde{R}^s_i \right)$. Since $\varphi$ is decomposable and $x^s \in \varphi^s(\tilde{R}^s)$ and since $\tilde{R} \in \mathcal{R}^{sep}_I$, weak no veto-power implies that $h^s(m^s) \in \varphi^s(\tilde{R}^s)$.

Consider any $m^s \in L^3(m)$. Then, the definition of $h^s$ and the fact that $m^s$ is a Nash equilibrium of $\left( \Gamma^s, \tilde{R}^s \right)$ imply that $X^s \subseteq L \left( h^s(m^s), \tilde{R}^s_i \right)$ for all $i \in I$. Again, since $\varphi$ is decomposable and $\tilde{R} \in \mathcal{R}^{sep}_I$, unanimity implies that $h^s(m^s) \in \varphi^s(\tilde{R}^s)$.

Given that $L^1(m)$ is empty, we conclude that decomposability combined with the fact that $h^s(m^s) \in \varphi^s(\tilde{R}^s)$ for all $m^s \in L^2(m) \cup L^3(m)$ guarantees that $h(m) \in \varphi(\tilde{R})$.

**Case 2:** $L^1(m)$ is not empty.

First, suppose that $m^s \in L^1(m)$ for all $s \in S$. Thus Rule 1 applies to each $m^s$, so that for each $s \in S$, $\left( (\tilde{R}^s)^i, (x^s)^i \right) = (\tilde{R}^s, x^s)$ for all $i \in I$ and $x^s \in \varphi^s(\tilde{R}^s)$. Given that $\mathcal{R}_I$ satisfies Property A, for each $i \in I$ there exists $\tilde{R}_i \in \mathcal{R}^{sep}_I$ such that the $s$ conditional ordering induced by $\tilde{R}_i$ coincides with $\tilde{R}_i^s$. Therefore, there exists a profile $\tilde{R} \in \mathcal{R}^{sep}_I$ such that the profile of $s$ conditional orderings by the profile $\tilde{R}$ coincides with the profile announced by agents, that is, with $\tilde{R}^s$. Since $x^s \in \varphi^s(\tilde{R}^s)$ for all $s \in S$ and since, moreover, the SCR $\varphi$ is decomposable, it follows that $x \in \varphi(\tilde{R})$, where $x = (x^s)_{s \in S}$.

Consider any $s \in S$. Then, since each agent $i \in I$ can alter the current choice of sector $s$ to an outcome in $L \left( x^s, \tilde{R}_i^s \right)$ by a unilateral deviation, and $m^s$ is a Nash equilibrium of $\left( \Gamma^s, \tilde{R}^s \right)$, we must have $L \left( x^s, \tilde{R}_i^s \right) \subseteq L \left( x^s, \tilde{R}_i^s \right)$ for all $i \in I$.

Given that $\tilde{R}_i^s$ and $\tilde{R}_i^s$ are the $s$ conditional orderings induced respectively by $\tilde{R}_i$ and $\tilde{R}_i$, we have that $L \left( x^s, \tilde{R}_i^s \right) \subseteq L \left( x^s, \tilde{R}_i^s \right)$ is equivalent to $L^s \left( x, \tilde{R}_i \right) \subseteq L^s \left( x, \tilde{R}_i \right)$. Therefore, for all $i \in I$, it holds that

$$\text{for all } s \in S : L^s \left( x, \tilde{R}_i \right) \subseteq L^s \left( x, \tilde{R}_i \right).$$

We conclude that decomposable monotonicity together with $x \in \varphi(\tilde{R})$ ensures that $x \in \varphi(\tilde{R})$. 

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Finally, assume that $L^1(m)$ is not an empty set and that for some $s \in S$, $m^s \in L^2(m) \cup L^3(m)$. Consider any $m^s \in L^2(m) \cup L^3(m)$. Reasoning like that used in Case 1 shows that $h^s(m^s) \in \varphi^s(\hat{R}^s)$. But then, given that $m^s$ is a Nash equilibrium of $(\Gamma^s, \hat{R}^s)$ and $h^s(m^s) \in \varphi^s(\hat{R}^s)$, it is also an equilibrium for each agent $i \in I$ to play the strategy choice $\hat{m}^s_i = (\hat{R}^s, h^s(m^s), z^i)$ in sector $s$. Therefore, the corresponding strategy profile $\hat{m}^s = (\hat{m}^s_1, \ldots, \hat{m}^s_n)$ falls into Rule 1 and $h^s(\hat{m}^s) = h^s(m^s)$.

Let $\hat{m} = (\hat{m}^s) \in \prod_{s \in S} M^s$ be a strategy profile such that the strategy profile $\hat{m}^s \in M^s$ coincides with $m^s$ if $m^s \in L^1(m)$ or else, $\hat{m}^s$ coincides with $\hat{m}$'. Since $\hat{m}^s$ is a Nash equilibrium of $(\Gamma^s, \hat{R}^s)$ for each $s \in S$, Lemma 1 implies that $\hat{m} \in NE(\Gamma, \hat{R})$. Furthermore, $\hat{m}$ is such that Rule 1 applies to each $\hat{m}^s$ and $h^s(m^s) = h^s(\hat{m}^s)$ for all $s \in S$. Reasoning like that used in the first three paragraphs of Case 2 shows that $h^s(\hat{m}) = h^s(m) \in \varphi(\hat{R})$.

It remains to show that $h^s(m) \in \varphi(R)$. Because the SCR $\varphi$ is monotonic, $(15)$ and the fact that $h^s(m) \in \varphi(\hat{R})$ imply that $h^s(m) \in \varphi(R)$. Thus, the product set of partial equilibrium mechanisms $\Gamma$ implements the SCR $\varphi$ in partial equilibrium. ■

6 Implications

In this section, we briefly discuss the implications of Theorem 5.

In sub-section 6.1, we follow Sen (1995) and Thomson (1999). Specifically, we consider a sequence $(\varphi^s)_{s \in S}$ of sector $s$ (Maskin) monotonic SCRs where the domain of $\varphi^s$ is the set of profiles of orderings on $X^s$, and we look for the minimal way in which that sequence has to be enlarged so as to satisfy (Maskin) monotonicity on the domain $R_I$.

In sub-section 6.2, we provide some examples of SCRs that are implementable in partial equilibrium. Moreover, given that Property A imposes no restrictions of domains of interest, we identify below a domain condition that is necessary for Property B. That domain condition is also sufficient provided that the set of outcomes $X$ is finite and the domain of (allowable) orderings on $X$ for agent $i$ includes the set of separable orderings on $X$. Finally, we reconsider the auction environment described in section 2.2 and show that for the class of orderings that can be represented by a utility function of the form given in (2), Property B restricts that domain to separable orderings.
6.1 Minimal monotonic extensions

Write $R^s_i(X^s)$ for the set of orderings on $X^s$ and $R^s_i(X^s)$ for the product set of $R^s_i(X^s)$’s, that is,

$$R^s_i(X^s) = \prod_{i \in I} R^s_i(X^s),$$

with $R^s$ as a typical profile. In this sub-section, we ask the following question: Given a sequence of (Maskin) monotonic SCRs $(\varphi^s)$, where $\varphi^s : R^s_i(X^s) \rightarrow X^s$, what does their extension to the whole domain $R^s$ look like provided that each agent behaves as if she had separable preferences? To answer that question, we first provide an extension of the sequence of SCRs $(\varphi^s)$ over the whole domain $R^s$ and then show that it is the smallest decomposable, monotonic and decomposable monotonic extension. If it is assumed that the sequence of SCRs under consideration satisfy unanimity and the condition of weak no veto-power, then it is shown that our extension is the smallest extension on $R^s$ that is implementable in partial equilibrium. This is reminiscent of the idea of minimal monotonic extension due to Sen (1995) and Thomson (1999). We first establish some notation and definitions.

For all $R_i \in \mathcal{R}_i$ and $x \in X$, recall that $R_i^s(x^{sc})$, on $X^s$, denote the $s$ conditional ordering, that is,

$$R_i^s(x^{sc}) = R^s_i(x^{sc}) R_i(z^s, x^{sc}).$$

To save notation, for any $R \in \mathcal{R}_I$, $x \in X$ and $s \in S$, write $R^s(x^{sc})$ for the profile of the $s$ conditional orderings corresponding to the profile $R$, that is, $R^s(x^{sc}) = (R_i^s(x^{sc}))_{i \in I}$. For any given $s \in S$, $\varphi^s : R^s_i(X^s) \rightarrow X^s$ is a sector $s$ SCR that associates a non-empty set $\varphi^s(R^s)$ of $X^s$ for every profile $R^s$.

A central property which is crucial to Maskin’s Theorem for sector $s$ is stated below. To introduce it, for any ordering $R_i^s \in R^s_i(X^s)$ and outcome $x^s \in X^s$, we write $L(x^s, R_i^s)$ for the weak lower contour set of $R_i^s$ at $x^s$, which can be defined by $L(x^s, R_i^s) = \{y^s \in X^s | x^s R_i^s y^s\}$. Then:

**Definition 11.** The SCR $\varphi^s : R^s_i(X^s) \rightarrow X^s$ is sector $s$ (Maskin) monotonic provided that for all $x^s \in X^s$, all $R^s, R^s_0 \in R^s_i(X^s)$, if $x^s \in \varphi^s(R^s)$ and $L(x^s, R_i^s) \subseteq L(x, R^s_i)$
for all $i \in I$, then $x^s \in \varphi^s(\bar{R}^s)$.

We next define our extension of the sequence of SCRs $(\varphi^s)_{s \in S}$ over the domain $\mathcal{R}_I$.

**Definition 12.** Given any sequence $(\varphi^s)_{s \in S}$, an extension of $(\varphi^s)_{s \in S}$ is a SCR $\varphi^*: \mathcal{R}_I \rightarrow X$ such that

$$x \in \varphi^s(R) \iff x^s \in \varphi^s(R^s(x^{sc})) \text{ for all } s \in S.$$ 

The examples presented in sub-section 6.2, sector-wise core solution and sector-wise stable solution are all indeed extensions of sequences of sector $s$ core solutions and of sector $s$ stable solutions, respectively.

Our first result is that the extension $\varphi^*$ of $(\varphi^s)_{s \in S}$ is a decomposable, monotonic and decomposable monotonic SCR provided that every $\varphi^s$ is a sector $s$ monotonic SCR.

**Theorem 6.** For each $s \in S$, let $\varphi^s: \mathcal{R}^s_1(X^s) \rightarrow X^s$ be a sector $s$ monotonic SCR. The extension $\varphi^*: \mathcal{R}_I \rightarrow X$ of $(\varphi^s)_{s \in S}$ is a decomposable, monotonic and decomposable monotonic SCR.

**Proof.** For each $s \in S$, let $\varphi^s: \mathcal{R}^s_1(X^s) \rightarrow X^s$ be a sector $s$ monotonic SCR. We show that the extension $\varphi^*: \mathcal{R}_I \rightarrow X$ of $(\varphi^s)_{s \in S}$ is a decomposable, monotonic and decomposable monotonic SCR. Since it is plain that $\varphi^*$ is decomposable, we show that it is monotonic and decomposable monotonic.

To show that $\varphi^*$ is monotonic, for some profile $R \in \mathcal{R}_I$ consider $x \in \varphi^s(R)$. Furthermore, consider any profile $\bar{R} \in \mathcal{R}_I$ such that for all $i \in I$, it holds $L(x, R_i) \subseteq L(x, \bar{R}_i)$. Then, for any $i \in I$ and any $s \in S$, it also holds that

$$\text{for all } y^s \in X^s: (x^s, x^{sc})R_i(y^s, x^{sc}) \Rightarrow (x^s, x^{sc})\bar{R}_i(y^s, x^{sc}),$$

so that

$$\text{for all } i \in I \text{ and all } s \in S: L(x^s, R_i^s(x^{sc})) \subseteq L(x^s, \bar{R}_i^s(x^{sc})).$$

Moreover, by definition of $\varphi^*$, it follows that $x^s \in \varphi^s(R^s(x^{sc}))$ for all $s \in S$. Then, for any given $s \in S$, we established that $x^s \in \varphi^s(R^s(x^{-s}))$ and $L(x^s, R_i^s(x^{-s})) \subseteq L(x^s, \bar{R}_i^s(x^{-s}))$. 

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Given that \( \varphi^s \) is a sector \( s \) monotonic SCR, we have that \( x^s \in \varphi^s(\bar{R}^s(x^c)) \) for all \( s \in S \), and so \( x \in \varphi^s(\bar{R}) \). Thus, \( \varphi^s \) is monotonic.

To show that \( \varphi^s \) is decomposable monotonic, for some profile \( R \in \mathcal{R}_I \) consider \( x \in \varphi^s(R) \). Furthermore, consider any profile \( \bar{R} \in \mathcal{R}_I^{sep} \) such that for all \( i \in I \), it holds that \( L^s(x,R_i) \subseteq L^s(x,\bar{R}_i) \) for all \( s \in S \). Reasoning like that used above shows that \( x \in \varphi^s(\bar{R}) \). Thus, \( \varphi^s \) is decomposable monotonic.

The next result asserts that the monotonic extension \( \varphi^s \) of the sequence of sector \( s \) monotonic SCRs \((\varphi^s)_s \in S\) is its minimal monotonic extension. Specifically, given a sequence of sector \( s \) monotonic SCRs \((\varphi^s)_s \in S\), we show that any monotonic SCR \( \psi \) that can be decomposed into the product of one-dimensional SCRs contains the monotonic extension \( \varphi^s \) provided that the \( s \)th component of \( \psi \) contains \( \varphi^s \) and that the domain \( \mathcal{R}_I \) satisfies Property B. Note that the trivial SCR \( \psi \) that maps each profile of orderings onto \( X \) is, of course, monotonic and decomposable and satisfies the requirement that \( \psi^s \supseteq \varphi^s \).

**Theorem 7.** Let \( \mathcal{R}_I \) satisfy Property B. For each \( s \in S \), let \( \varphi^s : \mathcal{R}_I^s(X^s) \to X^s \) be a sector \( s \) monotonic SCR. Let \( \psi : \mathcal{R}_I \to X \) be any decomposable SCR such that for all \( s \in S \), it holds that \( \psi^s \supseteq \varphi^s \). Let the SCR \( \varphi^s : \mathcal{R}_I \to X \) be an extension of \((\varphi^s)_s \in S\). Then, if the SCR \( \psi \) is monotonic, then \( \varphi^s \subseteq \psi \).

**Proof.** Let \( \mathcal{R}_I \) satisfy Property B. For each \( s \in S \), let \( \varphi^s : \mathcal{R}_I^s(X^s) \to X^s \) be a sector \( s \) monotonic SCR. Let \( \psi : \mathcal{R}_I \to X \) be any decomposable SCR such that for all \( s \in S \), it holds that \( \psi^s \supseteq \varphi^s \). Let the SCR \( \varphi^s : \mathcal{R}_I \to X \) be an extension of \((\varphi^s)_s \in S\). Suppose that the SCR \( \psi \) is a monotonic SCR.

To show that \( \varphi^s \subseteq \psi \), for some profile \( R \in \mathcal{R}_I \) consider \( x \in \varphi^s(R) \). Then, \( x^s \in \varphi^s(R^s(x^c)) \) for all \( s \in S \). We show that \( x \in \psi(R) \). Given that \( \mathcal{R}_I \) satisfies Property B, there exists a profile \( \bar{R} \in \mathcal{R}_I^{sep} \) such that for each \( i \in I \), it holds that

\[
L^s(x,R_i) \subseteq L^s(x,\bar{R}_i),
\]

for all \( s \in S \). Also, note that if \( \psi \) and \( \tilde{\psi} \) are two decomposable SCRs such that for each \( s \in S \), it holds that \( \psi^s \cup \tilde{\psi}^s \supseteq \varphi^s \), then their intersection \( \psi \cap \tilde{\psi} \) is a decomposable SCR provided that it is a well defined SCR.
Consider any \( s \in S \). It follows from (16) and (17) that \( L(x, \tilde{R}_i) \subseteq L(x, R_i) \) for all \( i \in I \). Since \( x^s \in \varphi^s(R^s(x^s)) \) and \( \varphi^s \) is a sector \( s \) monotonic SCR, we have that \( x^s \in \varphi^s(\tilde{R}^s(x^s)) \). Therefore, \( x^s \in \varphi^s(\tilde{R}^s(x^s)) \) for all \( s \in S \). Since by assumption \( \psi^s \supseteq \varphi^s \) for all \( s \in S \) and the SCR \( \psi \) is decomposable, it holds that \( x \in \psi(\tilde{R}) \). From (17) and the fact that the SCR \( \psi \) is monotonic, it follows that \( x \in \psi(R) \). Thus, \( \varphi^s \) is the minimal monotonic extension of \( (\varphi^s)_{s \in S} \). ■

6.2 Examples

The following examples give an idea of the range of applications covered by Theorem 5.

Example 5 (Sector-wise core SCR). Consider \( \ell \geq 2 \) distinct barter exchange markets with \( n \geq 3 \) agents, where in each market \( s \in S \) agents are allowed to exchange items of the same type \( s \). Each agent starts with some initial bundle of indivisible items, one for each type \( s \in S \). Let \( e^s_i \) denote item of type \( s \) owned by agent \( i \). Agents can consume exactly one item per type. The total of items of type \( s \in S \) available is

\[
\sum_{i \in I} e^s_i.
\]

The set of items of type \( s \) is denoted by \( X^s \). An allocation \( x^s \) of items of type \( s \in S \) is a list of items of type \( s \) consistent with the total initial available. That is,

\[
Y^s = \left\{ x^s | x^s_i \in X^s \text{ for all } i \in I \text{ and } \sum_{i \in I} x^s_i = \sum_{i \in I} e^s_i \right\} \text{ for all } s \in S.
\]

The set of allocations is given by

\[
Y = \prod_{s \in S} Y^s,
\]
with $x$ as a typical element.

Consider any profile $R \in \mathcal{R}_I$ and allocation $x \in Y$. The allocation $x^s$ is Pareto efficient for the profile of the $s$ conditional orderings $R^s(x^{sc}) = (R^s_i(x^{sc}))_{i \in I}$ if it is not Pareto dominated by any other allocation $y^s$. The allocation $x^s$ is individually rational for $R^s(x^{sc})$ if it leaves each agent $i$ as well off as her endowment of type $s$, that is, $x^s_iR^s_i(x^{sc})e_i^s$ for all $i \in I$. The allocation $x^s$ is a core allocation for $R^s(x^{sc})$ if it is Pareto efficient and individually rational for $R^s(x^{sc})$.

The SCR $\varphi_C^s : \mathcal{R}_I^{s}(Y^s) \rightarrow Y^s$ is the sector $s$ core SCR provided that for all $R^s \in \mathcal{R}_I^{s}(Y^s)$ and all $x^s \in Y^s$,

$$x^s \in \varphi_C^s(R^s) \iff x^s \text{ is a core allocation for } R^s.$$  

The SCR $\varphi : \mathcal{R}_I \rightarrow Y$ is the sector-wise core SCR provided that for all $R \in \mathcal{R}_I$ and all $x \in Y$,

$$x \in \varphi_{SC}(R) \iff x^s \in \varphi_C^s(R^s(x^{sc})) \text{ for all } s \in S.$$  

It can be checked that the sector-wise core SCR is decomposable by construction and it satisfies sector $s$ monotonicity, unanimity and the condition of weak no veto-power. Furthermore, in light of Theorem 6, $\varphi_{SC}$ also satisfies decomposable monotonicity and monotonicity. Thus, if $\mathcal{R}_I$ on $Y$ satisfies Property A and Property B, $\varphi_{SC}$ is implementable in partial equilibrium.

**Example 6 (Sector-wise stable SCR).** Consider $\ell \geq 2$ distinct Gale-Shapley (1962) (marriage) markets with $n \geq 3$ agents. Consider the ordered triplet $(M, W, \succ^s)$, where:

- $M$ is a finite non-empty set of men, where a generic man is denoted by $m$.
- $W$ is a finite non-empty set of women, where a generic woman is denoted by $w$.
- $\succ^s_m$ is a linear order for the man $m$ in market $s \in S$, i.e., a complete, transitive and antisymmetric (binary) relation, on $W \cup \{m\}$.
• \(\succ^s_w\) is a linear order for the woman \(w\) on \(M \cup \{w\}\) in market \(s \in S\).

• \(\succ^s\) is a profile of linear orders for men and women in market \(s \in S\).

A market \(s \in S\) matching \(\mu^s\) is a function \(\mu^s : M \cup W \to M \cup W\) such that:

- For all \(m \in M\), \(\mu^s(m) \notin W \implies \mu^s(m) = m\).
- For all \(w \in W\), \(\mu^s(w) \notin M \implies \mu^s(w) = w\).
- For all \(m \in M\) and all \(w \in W\), \(\mu^s(m) = w \iff \mu^s(w) = m\).

Let us denote the set of market \(s\) matchings on \(W \cup M\) by \(\mathcal{M}^s\). Given a linear order \(\succ^s_m\) of the man \(m\), we extend it from \(W \cup \{m\}\) to the set of matchings \(\mathcal{M}^s\) in the following way:

\[
\text{for all } \mu^s, \tilde{\mu}^s \in \mathcal{M}^s : \mu^s R^s_{\mu^s \tilde{\mu}^s} \iff \mu^s(m) \succ^s_m \tilde{\mu}^s(m) \text{ or } \mu^s(m) = \tilde{\mu}^s(m).
\]

Simply put, a man’s preferences regarding alternative matchings correspond exactly to his preferences regarding his mates at the two matchings. The ordering \(R^s_w\) of a woman \(w\) is defined likewise. Let \(\mathcal{R}^s_i(\mathcal{M}^s)\) denote the set of market \(s\) orderings for agent \(i \in M \cup W\) and write \(\mathcal{R}^s_i(\mathcal{M}^s)\) for the profile of market \(s\) orderings for women and men, with \(R^s\) as a typical element.

A market \(s\) matching \(\mu^s\) is individually rational at \(R^s\) if for all \(i \in W \cup M\), \(\mu(i) R^s_i\). A market \(s\) matching \(\mu^s\) is blocked by a pair \((m, w)\) under \(R^s\) if \(w P(R^s_m) \mu(m)\) and \(m P(R^s_w) \mu(w)\). A market \(s\) matching \(\mu^s\) is market \(s\) stable at \(R^s\) if it is individually rational at \(R^s\) and it is not blocked by any pair \((m, w)\) under \(R^s\).

The SCR \(\varphi^s_{St} : \mathcal{R}^s_i(\mathcal{M}^s) \to \mathcal{M}^s\) is the market \(s\) stable SCR provided that for all \(R^s \in \mathcal{R}^s_i(\mathcal{M}^s)\) and all \(\mu^s \in \mathcal{M}^s\),

\[
\mu^s \in \varphi^s_{St}(R^s) \iff \mu^s \text{ is a market } s \text{ stable matching at } R^s.
\]

Note that \(\varphi^s_{St}\) is market \(s\) (Maskin) monotonic.
The SCR \( \varphi : \mathcal{R}_I \to \prod_{s \in S} \mathcal{M}^s \) is the sector-wise stable SCR provided that for all
\( R \in \mathcal{R}_I \) and all \( \mu \in \prod_{s \in S} \mathcal{M}^s \):

\[
\mu \in \varphi_{SSt}(R) \iff \mu^s \in \varphi_{St}^s(R^s(\mu^s)) \text{ for all } s \in S.
\]

Reasoning like that used in the preceding example shows that \( \varphi_{SSt} \) is decomposable, monotonic, decomposable monotonic and, moreover, it satisfies unanimity and the condition of weak no veto-power. Thus, if \( \mathcal{R}_I \) on \( \prod_{s \in S} \mathcal{M}^s \) satisfies Property A and Property B, \( \varphi_{SC} \) is implementable in partial equilibrium.

A domain condition which is implied by Property B can be defined as follows:

**Definition 13.** The domain \( \mathcal{R}_i \subseteq \mathcal{R}(X) \) satisfies Property B* if for all \( R_i \in \mathcal{R}_i \) and all \( x, y \in X \) it holds that

\[
xR_i(y^s, x^{sC}) \text{ for all } s \in S \implies xR_iy.
\]

Property B* is easier to check than Property B, and it rules out “too much complementarity.” The next result shows that Property B* is necessary for Property B and that the two properties are equivalent where the sector \( s \in S \) set \( X^s \) is finite and the domain \( \mathcal{R}_i \) includes the set of separable orderings on \( X \).

**Theorem 8.** If \( \mathcal{R}_i \subseteq \mathcal{R}(X) \) satisfies Property B, then \( \mathcal{R}_i \) satisfies Property B*. The converse is true provided that \( X^s \) is finite for all \( s \in S \) and that \( \mathcal{R}^{sep}(X) \subseteq \mathcal{R}_i \).

**Proof.** Consider any \( R_i \in \mathcal{R}_i \) and \( x, y \in X \) such that \( xR_i(y^s, x^{sC}) \) for all \( s \in S \). Suppose that \( \mathcal{R}_i \) satisfies Property B. Then, there exists a separable ordering \( \bar{R}_i \in \mathcal{R}_i^{sep} \) such that (9) and (10) hold. Since, by hypothesis, \( xR_i(y^s, x^{sC}) \), it follows from (9) that \( x\bar{R}_i(y^s, x^{sC}) \), and so the \( s \) conditional ordering is such that \( x^s\bar{R}_i^s y^s \). Given that \( \bar{R}_i \) is a separable ordering, we have that

\[
x\bar{R}_i(y^1, x^{1C}),
\]
that

for all \( s \in S \setminus \{1, \ell\} : \left( (y^q)_{q=1,\ldots,s-1} ; (x^q)_{q=s,\ldots,\ell} \right) \backsim_i \left( (y^q)_{q=1,\ldots,s} ; (x^q)_{q=s+1,\ldots,\ell} \right) \)

and that

\[
\left( y^c, x^c \right) \backsim_i y.
\]

Since \( \backsim_i \) is transitive, it follows that \( x \sim y \). Given that (10) holds, we have that \( xR_i y \).

Thus, \( \mathcal{R} \) satisfies Property B*.

To show the converse, suppose \( X^s \) is finite for all \( s \in S \) and that \( \mathcal{R}_{\text{sep}}(X) \subseteq \mathcal{R}_i \).

Moreover, suppose that \( \mathcal{R}_i \) satisfies Property B*.

Assume, to the contrary, that Property B is violated. Fix any \( R \in \mathcal{R}_i \) and \( x \in X \).

For each \( s \in S \), fix a representation of the \( s \) conditional ordering \( R_i^s(x^{sc}) \), which is denoted by \( v_i^s \). Then, for any \( \lambda > 0 \), let \( \bar{u}_i^C \) be a separable ordering represented in the form

\[
\bar{u}_i^C(y) = \sum_{s \in S} \exp \lambda(v_i^s(y^s) - v_i^s(x^s)).
\]

For \( \lambda \) sufficiently large it holds that

\[
x \bar{R}_i^C y \implies xR_i(y^s, x^{sc}) \text{ for all } s \in S.
\]

This is because if \( x \bar{R}_i^C y \) but \( (y^s, x^{sc})P(R_i) x \) for some \( s \in S \), then for \( \lambda \) sufficiently large the term \( \exp \lambda(v_i^s(y^s) - v_i^s(x^s)) \) becomes arbitrarily large, which leads to \( yP(\bar{R}_i^C) x \).

Fix any \( s \in S \). Suppose that \( xR_i(y^s, x^{sc}) \) for some \( y^s \in X^s \). Then, \( v_i^s(x^s) \geq v_i^s(y^s) \) given that \( x^s \backsim_i R_i^s(x^{sc}) y^s \). We need to rule out the case that \( (y^s, x^{sc})P(\bar{R}_i^C) x \) to conclude that \( x \bar{R}_i^C (y^s, x^{sc}) \). Thus, suppose that \( (y^s, x^{sc})P(\bar{R}_i^C) x \). By definition of \( \bar{u}_i^C \), it must hold that \( \bar{u}_i^C(y^s, x^{sc}) > \bar{u}_i^C(x) \) or, equivalently, it must be the case that

\[
\exp \lambda(v_i^s(y^s) - v_i^s(x^s)) > 1,
\]

which is false given that \( v_i^s(x^s) \geq v_i^s(y^s) \) and \( \lambda > 0 \).

Suppose that there exists \( y \in X \) such that \( x \bar{R}_i^C y \) but \( yP(R_i) x \). Since \( x \bar{R}_i^C y \), then for \( \lambda \) sufficiently large it holds that \( xR_i(y^s, x^{sc}) \) for all \( s \in S \). Property B* implies that
$x R_i y$, which is a contradiction. Thus, $\mathcal{R}_i$ satisfies Property $B^*$. ■

**Example 7.** In this example we provide an ordering $R_i \in \mathcal{R}_i$ that violates Property $B$. To this end, let $S = \{1, 2\}$. Moreover, suppose that $X^s = \{x^s, y^s\}$, with $x^s \neq y^s$, for all $s \in S$. Consider the following ordering $R_i$ on $X^1 \times X^2$:

$$(y^1, y^2) P (R_i) (x^1, x^2) P (R_i) (y^1, x^2) I (R_i) (x^1, y^2).$$

The ordering $R_i$ violates Property $B^*$ since

$$(x^1, x^2) P (R_i) (y^1, x^2), (x^1, x^2) P (R_i) (x^1, y^2) \text{ but } (y^1, y^2) P (R_i) (x^1, x^2).$$

In light of Theorem 8, $R_i$ violates Property B.

The above example also shows that Property B is indispensable for our sufficiency result, since its violation leaves room for profitable deviations of agent $i$. The reason is that the second rule of the canonical mechanism of sector $s$ is used to give incentives to whistle-blowers so as to rule out the possibility that a unanimously false announcement could constitute a Nash equilibrium of the mechanism. To be considered credible, the dissenter must have nothing to gain by untruthfully dissenting, that is, the dissenter’s announced outcome must not be strictly better for her according to the untruthful profile announced by the others. This incentive only works if the SCR is (Maskin) monotonic. However, if agents’ preferences are not separable and they are required to act as if they were separable, agents can never announce the true environment. Then, once agents have made in each sector $s$ a unanimously false announcement of their conditional orderings and unanimously announced the $\varphi^s$-optimal outcome $x^s$ at that profile, agent $i$ could induce the ‘worst’ outcome $y^s$ in each sector $s$ by unilaterally deviating to the second rule, and according to the mechanism that deviation is credible. Now, if in each sector, no one of the other agents objects to agent $i$’s deviations, by unilaterally inducing the third rule, agent $i$ attains the most preferred outcome $y$ according to her true non-separable ordering.
Example 8. In this example we provide a preference domain which satisfies Property B. Let $S = \{1, 2\}$. Moreover, suppose that $X^s = \{x^s, y^s\}$, with $x^s \neq y^s$, for all $s \in S$. Define $\mathcal{R}_i$ as follows: $R_i \in \mathcal{R}_i$ if either $R_i \in \mathcal{R}^{sep}(X)$ or if $R_i \notin \mathcal{R}^{sep}(X)$ and for all $x^1, y^1 \in X^1$ and $x^2, y^2 \in X^2$ it holds that

\begin{equation}
(x^1, x^2) I(R_i) (y^1, y^2) P(R_i) (y^1, x^2) R_i(x^1, y^2).
\end{equation}

One can check that if $R_i$ satisfies (18), then it is not a separable ordering given that the sector 1 conditional ordering $R_i^1(x^2)$ differs from $R_i^1(y^2)$. As in sub-section 2.1, items of sector 1 can be viewed as school seats and items of sector 2 as houses. Suppose that houses $x^2$ and $y^2$ are equally sufficiently close to respective schools $x^1$ and $y^1$. Therefore, an interpretation of (18) is that agent $i$ strictly prefers the bundles that minimize the distance school-home to other available bundles and she finds the bundles $(x^1, x^2)$ and $(y^1, y^2)$ equally good.

Consider the following separable orderings:

\begin{align*}
given (x^1, x^2) : & (x^1, x^2) P (\bar{R}_i) (y^1, x^2) P (\bar{R}_i) (x^1, y^2) P (\bar{R}_i) (y^1, y^2) \\
given (y^1, y^2) : & (y^1, y^2) P (\bar{R}_i) (x^1, y^2) P (\bar{R}_i) (y^1, x^2) P (\bar{R}_i) (x^1, x^2) \\
given (x^1, y^2) : & (y^1, x^2) P (\bar{R}_i) (y^1, y^2) I (\bar{R}_i) (x^1, x^2) P (\bar{R}_i) (x^1, y^2) \\
given (y^1, x^2) : & (x^1, y^2) P (R_i^0) (x^1, x^2) I (R_i^0) (y^1, y^2) P (R_i^0) (y^1, x^2).
\end{align*}

One can check via Property B* or directly using Property B that $\mathcal{R}_i$ satisfies Property B.

Below we reconsider the auction environment described in section 2.2 and show that for the class of orderings that can be represented by a utility function of the form given in (2), Property B restricts that domain to non-separable orderings which exhibit complementarities between commodities via the commodity money.

Example 9. Assume that preferences belonging to $\mathcal{R}_i$ are represented in the form given in (2).\footnote{To assure that agent $i$’s willingness to pay/accept is well defined, we also assume that $U_i$ satisfies} We show that Property B is equivalent to the following property: for all
$d^1, d^2 \in D^1$, $d^3, d^2 \in D^2$ and $t^1, t^2 \in T$, if

$$U_i(d^1, d^2, t^1_i + t^2_i + e_i) = U_i(d^3, d^2, t^1_i + \Delta t^1_i + t^2_i + e_i)$$

$$= U_i(d^1, d^2, t^1_i + t^2_i + \Delta t^2_i + e_i),$$

then

$$U_i(d^3, d^2, t^1_i + \Delta t^1_i + t^2_i + \Delta t^2_i + e_i) = U_i(d^3, d^2, t^1_i + t^2_i + e_i).$$

This means that there is no complementarity between pure social decisions in the two sectors but those together exhibit income effects, and, therefore, the pure social decisions are not separable from each other because of them. Simply put, it means that there are no ‘direct’ complementarities between pure social decisions but that, instead, the commodity money enables ‘indirect’ complementarities between them.

To show that the above property is implied by Property B, pick any $d^1, \bar{d}^3 \in D^1$, $d^2, \bar{d}^2 \in D^2$ and $t^1, t^2 \in T$. Take any $\Delta t^1_i$ and $\Delta t^2_i$ such that the equalities in (19) hold. We need to show (20). Since agent $i$’s willingness to pay/accept is well defined, by assumption, there exists $\overline{\Delta t^2_i}$ such that

$$U_i(\bar{d}^3, \bar{d}^2, t^1_i + \Delta t^1_i + t^2_i + \overline{\Delta t^2_i} + e_i) = U_i(d^3, d^2, t^1_i + \Delta t^1_i + t^2_i + e_i),$$

and so, from (19), it follows that

$$U_i(\bar{d}^3, \bar{d}^2, t^1_i + \Delta t^1_i + t^2_i + \Delta t^2_i + e_i) = U_i(d^3, d^2, t^1_i + \Delta t^1_i + t^2_i + e_i).$$

Then, by applying Property B* to the equalities (21) and (22), we have that

$$U_i(\bar{d}^3, \bar{d}^2, t^1_i + \Delta t^1_i + \Delta t^2_i + e_i) = U_i(d^3, d^2, t^1_i + \Delta t^1_i + \Delta t^2_i + e_i),$$

the following property: For all $d^1, \bar{d}^3 \in D^1$, all $\bar{d}^2, \bar{d}^2 \in D^2$, all $t^1, t^2 \in T$, there exist $\bar{t}^1, \bar{t}^2 \in T$ such that

$$U_i(d^1, d^2, t^1_i + t^2_i + e_i) = U_i(d^1, d^2, t^1_i + t^2_i + e_i).$$
which, in turn, implies $\Delta t_i^2 = \Delta t_i^2$. Therefore, combining (19) and (21) with (23), we obtain (20). Thus, $\mathcal{R}_i$ satisfies the above property if it satisfies Property B.

The converse is true, because the indifference surface passing through $(d^1, d^2, t_1^i + t_2^i + e_i)$ coincides exactly with the indifference surface of the corresponding separable preference.

As a final application covered by Theorem 5, we consider the Marshallian SCR.

**Example 10 (Sector-wise Marshallian SCR).** There are $\ell \geq 2$ sectors. The task of each sector $s \in S$ authority consists in allocating a single commodity with closed transfers. We assume that transfers are made by means of a commodity money, which is used commonly across sectors. Let $H = \{ t \in [-\bar{t}, \infty)^n : \sum_{i \in I} t_i \leq 0 \}$ denote the set of closed net transfers or trades, where $\bar{t} > 0$ denotes some predetermined upper-bound for payments. Then the set of outcomes of sector $s \in S$ is given by $X_s = H \times H$, with $(q_s^s, t_s^s)$ as a typical element. $(q_s^s, t_s^s)$ is a pair of net trade of the $s$ commodity, $q_s^s$, and closed net transfers of the commodity money, $t_s^s$. We assume that there are at least $n \geq 3$ agents and that each agent $i \in I$ is endowed with an amount of commodity money, denoted by $e_i$, which is assumed to be $\geq \ell \bar{t}$. Let $\omega_i \in [\bar{t}, \infty)^\ell$ denote the $\ell$-vector of initial endowment.

The SCR $\varphi_{SM} : \mathcal{R}_I \to \prod_{s \in S} X_s$ is the sector-wise Marshallian SCR provided that for all $R \in \mathcal{R}_I$, $(q, t) \in \varphi_{SM}(R)$ if there exists a vector $(p^1, \cdots, p^\ell)$ such that $t_i^s = -p^s q_i^s$ for all $i \in I$ and all $s \in S$ and, moreover, for all $i \in I$, all $s \in S$ and all $z_i^s$ such that

$$p^s z_i^s \in \left\{ t_i^s \in [-\bar{t}, \infty) : t_i^s + \sum_{j \neq i} t_j^s \leq 0 \text{ for some } t_{-i}^s \in [-\bar{t}, \infty)^{n-1} \right\},$$

it holds that

$$u_i \left( \omega_i + q_i, e_i - p^s q_i^s - \sum_{\tilde{s} \in S_C} p^\tilde{s} q_{\tilde{s}}^i \right) \geq u_i \left( \omega_i + (z_i^s, q_i^s), e_i - p^s z_i^s - \sum_{\tilde{s} \in S_C} p^\tilde{s} q_{\tilde{s}}^i \right).$$

Note that this SCR cannot be given in the form of minimal extension, in the sense that every sector $s$ SCR is defined on the domain $\mathcal{R}_I^s(X)$. The reason is that
separable orderings in this environment are represented by the sum of quasi-linear functions, and so every sector $s$ conditional ordering must be quasi-linear. If the sector $s$ SCR was defined on $\mathcal{R}^s_f(X)$, the authority of sector $s$ would realize that there are non-quasi-linear conditional orderings and therefore it could infer that some of agents have non-separable orderings, which is in conflict with our idea that every sector authority acts as if agents’ orderings were separable.

This rule is decomposable, because on the domain of separable preferences it induces a sector $s$ Marshalian SCR $\varphi^s_M : \mathcal{D}^s_f \rightarrow X^s$ for each $s \in S$, where $\mathcal{D}^s_f$ consists of quasi-linear orderings: for all $R^s \in \mathcal{D}^s_f$, $(q^s, t^s) \in \varphi^s_M(R^s)$ if and only if there exists $p^s \in \mathbb{R}$ such that for all $i \in I$:

$$v^s_i(\omega^s_i + q^s_i) - p^s q^s_i \geq v^s_i(\omega^s_i + z^s_i) - p^s z^s_i$$

for all $z^s_i$ such that $p^s z^s_i \in \{t^s_i \in [-\ell, \infty) : t^s_i + \sum_{j \neq i} t^s_j \leq 0 \text{ for some } t^s_{-i} \in [-\ell, \infty)^{n-1}\}$.

It is Maskin monotonic since it is coming from price-taking optimization under feasibility constraints. It is decomposable monotonic since it is coming from price-taking optimization under the feasibility constraint in each sector, given that the allocations in other sectors are fixed. Finally, it satisfies the conditions of weak no veto-power and unanimity. Thus, if $\mathcal{R}_f$ satisfies Property A and Property B, $\varphi_{SM}$ is implementable in partial equilibrium.

7 Concluding comments

A product set of partial equilibrium mechanisms is a mechanism in which its participants are constrained to submit their rankings to sector authorities separately and, moreover, sector authorities cannot communicate with each other, due to misspecification by the central designer that preferences are separable or due to technical/institutional constraints. Therefore, a key property of a single partial equilibrium mechanism is that participants are required to behave as if they had separable preferences.

We identify a set of necessary conditions for the implementation of SCRs via a product set of partial equilibrium mechanisms, that is, for the implementation in partial
equilibrium. Furthermore, under mild auxiliary conditions, reminiscent of Maskin’s Theorem (1999), we have also shown that they are sufficient for the implementation in partial equilibrium.

We conclude by discussing future research directions. The first thing to come next will be to quantify how much we lose by the type of misspecification considered in this paper. Theoretical, empirical and experimental studies will be helpful there.

It is also worth investigating what can be implemented when an incomplete yet not negligible communication is allowed among sector authorities, while the central designer has to make some modeling choice about how sector authorities communicate.

Another direction will be to study how we can improve the mechanism in a sector while keeping fixed the mechanisms in other sectors and given such change how we can improve the mechanism in another sector while keeping fixed those in other sectors, and so on. There is no obvious way do it, because under general equilibrium effects it is not obvious whether or not a change regarded as an "improvement" from the point of view of partial equilibrium mechanism design is indeed an improvement. That research direction will answer the question of how we should change the partial equilibrium mechanism in an improving manner.

References

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