How to Use Markets to Make Predictions

Filippo Massari

UNSW Australia

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Motivations: Feldman et al.(2015)

- “In making a current decision, how should the policymaker weigh the net benefits of that decision in one possible future against the net benefits of that decision in another possible future?”

Federal Reserve Bank of Minneapolis:

- “Policymakers should typically weigh the net benefit in a given possible future using what we term the market-based probability of that future occurring.”
Some natural questions:

- How market-based probabilities evolve over time?
- How efficiently do market-based probabilities aggregate private information?
  - Social planner aggregator Vs market aggregator.
Preview of the results

• How market-based probabilities evolve over time?
  • Generalized Bayesian Method.
    • Does not necessarily require application of Bayes’ rule.
    • Consistent with known behavioral biases.
    • Special cases include Bayesian updating, NML and SNML.

• How efficiently do market-based probabilities aggregate private information?
  • Unique most accurate trader: as efficient as using Bayes’ rule.
  • Non-unique most accurate trader: can be even more efficient than using Bayes’ rule.
Intuition:
Markets "learn" through selection

- Market Selection Hypothesis (Friedman (1953)): investors with incorrect beliefs will eventually be driven out of markets by those with correct beliefs.
  - As the Market selects for the most accurate trader, the weight of his beliefs on equilibrium prices increases.
  - Asymptotically only the most accurate trader survives ⇒ prices reflect the beliefs of most accurate trader.
Environment

- Discrete time infinite horizon Arrow-Debreu exchange economy with complete markets.
- A finite set $\mathcal{I}$ of traders $i$, with *dogmatic* iid beliefs $p^i$.
- In each period we have a finite state space $\mathcal{S}$, $|\mathcal{S}| = S$.
- The set of all infinite sequences of realizations is $\mathcal{S}^\infty$ with representative sequence $\sigma = (\sigma_1, ...)$. 
- $p(\sigma^t) \equiv p(\{\sigma_1 \times ... \times \sigma_t\} \times \mathcal{S} \times \mathcal{S} ...)$ is the probability of the partial history $\sigma^t$. 
Assumptions for the talk:

- Constant aggregate endowment $\forall t, \sum_{i \in I} e^i_t = 1$.
- Common support: $\forall i, \forall \sigma^t, p^i(\sigma^t) > 0 \iff P(\sigma^t) > 0$.
- The economy is iCRRA: $\forall i \in I, u^i(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}$.
- The economy is HDF: $\forall i \in I, \beta^i = \beta$. 
Competitive Equilibrium

- a sequence of prices \( \{ q(\sigma^t) \} \) and of consumption choices \( \{ c^i_t(\sigma) \} \) such that, \( \forall i, \{ c^i_t(\sigma) \} \), solves:

\[
\max_{\{ c_t(\sigma) \}} \sum_{t=0}^{\infty} \beta^t u^i(c^i(\sigma^t))
\]

S.t.,
- Affordability: \( \sum_{t=0}^{\infty} \sum_{\sigma^t \in S^t} q(\sigma^t) (c^i(\sigma^t) - e^i(\sigma^t)) \leq 0 \)
- Market Clearing: \( \forall t, \sum_{i \in I} e^i_t = \sum_{i \in I} c^i_t \)
Equilibrium prices

In a HDF-iCRRRA-economy, $\forall \sigma^t$,

- the time 0 equilibrium price of an Arrow security is given by:

$$q_{\gamma}(\sigma^t) = \beta^t \left( \sum_{i \in \mathcal{I}} p^i(\sigma^t)^{\frac{1}{\gamma}} c_0(i) \right)^{\gamma}$$

- next period equilibrium price of an Arrow security:

$$q(\sigma_t | \sigma^{t-1}) = \frac{q(\sigma^t)}{q(\sigma^{t-1})}$$
Market-based probabilities

• Market Beliefs (MB) on sequences of length $t$ are defined as:

$$p^q(\sigma^t) = \frac{q(\sigma^t)}{\sum_{\hat{\sigma}^t \in S^t} q(\hat{\sigma}^t)} = \frac{\left(\sum_{i \in I} p^i(\sigma^t)^{\frac{1}{\gamma}} c_0(i)\right)^{\gamma}}{\sum_{\hat{\sigma}^t \in S^t} \left(\sum_{i \in I} p^i(\hat{\sigma}^t)^{\frac{1}{\gamma}} c_0(i)\right)^{\gamma}}$$

• Market Forward Forecasts (MFF) are recursively defined as:

$$p^f(\sigma_t | \sigma^{t-1}) = \frac{q(\sigma_t | \sigma^{t-1})}{\sum_{\hat{\sigma}_t \in S} q(\hat{\sigma}_t | \sigma^{t-1})} = \frac{\left(\sum_{i \in I} p^i(\sigma_t | \sigma^{t-1})^{\frac{1}{\gamma}} c_{\frac{1}{\gamma}, t-1}^1(i)\right)^{\gamma}}{\sum_{\hat{\sigma}_t \in S} \left(\sum_{i \in I} p^i(\hat{\sigma}_t | \sigma^{t-1})^{\frac{1}{\gamma}} c_{\frac{1}{\gamma}, t-1}^1(i)\right)^{\gamma}}$$

with $c_{\frac{1}{\gamma}, t-1}^1(i) = \frac{p^i(\sigma^{t-1})^{\frac{1}{\gamma}} c_0(i)}{\sum_{i \in I} p^i(\sigma^{t-1})^{\frac{1}{\gamma}} c_0(i)}$

• $\gamma = 1 \Rightarrow$ MB and MFF coincide with Bayesian updating with prior $C_0$
Criterion for asymptotic efficiency:

Let $\mathcal{P}$ be a set of probabilistic models and $p^B(.)$ be a Bayesian model average with regular prior on $\mathcal{P}$.

- A probability $p$ are asymptotically $\mathcal{P}$-efficient if:
  $$\forall \sigma \in \mathcal{S}^\infty, \lim_{t \to \infty} \log \frac{p^B(\sigma_t)}{p(\sigma_t)} \in (-\infty, +\infty).$$

- A probability $p$ are asymptotically $\mathcal{P}$-super-efficient if:
  $$\forall \sigma \in \mathcal{S}^\infty, \lim_{t \to \infty} \log \frac{p^B(\sigma_t)}{p(\sigma_t)} < \infty, \text{ and } \exists \hat{p} : \lim_{t \to \infty} \log \frac{p^B(\sigma_t)}{p(\sigma_t)} \to \hat{p}\text{-a.s.} -\infty.$$
How efficiently do market-based probabilities aggregate private information?

Market-based probabilities aggregate private information very well!

**Theorem 1**

In a HDF-iCRAA-economy,

- $\forall \gamma \in [0, \infty) \ MB$ are asymptotically efficient;
- if $\gamma > 1$ MFF are asymptotically super-efficient;
- if $\gamma = 1$ MFF are asymptotically efficient;
- if $\gamma < 1$ MFF are asymptotically sub-efficient.
Marked-based probabilities are not Bayesian: Some (non)-fundamental properties

- A forecasting scheme $p$ is prequential if it satisfies the Law of Total Probability:

  $$\sum_{\hat{\sigma}^t \in S^t} p(\hat{\sigma}^t \cap \sigma^{t-1}) = p\left(\bigcup_{\hat{\sigma}^t \in S^t} \hat{\sigma}^t \cap \sigma^{t-1}\right) = p(\sigma^{t-1})$$

- A forecasting scheme $p$ is exchangeable if and only if, for any permutation $\pi$ of the indices,

  $$p(\sigma^t) = p(\sigma_{\pi(1)}, \ldots, \sigma_{\pi(t)})$$
Marked-based probabilities are not Bayesian

Proposition

In an iid-iCRRA-HDF-economy with no aggregate risk and $\gamma \neq 1$,

- $MB$ are exchangeable but not prequential;
- $MFF$ are prequential but not exchangeable;

Moreover,

- $\gamma = 1$ (log) $\Rightarrow$ $MB=MFF=\text{Bayesian Model Average}$;
- $\gamma \rightarrow 0$ (linear) $\Rightarrow$ $MB = NML$ (Rissanen 1986, Grünwald 2007);
- $\gamma \rightarrow 0$ (linear) $\Rightarrow$ $MFF = SNML$ (Roos-Rissanen 2008).
Example

- iCRRRA-HDF-economy, with two states: $S = \{L, R\}$.
- 2 traders, $p^1 = \left[\frac{1}{3}, \frac{2}{3}\right]$, $p^2 = \left[\frac{2}{3}, \frac{1}{3}\right]$
- Trader $i$’s maximization problem is:

$$
\max_{E_{p^i}} \sum_{t=0} \frac{\beta^t c_t^i(\sigma)^{1-\gamma}}{1-\gamma} \quad \text{s.t.} \quad \sum_{t=0} \sum_{\sigma^t \in S^t} q(\sigma^t) \left( c_t^i(\sigma) - e_t^i(\sigma) \right) \leq 0.
$$
Example:
LOG-economy

\[ p^q(\sigma^t) = p^f(\sigma^t) = \sum_{i \in \mathcal{I}} p^i(\sigma^t) c^i_0 \]

MB=MFF which are *prequential and exchangeable.*
Example:
Market Beliefs in a LIN-economy

\[ p^q(\sigma^t) = \frac{\max_i p^i(\sigma^t)}{\sum_{\hat{\sigma}^t} \max_i p^i(\hat{\sigma}^t)} \]

MB are exchangeable but not prequential:
Example:

Market Forward Forecasts in a LIN-economy

\[ p^f(\sigma^t) = \prod_{\tau=1}^{t} \frac{p^i(\sigma_\tau|\sigma_{\tau-1})}{\sum_{\sigma_\tau} p^i(\sigma_\tau|\sigma_{\tau-1})} \]

with \( i = \arg \max \{ p^1(\sigma^{\tau-1}), p^2(\sigma^{\tau-1}) \} \).

MFF are prequential but not exchangeable:

\[
\begin{align*}
\text{LIN}_{p^f(\sigma^t)} & \\
\text{LIN}_{p^f(\sigma^t|\sigma^{\tau-1})} & \\
\end{align*}
\]
Conclusions:

- Markets make use private information efficiently;
- Market-Based probabilities:
  - If $\gamma > 1$ MFF are even more efficient than Bayesian forecasting;
  - In most cases converge fast to the best beliefs in the economy;
  - Constitute a general class of probabilistic assessments that contain Bayesian updating, NML and SNML as special cases.