Uncertainty in the Housing Market: A Learning Model and Evidence from Household Survey Data

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Abstract

This paper argues that uncertainty in the housing market can lead to higher price volatility and persistent estimation errors. Empirically, I construct a Household Sentiment Index (HSI) by applying the Case-Shiller repeat-sales estimation method to the households’ survey responses in the Panel Studies of Income Dynamics (PSID). I find that compared to the actual housing prices in the same state, the HSI consistently overestimated the growth rate after mid-1990s and exhibited more volatility over the entire period from 1968 to 2011. Theoretically, I show that these facts can result from the households’ uncertainty regarding whether a recent change in the market is temporary or permanent. A housing model incorporating this assumption and a Bayesian learning mechanism is able to match the volatility of housing prices. As a proxy for the sentiment index, the one-period forecast from the model shows persistent under-estimation and over-estimation of housing prices, resembling the behavior of the PSID responses.

Key words: Household behavior, Adaptive learning, Price volatility, Housing market.

JEL Classification: D12, D81, D84, E32, E37, O41, R31.

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1 Introduction

In the United States, fluctuations in the city or state-level housing prices at the beginning of the 1980s and 1990s cannot compare with the national boom and burst episode witnessed in the 2000s. A substantial housing literature attributes this last increase in housing prices to overly optimistic projections of future growth by all players in the market\(^1\). However, consensus on the underlying cause and a method to quantify and forecast this over-optimism is elusive. This paper argues that home buyers’ over-optimism can result from their uncertainty towards the permanence and persistence of exogenous changes in the market. Suppose a household sees an increase in the rental rates or the selling prices of houses in the same neighborhood, does the household consider this an isolated incident or the beginning of a long term trend? Does this uncertainty lead to a difference between the housing prices predicted by a rational expectation model, the actual market housing price and the household’s perception? This paper answers these two questions by first exploring the Panel Studies of Income Dynamics (PSID) data and second providing a two-sector model with a learning mechanism to replicate the features of the housing price data and the household estimates.

First, I use the answers provided by each household over time in the PSID to construct an index that measures the households’ perspectives on the housing market. I focus on one particular question in the survey, which is “Could you tell me what the present value of your (house/apartment) is? About how much would it bring if you sold it today?” Each household provides an estimate for the potential selling price of their house. I treat this estimate over time as repeat-sales data. Using the Case-Shiller method, I construct a time series capturing the households’ estimates of the market, henceforth referred to as the Household Sentiment Index (HSI). To be able to compare the HSI with the actual housing price index, using the information on the location of the households, I assign to each household the price index in their state and again apply the Case-Shiller method on the state housing price index (HPI). While one may expect the two series to be similar, the overall trend of the HSI lags behind the HPI, underestimating the long-term growth for a period of two decades. The cyclical component of the HSI, however, is more volatile. Compared to the

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\(^1\)One of the earlier papers to explicitly discuss and model how over-confidence can cause asset bubbles is Scheinkman and Xiong (2003).
market index, the households often overshoot and then quickly adjust.

Second, I now consider an economy full of households who share the same uncertainty. If there is an initial small increase in housing prices or rental rates, they may not decide to take action. However, after a longer period of steady increases, the households may think there is a higher probability of a permanent trend. Expecting an even higher price in the future, their decisions to purchase houses at this point will further increase prices and generate a housing boom. This amplification mechanism driven by uncertainty is the main feature of the model in this paper.

The purpose of the model is not just to explain the historical housing price data but to serve as a laboratory for policy experiments, policies that aim to revive the housing market or to prevent future bubbles. If this learning mechanism can reflect how households interpret recent developments in the housing market, the model can provide better forecasts under different policies. This is particularly relevant because many households or real estate investors are pondering over the latest housing price increase of 7.7% over 87% of US cities in the first half of 2013\(^2\). The numbers are more striking in several West Coast cities such as 34% from a year ago in the Sacramento area, 33% in Las Vegas, and 31% in Oakland\(^3\). In many investors’ view, the housing market has become an increasingly cyclical, market-timing business, one more similar to a stock market than the traditional brick-and-mortar business. There is no doubt that the debate on the next potential housing boom can benefit from a good model to capture the evolution of household’s expectations.

The baseline model has two sectors: the consumption goods and the housing sector. Capital is sector-specific while labor can move between two sectors. There is a social planner who maximizes utility by choosing the optimal level of consumption, housing services, and the allocation of labor to each sector. There are two shock processes to the model: productivity shock and housing preference shock. Housing preference refers to the weight of utility drawn from housing services compared to the utility from consumption goods. In the baseline model, there is no uncertainty about the nature of these two shocks. Alternatively, in the learning case, both shocks follow an AR(p) process but the social planner does not know if the housing preference shock is stationary or not. In other words, there are two sub-models: one with a stationary and one with a non-stationary housing


\(^3\)http://www.businessinsider.com/is-there-a-housing-bubble-in-the-western-us-2013-8
preference shock\textsuperscript{4}. In the Bayesian learning process, the social planner updates both the probability of each model and the auto-regressive coefficients every period. This learning mechanism was first derived by Cogley and Sargent in 2005 in the context of a central banker evaluating three different Phillips curve specifications. Tortorice (2012) applies this learning mechanism to the case of stationary versus non-stationary productivity shock in explaining consumption volatility. I then cast the two alternative models as one optimal linear regulator problem and solve for the optimal consumption, purchase of housing and labor supply in each period. From these optimal decision rules, I obtain a simulated time series of housing prices in the baseline model and the one with a learning mechanism. Since in the baseline case, the agent knows the true underlying data generating process, their expectations do not deviate from the true model in the face of an exogenous shock. Therefore, a small stationary exogenous shock will only translate into a small temporary deviation that will quickly revert to the steady state. One advantage of this learning model is that it does not assume any structural break in the shock process. Independent and identical shocks from a normal distribution can generate significantly more volatility under uncertainty.

There are two sets of results in this paper. First, I calibrate the standard deviation and AR(1) coefficients of the shocks to match the moments of the housing inventory series. Using the same calibrated shocks, I simulate and obtain a distribution of the housing prices under the baseline model and the learning mechanism. Over the long run, the learning mechanism can double the volatility in predicted housing prices and also match the second moments of the housing price data. Second, I construct a sentiment index from each model using a one-period forecast. The sentiment index under the learning model shows periods of persistent under-estimation and over-estimation of housing prices. Under learning, the sentiment tends to overshoot in response to the same housing preference shocks.

This paper complements the intersection between the housing literature and learning models. A number of papers have incorporated the assumption of near-rational expectation to explain the last housing bubble in the United States. Eusepi and Preston (2011) provides a theoretical real

\textsuperscript{4}We can also have stationary and non-stationary cases for both productivity shock and housing preference shock. This will lead to four combinations and four sub-models. While technically feasible, the interpretation of each model’s weight is not as intuitive.
business cycle model in which agents do not know the mapping between primitive disturbances, the aggregate state of the economy and market clearing prices. Focusing on the changing persistence of inflation since the 1970s, Slobodyan and Wouters (2012) use a dynamic stochastic general equilibrium model embedded with an auto-regressive forecasting rule and a Kalman Filter updating algorithm. Since the forecasting rule does not change, when applied to a housing model, this learning mechanism can generate the persistence but not the volatility in the data. Adam, Kuang and Marcet (2012) use an open economy model with housing in which agents do not know the model that links prices to the fundamentals. They treat price as a random exogenous variable and use Bayes’ rule to update the joint distribution of prices and fundamentals. After an initial reduction in interest rate, the near rational expectation beliefs with a constant-gain learning algorithm can generate a larger increase in housing prices compared to the fully rational expectation case. Additionally, if the realized price growth does not meet the expected price growth, the beliefs get revised downward, which leads to a price burst. Citing evidence from the U.S. and Norway household survey data, Gelain and Bank (2013) propose a simpler strategy to match the distributions of price-rent ratios and housing returns. Using a Lucas-type asset-pricing model with exogenous housing price, Gelain and Bank (2013) assumes that households use a four-year moving average rule to estimate and update their beliefs about the mean, persistence and volatility of the growth rate of rent. Their model can therefore generate the momentum in housing returns observed in historical data. Compared to these previous papers, the learning model in this paper is tractable with assumptions based on household survey data.

2 Data

Housing price index

The housing price index (HPI) is from the Federal Housing Finance Agency. The index is a broad measure of the movement of single-family house prices. The HPI is a weighted, repeat-sales index, meaning that it measures average price changes in repeat sales or refinancing on the same properties. This information is obtained by reviewing repeat mortgage transactions on single-family properties whose mortgages have been purchased or securitized by Fannie Mae or Freddie
Mac since January 1975. The HPI serves as a timely, accurate indicator of house price trends at various geographic levels. Because of the breadth of the sample, it provides more information than is available in other house price indexes. It also provides housing economists with an improved analytical tool that is useful for estimating changes in the rates of mortgage defaults, prepayments and housing affordability in specific geographic areas. The HPI includes house price figures for the nine Census Bureau divisions, for the 50 states and the District of Columbia. In order to match with the household data, I use the all transaction state-level HPI.

Household survey data from PSID

The household survey data are from Panel Study of Income Dynamics by the Institute for Social Research, Survey Research Center, University of Michigan, Ann Arbor. The survey starts with 5000 families in 1968 and follows these individuals and their descendants until 2011. As the children get married and move out of the house, they enter the sample as another household. The previous individuals can also drop out of the sample. In the sample, 68.93% of the households are home owners. They are less likely to move in a couple of years compared to renters. From the data on mortgages in 2009 and 2011, I document that 9.28% of households with mortgages have already worked with their banks or lenders to modify their mortgages. Among the households that are late on their payments, 21.5% are under the procedure of foreclosing. As shown in Table 2, home-owners that intend to move in a few years estimate a lower growth rate in the value of their property.

Table 1: Summary Statistics

<table>
<thead>
<tr>
<th></th>
<th>No. of Observations</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Own a house</td>
<td>46,065</td>
<td>68.93</td>
</tr>
<tr>
<td>Rent</td>
<td>20,771</td>
<td>31.07</td>
</tr>
<tr>
<td>Might move in a few years</td>
<td>renting</td>
<td>11,692</td>
</tr>
<tr>
<td>Might move in a few years</td>
<td>owning</td>
<td>7,967</td>
</tr>
<tr>
<td>Has at least one mortgage</td>
<td>owning a house</td>
<td>25,476</td>
</tr>
<tr>
<td>Behind on mortgage payments</td>
<td>93</td>
<td>3.82</td>
</tr>
<tr>
<td>Worked with bank or lender to structure or modify mortgage</td>
<td>226</td>
<td>9.28</td>
</tr>
<tr>
<td>Bank or lender has started the process of foreclosing</td>
<td>late payments</td>
<td>20</td>
</tr>
</tbody>
</table>

Note: (1) No. of observations = households*years. Unique households in the whole sample = 6421. (2) Detailed data on late payments, restructuring mortgages and foreclosure are only available in 2009 and 2011.
Figure 1: Percentage of households who rent, intend to move and hold at least one mortgage over the years. 43% of households in the sample are renters. On average, about 35.7% of households intend to move within a few years and 67.6% hold at least one mortgage.

Table 2: Factors correlated with the growth rate in each household’s estimates

<table>
<thead>
<tr>
<th></th>
<th>(1) %ΔHSI</th>
<th>(2) %ΔHSI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Move</td>
<td>-0.151**</td>
<td>-0.147**</td>
</tr>
<tr>
<td></td>
<td>(-16.00)</td>
<td>(-16.15)</td>
</tr>
<tr>
<td>Age</td>
<td>0.00246**</td>
<td>0.00303**</td>
</tr>
<tr>
<td></td>
<td>(5.91)</td>
<td>(6.92)</td>
</tr>
<tr>
<td>Age²</td>
<td>-0.00000104</td>
<td>-0.00000158</td>
</tr>
<tr>
<td></td>
<td>(-0.66)</td>
<td>(-1.00)</td>
</tr>
<tr>
<td>Constant</td>
<td>0.00616**</td>
<td>0.00152</td>
</tr>
<tr>
<td></td>
<td>(5.28)</td>
<td>(0.89)</td>
</tr>
<tr>
<td>Year FE</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Observations</td>
<td>853072</td>
<td>853072</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.0290</td>
<td>0.0455</td>
</tr>
</tbody>
</table>

Note: Dependent variable: Percentage change in Household Sentiment Index. t statistics in parentheses. ** p<0.01, * p<0.05, + p<0.1. All standard errors are corrected for heteroskedasticity and clustered around households.

**Merging household survey data with the corresponding state index**

From the PSID data, I can obtain the information on each household’s location at the state level. I then match the household’s estimation with the housing price index in the same state. While the last boom - bust episode in the real estate market was prevalent nation-wide, there was significant variation in the housing price series across states and cities.\(^5\)

\(^5\) This cross-sectional variation in past housing price series allow me to address a major challenge of distinguishing age, time and cohort effect in panel data as thoroughly discussed by John Ameriks and Stephen Zeldes (2004).
3 Repeat-Sales Index Methodology

The micro household survey data are from the Panel Studies of Income Dynamics. The survey asks the households the following questions: “Could you tell me what the present value of your (house/apartment) is? About how much would it bring if you sold it today?” I treat their answers as the households’ estimates of the value of their own properties. Their estimates also reflect their sentiments of the housing market in that state. To compare the household’s estimates with the actual housing price index, I apply the Case-Shiller weighted repeat-sales (WRS) estimation method to construct an index measuring households’ repeated evaluation of their own houses, called the Household Sentiment Index. Unlike a simple median price, the WRS can capture the experience of the homeowners. The median housing price in a neighborhood can rise if more expensive houses get built, even if the value of existing houses remains constant. The WRS method follows the same set of houses over many years, producing a measure of the actual financial returns for the owners. Another method to construct a price index is the hedonic approach. Case and Shiller (1987) argue that the hedonic approach requires information on a large quantities of houses with all the relevant features such as number of rooms, square feet of interior space, quality of construction, and so forth. The disadvantage of the WRS is that we have to exclude single “sales.” The WRS is also not optimal for small transaction volume. Figure 2 shows the frequency of house value observations in the PSID. Baroni et al. (2009) and Nagaraja et al. (2014) compare recent improvements on the WRS method. I provide in the Appendix the results from several approaches. Since some states have more households than another, I apply the same method to construct the state housing price index. This is to ensure that the two indices from the state-level housing price and the household estimates are comparable.

4 Empirical facts

4.0.1 Underestimate or Overestimate?

The households underestimated the index during the earlier periods but overestimated the index after mid-1990s. As shown in figure (1a), the index constructed from the household’s estimates
grows at a lower rate during the period prior to 1993, straying further away from the actual housing prices. However, during the late 1990s and the following decade, the household’s estimates grew at a faster rate to “overshoot” the actual housing prices.

The cyclical component of the Household Sentiment Index shows larger swings than the Housing Price Index. I apply the HP-filter with $\lambda = 100$ to both annual series to investigate their cyclical components. In response to the actual housing price index, the household’s estimates tend to overshoot, creating more volatility. The relative variability measured by the ratio of standard deviation over the mean of the two series are $\sigma/\mu_{HSI} = 13.29\%$ and $\sigma/\mu_{HPI} = 9.8\%$.

In order to further investigate the households’ responses to the changes in the state price, I perform a dynamic regression for each household:

$$y_{it} = \beta_{i,0} + \beta_{i,1} y_{i,t-1} + \beta_{i,2} x_{i,t-1} + u_{it}$$

where $y_{it}$ is the appreciation rate on individual household’s evaluation, $x_{i,t}$ is the appreciation rate on the state housing price index. $\beta_{i,0}$ will capture any fixed characteristics of the house. The distribution of $\beta_{i,2}$ shown in figure 4 suggests that household estimation is elastic to changes in the state index. 74% of households have coefficients greater than 1.
Figure 3: (a) The Housing Price Index and Household Sentiment Index series are constructed by applying the Case-Shiller repeat-sales estimation method to the survey responses of 3393 families from 51 states; (b) HP-filtered annual series of the HSI and the HPI with $\lambda = 100$; (c) The gap between the cyclical components of HSI and HPI; (d) Annual growth rate in each index series; (e) Gap in annual growth rates between HSI and HPI.
Figure 4: Elasticity of household evaluation to state index. The majority of the elasticities are greater than 1 suggesting that the household estimates often overshoot in response to changes in the housing market.

4.0.2 Leading or lagging?

Here I want to investigate whether the households’ estimates are leading or lagging indicators of the housing prices and potentially the business cycles. For each year, I calculate the correlation coefficients between the growth rate in each household’s estimate of their own property and the growth rate of the same housing price index in that state. Figure 5 shows the magnitude and significance of these correlation coefficients over the years. Sub-figure (a) shows the average correlation between the growth rate in each household’s estimate with the next period’s housing price growth rate ($\text{corr}(%\Delta PSID_{i,t}, %\Delta FHFA_{i,t+1})$). Sub-figure (b) shows the average correlation between the growth rate in each household’s estimate with the last period’s housing price growth rate ($\text{corr}(%\Delta PSID_{i,t}, %\Delta FHFA_{i,t-1})$). From 1975 to early 2000s, the growth rate in the households’ estimates is positively correlated with the lag of the growth rate in the housing price index. However, after the 2000s, the growth rate in the households’ estimates have consistently and negatively predicted the growth rate in housing prices.
Figure 5: (a) Correlation coefficients between the growth rate in each household’s estimate and next period’s housing price index in the same state (b) Correlation coefficients between the growth rate in each household’s estimate and last period’s housing price index in the same state. The red bars show coefficients significant at $\alpha = 5\%$. The t-value is calculated as $t = \frac{\rho}{\sqrt{1-\rho^2}/(n-2)}$.

5 Housing model

5.1 Basic Set-up

I am solving a simple central planner problem with two sectors. The final goods sector uses labor $n_1$ and capital $k$ to produce final goods used for consumption $c$ and investment $i$. The housing sector uses land and labor $n_2$ to produce housing and provide housing services $h$. While capital is sector-specific, labor can move between two sectors. The supply of land is fixed and the labor share of housing output is 1. Total labor supply is fixed and normalized to 1 so we have $n_1 + n_2 = \bar{N} = 1$. The social planner’s objective is to maximize the following utility function:

$$E_t \sum_{t=0}^{\infty} \beta^t \left\{ \log c_t + j_t \log h_t - \chi (n_1^{\gamma_1} + n_2^{\gamma_2})^{\frac{1}{\gamma}} \right\}$$

subject to

$$k_{t+1} = (1 - \delta_k)k_t + i_t$$

$$c_t + i_t = Z_t n_1^{\alpha} k_t^{1-\alpha}$$

$$Z_t = Z_0 (1 + g)^t e^{\tau}$$

$$h_{t+1} = (1 - \delta_h)h_t + n_2 = (1 - \delta_h)h_t + (\bar{N} - n_1t)$$
where \( \beta \) is the discount factor, \( \chi \) is the weight of the dis-utility of labor and \( \gamma \) measures the preference for specialization between two types of labor, \( g \) is the constant growth rate of the total factor productivity \( Z \). As mentioned in Iacoviello and Neri (2010) and Horvath (2000), in order for the social planner to have a preference for differentiating labor across the two sectors, the value of \( \gamma \) has to be greater than 1. If \( \gamma \) is smaller than 1, the dis-utility of labor is minimized if all labor is allocated to the sector that pays higher wage. If \( \gamma \) is larger than 1, the social planner minimizes this dis-utility by allocating labor equally between two sectors. There are two shock processes to the model, total factor productivity shock \( z \) and housing preference shock \( j \). Since the production function in the housing sector does not include a productivity parameter, the productivity shock measures the relative productivity between the two sectors. I assume the productivity shock is always stationary and focus on the uncertainty about the housing preference shock. The housing preference shock \( j \) measures how much the social planner weighs the utility from housing services compared to consumption goods. The social planner does not know if the housing preference shock is stationary or non-stationary. In other words, the exogenous change in the housing preference can be either temporary or permanent. The two shocks follow AR(1) processes\(^6\). Both the error terms are i.i.d. \( \sim N(0,\sigma^2) \). We have the following two scenarios: the stationary and non-stationary case.

The stationary case

\[
z_t = \rho_z z_{t-1} + \epsilon_{1t} \tag{6}
\]

\[
j_t = \rho_j j_{t-1} + \epsilon_{2t} \tag{7}
\]

Non-Stationary case

\[
z_t = \rho_z z_{t-1} + \epsilon_{1t} \tag{8}
\]

\[
\Delta j_t = \rho_j \Delta j_{t-1} + \epsilon_{2t} \tag{9}
\]

\( ^6 \)I present in the paper the AR(1) process for the benefit of clarity. The shocks in the model can be easily extended to AR(p) processes.
State-space representation

We now write the model in a state space representation. The state variables in this model are the capital stock, the housing stock, the level of housing preference and productivity. The control variables are labor supply in the final goods sector $n_{1t}$ and investment $i_t$.

By substituting the constraint into the utility function, we have

$$E_t \sum_{t=0}^{\infty} \beta^t \left\{ \log \left[ Z_t n_{1t}^{\alpha} k_{1t}^{1-\alpha} - i_t \right] + j_t \log h_t - \chi (n_{1t}^\gamma + (\bar{N} - n_{1t})^\gamma)^{\frac{1}{\gamma}} \right\}$$ (10)

In order to use the optimal linear regulator solution method for the model, I approximate the utility function by a linear quadratic equation. The derivation is in Appendix A4.

$$\hat{U} = x'Rx + u'Qu + 2u'W'x$$ (11)

where $\hat{U}$ is the approximated utility function, $x$ is the matrix of state variables, and $u$ is the vector of control variables.

The law of motions in the model can be written as:

$$x_{t+1} = A \cdot x_t + B \cdot u_t + C \cdot \epsilon_{t+1}$$ (12)

The matrices for each the stationary and non-stationary case are presented in Appendix A3.

Stacking

Once I have specified all the matrices for both cases, stationary and non-stationary, we have a quadratic return function and a linear law of motion. I can then cast the two models as one optimal linear regulator problem. Since the social planner does not know which shock process is the true model, he or she will put some weight on each possibility. The agent treats its probability weights as constants when calculating the optimal policy. This is what Kreps (1998) calls an “anticipated utility” model. Sargent (2005) and Tortorice (2012) employ the same approach, which allows a tractable solution to the problem.

The weight for the stationary model is $w_t$. This weight will be updated every period using
Bayesian updating algorithm discussed in section 5 within the learning process. From here, I can solve this optimal linear regulator problem and obtain the linear policy function $F$ such that

$$u_t = -F \cdot x_t$$

For the law of motion, we need the vector of state variables $x_t$ and the matrix $A_t$, B and C.

$$x_t = \begin{bmatrix} x_t^s' \\ x_t^{ns'} \end{bmatrix}$$

$$A_t = \begin{bmatrix} A_t^s & 0 \\ 0 & A_t^{ns} \end{bmatrix} \quad B = \begin{bmatrix} B^s \\ B^{ns} \end{bmatrix} \quad C = \begin{bmatrix} C^s \\ C^{ns} \end{bmatrix}$$

For the return function, we now have $\hat{U}_t = w_t \cdot \hat{U}_s + (1 - w_t) \cdot \hat{U}_{ns}$ which translates into the new $R_t$ and $W_t$ matrices below.

$$R_t = \begin{bmatrix} w_t \cdot R^s & 0 \\ 0 & (1 - w_t) \cdot R^{ns} \end{bmatrix} \quad W_t = \begin{bmatrix} w_t \cdot W^s \\ (1 - w_t) \cdot W^{ns} \end{bmatrix}$$

Setting up the housing model as a social planner problem allows the convenient stacking of the two sub-models and the optimal linear regulator solution method. The model can also take the form of a decentralized problem consisting of four markets: the market for consumption goods, the market for housing services, the market for rental services of capital and the market for labor.

A competitive equilibrium is a set of quantities: consumption goods $c$, labor in final goods sector $n_1$, labor in housing sector $n_2$, housing services $h$ and capital $k$, and prices: wages in final goods sector $w_1$, wages in housing sector $w_2$, housing prices $q$ and rental rate of capital $r$ which satisfy the following properties:

1. The representative household chooses $c$, $h$, $n_1$ and $n_2$ given wage rates $w_1$ and $w_2$ in each sector, housing prices $q$ and rental rate $r$.

2. The representative firm in the final goods sector chooses $n_1$ and $k$ optimally given $w_1$ and $r$.

3. The representative firm in the housing sector chooses $n_2$ and $l$ optimally given $w_2$ and $q$.

When layering the learning model on top of the housing model, the source of uncertainty needs some clarification. In the social planner problem, the social planner is uncertain about (1) whether the housing preference shock is stationary or non-stationary and (2) the autocorrelation coefficients of both the two shocks. In a decentralized problem, the household is uncertain about the persistence and permanence of the shocks. It is the household that undergoes the learning process. The firms in both sectors only respond to the demand for consumption goods and housing services by the households.

5.2 Housing prices

In this model, there is not an explicit variable for housing price. Instead, housing price $q$ is the relative cost of producing a unit of housing over a unit of consumption goods. The cost of producing a unit of each good is the product of the amount of labor needed per unit and the wage in that sector. The wage in each sector is the dis-utility of the social planner in supplying labor in that sector. First, we calculate the marginal costs in the final goods sector in equation (13) and the housing sector in equation (14).

\[
MC_y = \frac{d \text{ Cost}}{d \text{ Final goods}} = \text{wage}_y \cdot \frac{dn_1}{dy} = \frac{\partial u}{\partial n_1} \cdot \frac{dn_1}{dy} = \frac{\partial u}{\partial n_1} \cdot \alpha Z n_1^{\alpha-1} k^{1-\alpha} \tag{13}
\]

\[
MC_h = \frac{d \text{ Cost}}{d \text{ Housing}} = \text{wage}_h \cdot \frac{dn_2}{dh} = \frac{\partial u}{\partial n_2} \cdot \frac{dn_2}{dh} = \frac{\partial u}{\partial n_2} \tag{14}
\]

The housing price in each period would be the relative marginal cost in the housing sector over the marginal cost in the final goods sector as shown in equation (15).

\[
q = \frac{MC_h}{MC_y} = \frac{\frac{\partial n_2}{\partial n_1}}{\frac{\partial n_2}{\partial n_1}} \cdot Z \alpha n_1^{\alpha-1} k^{1-\alpha} = \left(\frac{n_2}{n_1}\right)^{\gamma-1} \alpha Z n_1^{\alpha-1} k_t^{1-\alpha} \tag{15}
\]

**Proposition 1a:** For $\gamma < 1$, $\frac{\partial q}{\partial j} < 0$. For $\gamma > 1$, $\frac{\partial q}{\partial j} > 0$.

A positive shock to the housing preference will lead to an increase in the production of housing and a higher demand for labor in the housing sector. This will reduce the marginal product of
labor in the housing sector, or in other words, increase the cost of producing an additional unit of housing and the housing price. At the same time, depending on how strong the social planner prefers to diversify labor, the net impact of a housing preference shock can either be positive or negative.

**Proposition 1b:** For \( \gamma < 1 \), \( \frac{\partial q}{\partial z} > 0 \). For \( \gamma > 1 \), \( \frac{\partial q}{\partial z} < 0 \).

A positive productivity shock in the final goods sector affects the housing prices in two ways. It will directly increase the opportunity cost of producing one unit of housing and therefore, increase housing price. At the same time, labor will relocate from the housing sector to the final goods sector, which decreases the marginal product of labor in the final goods sector and decreases the housing price. Depending on how strong the social planner prefers to diversify labor, the net impact of a productivity shock can either be positive or negative.

In this paper, given the parameter value \( \gamma = 1.2 \), the housing preference shock increases housing price while the productivity shock in the final goods sector has the opposite effect. The impulse responses to these two shocks are presented in Appendix D.

6 Learning model

The social planner learns about both the AR(1) coefficients of the shocks and the weights of the two sub-models: stationary and non-stationary.

6.1 Updating the parameters

We can write the shock processes as

\[
\Xi_t = \theta_t \cdot \Xi_{t-1} + \epsilon_t
\]  

(16)

where \( \Xi_t \) is either \( z_t \), \( j_t \), or \( \Delta j_t \). Here \( \theta_t \) is the AR(1) coefficient for each of these series. The social planner updates this coefficient every period.
The notation and assumptions for updating the parameters are based on Cogley and Sargent (2005). The social planner has a normal-inverse gamma prior.

\[ p(\theta, \sigma^2) = p(\theta | \sigma^2)p(\sigma^2) \]  

(17)

where \( \sigma^2 \) is the variance of the residual of equation (16). The marginal prior \( p(\sigma^2) \) makes the error variance an inverse gamma variate, and the conditional prior \( p(\theta | \sigma^2) \) makes the regression parameters a normal random vector. As proven in Cogley and Sargent (2005), these assumptions make the conditional likelihood function normal. With a normal-inverse gamma prior and a Gaussian conditional likelihood, the posterior also belongs to the normal-inverse gamma family, and its parameters can be updated recursively. The assumptions about the distributions of the prior and posterior beliefs allow the learning model to be tractable with an analytical solution.

Let \( Z^t \) summarize the history of these two shocks up to date \( t \). Before seeing the data at \( t \), the social planner’s prior is:

\[ p(\theta | \sigma^2, Z^{t-1}) = N(\theta_{t-1}, \sigma^2 P_{t-1}^{-1}) \]  

(18)

\[ p(\sigma^2 | Z^{t-1}) = IG(s_{t-1}, v_{t-1}) \]  

(19)

The last period’s variable are the precision matrix \( P_{t-1} \), a scale parameter for the inverse-gamma density \( s_{t-1} \), and the degree of freedoms \( v_{t-1} \). The estimate of \( \sigma^2 \) is \( s_{t-1}/v_{t-1} \). The larger the precision matrix, the smaller the variance of the distribution from which the social planner draws the new beliefs. This will slow down the updating process.

After seeing outcomes at time \( t \), the social planner updates their beliefs. Due to the assumptions about the priors, the updating process is similar to the method of recursive ordinary least squares.

\[ p(\theta | \sigma^2, Z^t) = N(\theta_t, \sigma^2 P_t^{-1}) \]  

(20)

\[ p(\sigma^2 | Z^t) = IG(s_t, v_t) \]  

(21)
where:

\[ P_t = P_{t-1} + \Xi_{t-1} \Xi'_{t-1} \]  
(22)

\[ \theta_t = P_t^{-1}(P_{t-1}\theta_{t-1} + \Xi_{t-1}\Xi'_{t}) \]  
(23)

\[ s_t = s_{t-1} + \Xi'_{t}\Xi_{t} + \theta'_{t-1}P_{t-1}\theta_{t-1} - \theta'_{t}P_{t}\theta_{t} \]  
(24)

\[ v_t = v_{t-1} + 1 \]  
(25)

### 6.2 Updating the weight

The social planner updates the weight of each model by evaluating their relative likelihood functions. The un-normalized weight of each model evolves according to the Bayesian rule. The subscript \( i \) represents each model. After some derivations, the evolution of the weight is as follows:

\[
\ln w_{i,t+1} = \ln w_{i,t} + \ln p(\Xi_{i,t+1} | \Xi_{i,t}, \theta_{i}, \sigma^2_{i}) - \ln \frac{p(\theta_{i}, \sigma^2_{i}|Z_{i}^{t+1})}{p(\theta_{i}, \sigma^2_{i}|Z_{i}^{t})} 
\]  
(26)

where \( \ln p(\Xi_{i,t+1} | \Xi_{i,t}, \theta_{i}, \sigma^2_{i}) \) is the conditional log-likelihood for observation \( t + 1 \) assuming normal distribution and \( \ln \frac{p(\theta_{i}, \sigma^2_{i}|Z_{i}^{t+1})}{p(\theta_{i}, \sigma^2_{i}|Z_{i}^{t})} \) is the change in the log posterior that results from a new observation. Detailed expressions are in Appendix F.

The normalized weight of the stationary model would be

\[
w_t = \frac{w_{t,s}}{w_{t,ns} + w_{t,s}}
\]  
(27)

where \( w_{t,s} \) is the weight of the stationary case and \( w_{t,ns} \) is the weight of the non-stationary in each period.

### 6.3 Construction of the Sentiment Index

Household Sentiment in the model is approximated by a one-period forecast. For a multi-period forecast, I substitute the result from equation (29) for the next period’s control variables and then keep iterating equation (28) forward keeping the same matrices \( A, B \) and \( F \). After calculating the control variables on investment and labor supply, I obtain the forecast of housing prices from the
baseline model and the learning mechanism. The results are in Figure 8 and 9.

\[ E_t x_{t+1} = A_t \cdot x_t + B_t \cdot u_t + C \cdot E_t e_{t+1} = A_t \cdot x_t + B_t \cdot u_t \]  

(28)

\[ E_t u_{t+1} = -F_t \cdot E_t x_{t+1} \]  

(29)

7 Parameters Calibration and Simulation Results

7.1 Parameters Calibration

Choices for some of the parameters are based on Iacoviello and Neri (2010). In their paper, they set \( \beta = 0.9925 \) implying a steady-state real interest rate of 3 percent on an annual basis. Harding, Rosenthal and Sirmans (2007) use the repeat-sales model to estimate the depreciation rate of housing capital controlling for maintenance. Their estimates show that housing depreciates at roughly 2.5% per year over the 1983 to 2001 period. Iacoviello and Neri (2010) also set the depreciation rate for capital in the consumption goods sector to also be 2.5% per year.

From the Bureau of Labor Statistics data on labor allocation in the U.S. in 2000\(^7\), construction makes up 4.6% and manufacturing makes up 11.8% of the total labor force. If the economy only consisted of final goods manufacturing and housing, the share of labor on manufacturing \( n_1 \) would be 72%. Jointly, the parameter \( \chi \) and \( \gamma \) are calibrated to match the steady state share of the labor in manufacturing. In this model, given the parameters, the steady state values are:

\[ [k_{ss} = 13.88, h_{ss} = 59.38, j_{ss} = 0, z_{ss} = 0, n_{ss} = 0.72, i_{ss} = 0.0867] \]

\(^7\)http://www.bls.gov/emp/ep_table_201.htm
Table 3: Quarterly parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>Discount factor</td>
<td>0.9925</td>
</tr>
<tr>
<td>$\delta_c$</td>
<td>Depreciation rate for final goods</td>
<td>0.00625</td>
</tr>
<tr>
<td>$\delta_h$</td>
<td>Depreciation rate for housing</td>
<td>0.00625</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Output share of labor in final goods sector</td>
<td>2/3</td>
</tr>
<tr>
<td>$\bar{N}$</td>
<td>Normalized total labor supply</td>
<td>1</td>
</tr>
<tr>
<td>$\chi$</td>
<td>Weight on dis-utility from labor</td>
<td>0.72</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Preference for diversity of labor</td>
<td>1.2</td>
</tr>
</tbody>
</table>

There are two main parameters that determine the shock, the auto-regressive coefficients and the standard deviation of its distribution. These parameters are calibrated so that the model can match the standard deviation and auto-correlation of the housing stock series\(^8\).

\[
z_t = 0.8 \cdot z_{t-1} + \epsilon_{1t} \tag{30}
\]

Stationary case:

\[
\dot{j}_t = 0.8 \cdot \dot{j}_{t-1} + \epsilon_{2t} \tag{31}
\]

Non-stationary case:

\[
\Delta \dot{j}_t = 0.29 \cdot \Delta \dot{j}_{t-1} + \epsilon_{2t} \tag{32}
\]

where $\epsilon_1$ and $\epsilon_2$ come from $N(0, 0.0312)$.

Table 2 presents the moments of the cyclical component of the housing stock series from the data, the baseline model with Rational Expectation and the model with adaptive learning. The parameters allow both the two models to match the standard deviation of the housing stock. In both the baseline model and the adaptive learning case, the simulated housing stock is relatively more persistent for the first quarter but quickly reverts to the level of the data after four lags. In the data, the housing stock series fluctuate much less than the housing price series, suggesting that price movements are mainly driven by the changes in demand. In parallel, the housing stock series show the same characteristics under both the baseline Rational Expectation model and the

\(^8\)I use the identity matrix as the weighting matrix between the parameters and the moments. Four parameters are \(\{\theta_j, \theta_{jns}, \theta_z, \sigma_j\}\) and the moments are \(\{\sigma_h, \ Corr(h_t, h_{t-1}), \ Corr(h_t, h_{t-4})\}\) for the two cases: with and without learning.
learning model.

Table 4: Moments of Housing Stock

<table>
<thead>
<tr>
<th>Parameters Data</th>
<th>Rational Expectation</th>
<th>Learning model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard deviation</td>
<td>0.0052</td>
<td>0.0051</td>
</tr>
<tr>
<td>$\text{Corr}(h_t, h_{t-1})$</td>
<td>0.6215</td>
<td>0.8737</td>
</tr>
<tr>
<td>$\text{Corr}(h_t, h_{t-4})$</td>
<td>0.2732</td>
<td>0.2235</td>
</tr>
</tbody>
</table>

7.2 Prior beliefs

The prior beliefs are as followings. I assume the initial weight of the stationary model $w_0$ is 0.5 so that the social planner is indifferent between the two models. The precision matrix is $P_0 = .2$. For a larger precision matrix, the estimated parameters of the preference processes will change slowly. This will slow down the learning process about the parameters. The process of updating the weight is not significantly affected. As a robustness check, I simulate the model using consecutive values for the precision matrix between 0 and 1. The relative volatility in housing prices under rational expectation and uncertainty does not significantly depend on the precision matrix. The initial values for the AR(1) coefficient are $\theta = [0.5, 0.5, 0.5, 0.5]$.

7.3 Results on housing price series

First, I investigate the long-run behavior of the learning model. For this simulation, I assume the true data generating process for the housing preference shock is stationary. However, the social planner considers both stationary and non-stationary cases, each with the probability of 0.5. Once the social planner figures out the true model then the weight for the non-stationary case reaches 0 and remains at that value. The true coefficient in the AR(1) shock process is 0.8 while the social planner has two priors: 0.5 and 1.29. Since 0.8 is sufficiently far from 1, the social planner figures out the weight after 35 periods. After the weight reaches its true value, the social planner now thinks process is stationary and both the two coefficients should go back to 0.8. Since the non-stationary case is written in first-difference, the prior coefficient 0.29 should go to its true value of approximately -0.09. The paths for the weight and the coefficients are shown in Figure 6.
Figure 6: This figure shows the long-run behavior of one simulation. The true model is stationary with the AR(1) coefficient of 0.8. The learning model quickly identifies the weight of the true model after 35 periods. However, the coefficients take a much longer time to reach their true values. The stationary AR(1) coefficient in the learning model takes more than 500 periods to reach the value of 0.8 and the non-stationary AR(1) coefficient takes more than 1000 periods to reach the value of approximately -0.09.

Next, I simulate 1000 housing price series for 200 periods using the calibrated parameters. The true model is the stationary case while the social planner continues updating his or her beliefs about the probability of equation (31) and equation (32). I apply the HP filter with $\lambda = 1600$ to all three housing price series in logarithms. Table 5 reports the key moments of the three series.

<table>
<thead>
<tr>
<th></th>
<th>Housing Price Data</th>
<th>Rational Expectation</th>
<th>Learning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard deviation</td>
<td>0.0219</td>
<td>0.0146 (0.0021)</td>
<td>0.0297$^\dagger$ (0.0048)</td>
</tr>
<tr>
<td>(95% Confidence Interval)</td>
<td>(0.0105; 0.0187)</td>
<td>(0.0203; 0.0391)</td>
<td></td>
</tr>
<tr>
<td>Moving volatility (4 quarters)</td>
<td>0.0111</td>
<td>0.0105$^\dagger$ (0.0011)</td>
<td>0.0229$^*$ (0.0035)</td>
</tr>
<tr>
<td>Moving volatility (20 quarters)</td>
<td>0.0197</td>
<td>0.0143 (0.002)</td>
<td>0.0292$^*$ (0.0047)</td>
</tr>
<tr>
<td>Moving volatility (40 quarters)</td>
<td>0.0206</td>
<td>0.0145 (0.002)</td>
<td>0.0296$^\dagger$ (0.0048)</td>
</tr>
<tr>
<td>$\text{Corr}(q_t, q_{t-4})$</td>
<td>0.9058</td>
<td>0.8077</td>
<td>0.7541</td>
</tr>
<tr>
<td>$\text{Corr}(q_t, q_{t-4})$</td>
<td>0.5560</td>
<td>0.2186</td>
<td>0.0953</td>
</tr>
</tbody>
</table>

Note: $^*$ denotes 95% significant difference between the baseline model and the adaptive learning model. $^\dagger$ signifies that the moment of the data falls within the 95% confidence interval.
The volatility of the price series under the learning model particularly matches for the long-term volatility in the data. In terms of persistence, both the baseline and the learning model cannot capture the persistence measure in the data with AR(1) shock processes. There is a trade-off between adding more lags to capture the persistence and having more calibrated parameters.

7.4 Dynamics between the housing price index and the sentiment index

From the model simulations, I calculate the percentage deviation of the forecast from the prices series called the “gap.” Figure 8 shows one draw from the simulations. As illustrated in Figure 8, the relationship between the housing prices and the forecasts under the learning model resembles the dynamics between the HSI and HPI. Given the same housing preference shock, the price series under the learning model tends to overshoot and fluctuate more. The forecast or sentiment index therefore shows periods of persistent under-estimation and over-estimation. The bottom plot in Figure 8 shows the magnitude and Figure 9 shows the distributions of this “gap” between the sentiment and the price series under the baseline Rational Expectation model and the Learning model. The learning model is able to produce much larger gaps between the price series and the forecasts. Table 6 reaffirms that the gaps are wider and more persistent under the learning model than under the Rational Expectation case.
Figure 8: From one draw of the simulation, the forecast series from the learning model shows periods of persistent under-estimation and over-estimation. In this simulation, the productivity in the final goods sector is growing at 1% per period.

Table 6: The “gap” : the percentage deviation of the forecast from the price series

<table>
<thead>
<tr>
<th></th>
<th>Rational Expectation</th>
<th>Learning model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard deviation</td>
<td>0.6438</td>
<td>2.0941</td>
</tr>
<tr>
<td>$Corr(gap_t, gap_{t-1})$</td>
<td>0.1155</td>
<td>0.5828</td>
</tr>
<tr>
<td>$Corr(gap_t, gap_{t-4})$</td>
<td>0.0880</td>
<td>0.4466</td>
</tr>
</tbody>
</table>

Figure 9: The distribution of percentage difference between the forecast and the price under the learning model is significantly wider than under the Rational Expectation mode.
8 Conclusion

This paper uses the Panel Studies of Income Dynamics to investigate the potential deviation of the household's expectation from the actual housing price. The data show that household’s estimates lagged behind the price index, exhibited slower growth trend and more volatility. The cyclical component of the Household Sentiment Index fluctuated more compared to the housing price index. A hypothesis for this behavior is that households are uncertain if the information they receive from the market indicates a temporary or permanent increase in housing prices. Every period, they obtain more information from the market and update their beliefs about the probability of each scenario. This uncertainty causes more fluctuations in the household’s expectation of housing prices. It can also cause more volatility in consumption, labor supply and investment in the business cycles.

To test this hypothesis, the paper incorporates a learning mechanism based on Cogley and Sargent (2005) into a two-sector model with housing. The social planner in the model is uncertain whether a recent housing preference shock is temporary (stationary) or permanent (non-stationary). The parameters of the model are calibrated to match the moments of the housing stock series. Compare to the baseline model, housing prices under the learning mechanism are significantly more volatile, matching the long-term volatility of the housing price series in the data. The learning model is tractable and easily scaled up to incorporate more lags and more possibilities. Then from both the baseline model and the learning model, I construct a one-period forecast as the equivalent measure of the Household Sentiment Index. In the adaptive learning case, the forecasts fluctuate more and show periods of under-estimation and then over-estimation compared to the housing price series.

This experiment is particularly relevant for policy makers because many other households decisions such as labor supply, consumption and investment also depend on their expectations of the housing market. Empirical papers have estimated the effect of an increase in housing prices on consumption (Campbell and Cocco, 2007), investment (Chaney, Sraer and Thesmar, 2012), or investment in the education of children and their future income (Cooper and Luengo-Prado, 2011). There are two main channels for this effect: higher housing prices increase life-time wealth and
loosen the borrowing constraint. However, it is difficult to empirically estimate the effect of an expected future increase in housing prices on contemporaneous decisions. The learning model in this paper shows that with uncertainty, their expectations of housing prices are even more volatile and estimated with persistent errors. This finding would provide very different implications for policies aiming at reviving the housing market or preventing another housing bubble.
References


A Data

- **House Price Index**: All-Transactions Indexes Estimated using Sales Prices and Appraisal Data, (quarterly, 1975 - 2013).

- **Household Survey**: Panel Studies of Income Dynamics.
  - Link: http://simba.isr.umich.edu/VS/i.aspx

- **Productivity**: Total Factor Productivity, (quarterly, 1947 - 2013 Q2).
- Source: Federal Reserve Bank of San Francisco.
- Link: http://www.frbsf.org/economic-research/total-factor-productivity-tfp/


  - Link: http://www.census.gov/housing/hvs/data/histtabs.html (Table 8)
  - The CPS/HVS is administered by the Census Bureau using a probability selected sample of about 72,000 housing units, both occupied and vacant. The fieldwork is conducted during the calendar week that includes the 19th of the month. The questions refer to activities during the prior week; that is, the week that includes the 12th of the month. Households from all 50 states and the District of Columbia are in the survey for 4 consecutive months, out for 8, and then return for another 4 months before leaving the sample permanently.

B **Case-Shiller repeat-sales index method**

Case and Shiller (1987) improved on the Bailey, Muth, and Nourse method by allowing the variance of the error term to differ across houses. They assume the log price of house $i$ at time $t$ to follow this process:

$$P_{it} = C_t + H_{it} + N_{it}$$  \hspace{1cm} (33)$$

where

- $C_t$ is the log of the city-wide level of housing prices at time $t$
- $H_{it}$ is the Gaussian random walk that is uncorrelated with $C_t$ and $H_{jt}$ ($i \neq j$) for all $t$
- $N_{it}$ is the sale specific random error that has zero mean and variance $\sigma^2_N$ for all $i$ and serially uncorrelated.

The three stages to construct the Case-Shiller index are:
Stage 1: Regress the log differences in price on the dummy $D_t$ and calculate the squared residuals.

Stage 2: Run a weighted regression of the squared residual on the time interval between sales $t$ and $t'$. The constant term in this regression is an estimate of $2\sigma_N^2$, and the slope term is the estimate of $\sigma_h^2$.

$$\hat{\varepsilon}^2_{itt'} = \beta_0 + \beta_1 (t' - t) + \eta_{itt'}$$

Stage 3: Run a GLS (weighted) regression by first dividing each observation in the first stage by the square root of the fitted value in the stage two regression and repeat the stage one regression.

C Transition functions

Below are the matrices for the law of motion. For the stationary case, we have:

$$\begin{bmatrix} 1 \\ k_{t+1} \\ h_{t+1} \\ z_{t+1} \\ j_{t+1} \\ z_t \\ j_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1 - \delta_k) & 0 & 0 & 0 & 0 & 0 \\ \bar{N} & 0 & (1 - \delta_h) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_{z,t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_{j,t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ k_t \\ h_t \\ z_t \\ j_t \\ z_{t-1} \\ j_{t-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} n_{1t} \\ i_t \end{bmatrix} + \begin{bmatrix} \epsilon_{1t+1} \\ \epsilon_{2t+1} \end{bmatrix}$$

(34)
Meanwhile, for the non-stationary case, we have:

\[
\begin{bmatrix}
1 \\
k_{t+1} \\
h_{t+1} \\
z_{t+1} \\
j_{t+1} \\
z_t \\
\Delta j_{t+1}
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (1 - \delta_k) & 0 & 0 & 0 & 0 & 0 \\
\bar{N} & 0 & (1 - \delta_h) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \rho_{z,t} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \rho_{jns,t} & 0
\end{bmatrix}
\begin{bmatrix}
k_t \\
h_t \\
z_t \\
j_t \\
z_{t-1} \\
\Delta j_t
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
n_{1t} \\
i_t \\
e_{1t+1} \\
e_{2t+1}
\end{bmatrix}
\]

(35)
D  Impulse responses in the baseline model

Figure 10: Impulse responses to productivity shock

Figure 11: Impulse responses to housing preference shock
E Linear quadratic approximation

After I calculate all the steady state values, I can evaluate the Hessian and Jacobian matrices at the steady states. I approximate the utility function with the first two terms of a Taylor series.

\[
\hat{U}(\omega) = U(\bar{\omega}) + (\omega - \bar{\omega})' \frac{\partial U(\omega)}{\partial \omega} + \frac{1}{2} (\omega - \bar{\omega})' \frac{\partial^2 U(\omega)}{\partial \omega \partial \omega'} (\omega - \bar{\omega})
\] (36)

where \(\omega\) is an \((n + k)\) vector. In this case: \(\omega_t = \begin{bmatrix} k_t & h_t & z_t & j_t & n_{1t} & i_t \end{bmatrix}^T\). After obtaining the steady state vector \(\bar{\omega}\) and evaluate the Jacobian and Hessain matrices at the steady states, we can compute matrix \(M\) such that

We can express equation 36 as:

\[
\hat{U}(\omega) = \omega' M \omega
\]

Note that at this point I add a constant 1 to the \(\omega\) vector. Now \(\omega_t = \begin{bmatrix} 1 & k_t & h_t & z_t & j_t & n_{1t} & i_t \end{bmatrix}^T\).

where

\[
M = e \left[ U(\bar{\omega}) - (\frac{\partial U(\bar{\omega})}{\partial \omega})' \bar{\omega} + \frac{1}{2} \bar{\omega}' \frac{\partial^2 U(\bar{\omega})}{\partial \omega \partial \omega'} \bar{\omega} \right]' + \frac{1}{2} \frac{\partial^2 U(\bar{\omega})}{\partial \omega \partial \omega'}
\]

Partition \(M\), we have

\[
\omega' M \omega = \begin{pmatrix} x & u \end{pmatrix}' \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}
\]

\[
= \begin{pmatrix} x & u \end{pmatrix}' \begin{pmatrix} R & W \\ W' & Q \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}
\]

\[
= x'Rx + u'Qu + 2u'W'x
\]
E.1 Linear quadratic approximation figures

Figure 12: Utility functions approximated around steady states

\[ \ln w_{i,t+1} = \ln w_{it} + \ln p(\Xi_{i,t+1} | \Xi_{i,t}, \theta_i, \sigma_i^2) - \ln \frac{p(\theta_i, \sigma_i^2 | Z_{i}^{t+1})}{p(\theta_i, \sigma_i^2 | Z_{i}^{t})} \]  \hspace{1cm} (37)

where \( \ln p(\Xi_{i,t+1} | \Xi_{i,t}, \theta_i, \sigma_i^2) \) is the conditional log-likelihood for observation \( t + 1 \) assuming normal distribution; \( \ln \frac{p(\theta_i, \sigma_i^2 | Z_{i}^{t+1})}{p(\theta_i, \sigma_i^2 | Z_{i}^{t})} \) is the change in log posterior that results from a new observation.

\[ p(\Xi_{i,t+1} | \Xi_{i,t}, \theta_i, \sigma_i^2) = \frac{1}{\sqrt{2\pi} \sigma_i} \exp \left[ -\frac{1}{2} \frac{(\Xi_{it} - \Xi_{i,t-1} \theta_i)^2}{\sigma_i^2} \right] \]  \hspace{1cm} (38)

The posterior probability density function is

\[ p(\theta_i, \sigma_i^2 | Z_i^t) = C_{N_g}^{-1} \sigma_i^{-2(v_{it} + b_i + 2)} \exp \left[ -\frac{s_{it} + (\theta_i - \hat{\theta}_i)^' \Xi_{it-1} \Xi_{it-1}' (\theta_i - \hat{\theta}_i)}{2\sigma_i^2} \right] \]  \hspace{1cm} (39)

F Updating model weight

The unnormalized weight of each model evolves according to the Bayesian rule. The subscript \( i \) represents each model. After some derivations, the evolution of the weight is as follows:

\[ \ln w_{i,t+1} = \ln w_{it} + \ln p(\Xi_{i,t+1} | \Xi_{i,t}, \theta_i, \sigma_i^2) - \ln \frac{p(\theta_i, \sigma_i^2 | Z_{i}^{t+1})}{p(\theta_i, \sigma_i^2 | Z_{i}^{t})} \]  \hspace{1cm} (37)

where \( \ln p(\Xi_{i,t+1} | \Xi_{i,t}, \theta_i, \sigma_i^2) \) is the conditional log-likelihood for observation \( t + 1 \) assuming normal distribution; \( \ln \frac{p(\theta_i, \sigma_i^2 | Z_{i}^{t+1})}{p(\theta_i, \sigma_i^2 | Z_{i}^{t})} \) is the change in log posterior that results from a new observation.

\[ p(\Xi_{i,t+1} | \Xi_{i,t}, \theta_i, \sigma_i^2) = \frac{1}{\sqrt{2\pi} \sigma_i} \exp \left[ -\frac{1}{2} \frac{(\Xi_{it} - \Xi_{i,t-1} \theta_i)^2}{\sigma_i^2} \right] \]  \hspace{1cm} (38)

The posterior probability density function is

\[ p(\theta_i, \sigma_i^2 | Z_i^t) = C_{N_g}^{-1} \sigma_i^{-2(v_{it} + b_i + 2)} \exp \left[ -\frac{s_{it} + (\theta_i - \hat{\theta}_i)^' \Xi_{it-1} \Xi_{it-1}' (\theta_i - \hat{\theta}_i)}{2\sigma_i^2} \right] \]  \hspace{1cm} (39)

and
\[ C_{Ng} = \Gamma \left( \frac{v_{it}}{2} \right) \left( \frac{2}{s_{it}} \right)^{v_{it}/2} (2\pi)^{k_i/2} |\Xi_{it-1}^\prime \Xi_{it-1}|^{-1/2} \]  

(40)

The normalized weight of the stationary model would be

\[ w_t = \frac{w_{t,s}}{w_{t,ns} + w_{t,s}} \]  

(41)