Optimal Signals in Bayesian Persuasion Mechanisms*

Maxim Ivanov†
McMaster University

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Abstract

This paper offers a novel approach to solving Bayesian persuasion games and mechanisms which is based on the majorization theory and positive-dependence stochastic orders. We consider setups in which actions of the receiver (the mechanism) depend on the signal realization and the posterior mean of an increasing utility function of the state. In these setups, the sender’s ex-ante payoff is represented as an average of an ordered sequence of (possibly non-convex) functions of posterior means. For such mechanisms, we provide necessary and sufficient conditions of the optimality of monotone partitional signal structures. Our characterization results provide solutions to a variety of Bayesian persuasion applications: optimal selling mechanisms, information provision by the monopolist, and lobbying.

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1 Introduction

The main focus of this paper is to develop a tractable approach to derive optimal signal structures, i.e., joint distributions of the prior information (the state) and the observed information (the signal), in Bayesian persuasion games. We consider a model of Bayesian persuasion by Kamenica and Gentzkow (2011) with two players, an agent (the sender) and a decision maker’s (the receiver) whose preferences depend on an unknown state and the receiver’s decision. The players’ information about the state is initially limited

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†Department of Economics, McMaster University, 1280 Main Street Hamilton, ON, Canada L8S 4M4. E-mail: mivanov@mcmaster.ca. Phone: (905) 525-9140 x24532. Fax: (905) 521-8232.
by a common prior distribution. At the beginning of the game, the sender selects a signal structure. Then, the receiver observes the sender’s choice and a realization of the signal generated by the signal structure. Given this information, the receiver updates his posterior belief about the state by the Bayes rule and makes a decision. In this setup, the problem of the sender is to select the optimal signal structure, i.e., the one that maximizes her ex-ante payoff.

Kamenica and Gentzkow (2011) offer a general approach which treats the sender’s problem as the problem of the constrained optimal decomposition of the prior distribution into a distribution over posterior beliefs. The optimal decomposition must provide the ex-ante payoff to the sender on the lowest concave envelope of the sender’s interim payoff (that is, the payoff conditional on the receiver’s posterior belief). However, there are two main difficulties with using this approach for finding the optimal distribution of posterior beliefs in Bayesian games. First, the space of distributions of posterior beliefs is not a simple object, especially if the prior distribution is continuous. In particular, it is a set of probability distributions over probability distributions over the state (posterior beliefs), which are interdependent due to the Bayes-plausibility constraint. As a result, optimizing over the space of Bayes-plausible posterior beliefs is a computationally difficult problem. Second, in many economic applications posterior beliefs contain redundant information. For example, a risk-neutral buyer (the receiver) is interested only in the posterior mean value of the product before making a decision about the purchase. Because the payoff to the seller (the sender) depends on the buyer’s decision only, all other information contained in posterior beliefs is irrelevant for players. However, the derivation of a distribution of posterior beliefs and hence, the signal structure from an (optimal) distribution of posterior means is a non-trivial problem. Due to these difficulties, the approach of Kamenica and Gentzkow (2011) does not always result in precise solutions of optimal signal structures.

In addition, Kamenica and Gentzkow (2011) restrict attention to setups in which the receiver’s action solely depends on the posterior distribution induced by the signal realization, but not on the signal realization itself. That is, the action of the receiver is identical for all signals that induce the same posterior distribution. This class of setups does not include a range of economic applications in which an action of the mechanism is affected by the signal realization. In selling mechanisms, for example, Bergemann and Pesendorfer (2007) show that the expected probability of allocating the object and the transfer of each buyer are affected not only by the posterior distribution, namely, the mean value of the object by the buyer, but also by the index (that is, the signal) of the posterior mean in the ordered sequence of generated posterior means. Because the sender’s interim payoff in these mechanisms is affected by both the signal realization and the posterior distribution, it is impossible to derive the optimal signal structure by optimizing over the set of posterior distributions only.

In this paper, we offer an alternative approach to derive optimal signal structures in Bayesian persuasion mechanisms in which an action of the mechanism (that is, the receiver) depends on both the posterior mean of an increasing utility function of the state and the signal realization. Our approach is briefly described as follows. Instead of optimizing over distributions of posterior beliefs directly, we start with the ranking

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1This approach has been extended by Alonso and Camara (2013) to the setup with heterogeneous prior beliefs of the players, and by Gentzkow and Kamenica (2014) to the case of costly signal structures, where the cost is proportional to expected reduction in entropy.
of joint distributions of the state and the signal based on the positive dependence of these variables. Second, we demonstrate how the ranking of signal structures can be transformed into the ranking of distributions over generated posterior means by the weighted majorization. In this light, it is important to note that the ranking of distributions of posterior means holds not only for posterior means of states, but also for increasing utility functions of the state. This extension is important, because some common informativeness orders of distributions of posterior means are not invariant to monotone transformations of the state (Brandt et al., 2014). Third, we provide the necessary and sufficient conditions under which the ranking of distributions over posterior means by the weighted majorization preserves the ranking of the sender’s ex-ante payoffs in our class of Bayesian persuasion mechanisms.

Our first main result shows that for a sequence of posterior means generated by an arbitrary signal structure there exists a monotone partitional signal structure that generates a sequence of posterior means which dominates the original sequence by the weighted majorization (or $g$-majorization). The proof of this result combines three results from the statistical literature.\(^2\) The weighted majorization is a special case of the convex stochastic order, or second-order stochastic dominance, such that the elements of the majorizing sequence are more dispersed than those of the original sequence while the weights (probabilities) of associated elements of both sequences are identical. Then, the ranking of distributions of posterior means by the weighted majorization preserves the ranking of sender’s ex-ante payoffs which are weighted Schur convex functions of a distribution of posterior means. In our class of Bayesian persuasion mechanisms, the sender’s ex-ante payoff can be expressed as an average of the sequence of functions of posterior means (that is, interim payoffs) indexed by signal realizations.\(^3\) Next, we provide necessary and sufficient conditions on the sequence of functions of posterior means under which the ex-ante payoff is a weighted Schur convex functions of a distribution of posterior means. These conditions highlight the difference in the requirements on the interim payoff functions in Bayesian persuasion games and mechanisms for the optimality of monotone partitions. In particular, in Bayesian persuasion games the sender’s interim payoff is a single function of the posterior mean. Then, the sender’s ex-ante payoff is an average of this function over posterior means, which is Schur convex if and only if the interim payoff is convex in posterior means. In Bayesian persuasion mechanisms the ex-ante payoff is average of the sequence of functions of posterior means. In this case, the sender’s ex-ante payoff can be Schur convex if some or all functions are non-convex in posterior means.

\(^2\)First, we note that among all signal structures with fixed marginal distributions, the monotone partitional structures generate signals which are most positively dependent on states by the positive-quadrant dependent (PQD) stochastic order. Second, we apply the result by Tchen (1980) who shows that the conditional mean of posterior means given that the signal is below some cutoff is lower for signal structures which are dominant by the PQD order. Finally, for discrete distributions of posterior means, this result implies the ranking of distributions of posterior means by the weighted majorization. It is important to note that the second and the third steps are crucial in Bayesian persuasion games. In particular, the ranking of joint distributions by the PQD order requires identical marginal distributions for single variables. However, different signal structures induce different marginal distributions of posterior means, which severely restricts the applicability of the PQD order to ranking joint distributions over states and posterior means.

\(^3\)Despite this condition, the sender’s interim payoff can be a function of other characteristics of a posterior belief. As we show below, however, the ex-ante payoff can sometimes aggregate these characteristics into the distribution over posterior means only.
Finally, we apply our results to demonstrate the optimality of partitional signal structures in various economic applications. First, we demonstrate that the main result of Bergemann and Pesendorfer (2007, Theorem 1) about the optimality of monotone partitional signal structures of risk-neutral bidders with independent values in mechanisms of selling a single indivisible object is a corollary of one of our results. It establishes that for any mechanism in which the sender’s ex-ante payoff is an average of functions which are linear in the product of posterior means and actions, there is a mechanism with a (weakly) smaller number of signals and a monotone partitional signal structure which is ex-ante superior to the original mechanism. Second, we derive optimal signal structures in the problem of information provision by the monopolist with both exogenous and endogenous prices. Finally, we demonstrate the optimality of monotone partitional signal structures in the model of lobbying by Kamenica and Gentzkow (2011).

Regarding the economic literature, our approach attempts to generalize methods used in more structured models, which focus on characterizing the optimal mapping of the state into signals. For example, in the auction design with the endogenous precision of information released by the seller to bidders, Bergemann and Pesendorfer (2007) demonstrate the sub-optimality of non-monotone partitional signal structures in optimal actions by variations of probabilities across (adjacent) posterior means. A similar method has been used by Ivanov (2010) for a partial characterization of optimal signal structures in Crawford and Sobel’s (1982) model of cheap-talk communication. In addition, Ivanov (2010) uses the majorization to prove the optimality of monotone partitions in special cases. Li and Shi (2013) use a convex order for the analysis of the optimal signal structure in dynamic price discrimination. In this light, our approach generalizes the stochastic order approaches by Ivanov (2010) and Li and Shi (2013) and applies it to a broad class of Bayesian persuasion mechanisms. This is because the weighted majorization is a special case of the convex stochastic order, which excludes the dominance of sequences by the convex order with different weights of elements of the sequences. At the same time, considering the more restrictive stochastic order allows us to extend the class of sender’s ex-ante problems to non-convex problems.

The rest of the paper is structured as follows. Section 2 presents the formal model. Section 3 focuses on the ranking of signal structures. Section 5 analyzes the relationship between signal structures and sender’s ex-ante payoffs. Economic applications of our results are provided in Section 5. Section 6 concludes the paper.

2 Model

We consider an extended version of the model by Kamenica and Gentzkow (2011) with two initially uninformed players, the sender and the receiver (the mechanism). The sender’s ex-post payoff function $v(a, \theta)$ depends on the random state $\theta$ and the mechanism’s action $a \in \mathcal{A}$, where $\mathcal{A} \subset \mathbb{R}$ is compact.

4In a recent paper, Li et al. (2013) investigate a persuasion model with a privately informed receiver such that the signal structure depends on a receiver’s report about his private information. They show that the structure and the methods of the analysis of such discriminatory persuasion mechanisms are similar to those used in the literature on constrained delegation.
2.1 Information

The state $\theta$ is distributed according to a prior distribution $F(\theta)$ with a density function $\mu_0 = f(\theta)$ and the support $\Theta = [\underline{\theta}, \overline{\theta}] \subset \mathbb{R}$. The maximum number of signals is exogenously given by $n$ such that $1 \leq n < \infty$. It determines a finite and ordered signal set $S_n = \{1, \ldots, n\}$. A signal structure $\sigma$ is an ordered collection of $n$ integrable functions $\sigma = \{\sigma_s(\theta)\}_{s=1}^n$. Technically, $\sigma_s(\theta) = \Pr\{s|\theta\} \in [0,1], s \in S_n$ is the probability of generating signal $s$ conditional on state $\theta$. We restrict attention to plausible $\sigma$, such that $\sigma_s(\theta) \in [0,1]$ and $\sum_{s=1}^n \sigma_s(\theta) = 1$ for all $\theta \in \Theta$. Let $\Pi_n$ be the set of plausible $\sigma$ of size $n$.

Clearly, a plausible $\sigma$ of size $k < n$ is in $\Pi^n$ once we put $\sigma_s(\theta) = 0$, $k < s \leq n$. A signal realization $s$ induces the posterior belief (the density function) $\mu_s$ by Bayes rule:

$$
\mu_s(\theta) = \frac{\sigma_s(\theta) f(\theta)}{g_s},
$$

where

$$
g_s = \Pr\{s\} = \int_{\Theta} \sigma_s(\theta) f(\theta) \, d\theta,
$$

is the marginal probability of signal $s$. Thus, a plausible signal structure $\sigma \in \Pi_n$ induces a Bayes-plausible distribution of posterior beliefs $\{g_s, \mu_s\}_{s=1}^n$, i.e., the distribution which satisfies the Bayesian-plausibility constraint

$$
E_{g_s}[\mu_s] = \sum_{s=1}^n g_s \mu_s = \mu_0.
$$

When $n$ is clear from the context, we use notation $\{g, \mu\}$ instead of $\{g_s, \mu_s\}_{s=1}^n$ hereafter. Conversely, for a Bayes-plausible distribution of posterior beliefs $\{g, \mu\}$, the signal structure that generates $\{g, \mu\}$ is given by $\sigma = \{\sigma_s(\theta)\}_{s=1}^n = \left\{ \frac{g_s \mu_s(\theta)}{\int \mu_s(\theta) \, d\theta} \right\}_{s=1}^n$.

Posterior means. For a posterior belief $\mu_s$, let $\omega_s$ be a posterior mean of a right-continuous increasing and integrable utility function $v(\theta), \theta \in \Theta$:

$$
\omega_s = E[v(\theta) | \mu_s] = \int_{\Theta} v(\theta) \mu_s(\theta) \, d\theta.
$$

Thus, a Bayes-plausible distribution of posterior beliefs $\{g, \mu\}$ generates a distribution of posterior means $\{g, \omega\}$, where posterior means $\omega = \{\omega_s\}_{s=1}^n$ are given by (4). We call a distribution of posterior means $\{g, \omega\}$ plausible if there exists a Bayes-plausible distribution of posterior beliefs $\{g, \mu\}$ which induces it. Denote $G_n$ the set of Bayes-plausible distributions $\{g, \omega\}$ of size $n$ (and, thus, size $n^o \leq n$).

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5That is, $\Pi_n = \left\{ \{\sigma_s(\theta)\}_{s=1}^n : \sigma_s(\theta) \in [0,1], \sum_{s=1}^n \sigma_s(\theta) = 1, \theta \in \Theta \right\}$.

6In other words, $\sigma$ determines the set of posterior beliefs $\{\mu_s\}_{s=1}^n$ and their probabilities $\{g_s\}_{s=1}^n$, where the signal realization $s$ randomly selects a particular belief in this set according to probabilities.

7That is, $\{g_s, \omega_s\}_{s=1}^n \in G_n$ if and only if $g_s = \int_{\Theta} \sigma_s(\theta) f(\theta) \, d\theta > 0$ and $\omega_s = \int_{\Theta} v(\theta) \frac{f(\theta) \sigma_s(\theta)}{g_s} \, d\theta$ for
\{g, \omega\} implies \(\sum_{s=1}^{n} g_s = 1\) and \(E_{g_s}[\omega_s] = E[v(\theta)]\). The converse, however, does not hold.\(^8\)

**Monotone partitions.** A special class of signal structures are monotone partitional ones. A signal structure in this class is specified by a collection of \(n\) non-overlapping non-degenerate intervals on \(\Theta\), such that it reveals only the interval that contains \(\theta\). Equivalently, it is given by a strictly increasing sequence of cutoff states \(\{\theta_s\}_{s=1}^{n}\) such that \(\theta_0 = \underline{\theta}, \theta_n = \bar{\theta}\), and the generated signals are \(s^\theta(\theta) = s\) if \(\theta \in [\theta_{s-1}, \theta_s]\). That is, \(\sigma^\omega \in \Pi_n^\omega\) is monotone partitional if \(\sigma^\omega_s(\theta) = 1\) for \(\theta \in [\theta_{s-1}, \theta_s]\), and \(\sigma^\omega(\theta) = 0\) otherwise.\(^9\) A monotone partitional \(\sigma^\omega\) determines a Bayes-plausible distribution of posterior beliefs \(\{g^\omega, \mu^\omega\}\) and a plausible distribution of posterior means \(\{g^\omega, \omega^\omega\}\) such that \(\mu^\omega_s = f(\theta | \theta \in [\theta_{s-1}, \theta_s]), g^\omega_s = \Pr[\theta \in [\theta_{s-1}, \theta_s]] = F(\theta_s) - F(\theta_{s-1})\), and \(\omega^\omega_s = E[v(\theta) | \theta \in [\theta_{s-1}, \theta_s]], s \in S_n\). Note that \(\omega^\omega\) generated by a monotone partitional \(\sigma^\omega\) is strictly increasing.

### 2.2 Bayesian persuasion mechanisms

We consider the class of settings in which the sender’s ex-ante payoff \(EV : \mathbb{R}_+^n \times \mathbb{R}^n \rightarrow \mathbb{R}\) is an average of the sequence of functions of posterior means \(\{V(\omega(s), (s))\}_{s=1}^{n}\), where \((s)\) is the \(s\)-th lowest order in sequence \(\omega\),

\[
EV = EV (g, \omega, n) = \sum_{(s)=1}^{n} g(s)V(\omega(s), (s)),
\]

where \(V(x, s)\) is upper semi-continuous in \(x\) for each \(s \in S_n\). The upper-semicontinuity of \(V\) is a mild regularity condition which is needed to guarantee the existence of the maximum of \(EV (g, \omega)\). Therefore, \(EV\) is symmetric with respect to permutations of pairs \(\{g_s, \omega_s\}\):

\[
EV (g, \omega) = V (g_{\eta_1}, ..., g_{\eta_n}; \omega_{\eta_1}, ..., \omega_{\eta_n}),
\]

for all permutations \(\eta\) of \(\{g, \omega\}\). Because of the symmetry of \(EV (g, \omega)\), we can restrict attention to distributions \(\{g, \omega\}\) such that \(\omega\) is weakly increasing. Denote \(G^+_n\) the set of plausible \(\{g, \omega\}\) of size weakly smaller than \(n\), such that \(\omega\) is weakly increasing. Then, the ex-ante payoff can be written as

\[
EV = EV (g, \omega) = \sum_{s=1}^{n} g_s V(\omega_s, s), \{g, \omega\} \in G^{n+}.\]

The justification for such a specification is as follows. At the beginning of the game, the sender publicly selects a Bayes-plausible signal structure \(\sigma \in \Pi_n\), which generates a distribution over posterior beliefs \(\{g, \mu\}\) and, thus, a distribution over posterior means \(\{g, \omega\} \in G_n\). Second, the mechanism derives the distribution of posterior means \(\{g, \omega\}\)

\(^8\)Suppose that \(\theta\) is distributed uniformly on \([0, 1]\) and \(v(\theta) = \theta\). Consider \(\{g, \omega\}\), such that \(g = \{\frac{1}{2}, \frac{1}{2}\}\), and \(\omega = \{0, 1\}\). Then, \(g_1 + g_2 = 1\) and \(g_1 \omega_1 + g_2 \omega_2 = E[\theta] = \frac{1}{2}\). However, \(\{g, \omega\}\) is not plausible since the variance of posterior means is higher than the variance of \(\theta\).

\(^9\)Up to the set of boundary points \(\{\theta_s\}_{s=1}^{n}\) of measure zero.
Bayesian persuasion games considered by Kamenica and Gentzkow (2011). In Kamenica's functions interim payoff to the seller from the buyer's type the buyer with posterior mean affected by both the signal realization rationality of the receiver. For example, the receiver can be a partially naive player whose decision is order of posterior means \( V(\text{payoff function}) \) both the value of a posterior mean mechanism makes a decision— the probability of winning and the transfer— according to \((\text{the mechanism}) \) orders players' posterior means in an increasing order first. Then, the auction and price-discriminating mechanisms. insufficient statistics in setups which are sensitive to the signal realization, for example, auctions and price-discriminating mechanisms. In auctions, for example, the auctioneer (the mechanism) orders players' posterior means in an increasing order first. Then, the mechanism makes a decision—the probability of winning and the transfer—according to both the value of a posterior mean \( \omega_s^o \) and its order \( s \) in sequence \( \omega^o \). At the same time, \( a \) does not depend directly on \( \sigma \) which generates \( \{g, \omega\} \).

Thus, an action \( a(\omega_s, s) \) determines the sender's interim payoff

\[
V(\mu_s, s) = E_{\mu_s}(\theta) \left[ v(a(\omega_s, s), \theta) \right].
\]

If the interim payoff is a function of the posterior mean \( \omega_s \) and signal \( s \), i.e.,

\[
V(\mu_s, s) = V(\omega_s, s),
\]

then the sender's ex-ante payoff is in the class of (7). Note that (8) is a sufficient, but not necessary condition for the ex-ante payoff to have the form (7). In general, the interim payoff \( V(\mu_s, s) \) can depend on other characteristics of \( \mu_s \), however, the ex-ante payoff function \( EV \) aggregates these characteristics into an average of modified functions \( \bar{V}(\omega_s, s) \) of pairs \( (\omega_s, s) \) only. Moreover, the interim payoff might depend on the set of posterior means \( \omega \), but the ex-ante payoff can be decomposed into an average over functions \( \bar{V}(\omega_s, s) \).

It is important to note that Bayesian persuasion mechanisms encompass the class of Bayesian persuasion games considered by Kamenica and Gentzkow (2011). In Kamenica 

\[a^0\] Another situation in which the sender's payoff depends on the signal realization is the bounded rationality of the receiver. For example, the receiver can be a partially naive player whose decision is affected by both the signal realization \( s \) and the posterior mean \( \omega_s \) induced by it. 

\[b^0\] For example, in price-discriminating mechanisms the incentive-compatibility constraints require that the buyer with posterior mean \( \omega_s \) cannot profitably deviate to reporting type \( \omega_{s-1} \). As a result, the interim payoff to the seller from the buyer's type \( \omega_s \) depends on both \( \omega_s \) and \( \omega_{s-1} \). However, the seller's ex-ante payoff in any incentive-compatible mechanism can be expressed as an average of functions \( V(\omega_s, s) \) that depend on posterior mean \( \omega_s \) and the order \( s \) of this mean in sequence \( \omega \). E.g., given a sequence \( \omega = \{\omega_s\}_{s=1}^n, n \geq 2 \), consider a function \( EV(\omega) = \sum_{s=2}^n \phi(\omega_s, \omega_{s-1}) \), where \( \phi(\omega_s, \omega_{s-1}) = \omega_s - b\omega_{s-1} \).

Then, \( EV(\omega) \) can be expressed as \( EV(\omega) = \sum_{s=1}^n V(\omega_s, s) \), where \( V(\omega_s, s) = \alpha_s \omega_s \), and \( \alpha_1 = -b, \alpha_n = 1, \) and \( \alpha_s = 1 - b, 2 \leq s \leq n - 1 \).
and Gentzkow (2011) the receiver’s action $a(\mu_s)$ is independent of signal realization $s$, which induces the posterior belief $\mu_s$, that is, $a(\mu_s) = a(\mu_{s'})$ for all $s$ and $s'$ such that $\mu_s = \mu_{s'}$. Hence, the sender’s interim payoff $V(\mu_s) = E_{\mu_s(\theta)}[v(\mu_s, \theta)]$ is a function of the posterior belief $\mu_s$ only, and the ex-ante payoff $EV = \sum_{s=1}^{n} g_s V(\mu_s)$ is an average of the function $V(\mu_s)$, which is independent of signal $s$. Therefore, Bayesian persuasion mechanisms include Bayesian persuasion games in which the sender’s interim payoff is a function of posterior means, $V(\mu_s) = V(\omega_s)$.

A Bayes-plausible signal structure $\sigma^* \in \Pi^n$ is optimal if it generates a distribution of posterior means $\{g, \omega\} \in G^n$ which achieves the maximum of the sender’s ex-ante payoff (5). That is, $\sigma^*$ is a solution to the problem

$$\max_{\sigma \in \Pi^n} \sum_{(s)=1}^{n} g(\sigma(s)) V(\mu(\sigma(s)), (s)) = \max_{\{g, \omega\} \in \mathcal{G}^n} EV(g, \omega), \quad (9)$$

where $\mu(\sigma_s)$ and $g(\sigma_s)$ are determined by (1) and (2), respectively. Thus, the Bayesian sender is interested in an ex-ante optimal decomposition of the prior belief $\mu_0$ into a Bayes-plausible distribution of posterior beliefs $\{g, \mu\}$ by selecting $n$ and $\sigma$.

Because the receiver’s action does not directly depend on $\sigma$, any mechanism with the sender’s interim payoff of the form (8) is ex-ante payoff equivalent to the mechanism in which signals that generate identical posterior means are collapsed. That is, partition first the signal set $S_n$ into an ordered set of subsets $\{S_{(k)}\}_{k=1}^{n^o} = \{s \in S_n | \omega_s = \omega^{(k)}_s\}$, where $(k) \in \{1, ..., n^o\}$ is $k$-th lowest order in sequence $\omega$, and all signals $s \in S_{(k)}$ induce the same posterior mean $\omega^{(k)}_s$. Then, replace the initial signal structure $\sigma$ by a signal structure $\sigma^{o}$, which collapses all signals $s \in S_{(k)}$ into a single $(k)$. Consider the mechanism which randomizes among actions in $A_{(k)} = \{a(\omega^{(k)}_s, s)\}_{s \in S_{(k)}}$ with probabilities $\frac{g_s}{g^{(k)}}$, where $g^{(k)} = \sum_{s \in S_{(k)}} g_s$ upon receiving a signal $(k) \in S_{(k)}$. In the modified mechanism, signal $(k) \in S_{(k)}$ is generated with probability $g^{(k)}_s$ and provides the sender’s interim payoff

$$V^{\omega^{(k)}}(\omega^{(k)}_s, (k)) = \frac{1}{g^{(k)}_s} \sum_{s \in S_{(k)}} g_s V(\omega^{(k)}_s, s).$$

Thus, the sender’s ex-ante payoff can be expressed as

$$EV(g, \omega) = \sum_{(k)=1}^{n^o} \sum_{s \in S_{(k)}} g_s V(\omega_s, s) = \sum_{(k)=1}^{n^o} g^{(k)}_s V^{\omega^{(k)}}(\omega^{(k)}_s, (k)) = EV^{\omega^{(k)}}(g^{o}, \omega^{o}),$$

where $n^o \leq n$ and $\omega^{o}$ is strictly increasing. Let $G^{++}_n$ the set of Bayes-plausible distributions.

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This setup is called Bayesian persuasion games because it captures interaction between the Sender and the Receiver with the ex-post payoff functions $v(a, \theta)$ and $u(a, \theta)$, respectively. Given the Sender’s choice of $\sigma$ and a signal realization $s$, Receiver forms the posterior $\mu_s$ by Bayes rule (1) and takes an interim-optimal action $a(\mu_s) \in \arg \max_{a \in A} E_{\mu_s(\theta)}[u(a, \theta)]$. Since $u(a, \theta)$ is independent of $s$, the receiver’s action solely depends on $\mu_s$. 

8
of posterior means of size (weakly smaller than) \( n \) such that \( \omega \) is strictly increasing. Then, the sender’s problem (9) is equivalent to
\[
\max_{\{g, \omega\} \in \mathcal{V}_n} EV (g, \omega) = \max_{\{g^o, \omega^o\} \in \mathcal{G}^+_n} EV^o (g^o, \omega^o).
\]

3 Ranking of distributions ofposteriors means

In order to solve the sender’s problem (10) and rank signal structures, our basic statistical tools are: the comonotonicity of distributions (Puccetti and Scarsini, 2010), the positive quadrant-dependent stochastic orders (Tchen, 1980; and Joe, 1997), and the majorization (Marshall, Olkin, and Arnold, 2011).

First, the comonotonicity of distributions means the perfect positive dependence between multiple random variables, which means that they can be represented as increasing functions of a single random variable. In particular, the joint distribution \( \bar{H} (s, \theta) \) is called comonotonic, if
\[
\bar{H} (s, \theta) = \min \{ G (s), F (\theta) \} \text{ for all } s \in S \text{ and } \theta \in \Theta, \tag{11}
\]
where \( G (s) \) and \( F (\theta) \) are marginal distributions of \( \theta \) and \( s \), respectively.

Second, the positive quadrant dependent (PQD) order (which is also called the concordance order by Joe, 1997), ranks joint distributions \( H (s, \theta) \) in the Fréchet class \( \mathcal{M}(G, F) \), which includes all bivariate distributions with fixed marginal distributions \( F (\theta) \) and \( G (s) \). Namely, we say that \( H^o (s, \theta) \) dominates \( H (s, \theta) \) by the PQD order if\(^{13}\)
\[
H^o (s, \theta) \succeq H (s, \theta) \text{ for all } s \in S \text{ and } \theta \in \Theta.
\]

Third, the weighted (or \( g \)-) majorization is a special case of the single-dimensional convex stochastic order, or second-order stochastic dominance, for discrete distributions such that the values of the majorizing distribution are more dispersed than those of the original distribution while the probabilities of associated values of both distributions are identical. In particular, let \( \{g, \omega\} \) and \( \{g, \omega^o\} \) be two discrete distributions with \( n \) values, where \( \omega \) and \( \omega^o \) are increasing. If
\[
\sum_{s=1}^{k} g_s \omega^o_s \leq \sum_{s=1}^{k} g_s \omega_s, k = 1, \ldots, n - 1, \text{ and } \sum_{s=1}^{n} g_s \omega^o_s = \sum_{s=1}^{n} g_s \omega_s,
\]
then we say \( \omega^o \) \( g \)-majorizes \( \omega \), or \( \omega^o \succ_g \omega \).

Given these preliminaries, we provide the main result about the ranking of distributions of posterior means generated by non-partitional and partitional signal structures.

\(^{13}\)For bivariate distributions, Müller and Scarsini (2000, Theorem 2.5) show that the PQD order is identical to the supermodular stochastic order defined as follows. We say that multidimensional random variable \( X^o \) dominates \( X \) by the supermodular order, or \( X^o \succeq_{SM} X \), if \( E [ \phi (X^o) ] \geq E [ \phi (X) ] \) for all supermodular functions \( \phi : \mathbb{R}^2 \to \mathbb{R} \), i.e., such that \( \varphi (x \land y) + \varphi (x \lor y) \geq \varphi (x) + \varphi (y) \).
Proposition 1 For any joint distribution $H(s, \theta)$ which induces a discrete distribution of posterior means $\{g, \omega\}$, there exists a monotone partitional $H^o(s, \theta)$ which induces $\{g, \omega^o\}$, such that $\omega^o \succ_g \omega$.

The proposition is proved by construction and combines three results from the statistical literature. First, for an initial joint distribution consider $H(s, \theta)$ in a Fréchet class $\mathcal{M}(G, F)$, consider a monotone partitional joint distribution $H^o(s, \theta)$ with cutoffs $\theta_s = F^{-1}(G(s)), s = 1, \ldots, n - 1$.

The choice of these cutoffs guarantees that $H^o(s, \theta)$ is in the same Fréchet class. Second, by construction, the signal generated by the partitional signal structure, $s(\theta) = s$ for $\theta \in [\theta_{s-1}, \theta_s]$, is an increasing function of $\theta$. This implies that $G(s)$ and $F(\theta)$ are comonotonic, that is, $\theta$ and $s$ are perfectly positively correlated.

Third, the comonotonicity implies that $H^o(s, \theta) = \min \{G(s), F(\theta)\}$, which is the upper bound on distributions in the Fréchet class $\mathcal{M}(G, F)$ (Joe, 1997). Fourth, Tchen (1980) shows that if $H^o(s, \theta)$ and $H(s, \theta)$ are in $\mathcal{M}(G, F)$ are such that $H^o(s, \theta) \succeq H(s, \theta)$ for all $s$ and $\theta$, and $v(\theta)$ is right-continuous increasing and integrable, then

$$E[E[v(\theta)\mid s^o] \mid s^o \leq x] \leq E[E[v(\theta)\mid s] \mid s \leq x] \text{ for all } x. \quad (12)$$

Finally, for discrete $G(s)$, (12) implies that $\omega^o$ $g$-majorizes $\omega$.

Two comments are necessary. First, extending the ranking of distributions of posterior means of $\theta$ to distributions of posterior means of an increasing utility function $v(\theta)$ is a valuable property of the weighted majorization. This is because informativeness orders of distributions of posterior means are generally not robust to monotone transformations of $\theta$ (see Brandt et al., 2014). Second, it is clear from the proposition that the PQD order cannot be used directly to rank joint distributions $F(a, \theta)$ over states and actions in Bayesian persuasion games. Because the receiver is a Bayesian player, he is interested in the meaning of a signal rather than the signal itself. As a result, the ex-ante payoff of the sender can be expressed as an average of the ex-post payoff over actions and states, where actions depend on the signal structure $\sigma$:

$$EV = \int_{A \times \Theta} v(a, \theta) \, dF(a, \theta|\sigma),$$

where

$$F(a, \theta|\sigma) = \int_{\{s, \theta' : a(\mu_s(\sigma), s) \leq a, \theta' \leq \theta\}} dH(s, \theta').$$

However, due to Kamenica and Gentzkow (2011, Proposition 1), in Bayesian persuasion games we can restrict attention to $S$ that satisfy the obedience constraint

$$a(\mu_s) = s \text{ for all } s \in S, \quad (13)$$

i.e., each signal recommends the action which coincides with the signal, and the receiver follows the recommendation. Imposing the obedience constraint (13) means that signal structures $\sigma$ and $\sigma^o$ can induce different ordered action sets $S$ and $S^o$ in equilibrium,
whereas the probabilities of associated actions (that is, actions with the same index in the action sets) are identical, \( g_{as} = g_{a\omega} \). That is, the constraint (13) conflicts with the necessary condition of identical marginal distributions for ranking signal structures by the PQD order. Because Bayesian persuasion games form a subclass of Bayesian persuasion mechanisms, this argument can be extended to mechanisms as well.

4 Ranking of payoff functions

Consider \( EV(g, \omega) \) of the form (6) which is differentiable in \( \omega \), symmetric with respect to permutation of pairs \( \{g_s, \omega_s\} \) and satisfies the following condition:

\[
(\omega_s - \omega_t) \left( \frac{1}{g_s} \frac{\partial EV(g, \omega)}{\partial \omega_s} - \frac{1}{g_t} \frac{\partial EV(g, \omega)}{\partial \omega_t} \right) \geq 0 \text{ for all } \omega \text{ and } s, t = 1, ..., n. \tag{14}
\]

These functions form a set of weighted (or \( g- \)) Schur-convex functions. The following proposition by Cheng (1977) demonstrates the relationship between finite sequences in \( \mathbb{R}^n \) (in our case, sequences of posterior means) ranked by the \( g- \)majorization and values of a function of these sequences.

**Proposition 2** (Cheng, 1977) If \( EV(g, \omega) \) is differentiable in \( \omega \) and satisfies (14), then \( \omega^o \succ_g \omega \) implies \( EV(g, \omega^o) \geq EV(g, \omega) \) if and only if (14) holds.

The key feature of (weighted) Schur-convex functions that plays a crucial role in our analysis is that a Schur-convex function is not necessarily convex, and a convex function is not necessarily Schur-convex. However, a convex function is weighted Schur-convex if and only if it is symmetric with respect to \( \{g_s, \omega_s\} \) pairs. Thus, \( EV(\omega, g) = \sum_{s=1}^{n} g_s V(\omega_s) \) is weighted Schur-convex if and only if \( V(\omega) \) is convex. However, in Bayesian persuasion mechanisms with the sender’s ex-ante payoff of the form (5), \( EV \) is an average of an ordered family of functions \( \{V(x, s)\}_{s=1}^{n} \). In this case, condition (14) is equivalent to

\[
V'_x(x, s) \geq V'_y(y, t) \text{ for all } x \geq y \text{ and } s > t, \tag{15}
\]

that is, the marginal payoff \( V'_x(x, s) \) from a higher posterior mean is isotone in \( (x, s) \). In order to see the equivalence between (14) and (15), note first that it is sufficient to consider \( \{g, \omega\} \in G^n_+ \) by (7). Thus, we can restrict (14) to \( \omega_s \geq \omega_t \) and \( s > t \). Then, (14) is satisfied if and only if

\[
(\omega_s - \omega_t) \left( \frac{1}{g_s} \frac{\partial EV(\omega, g)}{\partial \omega_s} - \frac{1}{g_t} \frac{\partial EV(\omega, g)}{\partial \omega_t} \right) = V'_{\omega_s}(\omega_s, s) - V'_{\omega_t}(\omega_t, t) \geq 0 \text{ for } \omega_s \geq \omega_t \text{ and } s > t.
\]

If \( x = y \), then (15) is equivalent to the supermodularity of \( V(x, s) \). Thus, the supermodularity of \( V(x, s) \) is a necessary condition for the preservation of the ranking of ex-ante payoff functions of sequences of \( \omega^o \) and \( \omega \) ordered by the \( g- \)majorization. Also, if \( V(x, s) = V(x, t) = V(x) \) for all \( x, s \) and \( t \), then (15) is equivalent to the convexity of \( V(x) \). Together, the supermodularity of \( V(x, s) \) and the convexity of \( V(x, s) \) in \( x \) for all
are sufficient conditions for (15) to hold, since they imply \( V'_x(x, s) \geq V'_y(y, s) \geq V'_y(y, t) \) for \( x \geq y \) and \( s > t \). However, if \( V(x, s) \neq V(x, t) \), then the convexity of \( V(x, s) \) in \( x \) is not required. Moreover, \( V(x, s) \) can be concave in \( x \) for all \( s \). The following lemma provides sufficient conditions for (15) to hold.

**Lemma 1** Condition (15) holds if: (A) \( V(x, s) \) is supermodular and in each pair \( V(x, s + 1) \) and \( V(x, s) \), \( s = 1, \ldots, n - 1 \) at least one function is convex in \( x \); or (B) \( \min_x V'_x(x, s + 1) \geq \max_x V'_x(x, s) \) for all \( s = 1, \ldots, n - 1 \).

The intuition behind Lemma 1 is as follows. Since \( \omega \) is increasing, then the supermodularity of \( V(x, s) \) implies that the marginal gains from increasing high values of \( \omega \) by the \( g \)-majorization exceed the marginal losses from decreasing low values of \( \omega \). As a result, the overall effect of a spread in \( \omega \) on \( EV(g, \omega) \) is positive. Conversely, if \( V(x, s) \) is concave in \( x \) for all \( s \) and \( V(x, s) \) is submodular, then the spread in \( \omega \) decreases \( EV(g, \omega) \). As a result, all posterior means in the optimal sequence are equal to the unconditional mean, \( \omega^o = \{ E[\theta] \}_{s=1}^n \). The first implication of this proposition is that if \( V(\omega_s, s) \) satisfies (15), then introducing the cost function of the signal structure \( C(\omega_s, s) \) which is submodular in \( (\omega_s, s) \) and concave in \( \omega_s \) for each \( s \) preserves the optimality of monotone partitional signal structures. Another implication of Propositions 1 and Lemma 1 is that the optimal signal structure is monotone partitional for mechanisms in which the sender’s ex-ante payoff is an average of a sequence of linear functions of \( \omega_s \), \( V(\omega_s, s) = a_s \omega_s + b_s \). As we show below, the class of such mechanisms includes, for example, selling mechanisms with risk-neutral buyers (Bergemann and Pesendorfer, 2007).

**Proposition 3** Consider a mechanism with action rule \( a(\omega_s, s) = \{a_s, b_s\} \) in which \( EV(g, \omega) = \sum_{s=1}^n g_s (a_s \omega_s + b_s), \{g, \omega\} \in G^+_n \). Then, there is a mechanism with monotone partitional \( \sigma^o \) of size \( n^o \leq n - t \), such that \( EV^o(g^o, \omega^o) \geq EV(g, \omega) \), where \( t \) is the number of pairs \( \{a_s, a_s\} \) such that \( a_s < a_{s-1} \).

This result is proved in three steps. First, if the action rule \( a_s \) is not monotonically increasing, then we construct an iterated sequence of mechanisms as follows. First, the signal structure \( \sigma^t \) in each iteration \( t \) of the initial mechanism \( (t-\text{mechanism}) \) collapses pairs of signals associated with decreasing pairs \( a_{s_{t-1}} < a_{s_{t-1}-1} \). Thus, the pair of posterior means \( \omega_{s_{t-1}} \) and \( \omega_{s_{t-1}-1} \) is collapsed into a single posterior mean \( \omega'_{s_{t-1}} = E[\omega_{s_{t-1}} | s \in \{s_{t-1}-1, s_{t-1}\}] \). Also, \( t-\text{mechanism} \) modifies the action rule \( a^t_s \) by inducing a mean action \( a'^{t-1}_{s_{t-1}} = E[a_{s_{t-1}} | s \in \{s_{t-1}-1, s_{t-1}\}] \) and preserving actions for \( s \notin \{s_{t-1}-1, s_{t-1}\} \). Since function \( V(x, s) = a_s x \) is linear in \( x \), then \( a_{s_{t-1}} < a_{s_{t-1}-1} \) implies that \( V(x, s) = a_s x \) is submodular for \( s \in \{s_{t-1}-1, s_{t-1}\} \). Thus, the inequality (1) is reversed. Also, because subsequence \( \{\omega'_{s_{t-1}}, \omega'_{s_{t-1}-1}\} \) is \( g \)-majorized by \( \{\omega_{s_{t-1}}, \omega_{s_{t-1}-1}\} \), then Proposition 2 means that each iteration of the mechanism increases the sender’s ex-ante payoff. Thus, after a finite number of iterations we obtain a \( T-\text{mechanism} \) which is ex-ante superior to the original mechanism and in which: 1) signal structure \( \sigma^T \) generates a distribution of posterior means \( \{g^T, \omega^T\} \in G^+_n \) of size \( n_T = n - T \); and 2) the action rule \( \{a^T_s\}_{s \in S^o} \) is increasing.

In the second step, we transform \( T-\text{mechanism} \) into a new mechanism. In this mechanism the signal structure \( \sigma^+ \) of size \( n^0 \leq n_T \) collapses all signals in \( S^+_k = \}
\( \{ s \in S, \omega_s = \omega^+_k \} \), i.e., ones that generate identical posterior means \( \omega_s = \omega^+_k \), into a single signal \( k \). Also, the mechanism’s action rule is \( a_k = E [ \alpha_s | s \in S^+_k ] \). By construction, the sender’s ex-ante payoffs in this mechanism and \( T \)-mechanism are identical. Finally, we modify the information structure in this mechanism into a monotone partitional \( \sigma^o \) of size \( n^o \), which generates \( \{ g^+, \omega^o \} \) and keep the action rule unchanged. Then, Propositions \ref{prop:1} and \ref{prop:2} imply that the mechanism with signal structure \( \sigma^o \) is ex-ante superior to the second mechanism, and thus to the initial mechanism.

The main implication of this section is that if the sender’s ex-ante payoff is a weighted sequence of ranked functions of \( \omega \), which can be represented in a form (7) which satisfies \( (15) \), then the set of optimal signal structures can be reduced to those of a simple interval form. The section below demonstrates how Propositions \ref{prop:1}–\ref{prop:3} can be used in order to either solve problems or generalize the existing solutions in several applications of persuasion games and mechanisms.

## 5 Applications

### 5.1 Auction design

Consider the problem of the auction design by Bergemann and Pesendorfer (2007). A seller offers a single object for sale to \( I \) potential risk-neutral bidders for the auction, indexed by \( i \in I = \{ 1, ..., I \} \). The valuation \( v_i \) of bidder \( i \) independently distributed with prior distribution function \( F_i(v_i) \) with a positive density \( f_i \) on \([0, 1] \), where \( F_i \) is common knowledge. The utility of the (winning) bidder is given by \( u_i = v_i - t_i \), where \( t_i \) is a monetary transfer. Prior to the auction, bidders do not know their valuations. The auctioneer can choose an arbitrary signal structure \( \sigma^i \) for every bidder \( i \) at zero cost, such that \( \sigma^i \) satisfies the Bayesian plausibility constraint and is common knowledge among the bidders. At the interim stage every agent observes privately a signal realization \( s_i \) and infers her posterior mean \( \omega_{s_i} = E [ \theta | s_i ] \).

The seller selects the bidders’ signal structures of the bidders and a revelation mechanism, which maximizes seller’s ex-ante revenue subject to the interim participation and interim incentive-compatibility constraints of the bidders. By the revelation principle, it is sufficient to consider direct revelation mechanisms, which consist of a quadruple \( M = \{ g^i, \omega^i, t_i, q_i \}_{i=1}^I \), where \( S_i \) is the signal set of bidder \( i \), \( \omega^i = \{ \omega^i_{s_i} \}_{s_i \in S_i} \) is the set of bidder \( i \)’s posterior means generated with probabilities \( g^i = \{ g^i_{s_i} \}_{s_i \in S_i} \), \( t_i \) is the transfer payment function of bidder \( i \) given the profile of reports of all bidders \( s = \{ s_i \}_{i \in I} \in S = \times_i S_i \), and \( t_i(s) \) is the probability of winning the object for bidder \( i \) given the report profile \( s \). Here, we assume that signal set of each bidder \( S_i = \{ 1, ..., n_i \} \) is finite. Then, \( T_i(s_i) = E_{s_{-i}} [ q_i(s_i, s_{-i}) ] \) is the expected transfer payment and \( Q_i(s_i) = E_{s_{-i}} [ q_i(s_i, s_{-i}) ] \) is the expected probability of winning of bidder \( i \) upon reporting his signal \( s_i \). Thus, the interim utility of bidder \( i \) with a signal \( s_i \) and announced signal \( \hat{s}_i \) is \( V_i(s_i, \hat{s}_i) = \omega_i Q_i(\hat{s}_i) - T_i(\hat{s}_i) \). The mechanism has to satisfy the interim participation constraints : \( V_i(s_i) = V_i(s_i, s_i) \geq 0 \) for all \( s_i \in S_i \), and the interim incentive-compatibility constraints \( V_i(s_i) \geq V_i(s_i, \hat{s}_i) \) for all \( s_i, \hat{s}_i \in S_i \).

As Bergemann and Pesendorfer (2007) show, the ex-ante revenue of the auctioneer
from a single bidder $i = 1, \ldots, I$ in any incentive-compatible mechanism is given by

$$EV_i (g^i, \omega^i, n_i) = \sum_{s=1}^{n_i} g^i_s \sum_{l=1}^{s} (Q^i_l - Q^i_{l-1}) \omega^i_l = \sum_{s=1}^{n_i} \omega^i_s \Delta Q^i_s \left(1 - G^i_{s-1}\right)$$

$$= \sum_{s=1}^{n_i} \omega^i_s \Delta Q^i_s \left(g^i_s + 1 - G^i_s\right) = \sum_{s=1}^{n_i} g^i_s \omega^i_s \Delta Q^i_s \left(1 + \frac{1}{\lambda^i_s}\right),$$

where $G^i_s = \sum_{l=1}^{s} g^i_l, s \in S_i$ is the marginal distribution function of signals of bidder $i$, $\Delta Q^i_s = Q^i_s - Q^i_{s-1} \geq 0$ is the difference in expected probabilities of winning $Q^i_s = Q^i(s), s \in S_i$ by bidder $i$’s types with signals $s$ and $s-1$, respectively; and $\lambda^i_s = \frac{g^i_s}{1 - G^i_s}$ is the hazard rate function of $G^i_s$. Therefore, $EV_i (g^i, \omega^i)$ can be expressed as

$$EV_i (g^i, \omega^i, n_i) = \sum_{s=1}^{n_i} g^i_s V^i (\omega^i_s, s),$$

where $V^i (\omega^i_s, s) = \alpha^i_s \omega^i_s$ and $\alpha^i_s = \Delta Q^i_s \left(1 + \frac{1}{\lambda^i_s}\right) \geq 0$. Because $V^i (\omega^i_s, s)$ is linear in $\omega^i_s$ for all $s$, it follows from Proposition 3 that for any incentive-compatible mechanism there is an ex-ante superior incentive-compatible mechanism with a monotone partitional signal structure. Hence, the main result of Bergemann and Pesendorfer (2007, Theorem 1) about the optimality of monotone partitional signal structures in selling mechanisms is a corollary of Proposition 3.

### 5.2 Provision of product information by the monopolist

The question of providing information by seller(s) to uninformed buyers has been investigated, for example, by Lewis and Sappington (1994), Johnson and Myatt (2006), and Ivanov (2013). Kamenica and Gentzkow (2011) provide a partial characterization of the optimal signal structure for the case of an exogenous price. We apply the results in the section above in order to characterize precise solutions to both problems with exogenous and endogenous prices.

**Exogenous price.** A firm (sender) wants to sell a product at price $p$ to a single risk-neutral consumer (receiver) who needs one unit of the product. The initially uninformed consumer gets utility $\theta$ from buying the product, which is distributed according to a distribution $F (\theta)$ with a density $f (\theta) > 0$ on $[\underline{\theta}, \overline{\theta}]$. Suppose that the $\theta$ reflects matching between consumer’s tastes and some product characteristics (instead of the quality overall quality of the product). If the buyer does not purchase the product, he receives an outside option that provides the utility $u \in [0, \theta - p]$. Since the consumer is risk neutral, he buys the product if and only if $\omega = E [\mu] - p \geq u$. Following Kamenica and Gentzkow (2011), we assume first that price $p$ is exogenous. Also, the seller incurs the unit costs of production $c > 0$. We assume that $p > \max \{c, E [\theta] - u\}$. Otherwise, the solution is trivial: seller either does not sell the product if $p \leq c$, or sells it for sure by generating a single posterior mean $\omega_1 = E [\theta]$ if $p \leq E [\theta] - u$.

Given a fixed number of signals $n$, the firm chooses the signal structure of the buyer,
which maximizes the ex-ante profit

$$EV(\omega, g) = (p - c) \sum_{i=1}^{n} g_{i}1_{\omega_{i} > p}. \quad (16)$$

Note that because $V(\omega) = 1_{\omega_{i} = p}$ is not convex, $EV(\omega, g)$ is not weighted Schur convex. Hence, $EV(g, \omega^{o}) \geq EV(g, \omega)$ does not generally hold for all $\omega^{o}$ and $\omega$ ranked by $g$–majorization.\(^{15}\) However, Proposition 1 still can be used in order to prove the optimality of monotone partitions. First, because the buyer’s decision is binary, we can restrict attention to signal structures that generate binary posterior means $\{\omega_{1}, \omega_{2}\}$ with probabilities $\{g_{1}, g_{2}\}$. (For $n > 2$, the sequence $\omega’ = \{\omega’_{i}\}_{i=1}^{n}$ can be collapsed into $\{\omega_{1}, \omega_{2}\} = \{E[\omega’|\omega’ < p], E[\omega’|\omega’ \geq p]\}$.) Thus, the problem of the firm becomes

$$\max_{\{g, \omega\} \in g_{2}} g_{2}(p - c) \text{ s.t. } \omega_{2} \geq p + u.$$ 

Kamenica and Gentzkow (2011) provide a partial characterization of optimal signal structures. They show that the support of $\mu_{1}$ is a subset of $[0, p + u]$, and the buyer with posterior value $\omega_{2}$ must be indifferent between buying and not buying, i.e., $\omega^{*}_{2} = p + u$. However, this characterization does not exclude signal structures that map $\theta < p + u$ into both $\omega_{1}$ and $\omega_{2}$. Proposition 1 allows to derive a precise solution. In particular, given any signal structure, which generates $\{g, \omega\}$, there is a monotone partitional signal structure, which generates $\{g, \omega^{o}\}$, such that $\omega^{o} \succ_{g} \omega$. This implies $\omega^{*}_{2} \geq \omega_{2}$. Therefore, the probability of selling the product under the new signal structure cannot decrease.\(^{16}\) This implies that there exists an optimal solution in the class of binary partitions, such that the optimal cutoff $\theta_{1} \in (\theta, p + u)$ is given by

$$\omega^{*}_{2} = p + u = E[\theta|\theta \geq \theta_{1}] = \frac{1}{F(\theta_{1})} \int_{\theta_{1}}^{\theta} \theta dF(\theta).$$

It exists, since $\varphi(\theta_{1}) = E[\theta|\theta \geq \theta_{1}] - p - u$ is continuous in $\theta_{1}$, and

$$\varphi(\theta) = E[\theta] - p - u < 0 < E[\theta|\theta \geq p + u] - p - u = \varphi(p + u).$$

Also, $\theta_{1}$ is unique, since $E[\theta|\theta \geq \theta_{1}]$ is strictly increasing in $\theta_{1}$.

It is also worth noting that the model with an exogenous price is isomorphic to the leading prosecutor-judge example in Kamenica and Gentzkow (2011) in which the degree of the defendant’s guiltiness is a continuous variable. As our analysis demonstrates, the

\(^{15}\)Suppose $c = 0, p = 0.55$, and $\theta$ is distributed uniformly on $[0, 1]$. Consider two signal structures. The first one is monotone partitional with 3 subintervals and cutoffs $\{\theta_{1}, \theta_{2}\} = \{1/3, 2/3\}$. It generates the sequence of posterior means $\omega^{o} = \{1/6, 1/2, 5/6\}$ with corresponding probabilities $g = \{1/3, 1/3, 1/3\}$. The second one is non-partitional, such that $[\theta_{1}, \theta_{2}]$ is mapped into signal $s_{2}$ and $s_{3}$ with probabilities $3/4$ and $1/4$, respectively, and $[\theta_{2}, 1]$ is mapped into $s_{2}$ and $s_{3}$ with probabilities $1/4$ and $3/4$, respectively. This signal structure generates the sequence of posterior means $\omega = \{1/6, 7/12, 3/4\}$ with the same corresponding probabilities. Then, $\omega^{o} \succ g \omega$, however, $U(\omega, g) = 11/30 > 11/60 = U(\omega^{o}, g)$.

\(^{16}\)Since probabilities of the associated posterior means are identical, the only way to decrease the probability of selling is to generate $\omega_{2} < p + u$. 

15

16
optimal investigation policy in this case must be monotone partitional. This contrasts with the situation in which the degree of guiltiness is a discrete variable.

**Endogenous price.** Proposition 1 can also be used in order to solve the problem with an endogenous price. That is, the seller determines the signal structure/price pair, which maximize his ex-ante profit (16). This means that the optimal price \( p \) varies with the seller’s choice of the signal structure. This problem is non-trivial if \( \bar{u} + c \in (\theta, \bar{\theta}) \). First, by the same logic as above we can restrict attention to binary distributions of posterior means. Second, for a sequence of posterior means \( \omega = \{\omega_1, \omega_2\} \), the optimal price \( p \in \{\omega_1 - u, \omega_2 - u\} \). This provides the upper bound on the seller’s profit.\(^{17}\)

\[
EV \left( g, \omega, 2 \right) \leq EV \left( g, \omega, 2 \right) = g_1 \max \{\omega_1 - u - c, 0\} + g_2 \max \{\omega_2 - u - c, 0\}.
\]

Because \( V \left( \omega_i \right) = \max \{\omega_i - u - c, 0\}, i = 1, 2 \) is convex, then by Propositions 1 and 2, for any signal structure, which generates a sequence of posterior means \( \omega \), there exists a monotone partitional signal structure that generates \( \omega^o \) such that \( EV \left( \omega^o, g \right) \geq EV \left( \omega, g \right) \). Also, the signal structure such that \( \omega_2^o > \omega_1^o \geq u + c \) and \( p = \omega_1 - u \) is suboptimal, since it is strictly less profitable than the non-informative signal structure, which generates a single posterior mean \( E[\theta] > \omega_1 \), and the price \( p = E[\theta] - u \). This implies that it is sufficient to consider \( EV \left( g, \omega^o, 2 \right) \) such that \( \omega_1^o < u + c \), which leads to

\[
EV \left( g, \omega^o \right) = EV \left( g, \omega^o \right) = g_2 \left( \omega_2^o - u - c \right) = \int_{\theta_1}^{\bar{\theta}} \theta dF \left( \theta \right) - \left( 1 - F \left( \theta_1 \right) \right) \left( u + c \right).
\]

Maximizing \( EV \left( g, \omega^o \right) \) over \( \theta_1 \) returns the unique \( \theta_1^* = u + c \). Intuitively, this signal structure and price \( \omega_2 = E[\theta|\theta \geq u + c] \) extract all social welfare. It is worth noting that Saak (2006) derives the solution to this problem by imposing stricter assumptions on the set of signal structures and using more involved arguments.

### 5.3 Lobbying

Here, we apply our results to the lobbying example by Kamenica and Gentzkow (2011). Consider a setup à la Crawford and Sobel (1982) with the sender (the lobbyist) and the receiver (the politician) who communicate about state \( \theta \), which is ex-ante unknown to both players. The state is distributed on \( \Theta = [\underline{\theta}, \bar{\theta}] \) according to a continuous distribution function \( F(\theta) \). The players’ payoff functions are quadratic:

\[
u_R \left( a, \theta \right) = - (a - \theta)^2 \quad \text{and} \quad v_S \left( a, b, \theta \right) = - (a - \theta - b(\theta))^2 ,
\]

where \( b(\theta) = \alpha \theta + \beta, (\alpha, \beta) \in \mathbb{R}^2 \) reflects the bias in players’ ideal actions \( a_R(\theta) = \theta \) and \( a_S(\theta) = \theta + b(\theta) \).

The timing of the game is as follows. Given a finite signal set \( S \), the sender selects the signal structure \( \sigma \) at the beginning of the game. Then, the signal realization \( s \) is generated

\(^{17}\)If \( \omega_1 < u + c \leq \omega_2 \), then the optimal price is \( p = \omega_2 - u \), which results in \( EV \left( \omega, g \right) = g_2 \left( \omega_2 - u - c \right) = EV \left( \omega, g \right) \). If \( \omega_1 \geq u + c \), then \( EV \left( \omega, g \right) = \omega_1 - u - c \leq g_1 \left( \omega_1 - u - c \right) + g_2 \left( \omega_2 - u - c \right) = EV \left( \omega, g \right) \) if \( p = \omega_1 - u \), and \( EV \left( \omega, g \right) = g_2 \left( \omega_2 - u - c \right) = U(\omega, g) \) if \( p = \omega_2 - u \). Finally, if \( \omega_2 < u + c \), then \( EV \left( \omega, g \right) \leq 0 = EV \left( \omega, g \right) \).
according to $\sigma$. Finally, $s$ is observed by the receiver who then takes an action $a$. Because of the quadratic preferences, the receiver is interested only in the posterior mean of $\theta$ upon observing a signal realization $s$, so that her best response is $a (\mu_s) = \omega_s$. This results in the sender’s interim payoff

$$V (\mu_s) = -E_\theta [(a (\mu_s) - \theta - b (\theta))^2 | \mu_s] = -E_\theta [(\omega_s - \theta - \alpha \theta - \beta)^2 | \mu_s]$$

$$= - (1 + \alpha)^2 \text{Var} [\mu_s] - (\alpha \omega_s + \beta)^2,$$

where $\text{Var}[\mu] = E_{\mu_s, \theta} [(\theta - \omega_s)^2]$ is the posterior variance. Thus, $V (\mu_s)$ is not a function of the posterior mean $\omega_s$ only. On the other hand, the sender’s ex-ante payoff in an equilibrium can be expressed as

$$EV (g, \omega, n) = -E_{g, \theta, \omega_s} [(\omega_s - \theta - b (\theta))^2]$$

$$= (1 + \alpha)^2 (\text{Var} [\omega_s] - \text{Var} [\theta]) - E_{g_s} [(\alpha \omega_s + b)]^2 = E_{g_s} [\hat{V} (\omega_s)],$$

where $\text{Var}[\theta] = E [(\theta - E [\theta])^2]$, $\text{Var}[\omega_s] = E_{g_s} [(\omega_s - E [\theta])^2]$, and

$$\hat{V} (\omega_s) = (1 + \alpha)^2 [(\omega_s - E [\theta])^2 - \text{Var} [\theta] - (\alpha \omega_s + b)^2].$$

Thus, $\hat{V} (\omega_s)$ depends on the posterior mean $\omega_s$ only. Then, $\hat{V} (\omega_s)$ is strictly convex in $\omega_s$ if $\alpha > \frac{1}{2}$, linear in $\alpha$ if $\alpha = -\frac{1}{2}$ and strictly concave in $\omega_s$ if $\alpha < -\frac{1}{2}$. As a result, given $\alpha \geq -\frac{1}{2}$ and $n \geq 2$ signals, Proposition 1 implies that the optimal signal structure is monotone partitional, whereas for $\alpha < -\frac{1}{2}$ the optimal signal structure is fully uninformative. Also, our analysis shows that the optimal signal structure solely depends on $\alpha$, that is, the alignment of the sensitivities of the player’s ideal actions with respect to $\theta$ rather than the absolute difference in the ideal actions.

6 Conclusion

This paper offers a novel approach to solve the problem of optimal signal structures in Bayesian environments in which the sender’s ex-ante payoff is a function of the distribution of posterior mean values. It adds to the literature by identifying the relationship between distributions of posterior means generated by partitional and non-partitional signal structures. This relationship allows us to characterize a class of games in which the optimal signal structures has monotone partitional character. This class includes Bayesian persuasion mechanisms in which the sender’s ex-ante payoff is an average of a ordered sequence of functions of posterior means. By applying these results to various economic applications of Bayesian persuasion setups, we demonstrate the optimality of monotone partitional signal structures in such settings.

This paper raises two natural questions. First, economic examples of our paper suggest that the class of games in which the optimal signal structure of the sender is monotone partitional is quite broad and not restricted to an average of convex functions. The more precise characterization of this class beyond those characterized in Lemma 1 is an important question which requires further investigation. Second, following Kamenica and Gentzkow (2011), our setup does not impose any restrictions on the set of signal
structures feasible to the sender. On one hand, it allows us to characterize the upper bound on the ex-ante payoff of the sender who can manipulate the prior information in an arbitrary way. On the other hand, an important question is whether our approach based on the ranking of signal structures via the majorization of generated posterior mean values is applicable to Bayesian persuasion setups in which the set of posterior beliefs is restricted because of the incentive-compatibility constraints. Thus, identifying the class of games with restrictions on signal structures in which optimal signal structures have the monotone partitional character remains an open question.

Appendix

Proof of Proposition 1. Let \( H(s, \theta) \) be a joint distribution of \((s, \theta)\), which generates a discrete distribution over \(n\) posterior means \(\{g, \omega\}\), such that \(\omega(s) = E[v(\theta)|s], k = 1, \ldots, n\), and \(v(\theta)\) is right-continuous increasing and integrable. Let \(H^o(s, \theta)\) be a monotone partitional joint distribution with a sequence of cutoff states \(\{\theta_s\}_{s=1}^{n-1}\), such that \(\theta_s = F^{-1}(G(s))\), where \(G(s) = \sum_{i=1}^{s}g_i\). By construction, \(F(\theta_s) = G^o(s)\), or equivalently, \(F(\theta_s) - F(\theta_{s-1}) = g_s\). This implies that the marginal distributions of signals \(G^o(s)\) and \(G(s)\) associated with \(H^o(s, \theta)\) and \(H(s, \theta)\), respectively, are identical, i.e., \(G^o(s) = G(s)\) for all \(s\). Also, the joint distribution of \((s, \theta)\) in the partitional signal structure is \(H^o(s, \theta) = \min \{G(s), F(\theta)\}\), i.e., \(s\) and \(\theta\) are comonotonic. This means \(H^o(s, \theta) \geq H'(s, \theta)\) for all \(H'(s, \theta)\) in the Fréchet class \(\mathcal{M}(G, F)\) (Joe, 1997). Because \(H^o(s, \theta)\) and \(H(s, \theta)\) are in \(\mathcal{M}(G, F)\), \(H^o(s, \theta) \geq H(s, \theta)\), and \(v(\theta)\) is right-continuous increasing and integrable, then

\[
E[E[v(\theta)|s^o]|s^o \leq x] \leq E[E[v(\theta)|s]|s \leq x] \text{ for all } x,
\]

by Tchen (1980). Since \(G(s)\) is discrete and \(G(s) = G^o(s)\) for all \(s\), this implies

\[
E[\omega^o(s)|s \leq k] = \frac{1}{G^o(k)} \sum_{s=1}^{k} g_s^o \omega_s^o = \frac{1}{G(k)} \sum_{s=1}^{k} g_s \omega_s^o \leq \frac{1}{G(k)} \sum_{s=1}^{k} g_s \omega_s = E[\omega(s)|s \leq k], k = 1, \ldots, n - 1,
\]

so that \(\sum_{s=1}^{k} g_s^o \omega_s^o \leq \sum_{s=1}^{k} g_s \omega_s, k = 1, \ldots, n - 1\). Finally, \(E[\omega^o(s)] = \sum_{s=1}^{n} g_s \omega_s^o = \sum_{s=1}^{n} g_s \omega_s = E[\omega(s)] = E[v(\theta)]\) results in \(\omega^o \succeq_g \omega\). ■

Proof of Lemma 1. Suppose that \(V(x, s)\) is supermodular in \((x, s)\), that is, \(V'_x(x, s) \geq V'_y(x, t)\) for all \(x\) and \(s > t\).

(A) Suppose that in each pair of functions \(\{V(x, s + 1), V(x, s)\}_{s=1}^{n-1}\) at least one function is convex in \(x\). Hence, if \(V(x, t), t \in \{1, \ldots, n\}\) is non-convex in \(x\), this means that \(V(x, t + 1)\) and \(V(x, t - 1)\) are convex in \(x\). This implies

\[
V'_x(x, s) \geq V'_x(x, t + 1) \geq V'_y(y, t + 1) \geq V'_y(y, t) \text{ for } x \geq y \text{ and } s > t,
\]
where the first and the last inequalities follow from the supermodularity of $V(x,s)$ and the second inequality follows from the convexity of $V(x,t+1)$ in $x$. Similarly,

$$V'_y(y,t) \geq V'_y(y,t-1) \geq V'_x(x,t-1) \geq V'_x(x,s)$$

for $y \geq x$ and $t > s$,

where the first and the last inequalities follow from the supermodularity of $V(x,s)$ and the second inequality follows from the convexity of $V(x,t-1)$ in $x$.

(B) If $\min_x V'_x(x,s+1) \geq \max_x V'_x(x,s)$ for all $s = 1, \ldots, n-1$, then

$$V'_x(x,s) \geq \min_x V'_x(x,s) \geq \max V'_x(x,s-1) \geq \min V'_x(x,s-1) \geq \ldots \geq \max V'_x(x,t) \geq V'_y(y,t)$$

for all $x, y$ and $s > t$.

Proof of Proposition 3. Consider a mechanism with a signal structure $\sigma$ in which the sender’s ex-ante payoff is linear in $a_s = a(\omega_s, s)$ and $\omega_s$, $EV(g, \omega, n) = \sum_{s=1}^n g_s(a_s\omega_s + b_s) = \sum_{s=1}^n g_sV(\omega_s, s)$, where $V(\omega_s, s) = a_s\omega_s + b_s$ and $\{g, \omega\} \in G^+_n$. We construct a modified mechanism with a monotone partitional $\sigma^o$ such that $n^o \leq n$, $\{g^o, \omega^o\} \in G^+_{n^o}$, and $EV(g^o, \omega^o) \geq EV(g, \omega)$.

First, suppose that $a_s$ is not weakly increasing. Consider the following finite sequence of mechanisms. Put $n_0 = n$, $\sigma^0 = \sigma$, $\omega^0 = \omega$, and $a^0_s = a_s$. For $t = 1, \ldots, n-1$, if there is $s_{t-1} \in S_{n_{t-1}} \setminus \{1\}$, such that $a_{s_{t-1}} < a_{s_{t-1}-1}$, then consider a $t-$mechanism with the signal set $S_{n_t} = S_{n_{t-1}} \setminus \{s_{t-1}\}$ of size $n_t = n - t$ and the signal structure $\sigma^t$, which is derived from $\sigma^{t-1}$ as follows:

$$\sigma^t_s(\theta) = \begin{cases} 
\sigma^{t-1}_{s-1}(\theta) & \text{if } s \notin \{s_{t-1}-1, s_{t-1}\}, \\
\sigma^{t-1}_{s_{t-1}-1}(\theta) + \sigma^{t-1}_{s_{t-1}}(\theta) & \text{if } s = s_{t-1}-1.
\end{cases}$$

That is, $\sigma^t = \{\sigma^t_s(\theta)\}_{s=1}^n$ is identical to $\sigma^{t-1}$ for signals $s \notin \{s_{t-1}-1, s_{t-1}\}$, but collapses signals $s_{t-1}-1$ and $s_{t-1}$ into a single signal $s_{t-1}-1$. Hence, $\sigma^t$ generates $\{g^t, \omega^t\}$ of size $n_t$, such that

$$g^t_s = \begin{cases} 
g^{t-1}_{s_{t-1}-1} & \text{if } s \notin \{s_{t-1}-1, s_{t-1}\}, \\
g_{s_{t-1}-1} + g_{s_{t-1}} & \text{if } s = s_{t-1}-1,
\end{cases}$$

and

$$\omega^t_s = \begin{cases} 
g^{t-1}_{s_{t-1}-1} & \text{if } s \notin \{s_{t-1}-1, s_{t-1}\}, \\
g_{s_{t-1}-1} + g_{s_{t-1}} & \omega^{t-1}_{s_{t-1}-1} + g_{s_{t-1}} & \text{if } s = s_{t-1}-1.
\end{cases}$$

By construction, $\{g^t, \omega^t\} \in G^+_{t-1}$ implies $\{g^t, \omega^t\} \in G^+_t$. Also, define the action rule in the $t-$mechanism as

$$a^t_s = \begin{cases} 
a^{t-1}_{s_{t-1}-1} & \text{if } s \notin \{s_{t-1}-1, s_{t-1}\}, \\
a^{t-1}_{s_{t-1}-1} & \text{if } s = s_{t-1}-1,
\end{cases}$$

with probabilities $\frac{g_{s_{t-1}-1}}{g_{s_{t-1}-1}+g_{s_{t-1}}}$ and $\frac{g_{s_{t-1}}}{g_{s_{t-1}-1}+g_{s_{t-1}}}$, respectively.

\[\text{Equivalently, an action } a^{t-1}_{s_{t-1}-1} \text{ in the } t-\text{mechanism can be a randomization between } a^{t-1}_{s_{t-1}-1} \text{ and } a^{t-1}_{s_{t-1}} \text{ with probabilities } \frac{g_{s_{t-1}-1}}{g_{s_{t-1}-1}+g_{s_{t-1}}} \text{ and } \frac{g_{s_{t-1}}}{g_{s_{t-1}-1}+g_{s_{t-1}}}, \text{ respectively.}\]
and
\[ b'_s = \begin{cases} 
  b'_{s-1} & \text{if } s \notin \{s_{t-1} - 1, s_{t-1}\}, \text{ and} \\
  E[b_s|s \in \{s_{t-1} - 1, s_{t-1}\}] & \text{if } s = s_{t-1} - 1. 
\end{cases} \]

Denote \( EV^t(g^t, \omega^t) \) the sender’s ex-ante payoff in \( t \)-mechanism with the generated distribution \( \{g^t, \omega^t\} \). Then, the marginal ex-ante payoff \( \Delta EV^t = EV^t(g^t, \omega^t) - EV^t(g^{t-1}, \omega^{t-1}) \) of \( t \)-mechanism is
\[
\Delta EV^t = g_{s_{t-1}-1}a_{s_{t-1}-1}^t\omega_{s_{t-1}-1}^t - (g_{s_{t-1}-1}a_{s_{t-1}-1}^{t-1}\omega_{s_{t-1}-1}^{t-1} + g_{s_{t-1}}a_{s_{t-1}}^{t-1}\omega_{s_{t-1}}^{t-1}) \\
= (g_{s_{t-1}-1}a_{s_{t-1}-1}^{t-1} + g_{s_{t-1}}a_{s_{t-1}}^{t-1})\omega_{s_{t-1}-1}^t - (g_{s_{t-1}-1}a_{s_{t-1}-1}^{t-1}\omega_{s_{t-1}-1}^{t-1} + g_{s_{t-1}}a_{s_{t-1}}^{t-1}\omega_{s_{t-1}}^{t-1}).
\]

By construction, \( \{\omega_{s_{t-1}-1}^t, \omega_{s_{t-1}}^t\} \supseteq \{\omega_{s_{t-1}-1}^{t-1}, \omega_{s_{t-1}}^{t-1}\} \). Also, since \( a_{s_{t-1}} < a_{s_{t-1}-1} \), then \( V(\omega, s) \) is submodular for \( s \in \{s_{t-1} - 1, s_{t-1}\} \), that is, the inequality (15) is reversed. Therefore, by Proposition 2, we have
\[
EV^t(g^t, \omega^t) \geq EV^{t-1}(g^{t-1}, \omega^{t-1}).
\]

Because \( n \) is finite, after \( T \in \{0, \ldots, n - 1\} \) iterations of the original mechanism we obtain a \( T \)-mechanism with signal structure \( \sigma^T \) of size \( n_T = n - T \), such that \( \{g^T, \omega^T\} \in \mathcal{G}^+_T \) and \( \{a_s^T\}_{s \in S_T} \) is weakly increasing. (If \( a_s \) is weakly increasing, this corresponds to \( T = 0 \).) Hence, (15) holds for \( V(\omega^T, s) = a_s^T\omega^T_s + b^T_s, s \in S_T \).

Next, consider a mechanism with the signal structure \( \sigma^+ \) of size \( n^+ \leq n_T \), which collapses subsets of signals that generate the same posterior mean \( \omega_s = \omega_k^+ \), i.e., \( S^+_k = \{s \in S_T | \omega_s = \omega_k^+\} \), into a single signal \( k \in S^+_k \), by transforming \( \sigma^T \) into \( \sigma^+_k(\theta) = \sum_{s \in S^+_k} \sigma^T_s(\theta) \). Also, we modify the action rule as \( a^+_k = E[a_s|s \in S^+_k] = \frac{1}{g^+_k} \sum_{s \in S^+_k} g_s a_s \) and \( b^+_k = E[b_s|s \in S^+_k], \) where \( g^+_k = \sum_{s \in S^+_k} g_s \).\(^{19}\) Since \( a^+_k \) is increasing, then (15) holds for \( V(\omega^+_s, s) = a^+_s\omega^+_s + b^+_s, s \in S^+_o \). The sender’s ex-ante payoffs \( EV^+(g^+, \omega^+) \) in this mechanism is
\[
EV^+(g^+, \omega^+) = \sum_{k \in S^+_o} g^+_k \left( a^+_k\omega^+_k + b^+_k \right) = \sum_{k \in S^+_o} g^+_k \frac{1}{g^+_k} \sum_{s \in S^+_k} g_s (a_s\omega_k + b_s) \\
= \sum_{s \in S_T} g_s (a_s\omega_s + b_s) = EV^T(g^T, \omega^T).
\]

Finally, consider a mechanism with the monotone partitional signal structure \( \sigma^o \) of size \( n^o \) which generates \( \{g^o, \omega^o\} \in \mathcal{G}^+_o \), and the action rule \( \{a^+_s, b^+_s\} = \{a^+_s, b^+_s\}, s \in S^+_o \). Since \( \sigma^o \) is monotone partitional, then \( \{g^+, \omega^o\} \in \mathcal{G}^+_o \) by construction. Therefore, Proposition

\(^{19}\)Similarly to the previous step, action \( a^+_k \) can be replaced by the randomization between actions in \( \{a_s\}_{s \in S^+_k} \).
1 implies $\omega^o \succ_{g^+} \omega^+$, and (15) and Proposition 2 imply

$$EV^o(g^+, \omega^o) = \sum_{s \in S_{\omega^o}} g_s^+ (a_s^+ \omega_s^o + b_s^+) \geq \sum_{s \in S_{\omega^o}} g_s^+ (a_s^+ \omega_s^+ + b_s^+) = EV^+(g^+, \omega^+)$$

$$= EV^T(g^T, \omega^T) \geq EV(g, \omega).$$

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