On Optimal Procurement with Budget Constrained Bidders

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Abstract

We consider auctions/tenders for the procurement of goods and services in the setting where the potential contractors face ex-post risks that may lead to cost overruns. The contractors have limited access to credit and are protected by limited liability. We are concerned with the optimal choice of the procurement scheme in this setting. The problem is non-trivial since ex-post risks are not verifiable and so cannot be contracted upon by a simple cost plus contract. We show that in our setting the procuring agency minimizing the expected costs of the project greatly benefits by allocating a share of the award ex-ante, at the time of contracting, with the remainder due ex-post, after the completion of the project. We identify the trade-offs that the procurement agency faces in such settings and characterize the optimal award sharing schemes for different types of budget constraints.

1 Introduction

Procurement of goods and services has attracted much attention in the economic literature because of the volume of transactions it generates. The procurement by federal, state and local governments in the USA accounts for more than 10% of GDP. The goods and services procured range in complexity and cost, as a result a variety of procurement mechanisms are utilized. The rights to provide complex goods, like new buildings, infrastructure or weaponry are allocated at government tenders. These involve several potential contractors engaged in the consultations about the objectives of the procurer, building prototypes, estimating production costs and finally submitting the offers at the tender where the price is determined by competitive bidding. In USA, Federal Acquisition Regulation strongly encourages the use of such tenders and particularly open competitive bidding whenever possible.1

Private sector engages in similar tender process when building or renovating homes.

1https://www.acquisition.gov/far/
Procurement tender can be analyzed along the same lines as an auction, except that the roles of the sellers and the buyers are reversed. The buyer in, say, real estate auction possesses private information about the value of the house. The contractor that participates in a government tender possesses private information about its own cost of providing the object, say constructing a new bridge.\(^2\) The seller seeks to maximize the sale price of the house, analogously, the agency responsible for the procurement minimizes the cost of construction. The methodology developed for the analysis of auctions can, indeed, be applied directly to the procurement settings if the cost of the project is known with certainty to the potential contractors at the time of the tender. In fact, quite to the contrary, neither the true value of an old house sold at an auction, nor the cost of constructing the bridge, is known at the tender and is discovered by the winner much after the bidding is over.

Here lies a distinction between the auction and the tender. If the winner of the real estate auction discovers later that the value of his house is lower because it is more expensive to renovate, the winner cannot ask the seller for a compensation. If the winner of the procurement tender learns later that the bridge is impossible or unprofitable to complete given the budget, the contractor abandons the project protected by the limited liability.\(^3\) The government is then left to finish the construction at an extra cost. The buyer in the procurement tender therefore suffers from the bidders flawed cost estimates, whereas the seller in the auction is impervious to similar mistakes of the bidders.

The flaws in the costs estimates, that lead to the cost overruns as the project unfolds can be very substantial. Peck and Scherer (1962) estimate that for U.S. defence programs development cost exceed the original predictions by 220 percent on average. Cost overruns are also prevalent in the building industry. “Big Dig” – a highway tunnel under the central part of Boston has been constructed at the cost of 1.6 billion dollars over the original estimate. Opera House in Sydney took extra ten years to complete, went through numerous design revisions with an intent to lower the cost and yet exceeded the original budget by more than 14 times. The responsibility for the cost overrun can lie with the procurer—unexpected change of objectives, the contractor—inadequate design, or “third party”—unexpected soil composition, shocks to the material prices, new environmental and building regulations. As is often the case, the responsibility is hard to assign and remains a subject of the heated debate for years, see Boston Globe on the issues surrounding Big Dig project.\(^4\)

We provide an original model of procurement process with asymmetric information about the cost of the project. The cost of the project is subject to the shock which

\(^2\)The motivation for why the cost of construction can be thought of as private information of the contractor is given in the description of the model.

\(^3\)In the US during 1990-1997 more than 80,000 contractors filed for bankruptcy leaving behind unfinished private and public construction projects with liabilities exceeding $21 billion. (Dun & Bradstreet Business Failure Record).

\(^4\)http://www.boston.com/news/specials/bechtel/part_1
is realized after the construction has started. The value of the cost overrun is private information of the winner of the tender. This precludes the use of cost plus contracts, where the burden of the cost overrun is transferred to the buyer, see Bajari and Tadelis (2001). Our analysis thus concentrates on so called fixed-price contracts, with the level of the price determined by competitive bidding. This form of the contract is most commonly used in the government defence procurement, see Fox (1974), McAfee and McMillan (1986) and in the public infrastructure procurement, see Tadelis and Bajari (2006). The government allocates the project via the second lowest bid tender. The overall amount of the award that the winner is set to receive is determined at the time of the tender. The novelty of the approach is that the award can be allocation in proportions, one part ex-ante—before the construction starts and the remainder ex-post—after the completion of the project. The proportion is known to the bidders at the time of the tender and certainly affects their bidding behavior. We consider two settings, in one the firms have unlimited budgets to cover any possible costs, in the more general one the firms have limited access to credit. The budget constraint constitutes another aspect of the contractor’s private information in the spirit of Che and Gale (1998). We assume that the contractors are protected by limited liability therefore our work also contributes to the new and growing field of mechanism design without commitment.

As we show, the following fundamental tensions arise. The higher is the proportion of the tender paid ex-ante, the lower are the bids. The bidders protected by the limited liability are not afraid to walk away from the project in case of the large cost overrun and bid quite aggressively. In the extreme case of full ex-ante payment, the bidders bid their private costs and abandon the project whenever the shock causes cost overruns. The buyer contracts to complete the project at the lowest possible price but faces the additional costs due to incomplete projects more often. We assume that it is more costly to complete the project for the buyer than for the winning contractor. With fully ex-post payment the contractors are more exposed to the risks associated with the shocks and inflate their bids accordingly. They, however, have the strongest incentives to complete the project. In case of unlimited budgets we show that it is optimal to pay the whole amount of the award ex-post. When the budgets are limited fully ex-post payment often leads to incomplete projects because the winner has no funds to finish the project. At the same time if the award is fully paid ex-ante the winner has no incentive to finish the project in case of the cost overrun. We show that in the case of limited budgets the optimal proportion is interior, it is optimal to pay a fraction \( r \in \left(0, \frac{1}{2}\right) \) of the tender ex-ante and the remainder \((1 - r)\) ex-post if and when the project is completed.

The rest of the paper is organized as follows. Section 2 summarizes the relevant literature, Section 3 presents the model, Section 4 deals with the special case of

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5While the second lowest bid format is rarely used for procurement, we use it as a notationally simpler equivalent to an open descending price format that is commonly used. The results qualitatively are likely to be the same.
unlimited budget, Section 5 with the case of “lean budget” where each bidder just has enough own funds for the initial investment. The special cases are not only of independent interest but also serve as building blocks of the equilibrium construction for the general two dimensional case of arbitrary budgets presented in Section 6. Section 6 also contains our main result. Section 7 extends the results for unlimited and lean budgets to the case where shocks $z$ arrive from distribution with arbitrary c.d.f. $F_z$. The equilibrium strategies then cannot be obtained in closed form but optimal sharing schemes are the same as for binary distribution of shocks in sections 4 and 5. We expect to generalize the main result in Section 6 to the general distribution of the shocks.

2 Related Literature

The asymmetric information literature on procurement is quite large, see the books by Laffont and Tirole (1993) and McAfee and McMillan (1988) for surveys. The modern approach to the procurement problem recognizes the elements of adverse selection, the contractor has better information about the production cost than the buyer, and of moral hazard, the contractor may reduce the cost of production by exerting more effort. The approach to the adverse selection is the usual screening, the buyer offers a menu of contracts from which the seller selects a particular one thereby reducing the informational asymmetry. To our knowledge, the screening stage is always modelled as one on one interaction, that is the buyer has already selected the contractor for the project and now the problem is reduced to providing incentives for this particular contractor.

Another approach to informational asymmetry is competitive bidding. Here of course, there are multiple potential contractors who posses private information about the production cost. The competitive bidding in the setting where the contractors know with certainty what the cost will be before they start production is equivalent to an auction and can be analyzed along the lines of Myerson (1981) or Riley and Samuelson (1981) optimal auction, (see also Manelli and Vincent (1995) for optimal procurement mechanism when goods vary in quality.)\(^6\) This setting is however, unrealistic. Ashley and Workman (1986) in a survey of contractors and buyers in USA building industry report that project engineering must be 40-60% complete to establish the reasonable estimate for the cost.

McAfee and McMillan (1986) consider a model similar to ours where the costs are subject to ex-post shocks but also may be reduced by the winner’s effort. The buyer therefore faces both adverse selection and moral hazard. McAfee and McMillan consider bidding for a linear incentive contract that factors in both the cost (assumed observable ex-post) and the winning bid and derive the optimal contract in this class.

\(^6\)The literature on auctions with budget constrained bidders is small, see Che and Gale (1998) and (2000), Che, Gale and Kim (2011), Pai and Vohra (2009).
The contract in McAfee and McMillan therefore combines the elements of the fixed price and cost plus contracts. Such contracts are infeasible in our setting as both the original estimate and the final value of the cost remain the private information of the winner. Parlane (2003) also deals with bidding in the model with potential cost overruns and bidders protected by limited liability. She shows that among all efficient mechanisms in which only the winner gets paid, the lowest bid tender leads to the highest expected price, thus minimizing the chances of bankruptcy. In our setting where the bidders are budget constrained neither the first nor the second lowest bid tender is efficient.

Piccione and Tan (1996) consider a model in which ex-ante symmetric potential contractors invest in the cost reduction technology and then compete for the procurement contract. The cost can be further reduced by exerting effort with no exogenous shocks. If the buyer is able to commit to the procurement mechanism before the investment stage the first best solution can be implemented by either first or second price auction under quite general conditions. If the buyer chooses the mechanism after the initial investment stage the level of investment is suboptimal. Arozamena and Cantillon (2004) consider a similar model, however, the firms are ex-ante heterogeneous, investment is only made by one firm and the level of investment is made observable introducing asymmetry in the competition. If investment affects which firm is the most efficient then the first-price auction will induce less investment than second-price auction. In all of these papers the contractors have sufficient budgets to cover all possible costs and it is assumed that the award is paid ex-post.

Burguet et. al. (2006) study procurement in the setting where the bidders are symmetric with respect to both ex-ante and ex-post costs, but have privately known budgets. With limited liability and budget being the only screening variable incentive compatibility implies that win probabilities are monotonically decreasing with the budgets, thus an auction allocates the contract to the least financially solvent contractor. In our setting, naturally, privately known budgets affect the bidding strategies but those mainly depend on the privately known costs, see the equilibrium construction in Section 6.

Contract theory literature that deals with one on one relationship between the buyer and the contractor is huge. Here we concentrate on the papers that deal with procurement and issues of cost overruns. Again with few exceptions all of this literature assumes no credit constraints. Bajari and Tadelis (2001) build the model in which there is no role for the ex-ante asymmetric information, however, the design may be subject to ex-post changes. The contractor possesses private information about the cost of the change of the original design. The buyer ex-ante decides on the completeness of the design. With more complete design the likelihood that it will need to be renegotiated ex-post is smaller, which reduces the ex-post cost of the buyer. Providing more complete designs is, however, costly ex-ante. Bajari and Tadelis compare cost plus and fixed price contracts and show that simple projects will be procured via fixed price contracts and will have a high level of design completeness.
More complex projects will be procured at cost plus contracts and will have low level of completeness. Crocker and Reynolds (1993) provide an empirical study of the effects of various types of contracts used in defence procurement.

In Riordan and Sappington (1988) the buyer either uses contingent prices or can commit to negotiate a price in the second period once uncertainty about the shock is realized. It is often difficult to provide a complete contract that will cover any contingency and impossible to do so if the costs are not verifiable like in our model. Committing to negotiate a price in the future may be suboptimal since the contracts will be submitting very low bids trying to lock in the contract before the shock is realized and take advantage of the situation to increase the price at the renegotiation stage.

Tirole (1986) in an incomplete contract setting examines the ex-ante investment in cost reduction by the contractor. Parties bound by an incomplete contract have an incentive to renegotiate after acquiring the information about the shock. If the contractor’s investment is not observed by the buyer, in a vast majority of bargaining schemes, the firm invests less than it would under symmetric information and a complete contract. If the investment can be observed by the two parties that jointly decide on the firm’s investment level (and on the investment sharing rule), the mutually agreed-on investment may be lower than its level under unobservability, higher than the first-best level, or intermediate between the two values.

3 Model

The government, later a buyer, needs to realize a new project that once completed it values at $V$. There is a number $N$ of contractors/firms that have a capacity to complete the project. Each contractor $i$ is given the right to examine the requirements of the project to be completed and forms its estimate of the cost of the initial investment, $c_i$. This cost is private information of firm $i$, it is unverifiable and cannot be contracted upon. The buyer, though, can ascertain if the necessary initial investment has been made.

Any contractor that is awarded the contract and invests in it faced ex post risk. Before completion, it may have to incur extra costs $Z > 0$ with probability of such an adverse event occurring $p > 0$. These costs can arise due to a management oversight,

\footnote{We implicitly assume here that each contractor is endowed with exogenously given amount of capital and labor and that contractors are symmetric in that respect. The asset of the contractor that is responsible for the private information in the model is the managerial talent of the contractor. This talent is intangible and unverifiable, cannot be therefore used as collateral, but can reduce the cost of the project. A large construction project involves coordination of work of hundreds of independent sub-contractors and managerial costs can be very substantial. Bajari and Tadelis (2001) provide an excellent description of construction industry in the USA and the work on Getty Art Centre in Los Angeles.}

\footnote{This assumption is needed for off equilibrium considerations. In equilibrium the contractor is provided with incentives to make an initial investment.}
an adverse shock to input costs, or some other unforeseen contingencies. We assume
that these costs are also not verifiable and so cannot be contracted upon and that they
are the same both in magnitude and probability for any firm. The latter assumption
is made solely for simplicity of exposition. The firm may decide not to cover these
extra costs, in which case the project will not be completed.\footnote{This is an extreme form of limited liability, a more realistic assumption that the buyer can
enforce a contract by incurring a cost $\sigma > 0$ leads to the same qualitative results.}

The buyer uses a second-lowest sealed bid auction to allocate the contract, that
is, it asks for each contractor to make a bid, awards the contract to the one with the
lowest bid, and sets the value of the award $\beta$ equal to the second lowest bid. The
crucial choice of the buyer is to decide on the share of the award to be transferred at
the initial investment stage, $r \in [0, 1]$. Thus, once the tender finishes and the award
is set at $\beta$, the buyer pays to the winner $r \beta$, and once the project is completed, it pays
the remainder $(1-r)\beta$. We assume that if the shock hits and the winner decides not
to cover extra costs, the buyer is not obligated to pay the ex-post part of the award.
If she decides to complete the project it has to incur extra costs of $X > Z$.\footnote{Implicitly we assume that there is another tender process that results in some contractor being
chosen to complete the project. It is important that the original contract is not renegotiated.
In addition, the winner of the original tender is excluded from participation in the second tender.}
The inequality reflects possible extra costs (in addition to $Z$) from time delays, inadequate
expertise, and other factors.

Each contractor faces a “hard budget” constraint when making each investment.
Contractor $i$ initially has $m_i$ in cash, which is her private information. For each
investment it wants to make, $i$ has to have enough cash to cover it. We assume that
each contractor has limited liability in the sense that it cannot be forced to make
an unprofitable initial or additional investment. Thus, each contractor has a two-
dimensional private information $(c_i, m_i)$, independently drawn from other firms from
the same cumulative distribution $F$. Thus, the common knowledge includes values
of $V$, $X$, $Z$, $p$, and c.d.f. $F[c, m]$ on $[C_{\min}, C_{\max}] \times [S_{\min}, S_{\max}]$.\footnote{At this stage we make no further assumptions on $F[c, m]$. Deriving the expected cost of the
project we will assume that $m_i \geq c_i$ for each contractor. We will discuss this assumption in more
detail after the equilibrium is derived. Roughly $m_i \geq c_i$ in our model is guaranteed if the contractors
have access to competitive banking sector.}

Finally, the objective of the buyer is to maximize its value from having the project
completed considering the payments to be made. Given our assumptions it amounts
to minimizing the expected cost of the project, inclusive or all the payments the buyer
faces. The objective of each contractor is to maximize its profits. Both the buyer
and contractor are risk-neutral and have the same value for money.

A trivial immediate observation is that it is optimal for the buyer to make the
incentives for the initial and additional investments the strongest, that is, pay $r \beta$ if
and only if the necessary initial investment occurs, pay $(1-r)\beta$ if and only if the
additional investment is made and the project is actually completed, and pay nothing otherwise. We will show that in equilibrium the winner indeed has the incentives to invest \( c_i \).

Thus, the timing of the game can be represented as follows. At time 0, the buyer announces \( r \), each contractor \( i \) learns \( (c_i, m_i) \). At time 1, the tender is being conducted, the contractors submit bids, the government selects the winner, say bidder \( j \), the size of the award \( \beta \) is determined. Shortly thereafter, the first installment of \( r\beta \) is made and the initial investment of \( c_j \) is made. At time 2, the shock—the need of an extra investment—may hit. If it does not, the project is completed and the ex post payment is made. If the shock hits, the winner decides whether to make the extra investment. If she does not, she does not receive an ex post payment, and the buyer pays \( X \) to the third party, otherwise the project is completed and the ex-post payment is made. The timing is therefore as follows.

\[
\begin{array}{c|c|c|c}
\text{all learn } r & \text{all learn } (c_i, m_i) & \text{winner gets } r\beta & \text{learns } Z \\
\text{and bid} & \text{winner invests } c_j & \text{finishes and gets } (1 - r)\beta \text{ or quits} & \\
0 & 1 & 2 & t
\end{array}
\]

Figure 1: Timeline

Our model therefore extends the conventional screening model of procurement in three dimensions. We add post-allocation shocks, limited liability and budget constraints. The buyer’s sole control variable is \( r \), the idea is that such modest departure from the standard ex-post payment scheme may be well received by the industry regulators.

We further provide some illuminating examples where the distributions \( F(c, m) \) is of special interest. The equilibria derived in these examples are later used as building blocks for the equilibrium construction in the general case.

## 4 Unlimited Budgets

In this section we examine the simplified problem where for each firm the budget constraint is never binding \( s_i > c_i + Z \). To gain the intuition we start with considering the regimes of fully ex-post and fully ex-ante payments. Then we consider the payment scheme that corresponds to arbitrary \( r \). We show that with unlimited budgets fully ex-post payment is optimal. Note that this case corresponds to degenerate distribution with \( s_i = S > C + Z \) for every bidder.
4.1 Full Ex-post Payment

We first examine the case of fully ex-post payment and derive the equilibrium bidding strategies. Consider bidder $i$ with cost $c_i$. Suppose this bidder submits bid $b$. The amount of the awarded tender, $\beta \geq b$, by the rules of the tender. To calculate the expected profit and the optimal value of $b$ consider the following realizations. If total cost $Z + c_i \leq \beta$, it is optimal to complete the project which leads to the profit of bidder $i$ : $\pi_i = \beta - Z - c_i$. If $Z + c_i > \beta$, then $\pi_i = \max\{-c_i, \beta - Z - c_i\}$. Since the amount $c_i$ is sunk before bidder $i$ learns $Z$; when $Z < c_i$ the winning bidder optimally abandons the project and the profit is again $\pi_i = \beta - Z - c_i \geq -c_i$. Thus if $b > Z$ the profit of bidder $i$ conditional on winning the tender is either $\beta - c_i$ or $\beta - c_i - Z$ depending on whether the shock hits or not. To derive the expected payoff of bidder $i$, consider all the bidders but bidder $i$; find among them the bidder with the lowest cost and denote with $\beta_1(c)$ his bidding function. Assume that $\beta_1$ is strictly increasing and continuous such that inverse $\beta_1^{-1}(b)$ is well defined. Denote with $G(\beta_1)$ the c.d.f. of $\beta_1$. Introduce

$$
\pi(c, b) = \int_{\beta_1^{-1}(c)}^{\beta_1^{-1}(\pi)} (\beta_1 - c) \, dG(\beta_1).
$$

If bidder $i$ with cost $c_i$ submits bid $b > Z$, her expected payoff

$$
E[\pi_i] = (1 - p) \pi_i(c_i, b) + p \int_{\beta_1^{-1}(c)}^{\beta_1^{-1}(\pi)} (\beta_1 - Z - c_i) \, dG(\beta_1) =
$$

$$
E[\pi_i] = \int_{\beta_1^{-1}(c)}^{\beta_1^{-1}(\pi)} (\beta_1 - pZ - c_i) \, dG(\beta_1)
$$

Therefore it is optimal to bid $b^* = c_i + pZ$ as long as $b^* \geq Z$. Note that at $b^* = Z$, $c_i = (1 - p)Z$ that is the optimal bid

$$
b^* = c_i + pZ \text{ for } c_i \geq (1 - p)Z
$$

Now, if bidder $i$ with cost $c_i$ submits bid $b < Z$ the awarded tender $\beta$ can be above or below $Z$. If $\beta \geq Z$ as before there are two realizations of the profit $\pi_i = \beta - c_i$ or $\pi_i = \beta - c_i - Z$ depending on whether the shock hits or not. If $\beta < Z$ and there happens to be a shock $Z$, extra investment will not be undertaken and bidder $i$ will optimally leave the project with the profit $\pi_i = -c_i$. The expected payoff of bidder $i$ who submits bid $b_i < Z$ :

$$
E\pi_i = (1 - p) \pi_i(c_i, b) + p \left[ \int_{b}^{Z} (-c_i) \, dG(\beta_1) + \int_{Z}^{\beta_1^{-1}(\pi)} (\beta_1 - Z - c_i) \, dG(\beta_1) \right]
$$
collecting the terms and simplifying:

\[ E\pi_i = (1 - p) \int_b^{\beta_1^{-1}(x)} (\beta_1 - c_i / (1 - p)) dG(\beta_1) + p \int_Z^{\beta_1^{-1}(x)} (\beta_1 - Z) dG(\beta_1) \]

The optimal bid is therefore

\[ b^* = c_i / (1 - p) \text{ for } b^* < Z. \]

Note again that at \( b^* = Z, c_i = (1 - p) Z \) so that for bidders with cost \( c < (1 - p) Z \) it is optimal to bid according to \( b^* = c_i / (1 - p) \). The optimal bid overall

\[ b^* = \begin{cases} 
  c_i / (1 - p) & \text{if } c_i < (1 - p) Z \\
  c_i + pZ & \text{if } c_i \geq (1 - p) Z 
\end{cases} \]

Note also that at \( c_i = (1 - p) Z \).

\[ c_i + pZ = c_i / (1 - p) \]

so that the derived optimal bidding function is increasing and continuous but involves a kink at \( c = (1 - p) Z \).

Now let’s examine the expected cost of the project. Denote with \( c_1 \) the cost of the winner, the lowest cost. When \( c_1 \geq (1 - p) Z \) in equilibrium the project is always completed. In case the shock hits, the contractor compares the extra cost of completing the project \( Z \) with the payment that will result only if the project is completed. This payment exceeds \( c_1 + pZ \) and the contractor optimally completes. The cost of the project in such realization is \( c_2 + pZ \), where \( c_2 \) is the second lowest cost. The expected cost over these realizations is \( E[c_2] + pZ \). In case \( c_1 < (1 - p) Z \) and \( \beta > Z \) the project is also completed by the winner. Note that since the bid of the winner \( c_1 / (1 - p) < c_1 + pZ \) the expected cost of the project is lower than \( E[c_2] + pZ \). In case \( \beta < Z \) the project is not completed, but \( c_1 \) is invested from bidder 1 funds. The buyer has to pay \( X \) to complete the project.

Overall the project costs at most \( E[c_2] + pZ \) when completed by the winner and \( X \) when completed by the buyer when the tender is paid ex-post.

### 4.2 Full Ex-ante Payment

Now suppose the payment is made ex-ante. The winning bidder invests \( c_i \) to carry on with the project until the shock is realized. When the realized shock is 0 the project is competed and the winner of the tender gets the payoff \( \beta - c_i \). When the realized shock is \( Z \), the project is not completed and the winner of the tender quits the project. Since the initial costs and the value of the shock are not verifiable the
winner optimally expropriates the residual $\beta - c_i$ no matter what the shock is. Thus $i$’s expected payoff conditional on winning is $\beta - c_i$. The optimal bid

$$b^* = c_i \text{ for all } c_i.$$  

The buyer obtains the project at the second highest cost but whenever the shock hits they have to complete the project for the cost $X > Z$.

$$E[\beta]^- = E[c_2] + pX,$$

In the realizations where with ex-post payment the project cost is $X$ it is $E[c_2] + X$ with ex-ante payment. In all other realizations the expected cost with ex-post payment

$$E[\beta]^+ \leq E(c_2) + pZ \leq E(c_2) + pX = E[\beta]^-. $$

Thus with unlimited budgets ex-post financing dominates ex-ante if $X > Z$.

### 4.3 Fixed Share Scheme

More generally suppose the fraction $r$ of the award $\beta$ is being paid ex-ante and the remainder upon completion of the project. Even though the bidders do not face hard budget constraints they may be unwilling to finish the project if the ex-post portion of the award $(1 - r)\beta < Z$. For bidder $i$ with cost $c_i$ in case the shock hits the payoff is $\pi_i = \beta - c_i - Z$ if $\beta (1 - r) \geq Z$. If $\beta (1 - r) < Z$ bidder $i$ will optimally quit the project with the payoff $\pi_i = r\beta - c_i$. Suppose bidder $i$ with cost $c_i$ bids $b \geq Z/(1 - r)$ and $\beta_1^{-1}(\cdot)$ is the inverse of the bidding function of the opponent with the lowest cost. Bidder $i$’s expected payoff is:

$$\int_b^{\beta_1^{-1}(\pi)} (\beta_1 - c_i - pZ) dG(\beta_1)$$

which is again optimized with $b^* = c_i + pZ$.

When bidder $i$ with cost $c_i$ bids $b \leq Z/(1 - r)$ her expected payoff is

$$E \pi_i = (1 - p) \pi_i(c_i, b) + p \int_b^{Z/(1 - r)} (r\beta_1 - c_i) dG(\beta_1) + p \int_{Z/(1 - r)}^{\beta_1^{-1}(\pi)} (\beta_1 - c_i - Z) dG(\beta_1).$$

Collecting the terms that are affected by $b$

$$E \pi_i = \int_b^{Z/(1 - r)} ((1 - p(1 - r))\beta_1 - c_i) dG(\beta_1) + \int_{Z/(1 - r)}^{\beta_1^{-1}(\pi)} (\beta_1 - c_i - pZ) dG(\beta_1).$$
For $b < Z/(1-r)$ with $q = 1-r$ the optimal bid is then $b^* = c_i/(1-pq)$.

Introduce

$$c^* = \frac{Z}{q} - pZ.$$

The optimal bid overall

$$b^* = \begin{cases} 
  c_i/(1-pq) & \text{if } c_i < c^* \\
  c_i + pZ & \text{if } c_i \geq c^* 
\end{cases}.$$

Note that at $c^*$, $c^* + pZ = c^*/(1-pq)$, so that the bidding strategy involves a kink at $c^*$, see Figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=.5\textwidth]{bidding_strategy.png}
\caption{Bidding strategy when the bidder faces “willing” to finish constraint}
\end{figure}

### 4.4 Expected Cost of the Project

Note that the buyer always finishes the project even if the contractor abandons the projects after the cost overrun. The government naturally only pays the ex-ante portion of the award in such event. Note that the second lowest cost $c_2$ determines the amount of the award. Recall that $X$ is the extra cost of completion if the winner abandons the project after the cost overrun. Suppose $c_2 < c^*$. Then in case of no cost overrun, with probability $1-p$, the project is completed and costs $c_2/(1-pq)$ to the government. In case of cost overrun, with probability $p$, the project is abandoned by the contractor, who receives only the ex-ante part of the award $rc_2/(1-pq)$. In addition, in these realizations the buyer invests $X > Z$ to finish the project. If $c_2 \geq c^*$ the project is completed regardless of the cost overrun and costs $c_2 + pZ$ to the government. Denote with $H(c)$ the distribution of the second lowest cost, the second lowest order statistics from $n$ draws. The expected cost of the project:

$$E[C] = H(c^*) \left[ E[c_2|c_2 < c^*] + pX \right] + (1-H(c^*)) \left[ E[c_2|c_2 \geq c^*] + pZ \right]$$
Since $H(c^*) E[c_2 | c_2 < c^*] + (1 - H(c^*)) E[c_2 | c_2 \geq c^*] = E[c_2]$ the expected cost

$$E[C] = H(c^*) pX + (1 - H(c^*)) pZ + E[c_2]$$

Since $H$ is increasing and $X > Z$ the expected cost is minimized when $c^*$ is minimized, by setting $r = 0$.

**Proposition 1** *With unlimited budgets it is optimal to pay the entire award ex-post.*

**Corollary 1** *With $X = Z$ any $r$ is optimal.*

Note that the case of $X = Z$ effective assumes that any contractor—the original or the replacement one can fix the cost overrun for the same price. Therefore it may seem that the optimal contract can be written as fixed price with a contingency that in case of the cost overrun the contract switches into cost plus. Indeed in equilibrium the fact that the contractor walks away from the project reveals to all the parties that the shock is $Z$ and with the cost being common knowledge cost plus contract is optimal. This intuition is false, since introducing such a contingency in the contract destroys the incentives of the original contractor to invest $c_i$. It is important for the result that the original contract is not renegotiated and remains fixed price. In case of the cost overrun it is better to let the contractor walk away and replace him with a new one. This may not always be feasible in defence procurement or major infrastructure projects, but is quite feasible in case of house renovation or car repair.

After the original contractor walks away from the house renovation there is normally more quotes solicited and the tender process (for the works left to do) is repeated. This tender process generates some price. We are not modelling this tender process, instead assume that it generates price $X$ to complete the project. Due to adaptation costs alone $X > Z$, which is what is needed for the result. Section 7 deals with the case when shock $z$ is drawn from an arbitrary distribution $F_z$. There the fact that the original contractor quits the project does not reveal to the buyer the true value of $z$. Any other contractor invited to the project knows $z$ but also knows that the buyer does not know it perfectly. We abstain from modelling the interaction between the new contractor and the buyer and assume instead that in case the original contractor left the project and the realized shock is $z$ it costs $z + \delta$ to complete the project. Out of these only the value of $\delta > 0$ is common knowledge and reflects adaptation costs, the effects of delays, etc.

Note that in the auction setting with iid private values the second price auction with correctly chosen reserve price is optimal, Myerson (1981). The analog of reserve price in our setting is a “award floor”. The award is constrained from the bottom so that the winner receives $\max \{ R, \beta \}$. This amount is also being paid in proportion $r$ ex-ante and $1 - r$ upon completion. Clearly the choice of $R$ will affect the expected cost of the project. It is suboptimal to choose $R$ such that the winner who is willing to finish the project and receive $(1 - r) \beta$ is getting extra compensation. That is
optimal \( R \leq Z/q \). It is also suboptimal to pay the award that does not provide enough incentives to finish the project, that is \( R \geq Z/q \). With \( R = Z/q \) if \( c_2 \geq c^* \) the project is completed by the winner and the award is \( c_2 + pZ \). If \( c_2 < c^* \) the project is also completed and the award is \( Z/q \). Since \( c^* = Z/q - pZ \), \( R = c^* + pZ \) the expected cost of the project can be written as

\[
E[C] = pZ + \int_{c^*}^c c^* dH(c_2) + \int_{c^*}^C c_2 dH(c_2).
\]

Clearly the loss is increasing in \( c^* \), therefore it is optimal to again choose \( r = 0 \) and \( R = Z \). The optimal expected cost with the award floor is \( E[C]^* = pZ + c^* H(c^*) + \int_{c^*}^C c_2 dH(c_2) \). Without the award floor with the same optimal choice of \( c^* \) the expected cost is

\[
E[C]^{**} = pZ + p(X - Z) H(c^*) + \int_{0}^{c^*} c_2 dH(c_2) + \int_{c^*}^C c_2 dH(c_2),
\]

so that it is optimal to introduce the award floor if

\[
p(X - Z) H(c^*) + \int_{0}^{c^*} c_2 dH(c_2) > c^* H(c^*)
\]

Recall that optimal \( c^* = (1 - p) Z \). It is then optimal to pay the entire award ex-post and introduce the award floor \( R = Z \) if

\[
E[c_2|c_2 \leq (1 - p) Z] > Z - pX.
\]

Note that in the second price action optimally set reserve price always improves expected revenue. In particular, a small reserve price always helps. This is not the case here. In our setting a small award floor will almost for sure lead to higher payment to the winner that does not, however, provide him any additional incentives to finish the project. Small award floors are therefore worse than none. Because of such inflexibility with the choice of the award floor they only improve expected revenue if the loss from incomplete projects are high enough.

5 Budget Constrained Bidders

In this section we assume that the bidders’ types are two dimensional. Bidder \( i \)'s type is \([c_i, m_i]\), where \( m_i \) is \( i \)'s financial reserves. This \( m_i \) constitutes a “hard” budget constraint. Bidder \( i \) in case she wins the tender can only use the portion of the tender allocated ex-ante and her own reserves \( m_i \) to cover the cost of the project. The initial investment of \( c_i \) is feasible if

\[
m_i + r \beta \geq c_i
\]
After initial investment in case shock $Z$ is realized the winner will continue with the project if two conditions are met. First, if $Z < (1 - r)\beta$ the ex-post part of the award covers the shock and the winner will prefer to finish the project. Second, after the initial investment the winner has to have enough funds to finish the project: 

$$m_i + r\beta - c_i \geq Z$$

$$Z \leq \min\{(1 - r)\beta, r\beta + m_i - c_i\}$$  \hspace{1cm} (1)$$

We will derive the equilibrium bidding strategies in this setting. Even though we restrict our attention to the second price auction, the resulting equilibrium is quite involved. We therefore start with a simplified case where $m_i = c_i$ for every bidder. The elements of the equilibrium construction will be later used as building blocks for the equilibrium in the general case.

5.1 Lean Budgets

Suppose $m_i = c_i$ for every bidder. Then condition (1) simplifies to

$$Z \leq \min\{(1 - r), r\}\beta$$

and there are two cases to consider. With $r < \frac{1}{2}$ only the “able” to finish constraint $Z \leq r\beta$ may be binding. With $r \geq \frac{1}{2}$ only the “willing” to finish $Z \leq (1 - r)\beta$ constraint may be binding.

Suppose first that $r < \frac{1}{2}$. In case of a negative shock if $\beta \geq Z/r$ then the bidder will be able to cover the shock, finish the project and receive $\beta - Z - c_i$ as payoff. If $\beta < Z/r$ the bidder will initially invest, will not be able to cover the shock, will lose original $c_i$, and further quit the project with payoff $r\beta - c_i$. If $b < Z/r$ both options are possible. If $b > Z/r$ only $\beta \geq Z/r$ is possible. Suppose bidder $i$ with cost $c_i$ submits bid $b_i \geq Z/r$. The project is always finished and the expected payoff of bidder $i$

$$E\pi_i = \int_b^{\beta_1^{-1}(\sigma)} (\beta_1 - c_i - pZ) \, dG(\beta_1),$$

The optimal bid $b^* = c_i + pZ$ if $b > Z/r$. Again comparing such $b^*$ with $Z/r$ it is optimal to bid according to

$$b^* = c_i + pZ \text{ if } c_i \geq \frac{Z}{r} - pZ.$$

Next suppose bidder $i$ with cost $c_i$ submits bid $b_i < Z/r$. The expected payoff

$$E\pi_i = (1 - p)\pi_i(c_i, b) + p \int_b^{Z/r} (r\beta_1 - c_i) \, dG(\beta_1) + p \int_{Z/r}^{\beta_1^{-1}(\sigma)} (\beta_1 - c_i - Z) \, dG(\beta_1)$$
Collecting the terms that are affected by the choice of $b$

$$E\pi_i = \int_b^{Z/r} ((1 - p(1 - r)) \beta_1 - c_i) dG(\beta_1) + \int_{Z/r}^{\beta_1^{-1}(\sigma)} (\beta_1 - c_i - pZ) dG(\beta_1)$$

Note that the second integral is positive as we just argued that bidders with $c \geq \frac{Z}{r} - pZ$ bid $c + pZ$. The first integral is maximized by setting

$$b^* = c_i / (1 - p + pr) \text{ if } b < Z/r$$

With $q = 1 - r$ if the optimal bid $b^* < Z/r$, $b^* = c_i / (1 - pq)$. Comparing such $b^*$ with $Z/r$ it is optimal to bid according to

$$b^* = c_i / (1 - pq) \text{ if } c_i < \frac{Z}{r} (1 - pq) = \frac{Z}{r} - \frac{Zp}{r} + pZ$$

Note that for any $r < \frac{1}{2}$, $c^- \equiv \frac{Z}{r} - \frac{Zp}{r} + pZ < \frac{Z}{r} - pZ \equiv c^+$. With $r < \frac{1}{2}$ the optimal bid is therefore

$$b^*(c_i, d_i) = \begin{cases} 
  c_i / (1 - pq) & \text{if } c_i < c^- \\
  Z/r & \text{if } c_i \in (c^-, c^+) \\
  c_i + pZ & \text{if } c_i > c^+.
\end{cases}$$

The bidding function above is increasing and continuous but incorporates a flat interval at the level $b^* = Z/r$ for $c_i \in (\frac{Z}{r} (1 - p) + pZ, \frac{Z}{r} - pZ)$. It may seem that bidder $i$ who is supposed to submit a bid $Z/r$ is facing a mass point in the bidding strategies of the opponents and will benefit from submitting a slightly lower bid. It is true that doing so will increase $i$’s chances of winning but at the same time increases her exposure to the risks of incompletion. After the deviation she may win and only receive $Z$ as payoﬀ. If $b < Z = (1 - r)$ both options are possible. If $b > Z = (1 - r)$ only $Z = (1 - r)$ is possible. As in the preceding analysis if bidder $i$ with cost $c_i$ submits bid $b_i < Z/(1 - r)$ the optimal bid is

$$b^* = c_i / (1 - p + pr) = c_i / (1 - pq)$$

If bidder $i$ with cost $c_i$ submits bid $b_i > Z/(1 - r)$ the optimal bid

$$b^* = c_i + pZ \text{ if } b > Z/(1 - r)$$
Note that at $c^*$

\[
c^* / (1 - p + pr) = c^* + pZ = Z / (1 - r)
\]

so that the derived bidding strategies are strictly increasing and continuous. Again note that when the bidding strategy passes through the level of “willing” to finish constraint it involves a kink.

5.2 Expected Cost of the Project

**Proposition 2** With lean budgets the expected cost of the project is minimized by setting $r = \frac{1}{2}$.

**Proof.** Since with $r \geq \frac{1}{2}$ the equilibrium bidding strategy is exactly the same as with unlimited budgets, with $r \geq \frac{1}{2}$ to minimize the expected cost it is optimal to set $r = 1/2$.

With $r < 1/2$ the project is completed when $\beta \geq Z/r$ despite of the cost overruns and is only completed with probability $1 - p$ when $\beta < Z/r$. Introduce $c'(r) = Z (1 - pq) / r$ and $c''(r) = Z/r - pZ$, note that $c''(r) > c'(r)$ for all $r < \frac{1}{2}$ and $c'' = c'$ for $r = \frac{1}{2}$. Note also that both $c'(r)$ and $c''(r)$ are decreasing in $r$. If $c_2$, the second lowest cost is distributed according to c.d.f. $H(\cdot)$, the expected cost of the project

\[
E[C] = H(c') [(1 - p + pr) E[c_2|c_2 \leq c'] / (1 - pq) + pX] +
\]

\[
+ (H(c'') - H(c')) Z/r + (1 - H(c'')) [E[c_2|c_2 \geq c''] + pZ]
\]

Collecting the terms

\[
E[C] = H(c') [E[c_2|c_2 \leq c'] + pX] + (H(c'') - H(c')) Z/r +
\]

\[
+ (1 - H(c'')) [E[c_2|c_2 \geq c'] + pZ],
\]
It is more convenient to minimize the expected cost after $c_2 + pZ$ is added and subtracted in every realization. Since

$$(H(c'') - H(c')) E[c_2 | c' < c_2 < c''] = \int_{c'}^{c''} c_2 dH(c_2)$$

the expected simplifies to

$$E[C] = E[c_2] + pZ + p(X - Z) H(c') + \int_{c'}^{c''} (Z/r - pZ - c_2) \, dH(c_2)$$

Suppose first that $X = Z$. Then the expected cost is minimized by minimizing

$$\int_{c'}^{c''} (Z/r - pZ - c_2) \, dH(c_2).$$

Note from the definition of $c''$ that $Z/r - pZ > c_2$ when $c_2 < c''$. The expected cost $E[C|X = Z]$ is minimized by setting $r = \frac{1}{2}$ so that $c'' = c'$. Now let’s consider the original $E[C]$. If $X > Z$, other things equal, the effect of $X$ on the expected cost is minimized by lowering $c'$, that is by increasing $r$. Thus $r = \frac{1}{2}$ minimizes the expected cost $E[C]$.

**Corollary 2** With $X = Z$ any $r \geq 1/2$ is optimal.

Note that with the optimal choice of $r$ the expected cost of the project is $E[C] = E[c_2] + pZ + p(X - Z) H((2 - p)Z)$ so that when $X$ is high it may be optimal to introduce the award floor. Similarly to the case of unlimited budgets with $r \geq \frac{1}{2}$ it is optimal to choose $R = Z/r$ and set $r = \frac{1}{2}$, so that $c^* = (2 - p)Z$ and $R = 2Z$. When $r < \frac{1}{2}$ the optimal award floor $R = Z/r$ and again it can be shown that optimal $r = \frac{1}{2}$. With optimal choice of $r$ and $R = 2Z$ the expected cost of the project is

$$E[C]^* = 2ZH(c^*) + \int_{c^*}^{C} (c_2 + pZ) \, dH(c_2).$$

The same choice of $c^* = (2 - p)Z$ is optimal without the award floor. The expected cost of the project is then

$$E[C]^{**} = \int_{0}^{c^*} (c_2 + pX) \, dH(c_2) + \int_{c^*}^{C} (c_2 + pZ) \, dH(c_2).$$

It is optimal to choose $r = \frac{1}{2}$ and install the award floor $R = 2Z$ if

$$E[c_2 | c_2 \leq (2 - p)Z] > 2Z - pX.$$
6 Arbitrary Budgets

Now we relax the constraint $m_i = c_i$ and deal with arbitrary $c_i$ and $m_i$. For each bidder $i$ it is convenient to introduce the deficit $d_i = c_i - m_i$. We need to consider the following constraints

$$r\beta \geq d_i,$$  \hspace{1cm} (2)

When award $\beta$ satisfies this constraint bidder $i$ will make the initial investment of $c_i$. If, in addition, $\beta$ satisfies

$$r\beta - d_i \geq Z,$$  \hspace{1cm} (3)

bidder $i$ will be able to cover the cost overrun due to the shock $Z$ and if, in addition, $\beta$ satisfies

$$(1 - r)\beta = q\beta \geq Z$$  \hspace{1cm} (4)

bidder $i$ will be willing to cover the cost overrun. Therefore for given award level $\beta$ and shock value $Z$ bidder $i$ will face either constraint (3) or (4) if

$$\beta \leq \max \{(Z + d_i) / r, Z/q\}$$

We will ignore constraint (2) for the moment and derive the equilibrium only taking into account the constraints (3) and (4). We will later argue that with competitive banking sector and $m_i$ representing the credit limit that the winner can secure $m_i \geq c_i$ so that constraint (2) will not be binding. Introduce $D$ that satisfies

$$(Z + D) / r = Z/q.$$  \hspace{1cm} (5)

Note that

$$D = \left(\frac{1}{q} - 2\right)Z = \frac{2r - 1}{1 - r}Z$$

is increasing in $r$ and that $D < c^* = \left(\frac{1}{q} - p\right)Z$. Note also that $d_i > D$ implies that $(Z + d_i) / r > Z/q$ for bidder $i$, that is only constraint (3) can be binding. Just like in the case of lean budgets for bidder $i$ with cost $c_i$ and deficit $d_i > D$ it is therefore optimal to bid according to

$$b^*_*(c_i, d_i) = \begin{cases} c_i / (1 - pq) & \text{if } c_i / (1 - pq) \leq (Z + d_i) / r \\ (Z + d_i) / r & \text{if } c_i / (1 - pq) > (Z + d_i) / r \text{ and } \\ c_i + pZ & \text{if } c_i + pZ < (Z + d_i) / r \\ c_i + pZ & \text{if } c_i + pZ \geq (Z + d_i) / r. \end{cases} \hspace{1cm} (5)$$

The equilibrium strategy for bidder $i$ with $d_i \leq D$, only constraint (4) can be binding, the strategy therefore changes regime at $c^* = Z/q - pZ$. For bidder $i$ with cost $c_i$ and deficit $d_i \leq D$ it is optimal to bid according to

$$b^* (c_i) = \begin{cases} c_i / (1 - pq) & \text{if } c_i < c^* \\ c_i + pZ & \text{if } c_i \geq c^* \end{cases}. \hspace{1cm} (6)$$
Note that $c^*$ solves $c^*/(1 - pq) = (Z + D)/r$. Therefore all the bidders with types $c_i \leq c^*$ are affected by either (3) or (4) and all bid $c_i/(1 - pq)$ irrespective of their budget $s_i$. For bidders with $c_i > c^*$ and $d_i \leq D$ neither of the constraints is binding and they bid $c_i + pZ$ again irrespective of their budget. For bidders with $d_i > D$ only constraint (3) can be binding. Budgets $m$ affect the bidding strategies only through the thresholds at which the regime changes from bidding $c/(1 - pq)$ to bidding $c + pZ$. Note that for bidder $i$ with $c_i$ and $d_i$ for the regime to change $c_i/(1 - pq) = (Z + d_i)/r$ must hold. Since $Z > 0$, $d_i = c_i - s_i$ and $r < 1 - pq$, for low enough $s_i$ the change of regime will never occur. Bidders with $m_i = 0$, in particular, always bid according to $c_i/(1 - pq)$.

Consider bidders with $d_i > D$. For given $c > c^*$ introduce $m^-(c)$, the solution to

$$\frac{Z + c - m^-(c)}{r} = c/(1 - pq).$$

Note that $m^-(c^*) = (2 - p)Z$ so that for type $(c^*, m^-(c^*))$ the corresponding $d = D$. Therefore ray $(c, m^-(c))$ originates at $(c^*, (2 - p)Z)$ on the line $d_i = D$. Since $r < 1 - pq$ ray $(c, m^-(c))$ has positive slope $\frac{\partial m^-(c)}{\partial c} = 1 - r/(1 - pq)$. Given $c$ all the bidders with deficits $d_i > D$ and budgets $m_i \leq m^-(c)$ bid according to $c/(1 - pq)$.

Again consider bidders with $d_i > D$ and for given $c > c^*$ introduce $m^+(c)$, the solution to

$$\frac{Z + c - m^+(c)}{r} = c + pZ.$$

Again $m^+(c^*) = (2 - p)Z$ so that $c^* - m^+(c^*) = D$. The slope of ray $(c, m^+(c))$ is $\frac{\partial m^+(c)}{\partial c} = 1 - r > \frac{\partial m^-(c)}{\partial c}$. Therefore the rays $(c, m^+(c))$ and $(c, m^-(c))$ have the same origin and $(c, m^+(c))$ always lies above $(c, m^-(c))$. Given $c$ all the bidders with deficits $d_i > D$ and budgets $m_i \geq m^+(c)$ bid according to $c + pZ$.

Bidders with $d_i > D$, $c_i > c^*$ and $m_i \in (m^-(c_i), m^+(c_i))$ such that $(Z + d_i)/r > c + pZ$ and $(Z + d_i)/r < c/(1 - pq)$ bid $(Z + d_i)/r$. For these types of bidders budgets $m_i$ affect the bids, however, only in a particular combination with $c_i$.

Figure 3 represents the isobid curves in the space $(c, m)$ with $D < 0$, $r < \frac{1}{2}$. Solid lines join the types that submit the same bid in equilibrium.

Note that for $c \leq c^*$ the isobid curves are vertical. To construct the isobid curve for a higher cost type fix $c' > c^*$, the cost of a type with $d_i < D$ that bids $c' + pZ$ in equilibrium. The corresponding type that submits the same bid but bids according to $c/(1 - pq)$ is $\sigma(c') = (1 - pq)(c' + pZ) < c'$. For such bidder $d_i > D$. Note that

$$0 < \frac{\partial \sigma(c)}{\partial c} = 1 - pq < 1,$$

that is when $c$ rises, $\sigma(c)$ rises at a lower rate. Therefore the types that submit bids $b'$ and $b'' = b' + \delta$ for any $\delta > 0$ “lie denser” for low $s$ where they bid according to $c/(1 - pq)$ and lie less dense for high $m$ where they bid according to $c + pZ$. Recall that $m^-(c)$ and $m^+(c)$ define the thresholds where the bidding changes the regime
Figure 4: Isobid curves. Note the change of regimes for $c > c^*$

for bidder with cost $c$ and deficit $d < D$. The isobid curve therefore is vertical for $m \geq m^+(c')$ and $m \leq m^-(c')$ and involves kinks at $(c', m^+(c'))$ and $(\sigma(c'), m^-(c'))$. In between the kinks the isobid curve slopes upwards along $d_i = \text{Const}$. Note also that

$$\frac{\partial m^+(c)}{\partial c} = q < 1 \text{ and } \frac{\partial m^-(c)}{\partial c} = \frac{q - pq}{1 - pq} < 1$$

so that the isobid curves that correspond to different $c$ do not intersect.

At this stage we would like to motivate an additional on the budget constraints. Suppose that budget $m_i$ represents the credit limit. Our paper is concerned with the situations in which the winning bidder has a signed contract with a buyer, the government or private party that stipulates the award that the contractor receives at the start and upon completion of the project. This contract can be viewed as collateral. Note that in equilibrium in every realization, regardless of whether the project is finished or not the contractor receives at least $c_i$ as a payment. Competitive banking sector will therefore lend at least $c_i$ to the winner of the tender prior to the initial investment stage. In what follows we assume that $m_i \geq c_i$ for every bidder. Note that our assumption does not remove budget constraints entirely since the winner of the tender may run into a cost overrun. The banks are naturally reluctant to extend his credit line in such situations. What is guaranteed though and what motivates our assumption is that in any realization of the shock in equilibrium whether the winner of the tender finishes the project or decides to quit he has enough cash to pay back $c_i$. Note also that $m_i$ is still to be understood as exogenously given.
budget of bidder $i$ and constitutes her private information.\footnote{Our intention is to abstain from explicitly modeling banking sector, since we realize that by introducing it formally we create avenues for off equilibrium behaviour. However, it would be unrealistic to say that the winner of the government procurement contract cannot borrow against the contracted award.}

**Assumption** For every contractor $m_i \geq c_i$.

With such additional assumption constraint (2) does not bind and our equilibrium construction is complete.

## 6.1 Expected Cost of the Project

The difficulty with calculating the expected cost with arbitrary budgets stems from the fact that the equilibrium just derived depends on the budgets and we are not guaranteed that the bid of the contractor with the second lowest cost determines the award. Indeed, it may be that the bidder with the second lowest cost has a large deficit but the bidder with, say, third lowest cost has smaller deficit and bids more aggressively. In fact, we are not even guaranteed that the winner is the contractor with the lowest private component of the cost.\footnote{Same applies to open descending price tender and results from the fact that the budget constraints are present in the model, not from our choice of the tender mechanism.} We however, will make an argument for the general case through a sequence of arguments for the cases each of which deals with degenerate $F(c,m)$. Consider again a procurement problem where $d_i = d$ for each bidder $i$. This corresponds to the case where costs and budgets for each bidder are distributed on a line parallel to $d = D$. This problem has one dimensional private information and therefore the award will be determined by the bid of the contractor with the second lowest cost just as in the case of lean budgets, Section 5.1. As is easy to see that the equilibrium in this degenerate distribution problem will be either of the type studied in Section 4 for unlimited budgets and involve a kink or the one from Section 5.1 and $r < \frac{1}{2}$, that is, involve a step. Which of the two cases arises depends on whether $d \geq D$ or $d < D$ applies, that is whether the line $d_i = d$ crosses the area in between the rays $(c, m^-(c))$ and $(c, m^+(c))$ or lies wholly to the left of it, see Figure 4 with the isobid curves.

For $d < D$ straightforward replication of Proposition 1 suggests that it is optimal to choose $r = 0$. For $d \geq D$ equally straightforward replication of Proposition 2 suggests that it is optimal to choose $r = \frac{1}{2}$. Note that neither of the arguments depends on the distribution of $c_2$ and therefore the corresponding optimal $r$ would remain optimal if the award were always determined by $c_3$ or $c_5$. Now consider a hypothetical procurement problem $\{c_2\}$ where the private information is two dimensional but the award is always determined by the bid of the contractor with the second lowest cost. Note that in equilibrium the project will be completed by the winner when neither of the constraints bind. The cost of the project is then $c_2 + pZ$ for $c_2 \geq c^*$.
and \( m \geq m^+(c_2) \). The project is also completed by the winner when \( c_2 \geq c^* \) and \( m^-(c_2) \leq m < m^+(c_2) \) and costs the government \( (Z + d_2)/r > c_2 + pZ \) in these realizations. In the realizations with \( c < c^* \) the bidder with the second lowest cost bids \( c_2/(1 - pq) \), the winner is unwilling to finish the project in case of a shock. The project is completed by the winner only with probability \( 1 - p \). The government pays only the ex-ante fraction of the award and finishes the project with probability \( p \). The expected cost in these realizations is \( c_2 + pX \). With \( c > c^* \) and \( m < m^-(c_2) \) the winner is willing to finish the project but is unable to because of the large deficit. The cost to the government is again \( c_2 + pX \). Figure 5 places the expected payments in the respective areas when \( r < \frac{1}{2} \).

![Figure 5: Expected cost of the project by realization](image)

With \( d_i \geq 0 \) only the upper part of the expected costs diagram matters. Assume \( r \geq \frac{1}{2} \), then \( D \geq 0 \) and \( d = D \) lies to the left of \( d = 0 \). Figure 6 represents the expected payments for this case.

It is obvious from the picture that with \( r \geq \frac{1}{2} \) and \( X > Z \) it is optimal to minimize \( c^* \) that is to choose \( r = \frac{1}{2} \). Now consider \( r < \frac{1}{2} \). We can further show that the optimal \( r \in \left(0, \frac{1}{2}\right) \). Note that on each of the lines \( d = Const \) the optimal \( r \) is either 0 or \( \frac{1}{2} \). Note also that the calculation of the expected cost of the project can be performed as integration of the costs over the area above \( d = 0 \), since \( m_i \geq c_i \) for every \( i \). This integration in our hypothetical case, where \( c_2 \) always determines the award, can be performed in a conventional way, first integrate over \( c_2 \) and for each \( c_2 \) over \( m_2 \) or in the system of coordinates where the outer integral is over a variable orthogonal to \( d = 0 \) and the inner integral along the lines parallel to \( d = 0 \). Such integration amounts to averaging out the cases where \( r = 0 \) is optimal with the cases where \( r = \frac{1}{2} \) is optimal. Therefore optimal \( r \) in the hypothetical procurement problem satisfies \( 0 < r < \frac{1}{2} \).
Figure 6: Expected cost of the project by realization $r \geq \frac{1}{2}$

Now consider another hypothetical procurement problem \{\(c_3\)\} where the award is always determined by the bid of the bidder with the third lowest cost. Since none of the arguments above explicitly depends on the distribution again optimal \(r\) satisfies \(0 < r < \frac{1}{2}\). Note that in our general procurement problem there are realizations where the award is determined by \(c_2\), realizations where it is determined by \(c_3\), etc. The optimal \(r\) in the general procurement problem is again the weighted average of the optimal \(r\)'s in hypothetical procurement problems.

**Proposition 3** With \(X > Z\) optimal \(r\) satisfies \(0 < r < \frac{1}{2}\).

**Corollary 3** With \(X = Z\) optimal \(r \geq \frac{1}{2}\).

Note that with \(X > Z\) fully ex-post payment is optimal with unlimited budgets and suboptimal when budget constraints may bind. The result for the Corollary follows from the fact that \(r \geq \frac{1}{2}\) is optimal for every \(d \leq D\) and any \(r\) is optimal for \(d > D\). Note that optimal \(r\) for \(X > Z\) is not a subset of the set of optimal \(r\) for \(X = Z\). Indeed with \(X = Z\) the bankruptcy problem is absent, the buyer can finish the project at the same cost as the contractor. Therefore it is optimal to motivate aggressive bidding for the right to invest initial \(c_i\). The award paid fully ex-ante does precisely that. With \(X > Z\) the buyer has to balance the incentives to complete the project with the ability to complete it. Intermediate \(r\) is more suitable for the task.
7 Arbitrary Shocks

In this section we generalize our findings to the case where shock \( z \) is drawn from distribution with arbitrary c.d.f. \( F_z \) on support \([0, Z_{\text{max}}]\)\(^{14}\) Apart from being more general such modification of the shocks process supports other aspects of the model. With the shock on two possible values 0, and \( Z \) the fact that the original contractor walked away from the project reveals to all the parties that the shock is indeed \( Z \). Therefore it can be argued that any contractor that replaces the original one should not be paid more than \( Z \), however, the assumption that \( X > Z \) is crucial for the results. In this section all that can be possibly revealed by original contractor quitting the project is that \( z \geq z \) where \( z \) is some endogenously determined threshold. In the spirit of our original model we do not model the process of tendering or contracting to fix the cost overrun, but rather assume that this process generates price \( z + \delta \) that the buyer is liable to pay for the cost overrun. The value of \( \delta > 0 \) is exogenously given.\(^{15}\) Again we stress that \( z + \delta \) is being paid not to the original contractor who is given no indication that his fixed price contract can be turned into cost plus contract in case of cost overruns.

We derive the equilibria and optimal \( r \) for the two leading examples, unlimited and lean budgets. We do not get closed form solutions and instead reason from the equations that describe the bidding strategies. Apart from introducing more generality in the model this modification of the shock process allows for more smooth bidding strategies, for given \( r \), the bidding strategy is not separated into regions anymore and make the assumption that the ex-post shock is not verifiable more plausible.

7.1 Unlimited budgets

Given the award \( \beta \) the project is finished if \( z \) is such that \[ z \leq (1 - r) \beta. \]

Then bidder \( i \) with cost \( c_i \) who bids \( b \) when the lowest bid submitted by the opponents is \( \beta_1 \) receives the payoff

\[ \pi_i (c_i, b, \beta_1) = (\beta_1 - c_i) F_z [(1 - r) \beta_1] + (r \beta_1 - c_i) (1 - F_z [(1 - r) \beta_1]) \]

This results in bidder \( i \)'s expected payoff

\[ E [\pi_i (c_i, b)] = \int_0^{\beta^{-1}(b)} (r \beta_1 + (1 - r) \beta_1 F [(1 - r) \beta_1] - c_i) dG (\beta_1), \]

\(^{14}\)The distribution \( F_z \) studied before was placing mass \( p \) on \( Z \) and the remainder on 0.

\(^{15}\)The buyer here acts as a price taker. Of course, when the replacement contractor asks for \( z + \delta \) the buyer learns the true value of the shock and may be strategic but all we need for the result is that \( \delta > 0 \) so for small \( \delta \) we believe it is reasonable to assume that the buyer acts as a price taker at this stage.
where $G$ is the distribution of $\beta_1$. Optimal $b(c_i, r)$ then solves
\[
rb + (1 - r) b F_z[(1 - r) b] = c_i. \tag{7}
\]
Note that $b(c_i, r)$ that satisfies the above is increasing in $c_i$ so that the solution to (7) is indeed an equilibrium. Note also that with $F_z[(1 - r) b] = 1$ for any $b$, that is in the model with no ex-post shocks, optimal $b_i = c_i$ for any $r$. Explicit computation suggests that $(1 - r) b(c_i, r)$ decreases in $r$. The buyer’s expected cost depends on $\delta$ and distribution of $z$ and $c_2$. The project is completed if $z \leq (1 - r) b_2$, $b_2$ is then paid to the contractor, and not completed otherwise with the total contracted payment and the payment for completion $rb_2 + z + \delta$. The value exact value of $z$ is not known at this stage, all that the buyer knows is that $z > (1 - r) \beta$, where award $\beta$ is endogenously determined by the second lowest bid. This results in the expected cost of the project
\[
E[C] = \int_{C_{\min}}^{C_{\max}} b_2 F_z[(1 - r) b_2] + rb_2 (1 - F_z[(1 - r) b_2]) + \int_{(1 - r)b_2}^{Z_{\max}} (z + \delta) dF_z \, dH,
\]
where $H$ stands for the distribution of the second lowest cost. The expected cost can be simplified to form
\[
E[C] = \int_{C_{\min}}^{C_{\max}} (1 - r) b_2 F_z[(1 - r) b_2] + rb_2 + \int_{(1 - r)b_2}^{Z_{\max}} (z + \delta) dF_z \, dH
\]
Note from equation (7) that the first two terms form exactly $c_2$ therefore the expected cost can be reduced to
\[
E[C] = E[c_2] + \int_{C_{\min}}^{C_{\max}} \int_{(1 - r)b_2}^{Z_{\max}} (z + \delta) dF_z dH \tag{8}
\]
Since $(1 - r) b_2 (r)$ decreases in $r$ the expected cost is minimized by setting $r = 0$. Note that $\delta > 0$ is important for the result as with $\delta < 0$ the integral includes negative values therefore increasing $(1 - r) b_2 (r)$ does not necessarily lower the expected cost.

### 7.2 Lean Budgets

When $d_i = 0$ the project will be completed if
\[
z \leq \min \{r, (1 - r)\} \beta,
\]
where again $\beta$ is the award which is endogenously determined.

**Case 1** $r \leq \frac{1}{2}$.  

26
When \( r \leq \frac{1}{2} \) then only the constraint \( z \leq r \beta \) can be binding and the payoff of bidder \( i \) with cost \( c_i \) who bids \( b \)

\[
\pi_i (c_i, b, \beta_1) = (\beta_1 - c_i) F_z [r \beta_1] + (r \beta_1 - c_i) (1 - F_z [r \beta_1])
\]

After collecting the terms the expected payoff

\[
E [\pi_i (c_i, b)] = \int_{b}^{\beta_1} [r \beta_1 + (1 - r) \beta_1 F_z [r \beta_1] - c_i] dG (\beta_1)
\]

Therefore for \( r \leq \frac{1}{2} \) it is optimal to submit bid \( b_i = (c_i, r) \) that solves

\[
rb + (1 - r) b F_z [rb] = c_i
\]  (9)

Note that \( b_i = (c_i, r) \) defined by such equation is increasing in \( c_i \). Note also that in the model with no ex-post shocks \( F_z [rb] = 1 \) and it is optimal to bid \( b_i = c_i \). For \( r = \frac{1}{2} \) equations (9) and (7) are identical. The expected cost of the project

\[
E [C] = \int_{c_{\text{min}}}^{c_{\text{max}}} \left[ b_2 F_z [(1 - r) b_2] + rb_2 (1 - F_z [rb_2]) + \int_{rb_2}^{z_{\text{max}}} (z + \delta) dF_z \right] dH
\]

which using (9) can be simplified to.

\[
E [C] = E [c_2] + \int_{c_{\text{min}}}^{c_{\text{max}}} \int_{rb_2}^{Z_{\text{max}}} (z + \delta) dF_z dH
\]

Again direct computation shows that \( rb (r) \) where \( b (r) \) solves (9) increases in \( r \). Therefore with \( \delta > 0 \) and \( r \leq \frac{1}{2} \) optimal \( r = \frac{1}{2} \).

**Case 2** \( r \geq \frac{1}{2} \).

With \( r \geq \frac{1}{2} \) as before only constraint \( z \leq (1 - r) \beta \) can bind and for bidder \( i \) with cost \( c_i \) it is optimal to submit \( b_i = (c_i, r) \) that solves (7) above. The expected cost of the project is given by (8) and is minimized by \( r = \frac{1}{2} \). Therefore with lean budgets optimal \( r = \frac{1}{2} \). We summarize the results form this Section so far in

**Proposition 4** With arbitrary distribution of shocks and unlimited budgets \( r = 0 \) is optimal, with lean budgets \( r = \frac{1}{2} \) is optimal.

### 7.3 Arbitrary Budgets

After initial investment of \( c_i \) bidder \( i \) with deficit \( d_i \) will complete the project

\[
z \leq \min \{ r \beta - d_i, (1 - r) \beta \},
\]
where $\beta$ is again the endogenously determined award in the tender. Introduce $D$, the solution to
\[ r\beta - D = (1 - r)\beta. \tag{10} \]

Note that $D$ is itself endogenously determined. If $d_i > D$ then for bidder $i$ only $z \leq r\beta - d_i$ may bind. If $d_i < D$ then only $z \leq (1 - r)\beta$ may bind.

Consider bidder $i$ with $d_i \geq D$. As before the expected payoff of bidder $i$ who bids $b$ and has private cost $c_i$ and deficit $d_i$
\[ E[\pi_i(c_i, d_i, b)] = \int_b^{\beta^{-1}(\sigma)} [r\beta_1 + (1 - r) \beta_1 F_z [r\beta_1 - d_i] - c_i] dG(\beta_1). \]

Optimal bidding strategy for such bidder $b^+(c_i, d_i, r)$ solves
\[ rb + (1 - r) b F_z [rb - d_i] = c_i \tag{11} \]

Next consider bidder $i$ with $d_i \leq D$. As before the expected payoff of such bidder $i$ does not depend on $d_i$ and only responds to the cost $c_i$:
\[ E[\pi_i(c_i, b)] = \int_b^{\beta^{-1}(\sigma)} [r\beta_1 + (1 - r) \beta_1 F_z [(1 - r) \beta_1] - c_i] dG(\beta_1). \]

Optimal bidding strategy for such bidder $b^-(c_i, r)$ solves
\[ rb + (1 - r) b F_z [(1 - r) b] = c_i \tag{12} \]

Note that at $d_i = D$, $r\beta - D = (1 - r)\beta$ therefore there is smooth transition from bidding according to (11) to bidding according to (11) and vice versa.

We maintain the assumption that $d_i \leq 0$ for every contractor. Therefore in the case of arbitrary budgets and arbitrary shock distribution the equilibrium is defined as follows.

**Definition 1** With arbitrary budgets and arbitrary distribution of the shocks the equilibrium in the procurement game is a tuple $(b^+(c_i, d_i, r), b^-(c_i, r), D, \beta)$ such that $b^+(c_i, d_i, r)$ solves (11) for each bidder with $d_i \geq D$, $b^-(c_i, r)$ solves (12) for each bidder with $d_i \leq D$, $\beta$ is determined given the collection of functions $\{b^+(c_i, d_i, r)\}_{i=1}^n$, $\{b^-(c_i, r)\}_{i=1}^n$ as the second lowest bid and $D$ is determined from (10) given $\beta$.

We continue to work on this case and conjecture that the main result of Section 6 generalizes to the case of arbitrary distributions of ex-post shocks.
8 Conclusions

This paper concerns with the issue of cost overruns in public procurement projects which rightfully attracts the attention of the regulators. We view the problem of cost overruns as an incentive problem. Those problems in procurement literature were traditionally approached from two sides: contract theory provides incentives at the after the tender stage, auction theory provides incentives at the tender stage. The procurement game in this paper combines both at the tender and after the tender stages. We stand on the premise that when private information needs to be revealed competitive bidding, where one buyer deals with multiple contractors, provides better incentives than a clever contract in which the buyer is by definition locked in one on one relationship with the contractor. The leverage that the buyer has at the tender stage simply because she still has a choice of which contractor to deal with dissipates once the contract is signed if the contractor retains private information. The fixed price contract is therefore superior to cost plus contract since the terms are “negotiated” at the stage where the buyer still has the leverage and then stay unchanged by the definition of the contract. We therefore concentrate on fixed price contracts and offer the regulators a modest departure to the contract terms, the fixed award determined by the competitive bidding is to be split in the ex-ante and ex-post parts. We show that this approach has a potential to reduce the cost of the procured projects and lower the likelihood of the contractor’s default.

References


