Estimation of operational risks using a semi-parametric approach
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Defining Operational Risk (OR)

OR is a measure that reflects with a high degree of certainty the amount of capital needed to cover unexpected losses.

- Financial institutions are required to reserve some capital to absorb unexpected losses.
- The meaning of "high degree of certainty" depends on the level of risk tolerance of the institution.
  - In practice, confidence levels tend to lie between 99.98% and 99.9% (Basel requirement).
- These levels of confidence require a high precision in modeling the tails of loss distributions.
Measuring OR

OR = unexpected loss - expected loss

- unexpected loss (e.g. 99.9% quantile of the loss distribution)
- expected loss (the mean of the loss distribution)

According to Basel committee OR losses are classified into eight Business Lines (BL) and seven Event Types (ET)

- OR has to be obtained for each BL and ET
**Motivation**

**OR measuring approaches**

**Implications of using normal distribution for measuring OR**

- Does not fit well the right tail of the loss distribution
  - OR’s can be underestimated
  - OR’s are biased and inconsistent
Objective: obtain consistent and unbiased OR estimates

Estimation methods

- Fitting Generalised Pareto Distribution (GPD) includes three stages
  1. Choosing the threshold (use method of Huisman et al., 2001 to estimate the optimal threshold loss)
  2. Obtaining the excess losses (use Peaks over threshold (POT) method of Embrechts et al., 1997)
  3. Estimate the shape and the scale parameters of GPD (use maximum likelihood)

Use estimated GPD parameters to compute
- the mean (Johansson, 2003)
- the quantile (Embrechts et al., 1997)
- the VaR and the ES
Objective

Contribution to the estimation methodology

In previous studies the threshold was selected

- either by using visual plots, e.g. the Hill plot and the mean excess function plot
- or it was set by the analyst

Our contribution

- To ensure that obtained risk measures are reliable, use statistical methods to estimate consistent and unbiased
  - tail index
  - mean of the heavy tailed loss distribution
The tail index reflects the thickness of the tails and the Hill index (Hill, 1975) is the most common estimator of the tail index.

For losses exceeding a given threshold, the Hill estimator is the average of the log of the losses minus the log of the threshold:

\[
\alpha_k = \left\{ \frac{1}{k} \sum_{i=1}^{k} \ln(X_{n,n-i+1}) - \ln(X_{n,n-k}) \right\}^{-1}
\]  

(1)

The Hill estimator is asymptotically normal, but in small samples it is biased. Huisman et al. (2001) note that this bias is approximately linear in \( k \).
Huisman et al. (2001) proposed to use an OLS regression for a given range of $k$.

$$\alpha(k) = \beta_0 + \beta_1 k + \epsilon(k)$$  \hspace{1cm} (2)

where $k = 1, \ldots, n/2$

This OLS estimator is unbiased, but inefficient due to two reasons:

- the variance of $\alpha(k)$ is not constant for different $k$ and thus the error term is heteroscedastic
- an overlapping data problem exists due to the way in which $\alpha(k)$ is constructed
Estimation of the tail index

Efficient and unbiased tail index estimator

A weighted least squares method (Huisman et al., 2001)

Multiplying (2) by the weight $w(k)$:

$$\alpha(k)w(k) = \beta_{0}^{wls}w(k) + \beta_{1}^{wls}kw(k) + u(k)$$

(3)

where $w(k)$ is defined in a vector form as $W = (\sqrt{1}, \sqrt{2}, \ldots, \sqrt{n/2})$ and $u(k) = \epsilon(k)w(k)$.

The new tail index estimator $\beta_{0}^{wls}$ is both unbiased and efficient.
Threshold analysis of the losses

Choosing the threshold

Optimal threshold

- The tail index estimated as weighted average of Hill estimators is robust to the choice of $k$, therefore Huisman et al., 2001 suggested to set $k = n/2$
- We suggest to select the threshold by minimising the distance between $\beta_0^{wls}$ and estimated $\hat{\alpha}$. 
Threshold analysis of the losses

Peaks over threshold (POT) approach

Extreme losses are more influential in determining the regulatory capital. (Embrechts et al., 1997) showed that as the threshold increases the limiting distribution of excess losses converges to the generalised Pareto distribution (GPD).

We select the threshold (loss) that corresponds to the optimal $k$ and then fit GPD to the excess losses.
Threshold analysis of the losses

GPD methodology (Pickands, 1975)

For a given threshold, the GPD:

\[ \Bar{G}_{\xi,\beta}(x) = \left(1 + \frac{x}{\beta} \right)^{-1/\xi} \]  

(4)

where \( \xi \) and \( \beta \) are the shape and scale parameters of the GPD, respectively.

- The shape parameter indicates the degree of the tail thickness
- The scale parameter represents the degree of the dispersion of the extremes
Estimating the mean of heavy-tailed distribution

the mean for the nontail region

\[
\hat{\mu}(1) = \frac{1}{n} \sum_{i=1}^{n-k} X_i,
\]  

(5)

the mean for the tail region

\[
\hat{\mu}(2) = \hat{p} \left( u + \frac{\hat{\beta}}{1 - \hat{\xi}} \right)
\]  

(6)

where \( u \) is the threshold, \( \hat{\xi} \) and \( \hat{\beta} \) are the maximum likelihood parameters of GPD.

Then the adjusted mean \( \mu_{adj} \) is \( \mu(1) + \mu(2) \).
Value at Risk (VaR) and its limitations

The quantile at the confidence level $p$:

$$\hat{x}_p = u + \frac{\hat{\beta}}{\hat{\xi}} \left( \left( \frac{n}{k} (1 - p) \right)^{-\hat{\xi}} - 1 \right).$$  \hspace{1cm} (7)

The VaR - the maximum loss that could occur at the confidence level $p$:

$$\hat{\text{VaR}}_p = \hat{x}_p - \mu_{adj}.$$  \hspace{1cm} (8)

**limitations of VaR**

- Measures the quantile of the distribution and disregards extreme losses beyond the VaR
- Does not satisfy the sub-additivity condition, therefore is not a coherent risk measure
- Expected Shortfall (ES) is a coherent risk measure
Consider a set $V$ of real-valued random variables. A function $\rho : V$ is called a coherent risk measure if it satisfies the following four conditions:

1. **monotone:** $X \in V, X \geq 0 \implies \rho(X) \leq 0$,
2. **positively homogeneous:** $X \in V, hX \in V, \implies \rho(hX) = h\rho(X)$,
3. **translation invariant:** $X \in V, a \in \mathbb{R}, \implies \rho(X + a) = \rho(X) - a$
4. **sub-additive:** $X, Y, X + Y \in V \implies \rho(X + Y) \leq \rho(X) + \rho(Y)$,

- For example, the risk measure of two banks after they have been merged should not be greater than the sum of their risk measures before they were merged.
**Expected Shortfall (ES) and its advantage**

The ES at confidence level $p$ is the conditional expectation of loss given that the loss exceeds the VaR

$$ES_p = E\{X_i | X_i > VaR\}$$

where VaR is the 99.9% OR.

**ES advantage**

- It is more sensitive to the shape of the loss distribution in the tail of the distribution
The main feature - severe right skewness

Figure: PDF plot for event type "Internal Fraud"

There are nine catastrophic losses exceeding $1 billion.
The Hill plot

![Hill plot](image)

**Figure:** Hill plot for event type ”Internal Fraud”

We choose an optimum $k$ by minimising the distance between $\alpha(k)$ and $\hat{\beta}_0^{wls} = 0.7777$. For $k = 98 \hat{\alpha} = 0.7793$, therefore 98 is the optimum $k$. 

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Goodness of fit of $k$: the mean excess function plot

**Figure:** Mean Excess Function (Davidson and Smith, 1990) of *Internal Fraud* losses excluding losses exceeding $65.7\text{mln.}$.
Excess losses are those above the threshold

Figure: *Internal Fraud* losses in mln USD. The threshold is equal to $65.7$ mln
POT application: fitting GPD to excess losses

**Figure:** CDF of *Internal Fraud* losses exceeding $65.7$ mln (solid line) and GPD with $\hat{\xi} = 0.75$ and $\hat{\beta} = 150.50$ (dashed line)
Shape parameters for different types of losses

Figure: Estimated shape parameters for three business lines and three event types

As the shape parameter approaches one the tails of the GPD become heavier.
## Estimated expected losses

<table>
<thead>
<tr>
<th>Loss type</th>
<th>Shape/scale parameters</th>
<th>Sample mean</th>
<th>Adjusted mean</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Operational losses: business line</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trading and Sales</td>
<td>0.96/35.14</td>
<td>93.0</td>
<td>405.9</td>
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<tr>
<td>Commercial Banking</td>
<td>0.84/71.63</td>
<td>83.4</td>
<td>128.2</td>
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<tr>
<td>Retail Banking</td>
<td>0.55/191.51</td>
<td>56.1</td>
<td>65.2</td>
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<tr>
<td><strong>Operational losses: event type</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Internal Fraud</td>
<td>0.75/150.50</td>
<td>84.3</td>
<td>123.9</td>
</tr>
<tr>
<td>Employment PWS</td>
<td>0.61/30.69</td>
<td>26.1</td>
<td>29.0</td>
</tr>
<tr>
<td>External Fraud</td>
<td>0.57/45.96</td>
<td>23.6</td>
<td>27.0</td>
</tr>
</tbody>
</table>
Estimates of the mean of the heavy right tailed distribution

- The simple sample average underestimates the true mean of the heavy right tailed loss distribution.
- The extent of this bias depends on the degree of the tail thickness: when the shape parameter is close to one the bias is the largest.

![Graph showing sample mean and adjusted mean vs shape parameter]

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## Estimated risk measures

<table>
<thead>
<tr>
<th>Loss type</th>
<th>Shape/scale parameters</th>
<th>OpVaR</th>
<th>ES</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Operational losses: business line</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trading and Sales</td>
<td>0.96/35.14</td>
<td>12,546.5</td>
<td>20,350.4</td>
</tr>
<tr>
<td>Commercial Banking</td>
<td>0.84/71.63</td>
<td>8,578.8</td>
<td>12,543.4</td>
</tr>
<tr>
<td>Retail Banking</td>
<td>0.55/191.51</td>
<td>4,523.0</td>
<td>5,569.5</td>
</tr>
<tr>
<td><strong>Operational losses: event type</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Internal Fraud</td>
<td>0.75/150.50</td>
<td>9,060.2</td>
<td>12,485.9</td>
</tr>
<tr>
<td>Employment PWS</td>
<td>0.61/30.69</td>
<td>1,417.3</td>
<td>1,796.2</td>
</tr>
<tr>
<td>External Fraud</td>
<td>0.57/45.96</td>
<td>1,360.8</td>
<td>1,682.8</td>
</tr>
</tbody>
</table>
Estimates of the risk measures

- VaR estimate is always below ES estimate
- The difference between VaR and ES is greater when the estimated shape parameter is close to one
The main objective is to estimate the reliable risk measures for heavy tailed loss distributions

- Used statistical methods designed specifically for heavy tailed distributions to obtain consistent and unbiased tail index estimates
- Obtained estimates of the mean and quantile of heavy tailed distributions that take into account the degree of tail thickness
- Obtained two risk measures, one of which is not a coherent risk measure and another which is a coherent risk measure
As the estimated shape parameter approaches one

- The size of bias for the mean and quantiles of heavy tailed distributions increases sharply
- The difference between estimated VaR and ES increases