Government and Private Employment, Wages, and Social Welfare in a Hotelling Oligopsony

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Abstract

Recent studies reconsider a new applications on Hotelling model by Bhaskar and To (2003, European Economic Review) and Kaas and Madden (2010, Economic Theory) which analyze the role of minimum wages in labor markets. In this article we further discuss the implications on the government employment in a Hotelling oligopsony. Consider a location-wage equilibrium with pure strategies in a bounded linear market, several analytical results are obtained as follows. Firstly, we show the existence of a collection of location-price equilibria which follow a unique type and provide equivalent employment and wage distributions. Secondly, endogenous unemployment rate is shown to be related to the marginal product and the number of firms. Wages are increasing in the marginal product of the oligopolistic firms. Thirdly, the optimal policies are discussed. The socially optimal minimum wage as well as the unemployment subsidy are solved analytically. Imposing a higher minimum wage driving out a lower productivity firm can be more welfare improving than lower minimum wages. The optimal unemployment subsidy is positively related to the transportation rate and negative to the productivity and the leisure utility from unemployment. Our results are robust either in a circular market.

The implication suggests that government employment shall be counter-cyclic to the private sectors.

Keywords: Hotelling model; Government and private employment, wages, Oligopsony, location-wage equilibrium

JEL Classification Numbers. D43, E24, J48
Government and Private Employment, Wages, and Social Welfare in a Hotelling Oligopsony

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1 Introduction

Several recent studies adopt Hotelling’s (1929) model with non-wage job characteristics to introduce new insights on labor markets with oligopsony. The purpose of this article is to develop an analytical framework of a Hotelling duopsony without full employment assumption and examine the minimum wage and government unemployment policies. Bhaskar and To (1999) first consider a circular space of workers and uniformly distributed oligoposonic firms who offer jobs to the workers nearby, and suggest that a rise in the minimum wage increases social welfare. Bhaskar and To (2003) further develop the implications on wage distribution, with the assumption of the uniform distribution of firms. While the above two papers treat locations as exogenous, Kaas and Madden (2010) make a great attempt to solve the location-wage game of the duopsony labor market and suggest that imposing minimum wage is always welfare-improving.

It is well-known that the transportation costs are essential for determining the equilibrium in Hotelling-type models. In this literature of the labor market with a Hotelling oligopsony, the transportation cost in Bhaskar and To (2003) is linear in distance, but firms are located exogenously, while Kaas and Madden (2010) use quadratic transportation rate with endogenous locations of firms. In our framework, the transportation rate is linear in distance and the locations of firms are endogenous. It is reasonable to use quadratic transportation in Kaas and Madden (2010), because Hotelling (1929) model with linear transportation rate has no pure strategy equilibrium (see d’Aspremont et al., 1979). However, a price-location equilibrium may exist under finite reservation prices, even with a linear transportation cost (see Economides (1984) and Hinloopen and MarMarrewijk (1999) for details). In this study a Hotelling duopsony (oligopsony) with linear transporation costs is considered, because the leisure utility (or government unemployment subsidies) of unemployed people can play a role as a limited reservation price in Hotelling model to ensure the existence of pure-strategy equilibrium in our framework. Moreover, using linear transportation cost may also avoid the personal arbitrage problem (see Aguirre and Paz Espinosa, 2004).1

1Since the quadratic transportation costs is convex, the costs can be shared by employee and employers to
Based on our framework, it is analytically to discuss two unemployment policies, minimum wage and unemployment subsidy. This article attempts to make several contributions on this literature: Firstly, it develops an analytical framework of the location-wage game on the duopsony model without full employment, while it is the major assumption in Bhaskar and To (2003) and Kaas and Madden (2010).\textsuperscript{2} Secondly, we provide the policy implications on both the minimum wage and the unemployment subsidy.

An optimal unemployment subsidy can improve social welfare and will be discussed in our framework. It is shown that the pure strategy location-wage equilibrium exists in that both duopsony firms both occupy disjointed ranges of labor market and other areas are unemployed. The equilibrium locations are not maximal differentiation in locations when unemployment is considered. The equilibrium unemployment rate is positive related to the transportation rate of the leisure utility from unemployment, while negative related to the productivity. The optimal minimum wage can either bind one or two firms, depends on the relative values of parameters, and leads to a welfare improving. However, imposing any arbitrary minimum may not be welfare improving in some cases. Moreover, imposing a higher minimum wage driving out a lower productivity firm can be more welfare improving than lower minimum wages. Another unemployment policy, unemployment subsidy, are also discussed and we find that when the economy is booming, then the optimal unemployment subsidy must decrease. As transportation rate increases, the optimal unemployment subsidy should also increase.

The rest of this paper is organized as follows: Section 2 is the model with minimum wage and Section 3 proves the existence of location-wage equilibria. Section 4 discusses the optimal minimum wage policy. Section 5 discusses another scenario such that government unemployment subsidy (without minimum wage) and the consequent welfare implications are further examined in Section 6. Our analysis can be easily applied in the model with a circular market except slighter different equilibrium locations, which will be shown in Section 7. The results are robust for the labor market with a Hotelling oligopsony and will be described in Section 8. Some concluding remarks are discussed in Section 9.

\textsuperscript{2}The effect of minimum wage on unemployment is concerned in empirical studies, see the review by Brown, Gilroy and Kohen (1982).
2 The Model with minimum wage

Suppose all labors are uniformly distributed on a linear market with a unit distance. Each one sells his labor either to one of the two private firms (1, 2) who are located at $x_1 \in [0,1]$, $x_2 \in [0,1]$, respectively, with $x_1 \leq x_2$, or stays unemployment. Suppose the wage offered by firm $i$ is $w_i$, $i = 1, 2$. The utility of a labor located at $x$ who are hired by firm 1, 2, respectively are:

$$u_i = w_i - k|x - x_i|, \quad i = 1, 2.$$  \hfill (1)

where $k$ is the unit transportation cost. Moreover, $u_l$ is the utility of any unemployed worker when enjoys her leisure. Each labor sells one unit of labor to the firm who makes him/her stay in the highest utility level. The game structure is as follows: In the first-stage, the government enacts a minimum wage $w_{\text{min}}$; In the second-stage, both firms decide their locations ($x_1$ and $x_2$) simultaneously; In the third-stage, firms decide their wages ($w_1$ and $w_2$) simultaneously and workers either choose their employers or stay unemployment.

3 The Equilibrium Wage and Locations under Laissez-Faire

The equilibrium is described as follows (see Figure 1). Suppose $n_1^L$ ($n_1^R$) is the left (right) indifferent labor between firm 1 and unemployment, and similarly for $n_2^L$ and $n_2^R$, where the superscript “$L$” represents the left side, while the superscript “$R$” represents the right side.

![Figure 1: The equilibrium locations and labors’ utilities](image-url)
In this case, $0 < x_1 < n_1^R < n_2^L < x_2 < 1$. Therefore, the utility functions $u_1$ and $u_2$ are defined as follows:

$$u_i = \begin{cases} u_i^L = w_i - k(x_i - x), & 0 \leq x \leq x_1, \ i = 1, 2, \\ u_i^R = w_i - k(x - x_i), & x_1 < x \leq 1, \ i = 1, 2. \end{cases}$$

Solving $u_1 = u_2$ yields $n_i^R = x_i + (w_i - u_i)/k$ and $n_i^L = x_i - (w_i - u_i)/k$, $i = 1, 2$. Therefore, the employment of firm $i$ is $N_i = (n_i^R - n_i^L) = 2(w_i - u_i)/k$, $i = 1, 2$, and the unemployment rate is $UN = 1 - N_1 - N_2$. Thus, the profit functions of firms are:

$$\pi_i(w_1, w_2) = (\phi_i - u_i)N_i = \frac{2(\phi_i - u_i)(w_i - u_i)}{k}, \ i = 1, 2,$$

where $\phi_1$ and $\phi_2$ are firms’ productivity for firm 1 and firm 2, respectively. Without loss of generality, assume that $\phi_2 \geq \phi_1$, indicating that 1 and 2 represent the low productivity firm and the high productivity firm, respectively. Note that $\pi_i$, $i = 1, 2$ is not related to $x_i$, $i = 1, 2$. This is because some unemployments exist in equilibrium and thus there exist disjointed ranges of employed labors for each firm. Therefore, the location equilibrium is a range of space instead of just only a point. Solving $\partial\pi_1/\partial w_1 = 0$ and $\partial\pi_2/\partial w_2 = 0$ yields\(^3\)

$$w_i^* = \frac{u_i + \phi_i}{2}, \ i = 1, 2. \tag{2}$$

Replacing the equilibrium wages in (2) into the firm’s employments and unemployment rate yield:

$$N_i^* = \frac{\phi_i - u_i}{k}, \ i = 1, 2, \tag{3}$$

$$UN^* = \frac{k + 2u_i - \phi_1 - \phi_2}{k}. \tag{4}$$

We shall prove the existence of pure strategy under laissez-faire as the following proposition.

**Proposition 1.** Given $\frac{\phi_1 + \phi_2 - k}{2} < u_i < \min\{\phi_1, \phi_2\}$, when two firms compete each other, the only possible equilibrium is that both firms occupy a disjointed range of labor market and other areas are unemployed. The equilibrium locations $(x_1^*, x_2^*)$ satisfy $x_1^* \geq (\phi_1 + \phi_2 - 2u_i)/(2k)$, $x_2^* \leq 1 - [(\phi_2 - u_i)/(2k)]$, and $x_2^* - x_1^* \geq (\phi_1 + \phi_2)/(2k)$, and the equilibrium wages are $w_1 = \frac{u_i + \phi_1}{2}$ and $w_2 = \frac{u_i + \phi_2}{2}$.

\(^3\)The above solution is consistent with the second order conditions, because

$$\begin{bmatrix}
\frac{\partial^2 \pi_1}{\partial w_1^2} & \frac{\partial^2 \pi_1}{\partial w_1 \partial w_2} \\
\frac{\partial^2 \pi_2}{\partial w_2 \partial w_1} & \frac{\partial^2 \pi_2}{\partial w_2^2}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{k} & 0 \\
0 & \frac{1}{k}
\end{bmatrix}
$$

is negative definite.
Proof. Plugging $w^*_1$, $w^*_2$ into $n^R_1$, $n^L_2$ yield

$$n^R_1 = x_1 + \frac{\phi_1 - u_l}{2k}, \quad n^L_2 = x_2 - \frac{\phi_2 - u_l}{2k}. \quad (5)$$

This solutions $(n^R_1, n^L_2)$ are feasible only for the following conditions are satisfied

$$n^L_2 - n^R_1 = -\frac{2u_l + \phi_2 - 2kx_2 + \phi_1 + 2kx_1}{2k} \geq 0$$

$$\Rightarrow \quad x_2^* - x_1^* \geq \frac{\phi_1 + \phi_2 - 2u_l}{2k} \equiv x^*_d, \quad (6)$$

$$n^L_1 = x_1 - \frac{\phi_1 - u_l}{2k} \geq 0$$

$$\Rightarrow \quad x_1 \geq \frac{\phi_1 - u_l}{2k} \equiv x^*_1, \quad (7)$$

$$n^R_2 = x_2 + \frac{\phi_2 - u_l}{2k} \leq 1$$

$$\Rightarrow \quad x_2 \leq 1 - \frac{\phi_2 - u_l}{2k} \equiv x^*_2 \quad (8)$$

Equation (6) represents the ranges of employed labors are disjointed. Equation (7) means that the left boundary of employed labors for firm 1 does not touch the left limit of labor. Equation (8) represents that the right boundary of employed labors for firm 2 does not over the right limit of labor.

Since $x_2 > x_1$ by assumption, using the above three inequities yield:

$$x^*_2 - x^*_1 \geq \frac{k - \phi_1 + 2u_l - \phi_2}{k} \geq 0,$$

which implies,

$$u_l \geq \frac{\phi_1 + \phi_2 - k}{2k} \equiv u^*_l. \quad (9)$$

Undercutting is not profitable as follows. Plugging equilibrium wages into profit functions yield

$$\pi_1^* = \frac{(-\phi_1 + u_l)^2}{2k}, \quad \pi_2^* = \frac{(-\phi_2 + u_l)^2}{2k}.$$

Since the above profit functions are independent to locations, a small change in locations does not alter the profit of firms. For firm 1, given any $w_2$ and $u_l$, any undercutting price $w'_1$ can not get a larger market share than $N_1(w'_1) = 2(w'_1 - u_l)/k$ and thus it can not yield a higher profit than $\pi_1^*$. Since

$$\pi'_1(w'_1, w^*_2) \leq N_1(w'_1)w'_1 = \frac{2(w'_1 - u_l)}{k}w'_1 \leq \frac{2(w^*_1 - u_l)}{k}w^*_1 = \pi_1^*.$$

5
The last inequality is because \( w^*_1 \) is a global maximal solution to \( \pi_1(w_1) = \frac{2(w_1-u_\ell)}{k} w_1 \). Similarly for firm 2.

The above proposition suggests a particular type of location-wage equilibrium. Each firm occupies a local labor market nearby its location. Remark that if \( u_\ell \leq \phi_1 + \phi_2 - k \), then the full employment emerges, since \( N^*_1 + N^*_2 = 1 \) at \( u_\ell = \frac{\phi_1 + \phi_2 - k}{2} \), which has been discussed in the previous literatures and that is not our focus in this paper. Note that if \( u_\ell \geq \min\{\phi_1, \phi_2\} \) then firm 1 cannot make profits from hiring any labor in the market.

**Corollary 1.** The equilibrium locations are not maximal location differentiation when unemployment is considered.

The additional implications of Proposition 1 suggest that maximum location differentiation in duopsony labor market (as Kaas and Madden, 2010) is no longer hold. This result shows that the unemployment can play a role on the location game. Allowing unemployment eliminates the incentive of both firms to move further apart to reduce wage competition. Instead, each firm enjoys the monopsony position on a local labor market near its location.

To ensure the existence of a reasonable equilibrium, \( \frac{\phi_1 + \phi_2 - k}{3} < u_\ell < \min\{\phi_1, \phi_2\} \) is hereafter assumed. From (2), the comparison between \( w^*_1 \) and \( w^*_2 \) is summarized as follows.

**Proposition 2.** The equilibrium wage \( w^*_1 \) (\( w^*_2 \)) is increasing in \( u_\ell \) and \( \phi_1 \) (\( \phi_2 \)) with \( \partial w_i / \partial \phi_i = \frac{1}{2}, i = 1, 2 \). Moreover, \( w^*_1 - w^*_2 = \frac{\phi_1 - \phi_2}{2} \), therefore, \( w^*_1 < w^*_2 \) if and only if \( \phi_1 < \phi_2 \).

**Proof.** It is clear from (2).

Our framework provides a simple analytical solution with pure strategy in the location-wage game. The intuition of Proposition 2 is that the wage rate is positive related to firm’s own productivity and the utility of unemployment. Moreover, the change in wages reflect one half of the changes of firm’s own productivities. The firm with a higher productivity will pay a higher wage rate in equilibrium. The next step is to discuss the equilibrium unemployment rate.

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4The maximal meaningful minimum wage is \( w_{\min} = \phi_2 \). Since \( \phi_2 > \phi_1 \), then to ensure the disjointed market areas are valid in equilibrium. Similar to the proof of Proposition 1, the following conditions must be satisfied:

- \( n^L_1 > 0 \Leftrightarrow x_1 \geq \frac{\phi_1 - u_\ell}{k} = x^L_{\min} \), \( n^R_1 < 1 \Leftrightarrow x_2 \leq x^R_{\max} = \frac{\phi_1 - u_\ell}{k} \), and \( n^L_1 - n^R_1 = x_2 - x_1 \leq \frac{4(\phi_2 - u_\ell)}{k} = x^d_{\min} \), and \( x^R_{\max} = x^L_{\min} - x_d > 0 \), implies \( u_\ell > \frac{4\phi_2 - k}{4} > \frac{\phi_1 + \phi_2 - k}{2} \).
Proposition 3. With a minimum wage $w_{\text{min}}$, the unemployment rate is $UN^* = 1 - (\phi_1 + \phi_2 - 2u_l)/k$. Moreover, the unemployment rate is increasing in $u_l$ and $k$, while it is decreasing in $\phi_i$, $i = 1, 2$.

Proof. It is clear from (4) and comparative statics,
\[
\frac{\partial UN}{\partial u_l} = \frac{2}{k} > 0, \quad \frac{\partial UN}{\partial \phi_i} = -\frac{1}{k} < 0, \quad \frac{\partial UN}{\partial k} = \frac{\phi_2 - 2u_l + \phi_1}{k^2} > 0.
\]

Proposition 3 suggests that the unemployment rate is negatively related to the productivities, but positive related to the unit transportation cost and the utility of unemployment.

Example 1: A numerical examples is given by the following values of parameters: $k = 0.64$, $u_l = 1.8$, $\phi_1 = 2$, $\phi_2 = 2.2$. The equilibrium locations, wages, market shares, and unemployment rate are: $x_1^* \geq x_1^{\text{min}} = 0.15625$, $x_2^* \leq x_2^{\text{max}} = 0.6875$, $x_2^* - x_1^* \geq x_d^{\text{min}} = 0.46875$, $w_1^* = 1.9$, $w_2^* = 2.0$, $N_1^* = 0.3125$, $N_2^* = 0.625$, and $UN^* = 0.0625$.

4 Social Welfare and Minimum Wage

This section will discuss the optimal minimum wage. First, consumer surplus of workers is defined as
\[
CS_e(w_1, w_2) = \int_{n_1^L}^{n_1^R} (w_1 - k|x - x_1|) \, dx + \int_{n_2^L}^{n_2^R} (w_2 - k|x - x_2|) \, dx
= \frac{w_1^2 + w_2^2 - 2u_l^2}{k}.
\]

Consumer surplus of unemployed people is defined as
\[
CS_u(w_1, w_2) = (1 - N_1 - N_2)u_l
= \frac{u_l}{k}(4u_l - 2(w_1 + w_2)).
\]

Social welfare is then defined as
\[
SW(w_1, w_2) = CS_e(w_1, w_2) + CS_u(w_1, w_2) + \pi_1(w_1, w_2) + \pi_2(w_1, w_2)
= \frac{2u_l^2 - w_1^2 - w_2^2 + 2(\phi_1 w_1 + \phi_2 w_2) + u_l k - 2u_l(\phi_1 + \phi_2)}{k}.
\]
4.1 The First-Best Solution

First, we discuss the first-best solution denoted by a superscript “o” as a benchmark case.

**Proposition 4.** The first-best locations, wages and unemployment rate are: \( x^{o}_1 \geq \frac{\phi_1-u_l}{k} \), \( x^{o}_2 \leq 1 - \frac{\phi_2+u_l}{k} \), and \( x^{o}_2 - x^{o}_1 \geq \frac{\phi_1+\phi_2-2u_l}{k} \); \( w^{o}_1 = \phi_1 \), \( w^{o}_2 = \phi_2 \); \( UN^{o} = \frac{k-2(\phi_1+\phi_2)+2u_l}{k} \).

**Proof.** See the appendix.

Proposition 4 shows that there is no duopsony power for firms in the first-best regime (because \( w_i = \phi_i \), \( i = 1, 2 \)). Moreover, the first-best unemployment rate is less than the equilibrium unemployment rate in the duopoly labor market. The first-best locations are more closer to each other than that of the equilibrium locations. These results are quite intuitively.

4.2 Social Welfare Function with Minimum Wage

Plugging equilibrium wages into (10) yields:

\[
CS_e(w_1^*, w_2^*) = \frac{1}{k} \left( \frac{\phi_1 + u_l}{2} \right)^2 + \left( \frac{\phi_2 + u_l}{2} \right)^2 - 2u_l^2.
\]

Plugging equilibrium wages into (11) yields:

\[
CS_u(w_1^*, w_2^*) = \frac{u_l}{k} (2u_l + k - \phi_1 - \phi_2).
\]

Plugging equilibrium wages into (12) yields:

\[
SW(w_1^*, w_2^*) = \frac{1}{4k} (3\phi_1^2 + 3\phi_2^2 - 6u_l(\phi_1 + \phi_2) + 4ku_l + 6u_l^2). \quad (13)
\]

If \( w_{\text{min}} \leq \min(w_1^*, w_2^*) = (u_l + \min(\phi_1, \phi_2))/2 \), the minimum wage is not binding and thus the minimum wage plays no role in the labor market. Therefore, we only consider the following two subcases.
**Case A:** \( w_1^* = (\phi_1 + u_l)/2 \leq w_{\text{min}} \leq w_2^* = (\phi_2 + u_l)/2 \)

In this case, \( w_1^* = w_{\text{min}} \), since \( \pi_1(w_1, w_2) \) is a strictly concave function of \( w_1 \). Replace the equilibrium wage \( w_1^*, w_2^* \) into firms’ profit and social welfare:

\[
\begin{align*}
\pi_1(w_{\text{min}}, w_2^*) &= \frac{2}{k}(\phi_1 - w_{\text{min}})(w_{\text{min}} - u_l), \\
\pi_2(w_{\text{min}}, w_2^*) &= \frac{2}{k}\left(\frac{\phi_2 - u_l}{2}\right)^2, \\
CS_c(w_{\text{min}}, w_2^*) &= \frac{1}{k}\left[\left(\frac{\phi_2 + u_l}{2}\right)^2 + w_{\text{min}}^2 - 2u_l^2\right], \\
CS_u(w_{\text{min}}, w_2^*) &= \frac{u_l}{k}(\phi_2 + 2w_{\text{min}} - k + 3u_l), \\
\text{SW}(w_{\text{min}}, w_2^*) &= \frac{7u_l^2 - 4w_{\text{min}}^2 - 6u_l\phi_2 + 3\phi_2^2 + 4u_l k - 8u_l\phi_1 + 8\phi_1 w_{\text{min}}}{4k}. \quad (14)
\end{align*}
\]

The first order condition of social welfare maximization is

\[
\frac{\partial \text{SW}(w_{\text{min}}, w_2^*)}{\partial w_{\text{min}}} = \frac{2(\phi_1 - w_{\text{min}})}{k}.
\]

Therefore, the optimal minimum wage is

\[ w_{\text{min}}^* = \phi_1. \quad (15) \]

The second order condition is satisfied. The necessary conditions of this case is that

\[ w_2^* > w_{\text{min}}^* \geq w_1^* \iff \frac{u_l + \phi_2}{2} - \phi_1 > 0, \]

where \( w_{\text{min}}^* - w_1^* = \frac{\phi_1 - u_l}{2} > 0 \) is derived from \( N_1^* > 0 \). Plugging \( w_{\text{min}}^* \) into social welfare function (14) yields

\[ \text{SW}^* \equiv \text{SW}(w_{\text{min}}^*, w_2^*) = \frac{7u_l^2 + 4\phi_2^2 - 6u_l\phi_2 + 3\phi_2^2 + 4u_l k - 8u_l\phi_1}{4k}. \quad (16) \]

**Case B:** \( w_2^* \leq w_{\text{min}} \)

In this case \( w_1^* = w_2^* = w_{\text{min}} \). Plugging \( w_1^* = w_2^* = w_{\text{min}} \) into profit functions yields

\[
\begin{align*}
\pi_1(w_{\text{min}}, w_{\text{min}}) &= \frac{2(-\phi_1 + w_{\text{min}})(u_l - w_{\text{min}})}{k}, \\
\pi_2(w_{\text{min}}, w_{\text{min}}) &= \frac{2(-\phi_2 + w_{\text{min}})(u_l - w_{\text{min}})}{k}.
\end{align*}
\]
The consumer surplus is
\[ CS_c(w_{\text{min}}, w_{\text{min}}) = \frac{-2u_l^2 - w_l^2 - w_k^2}{k}, \]
and the employment consumer surplus is
\[ CS_u(w_{\text{min}}, w_{\text{min}}) = \frac{(k + 4u_l - 2w_1 - 2w_2)u_l}{k}. \]
The social welfare in this case is
\[ SW(w_{\text{min}}, w_{\text{min}}) = \frac{2u_l^2 - 2w_k^2 + u_lk - 2u_l\phi_1 + 2\phi_1w_{\text{min}} - 2u_l\phi_2 + 2\phi_2w_{\text{min}}}{k}. \]
\[ (17) \]
The first order condition of social welfare is
\[ \frac{\partial SW(w_{\text{min}}, w_{\text{min}})}{\partial w_{\text{min}}} = -4w_{\text{min}} + 2\phi_1 + 2\phi_2. \]
The second order condition is satisfied. Solving the first order condition yields
\[ w_{\text{min}}^{**} = \frac{\phi_1 + \phi_2}{2} \geq \max\{w_1, w_2^*\} = \max\left\{ \frac{u_l + \phi_1}{2}, \frac{u_l + \phi_2}{2} \right\} \text{ since } \min\{\phi_1, \phi_2\} \geq u_l. \]
Plugging \( w_{\text{min}}^{**} \) into social welfare function (17) yields
\[ SW^{**} \equiv SW(w_{\text{min}}^{**}, w_{\text{min}}^{**}) = \frac{(\phi_1 + \phi_2)^2 - 4u_l(\phi_1 + \phi_2) + 2u_l(k + 2u_l)}{2k}. \]
\[ (18) \]

4.3 Minimum Wage Policy when the Productivity Difference is Relatively Large

In this subsection, we first consider a situation such that \( \phi_1 < w_2^* \) (see Figure 6), it implies \( \phi_2 - \phi_1 > \phi_1 - u_1 \). The situation can be divided into two sub-situations: when \( w_{\text{min}} \in [w_1, \phi_1] \), then \( \pi_1 > 0, \pi_2 > 0 \), when \( w_{\text{min}} \in [\phi_1, \phi_2] \), then \( \pi_1 < 0, \pi_2 > 0 \). It means the situation that the productivity difference is relatively large. It can be summarized as the following proposition

**Proposition 5.** Given \( \phi_2 - \phi_1 > \phi_1 - u_1 \), then (a) when \( w_{\text{min}} \leq \phi_1 \), then it is always welfare improving, and the maximal welfare improving is at \( w_{\text{min}} = \phi_1 \). (b) when \( \phi_1 \leq w_{\text{min}} < \frac{\phi_2 + u_1}{2} (= w_2^*) \), then it is social welfare improving iff \( (3\phi_1 - u_l)/2 > w_{\text{min}} > \frac{\phi_1 + u_1}{2} \). However, the maximal social welfare is at \( w_{\text{min}} = \phi_1 \). (c) when \( \frac{\phi_2 + u_1}{2} \leq w_{\text{min}} \), then it is social welfare improving iff \( w_{\text{min}} \leq \frac{\phi_1 + \phi_2}{2} + \frac{\sqrt{2\phi_1^2 - 2\phi_1^2 + 8u_1\phi_2 - 4u_l(\phi_1 + \phi_2) + 4u_1^2}}{4} \). The maximal social welfare is at \( w_{\text{min}} = \frac{\phi_1 + \phi_2}{2} \). (d) compare the maximal social welfare in (a), (b) and (c). The optimal minimum wage is \( w_{\text{min}} = \phi_1 \) if \( u_1 \geq \phi_2 - \sqrt{2}(\phi_2 - \phi_1) \) and \( w_{\text{min}} = \frac{\phi_1 + \phi_2}{2} \) if \( u \leq \phi_2 - \sqrt{2}(\phi_2 - \phi_1) \).
Proposition 5 states that imposing an optimal minimum wage $w^*_{\min}$ is always social welfare improving. However, for any minimum wage $w_{\min}$, there is not always welfare improving when $w_{\min} > (3\phi_1 - u_l)/2$. If $(3\phi_1 - u_l)/2 \geq w_2^* = (\phi_2 + u_l)/2$, that is $3\phi_1 - \phi_2 \geq 2u_l$, imposing any minimum wage in this case is always social welfare improving.

Proposition 5(d) suggests that imposing a higher minimum wage which binds the wages of these two firms (Case B) is better than a minimum wage which only binds the lower wage firm (Case A), if the difference of two firms’ productivities is relatively large.

Proposition 5 can be summarized as the following three figures. Figure 3 depicts that imposing any one-firm binding minimum wage is always a social welfare improving. If $u_l < \phi_2 - \sqrt{2}(\phi_2 - \phi_1)$, then the social welfare maximization policy is imposing the optimal minimum wage $w^*_{\min}$ which is binding in two firms. If $u_l > \phi_2 - \sqrt{2}(\phi_2 - \phi_1)$, then binding on a single firm is social optimum. The intuition is that the imposing of the minimum wage which binds on two firms yields some social cost (unemployment). However, when the leisure utility from unemployment $u_l$ is low, this social cost is relative low, and thus the benefit of minimum wage (decrease the duopoly power) is larger than the social welfare which only binding on one firm. Figure 4 depicts that imposing the minimum wage that binding only one firm, may be welfare worsen when $w_{\min} > (3\phi_1 - u_l)/2$. Note that $SW^*$ is still welfare superior to $SW(w_1^*, w_2^*)$, in other words, a high enough minimum wage is better than no binding minimum wage. Figure 5 depicts that $SW^{**}$ (binding on two firms) may lower than $SW^*$ (binding on one firm). This may happen when the leisure utility of unemployment is relative high.

Example 2: A numerical examples is given by the following values of parameters: $k = 0.64$, $\phi_1 = 2$, $\phi_2 = 2.2$. Comparing $SW^* - SW(w_1^*, w_2^*)$, $SW^{**} - SW(w_1^*, w_2^*)$, and $SW(w_2^*, w_2^*) - SW(w_1^*, w_2^*)$ and plot them in Figure 6. First, $SW^* - SW(w_1^*, w_2^*)$ is always positive, measure that imposing the optimal minimum wage $w^*_{\min}$ is always social welfare improving as shown in
Figure 3: The social welfare with minimum wage when the productivity difference is relatively large such that $3\phi_1 - \phi_2 \geq 2u_l$

Proposition 5(a). Second, $SW(w^*_2, w^*_2) - SW(w^*_1, w^*_2)$ may less than zero when $u_l$ is relatively large ($u_l > 1.9$) as shown in Proposition 5(b). As shown in Figure 4 and Figure 5, imposing an arbitrary minimum wage $w_{min}$ with one firm binding may be social welfare worsening. Third, for $SW^{**} - SW(w^*_1, w^*_2)$, it is social welfare improving (Figure 4) when $u_l$ is relatively small ($u_l < 1.927$), while it is welfare worsening (Figure 5) when $u_l$ is large ($u_l > 1.927$) as shown in Proposition 5(c). Finally, comparing $SW^{**} - SW(w^*_1, w^*_2)$ and $SW^* - SW(w^*_1, w^*_2)$, we have $SW^{**} - SW^* \geq 0$ when $u_l$ is relatively small ($u_l < 1.917$) as shown in Proposition 5(d).

The following proposition shows that it is possible that imposing a minimum wage may not be a social improvement.

4.4 Minimum Wage Policy when the Productivity Difference is Relatively Small

**Proposition 6.** Given $\phi_2 - \phi_1 < u_1 - u_l$, if a government imposes a minimum wage such that (a) when $w_{min} \leq \phi_1$, it is always welfare improving, and the maximal welfare is at
Figure 4: The social welfare with minimum wage when the productivity difference is relatively large such that $3\phi_1 - \phi_2 < 2u_1$ and $u_1 < (1 + \sqrt{3})\phi_2 + (1 - \sqrt{3})\phi_1$. 

\begin{align*}
 w_{\min} &= \phi_1. \ (b) \text{ when } \phi_1 \leq w_{\min}, \text{ then it is social welfare improving iff } w_{\min} \leq \frac{\phi_1 + \phi_2}{2} + \\
 &\quad \quad \sqrt{-2\phi_1^2 - 2\phi_2^2 + 8\phi_1\phi_2 - 4u_1(\phi_1 + \phi_2) + 4u_1^2}, \text{ and the maximal welfare is at } w_{\min} = \frac{\phi_1 + \phi_2}{2}. \ (c) \text{ compare the maximal welfare in (a) and (b), we find that the global welfare maximization is at } w_{\min} = \frac{\phi_1 + \phi_2}{2}.
\end{align*}

**Proof.** See the appendix. 

The condition in Proposition 6 is allowing negative profits. However, if non-negative profit condition is enacted, then the minimum wage $w_{\min} \leq \phi_1$, and (17) is increasing as $w_{\min}$ increase under this case, then the optimal minimum wage is $\phi_1$. Although both this case and part (c) of Proposition 5 has the same optimal minimum wage $\phi_1$. However, the major difference of these two cases is that $\phi_1 < w_2^*$ (Proposition 6(c)) and $\phi_1 > w_2^*$ (this case).
4.5 The Social Welfare with Minimum Wage when Drive Out the Low Productivity Firm

From Proposition 5 and 6, we know that \( w_{\text{min}} = \phi_1 \) if the non-negative profit condition is considered. Then

\[
\hat{SW}^* = \begin{cases} 
SW(\phi_1, w_2^*) & \text{when } \phi_1 < \frac{u_1 + \phi_2}{2}, \\
SW(\phi_1, \phi_2) & \text{when } \phi_1 > \frac{u_1 + \phi_2}{2},
\end{cases}
\]

where \( \hat{SW}^* \) is the optimal social welfare under no exist of firms. Now we compare (19) with the case that a government can enact a high minimum wage to drive out low productivity firms and whether the social welfare increases or decreases. Define

\[
\hat{SW} = \phi_2 N_2 + (1 - N_2)u_1 - \int_{n_2}^{n_2} k|x - x_2|dx.
\]

represent the case that \( w_{\text{min}} > \phi_1 \) and thus firm 1 is forced out of the industry (see Figure 7).

After some calculations,

\[
\hat{SW}(w_{\text{min}}) = \frac{u_1^2 - w_{\text{min}}^2 + u_1 k - 2\phi_2 u_1 + 2\phi_2 w_{\text{min}}}{k}.
\]
Proposition 7. (a) \( \hat{SW}(w_{\min}^{***}) > SW(x_1^*, w_2^*) \) if \( \phi_2 > u_l + \sqrt{3}(\phi_1 - u_l) \), (b) \( \hat{SW}(w_{\min}^{***}) > \hat{SW}^* \)
Figure 8: The social welfare with minimum wage when the productivity difference is relatively small such that \( \phi_2 - \phi_1 < \phi_1 - u_l \)

if \( \phi_2 > 2\phi_1 - u_l \).

**Proof.** \( \hat{SW}(w_{\min}^{**}) - SW(w_1^*, w_2^*) = \frac{\phi_2^2 + 6\phi_1 u_l - 3\phi_1^2 - 2u_l^2 - 2u_l \phi_2}{4k} > 0 \), iff \( \phi_2 > u_l + \sqrt{3}(\phi_1 - u_l) \).

\[
\hat{SW}(w_{\min}^{**}) - SW^* = \begin{cases} 
\frac{(2\phi_1 + \phi_2 - 3u_l)(-2\phi_1 + \phi_2 + u_l)}{4k}, & \text{when } \phi_1 < \frac{\phi_2 + u_l}{2}, \\
\frac{(\phi_2 - u_l)(-2\phi_1 + \phi_2 + u_l)}{k}, & \text{when } \phi_1 \geq \frac{\phi_2 + u_l}{2}.
\end{cases}
\]

The above proposition suggests that imposing minimum wage driving out the low productivity firm is social welfare improving when \( \phi_2 \) is relatively large, and can be better than the case with the minimum wage with partial binding \( (w_{\min}^{**} > \phi_1) \).

**Example 3:** Let \( k = 0.64 \), \( \phi_1 = 2 \), \( u_l = 1.7 \). Comparing \( SW^* - SW(w_1^*, w_2^*) \) and \( \hat{SW}(w_{\min}^{**}) - SW(w_1^*, w_2^*) \) yield Figure 9. First, \( \hat{SW}(w_{\min}^{**}) \) is welfare improving when \( \phi_2 > 2.2196 \). Second, \( \hat{SW}(w_{\min}^{**}) - SW(w_1^*, w_2^*) \) is better than \( SW^* - SW(w_1^*, w_2^*) \) when \( \phi_2 > 2.30 \). In other words, when \( \phi_2 \) is large enough, imposing a drive-low productivity-firm out policy is welfare improving.

Our model also provides an empirical implication on the influence of minimum wage on employment. From \( N_i = 2(w_i - u_l)/k \), \( i = 1, 2 \), the employment \( N_1 + N_2 \) is positively associated.
with the wages. If a binding (either partially or fully) minimum wage is imposed without driving out a low productivity firm, the employment will increase. That is, the employment $N_1 + N_2 \leq \frac{2(w_{min} + w^*_2 - 2u_l)}{k}$ in the partially binding case and $\frac{4(w_{min} - u_l)}{k}$ in the fully binding case. It implies a positive relationship between minimum wage and employment, because the minimum wage reduces the duopsony power of the employers. However, if a higher minimum wage drives out low productivity firms, the employment $N_2 = \frac{(w_{min} - u_l)}{k}$ will decrease. These two opposite effects can be mixed in empirical evidences. It provides an possible interpretation on why the effects of changes in the minimum wage on employment are limited in empirical studies (see, for instance, De Fraja (1999)).

5 The Model with unemployment subsidy

Instead of the minimum wage policy, suppose now the government can give a subsidy ($w_g$) to unemployed worker, and this subsidy can produce $\phi_3$ benefits (e.g. criminal rate reduced) in
this section. Other assumptions are being equal and thus equations (1) are still valid, and \( u_l \) should be replaced by \( u = w_g + u_l \). The same method can thus be applied to discuss the role of unemployment subsidy on this location-wage game.

5.1 The Equilibrium Wage and Locations

The equilibrium case is described as follows. Figure 10 is the utility of labor who are either

![Figure 10: The equilibrium locations and labors’ utilities with unemployment subsidy](image)

hired by firm 1 and 2, or unemployed.

Similarly, the profit functions of firms and the government surplus are:

\[
\begin{align*}
\pi_1 &= (\phi_1 - w_1)N_1 = -\frac{2(-\phi_1 + w_1)(-w_g - u_l + w_1)}{k}, \\
\pi_2 &= (\phi_2 - w_2)N_2 = -\frac{2(-\phi_2 + w_2)(-w_g - u_l + w_2)}{k}, \\
G &= (\phi_3 - w_g)UN \\
&= -\frac{(-\phi_3 + w_g)(k + 4w_g + 4u_l - 2w_1 - 2w_2)}{k},
\end{align*}
\]

Solving for \( \frac{\partial \pi_1}{\partial w_1} = 0, \frac{\partial \pi_2}{\partial w_2} = 0 \), yield

\[
w_1^* = \frac{w_g}{2} + \frac{u_l}{2} + \frac{\phi_1}{2}, \quad w_2^* = \frac{w_g}{2} + \frac{u_l}{2} + \frac{\phi_2}{2}.
\]

(20)
Replacing the equilibrium wages into market shares, then the firm's employments and unemployment rate become:

\[
N^*_1 = \frac{\phi_1 - w_g - u_l}{k}, \quad N^*_2 = \frac{\phi_2 - w_g - u_l}{k}, \quad UN^* = \frac{k - \phi_1 - \phi_2 + 2w_g + 2u_l}{k}.
\]

(21) (22)

The following proposition provides the existence of location-wage equilibria similar with Proposition 1.

**Proposition 8.** When two firms compete each other, the only possible equilibrium is that both firms occupy a range of market without overlapping and other areas are unemployed when \(w_g < w^*_{\text{max}}\). The equilibrium locations \((x^*_1, x^*_2)\) satisfy \(x^*_1 \geq \frac{\phi_1 - w_g - u_l}{2k}, x^*_2 \leq 1 - \frac{\phi_2 + w_g - u_l}{2k}\), and \(x^*_2 - x^*_1 \geq \frac{\phi_1 + \phi_2 - 2w_g - 2u_l}{2k}\), the equilibrium wage is \(w^*_1 = \frac{w_g + u_l + \phi_1}{2}\) and \(w^*_2 = \frac{w_g + u_l + \phi_2}{2}\).

**Proof.** See the appendix. \(\square\)

Note that if \(w_g > w^*_{\text{max}}\), then the total employment falls and it makes no sense to do that.

### 5.2 Comparative Statics and Social Welfare

To ensure the existence of a reasonable equilibrium, \(w_g < w^*_{\text{max}}\) is hereafter assumed. From (20), the comparison between \(w^*_1\) and \(w^*_2\) is summarized as follows.

**Proposition 9.** Under a government subsidy, \(w^*_1 = \frac{\phi_1 + w_l + w_g}{2}\) and \(w^*_2 = \frac{\phi_2 + w_l + w_g}{2}\) are increasing in \(k\). Moreover, \(w^*_1 - w^*_2 = \frac{\phi_1 - \phi_2}{2}\), therefore, \(w^*_1 < w^*_2\) if and only if \(\phi_1 < \phi_2\). Finally, \(\partial w_i / \partial \phi_i = 1/2\), which means that \(w^*_1\) is increasing (half) in \(w_g\), and \(w^*_2\) is decreasing (half) in \(t\).

**Proof.** It is clear from (20). \(\square\)

The first part of Proposition 9 is straightforward. The second part of Proposition 9 shows that a higher unemployment subsidy will result in a higher equilibrium wages. of private firms to reflect its convenience.

**Proposition 10.** Given a wage subsidy \(w_g\), the equilibrium unemployment rate is \(UN^* = \frac{k - \phi_1 + 2w_g + 2u_l - \phi_2}{k}\). Moreover, the unemployment rate is increasing in \(w_g\), \(u_l\) and \(k\), while it is decreasing in \(\phi_i\), \(i = 1, 2\).
Proof. By comparative statics,
\[
\frac{\partial U N^*}{\partial w_g} = \frac{2}{k} > 0, \quad \frac{\partial U N^*}{\partial u_l} = \frac{2}{k} > 0,
\]
\[
\frac{\partial U N^*}{\partial \phi_i} = -\frac{1}{k} < 0, \quad \frac{\partial U N^*}{\partial k} = \frac{\phi_1 + \phi_2 - 2w_g - 2u_l}{k^2} > 0.
\]
The above result implies that \( U N^* \) is increasing as \( w_g \) increases. \( \square \)

6 Social Welfare under Government Subsidy

To analyze a policy implication, we define a social welfare function as follows,

\[
\max_{w_g} SW = CS_e + CS_u + \pi_1 + \pi_2 + G,
\]

where \( G = (\phi_3 - w_g) \cdot UN \) is the government surplus and the government can use \( w_g \) to maximize social welfare.

\( CS_e \) (\( CS_u \)) is the consumer surplus for employed (unemployed) labor.

\[
CS_e = -\frac{-\phi_2^2 + 6w_g^2 + 12w_gu_l - 2w_g\phi_1 + 6u_l^2 - 2u_l\phi_1 - \phi_2^2 - 2w_g\phi_2 - 2u_l\phi_2}{4k},
\]
\[
CS_u = \frac{(k - \phi_1 + 2w_g + 2u_l - \phi_2)(w_g + u_l)}{k},
\]
\[
G = -\frac{(-\phi_3 + w_g)(k - \phi_1 + 2w_g + 2u_l - \phi_2)}{k}.
\]

\[
SW = (3\phi_2^2 - 2w_g^2 + 4w_gu_l - 2w_g\phi_1 + 6u_l^2 - 6u_l\phi_1 + 3\phi_1^2 - 2w_g\phi_2 - 6u_l\phi_2 + 4ku_l + 4\phi_3k - 4\phi_3\phi_1 + 8\phi_3w_g + 8\phi_3u_l - 4\phi_3\phi_2) / (4k).
\] (23)

We can discuss the optimal unemployment subsidy denoted by a superscript "s*" and unemployment rate as following propositions.

Proposition 11. The optimal unemployment subsidy \( w_{g^*} = 2\phi_3 + u_l - \frac{\phi_1 + \phi_2}{2} \). When the economy is booming (\( \phi_i \) increases, \( i = 1, 2 \)), then the optimal unemployment subsidy must decrease. Similarly, \( \partial w_{g^*} / \partial u_l > 0 \), and \( \partial w_{g^*} / \partial k = 0 \).

Proof. Solving \( \partial SW / \partial w_g = 0 \) yields \( w_{g^*} = 2\phi_3 + u_l - \frac{\phi_1 + \phi_2}{2} \). \( \square \)
Proposition 12. The optimal unemployment rate (under optimal unemployment subsidy) as the second-best solution is

\[ UN^* = \frac{k - 2\phi_1 + 4u_l - 2\phi_2 + 4\phi_3}{k} \geq 0. \]

The optimal unemployment rate is increasing in \( u_l, k \) and \( \phi_3 \), while it is decreasing in \( \phi_1, \phi_2 \).

Proof. It is clear from the optimal solution \( UN^* \).

The economic intuition is that the unemployment subsidy should adjust as per the situation of economic status: When the economy is booming (recession), the optimal subsidy should be decreased (increased); Similarly, the higher \( \phi_3 \), then the higher \( w_s^* \).

Example 4: A numerical examples with parameters: \( k = 0.64, u_l = 1.8, \phi_1 = 2, \phi_2 = 2.2, \phi_3 = 0.155 \), then the equilibrium locations, market shares, unemployment rate, and the optimal unemployment subsidy are:

- \( x_s^1 \geq x_{1\min} = 0.1484375 \)
- \( x_s^2 \leq x_{2\max} = 0.6953125 \)
- \( x_{d\min} = 0.453125 \)
- \( N_s^1 = 0.296875, N_s^2 = 0.609375, UN^* = 0.09375, w_s^* = 0.01. \)

Our model can also be extended to include both unemployment subsidy and a payroll tax. Consider that the government enacts a payroll tax \( t \) on firms for each hired worker, then the profit function for firm \( i \) is 

\[ \pi_i = (\phi_i - w_g - t)N_i, \quad i = 1, 2. \]

We can prove that no matter self-financing or non-self-financing systems, the optimal payroll tax \( t_s^* < 0 \), which implies an optimal payroll subsidy. The intuition behinds this result is because a payroll subsidy can reduce the distortion from duopsony labor market.

7 Circular Market

The current model can be easily extended to a circular market. Suppose there are two duopsony firms locate at a circular market with unit length (see Figure 11). Without loss of generality, assume \( x_1 = 0 \), and \( x_2 \in [0, y_2] \). \( u_1 \) and \( u_2 \) should be replaced by

\[
\begin{align*}
  u_1 &= \begin{cases} 
    u_{1L}^L = w_1 - k(1-x), & \forall \ x \in [\frac{1}{2}, 1], \\
    u_{1R}^L = w_1 - k(x-0), & \forall \ x \in [0, \frac{1}{2}], 
  \end{cases} \\
  u_2 &= \begin{cases} 
    u_{2L}^L = w_2 - k(x_2-x), & \forall \ x \in [0, x_2] \text{ and } [x_2 + \frac{1}{2}, 1], \\
    u_{2R}^L = w_2 - k(x-x_2), & \forall \ x \in [x_2, x_2 + \frac{1}{2}]. 
  \end{cases}
\end{align*}
\]

\( u_l \) is the same.

We thus can summarize the following proposition.
Proposition 13. Consider a duopsony-labor market in circular market, the location-wage equilibrium is \( x_1 = 0, x_2 \in [x_{2\min}, x_{2\max}] \). The equilibrium wages, profits, and market shares are all identical to the linear market.

Proof. See the appendix. \( \square \)

From Proposition 13, we find that the equilibrium of the duopsony circular market is identical to the linear duopsony market, except the equilibrium locations have a little different. Therefore, the optimal minimum wage and unemployment subsidy are still the same as the results in the linear market. In other words, our model is robust in the circular market.

8 Oligopsony Labor Markets and Minimum Wage

The duopsony model can be easily extended to an oligopsony-labor market. In order to simplify our model, suppose there are \( m_1 \) firms with productivity \( \phi_1 \), and \( m_2 \) firms with productivity \( \phi_2 \), and \( m_1 + m_2 = M \). Suppose firm \( i \in [1, 2, \ldots, M] \) locate at \( x_1 \leq x_2 \leq \ldots \leq x_M \), respectively, and the utility function are

\[
\begin{align*}
   u_i &= \begin{cases} 
   u_i^L = w_i - k(x_i - x), & \forall \ 0 \leq x \leq x_i, \ i = 1, 2, \ldots, M, \\
   u_i^R = w_i - k(x - x_i), & \forall \ x_i < x \leq 1, \ i = 1, 2, \ldots, M.
   \end{cases}
\end{align*}
\]
8.1 The Optimal Minimum Wage Policy

Solving $u_i = u_l$ yields

$$u_i^R = \frac{w_i - u_l + kx_i}{k}, \quad n_i^L = \frac{u_l - w_i + kx_i}{k}, \quad i = 1, 2.$$ 

By symmetry, the employment of firm $i$ is

$$N_i = \frac{2(w_i - u_l)}{k}, \quad i = 1, 2,$$

and the unemployment rate is $UN = 1 - \sum_{i=1}^{M} N_i$. The profit functions are

$$\pi_i(w_i) = (\phi_{j(i)} - w_i) \cdot N_i = \frac{2(\phi_{j(i)} - w_i)(w_i - u_l)}{k}, \quad i = 1, \ldots, M,$$

where

$$j(i) \begin{cases} = 1, & \text{for low productivity ($\phi_1$) firms,} \\ = 2, & \text{for high productivity ($\phi_2$) firms.} \end{cases}$$

It can summarized to one equilibrium case which is described as Figure 12.

![Figure 12: The equilibrium locations and labors utilities under oligopoly.](image)

Solving $\partial \pi_i(w_i)/\partial w_i = 0$ yield

$$w_i^* = \frac{w_i + \phi_{j(i)}}{2}.$$ 

Plugging $w_i^*$ into $n_i^R, n_i^L$ yield

$$n_i^{R*} = x_i + \frac{\phi_{j(i)} - u_l}{2k}, \quad n_i^{L*} = x_i - \frac{\phi_{j(i)} - u_l}{2k}.$$
Similarly we can find the pure-strategy equilibrium in the location-wage game with oligopsony labor market.

Therefore, we have proven the following proposition, which is corresponding to Proposition 1.

**Proposition 1’**. Given \( \frac{(m_1 \phi_1 + m_2 \phi_2) - k}{M} < u_l < \min\{\phi_1, \phi_2\} \), when there are \( M \) oligopsony firms compete each other, the only possible equilibrium is that each firm occupies a disjointed range of labor market, and other areas are unemployed. The equilibrium locations \( (x_1^*, x_2^*, \ldots, x_M^*) \) satisfy \( x_1^* \geq x_1^{\min}, x_M^* \leq x_M^{\max} \), and \( x_{i+1}^* - x_i^* \geq x_{d(i,i+1)}^{\min} \), the equilibrium wages are \( w_i = \frac{u_l + \phi_j(i)}{2} \).

**Proof.** See the appendix. 

We thus have

\[
N_i^* = \frac{\phi_j(i) - u_l}{k}, \quad i = 1, \ldots, M.
\]

and

\[
U N^* = 1 - \sum_{i=1}^{M} N_i = \frac{k - m_1 \phi_1 - m_2 \phi_2 - (m_1 + m_2) u_l}{k}.
\]

Therefore, Proposition 2 and Proposition 3 are still valid in this oligopsony case.

### 8.2 The First-Best Solution

Now we analyze the social welfare. The first-best solution is solved as follows. Social optimal (location and market share)

\[
\max_{\hat{N}_1, \hat{N}_2} \text{SW}(\hat{N}_1, \hat{N}_2) = \phi_1 \hat{m}_1 \hat{N}_1 + \phi_2 \hat{m}_2 \hat{N}_2 - \frac{k}{4}[m_1 \hat{N}_1 + m_2 \hat{N}_2] + (1 - m_1 \hat{N}_1 - m_2 \hat{N}_2) u_l,
\]

(24)

where \( \hat{N}_1 (\hat{N}_2) \) is the market share for the low (high) productivity firms. The first order condition of (24) are

\[
\frac{\partial \text{SW}(\hat{N}_1, \hat{N}_2)}{\partial \hat{N}_1} = \phi_1 - \frac{k}{2} \hat{N}_1 - u_l = 0,
\]

\[
\frac{\partial \text{SW}(\hat{N}_1, \hat{N}_2)}{\partial \hat{N}_2} = \phi_2 - \frac{k}{2} \hat{N}_2 - u_l = 0,
\]
then the first-best solution indexed by the superscript “$o$” becomes:

$$\hat{N}_i^o = \frac{2(\phi_i - u_i)}{k}, \quad i = 1, 2.$$ 

Thus, we have the optimal market shares

$$\hat{N}_1^o = \frac{2(\phi_1 - u_1)}{k} > N_i^*, \quad j(i) = 1,$$
$$\hat{N}_2^o = \frac{2(\phi_2 - u_1)}{k} > N_i^*, \quad j(i) = 2.$$ 

and

$$UN^o = \frac{k - 2(m_1 \phi_1 + m_2 \phi_2) + 2(m_1 + m_2)u_i}{k} > UN^*.$$ 

From above discussions, it is shown that the result in Proposition 4 can be extended to the oligopsony case. The first-best solution of unemployment rate is still less than the equilibrium unemployment rate.

### 8.3 The Social Welfare Function with Minimum Wage

If there is no binding for the minimum wage, $w_i = w_i^*$ and

$$\pi_i(w_i^*) = \frac{2 \left(\frac{w_i - \phi_i}{2}\right)^2}{k}.$$ 

The second order conditions are satisfied. The consumer surplus is

$$CS = \left(\int_{x_i}^{x_i} (w_i - k(x_i - x)) \, dx + \int_{x_i}^{x_i} (w_i - k(x_i - x)) \, dx\right)$$
$$= \frac{m_1 w_1^2 + m_2 w_2^2 - (m_1 + m_2)u_i^2}{k},$$

where $w_1$ ($w_2$) is the wage for low (high) productivity firms. For the unemployed workers,

$$CS_u = UN \cdot u_i = \frac{(k + 2(m_1 + m_2)u_i - 2(m_1 w_1 + m_2 w_2))u_i}{k}.$$ 

The social welfare is

$$SW(w_1, w_2) = CS_e + CS_u + m_1 \pi_1 + m_2 \pi_2$$
$$= \frac{1}{k} \left[(m_1 + m_2)u_i^2 + (k - 2\phi_1 m_1 - 2\phi_2 m_2)u_i + 2(m_1 \phi_1 w_1 + m_2 \phi_2 w_2) - (m_1 w_1^2 + m_2 w_2^2)\right].$$
If there is no binding for the minimum wage, then the equilibrium social welfare level is

\[ \text{SW}(w_1^*, w_2^*) = \frac{1}{4k} \left( 3(m_1 + m_2)u_l^2 + 4u_l k + 3(m_1 \phi_1^2 + m_2 \phi_2^2) - 6u_l (m_1 \phi_1 + m_2 \phi_2) \right). \] (25)

If the minimum wage is partial binding, then

\[ \text{SW}(w_{\text{min}}, w_2^*) = \frac{1}{4k} \left[ (4m_1 + 3m_2)u_l^2 + (4k - 6m_2 \phi_2 - 8m_1 \phi_1)u_l + 3m_2 \phi_2^2 + 8m_1 \phi_1 w_{\text{min}} - 4m_1 w_{\text{min}}^2 \right]. \] (26)

Solving \( \partial \text{SW}(w_{\text{min}}, w_2^*) / \partial w_{\text{min}} = 0 \) yield the optimal minimum wage

\[ w_{\text{min}}^* = \phi_1. \]

If the minimum wage is fully binding, then the social welfare is

\[ \text{SW}(w_{\text{min}}, w_{\text{min}}) = \frac{1}{k} \left[ (m_1 + m_2)u_l^2 + (k - 2m_1 \phi_1 + m_2 \phi_2)u_l + 2(m_1 \phi_1 + m_2 \phi_2)w_{\text{min}} - (m_1 + m_2)w_{\text{min}}^2 \right]. \]

Solving \( \partial \text{SW}(w_{\text{min}}, w_{\text{min}}) / \partial w_{\text{min}} = 0 \) yield

\[ w_{\text{min}}^{**} = \frac{\phi_1 m_1 + \phi_2 m_2}{m_1 + m_2}. \]

Plugging \( w_{\text{min}}^* \) into \( \text{SW}(w_{\text{min}}, w_2^*) \) and plugging \( w_{\text{min}}^{**} \) into \( \text{SW}(w_{\text{min}}^{**}, w_{\text{min}}^{**}) \) yield

\[ \text{SW}^* = \text{SW}(w_{\text{min}}^*, w_2^*). \]

and

\[ \text{SW}^{**} = \text{SW}(w_{\text{min}}^{**}, w_{\text{min}}^{**}). \]

8.4 The Minimum Wage Policy when the Productivity Difference is Relatively Large

It can be summarized as the following proposition (corresponding to Proposition 5)

**Proposition 5’** Given \( \phi_2 - \phi_1 \geq \phi_1 - u_l \), if a government imposes a minimum wage such that (a) when \( w_{\text{min}} \leq \phi_1 \), it is always a welfare improving, and the maximal welfare is at \( w_{\text{min}} = \phi_1 \). (b) when \( \phi_1 \leq w_{\text{min}} \leq w_2^* \), it is a welfare improving iff \( (3\phi_1 - u_l)/2 > w_{\text{min}} \).
(c) when \( w_{\text{min}} \geq w_2^* \), it is welfare improving iff \( w_{\text{min}} \leq \frac{\phi_1 m_1 + \phi_2 m_2 + \frac{1}{2} \sqrt{\lambda_1}}{m_1 + m_2} \), where \( \lambda_1 = (m_1 + m_2)^2 u_l^2 + 8\phi_1 \phi_2 m_1 m_2 + m_1^2 \phi_1^2 + m_2^2 \phi_2^2 - 3m_1 m_2 (\phi_1 + \phi_2) - 2(m_1 m_2 (\phi_1 + \phi_2) + m_1^2 \phi_1 + m_2^2 \phi_2). \) (d) summarize the above (a), (b) and (c), the optimal \( w_{\text{min}} = \phi_1 \), if \( u_l \geq \lambda_2 \), or \( w_{\text{min}} = \frac{\phi_1 m_1 + \phi_2 m_2}{m_1 + m_2} \). If \( u_l \leq \lambda_2 \), and \( \partial \lambda_2 / \partial m_1 < 0 \), \( \partial \lambda_2 / \partial m_2 > 0 \).

**Proof.** See the appendix.

Therefore, it is possible that imposing a minimum wage may not be a social improvement. Finally, we compare social welfare \( SW^* \) and \( SW^{**} \). Since \( \partial (SW^{**} - SW^*) / \partial u_l = \frac{m_2 (u_l - \phi_2)}{2k} \), it is a concave function in \( u_l \) and the maximum value is at \( u_l = \phi_2 \). However, the necessary condition in this case is \( u_l \leq \min \{ \phi_1, \phi_2 \} \). Thus, it is decreasing in \( u_l \) in the feasible range.

It suggests that imposing a higher minimum wage which binds the wages of these two firms is better than a minimum wage which only binds the lower wage firm, if the difference of two firms’ productivities is relatively large.

### 8.5 The Minimum Wage Policy when the Productivity Difference is Relatively Small

**Proposition 6’** Given \( \phi_2 - \phi_1 \leq \phi_1 - u_l \), if a government imposes a minimum wage such that (a) when \( w_{\text{min}} < \phi_1 \), then it is always a welfare improving, and the maximal welfare is at \( w_{\text{min}} = \phi_1 \). (b) when \( w_{\text{min}} \geq \phi_1 \), it is welfare improving iff \( w_{\text{min}} \leq \frac{\phi_1 m_1 + \phi_2 m_2 + \frac{1}{2} \sqrt{\lambda_1}}{m_1 + m_2} \), and the maximal welfare point is \( w_{\text{min}} = \frac{m_1 \phi_1 + m_2 \phi_2}{m_1 + m_2} \), and (c) compare the cases in (a) and (b), we find that the global minimal wage is at \( w_{\text{min}} = \frac{m_1 \phi_1 + m_2 \phi_2}{m_1 + m_2} \).

**Proof.** See the appendix.

The intuition of Proposition 6’ is that when \( m_1 \) increases, then \( \lambda_1 \) decreases, and therefore, \( SW^{**} - SW^* (w_1^*, w_2^*) < 0 \) is more possible to be emerged. The effect for \( \lambda_1 \) to \( m_2 \) is opposite.

### 8.6 The Social Welfare with Minimum Wage when the low Productivity Firms are Drive Out

From Proposition 5’ and 6’, we know that \( w_{\text{min}} = \phi_1 \) if the non-negative profit condition is considered. Then

\[
\tilde{SW}^* = \begin{cases} 
SW(\phi_1, w_2^*) & \text{when } \phi_1 < \frac{u_l + \phi_2}{2}, \\
SW(\phi_1, \phi_2) & \text{when } \phi_1 > \frac{u_l + \phi_2}{2},
\end{cases}
\]

(27)
when $\widehat{SW}^*$ is the optimal social welfare under no exist of firms. Now we compare (27) with the case that a government can enact a high minimum wage to drive out low productivity firms and whether the social welfare increases or decreases.

Consider the case that a government can impose a high minimum wage to force out the low productivity firms and define

$$\text{\widehat{SW}}(w_{\text{min}}) = m_2^2 u_l^2 - m_2^2 w_{\text{min}}^2 + w_k - 2m_2 \phi_2 (u_l - w_{\text{min}}).$$

represent the case that $w_{\text{min}} > \phi_1$ and all low productivity firms are driven out. Then,

$$\text{\widehat{SW}}(w_{\text{min}}) = m_2^2 u_l^2 - m_2^2 w_{\text{min}}^2 + w_k - 2m_2 \phi_2 (u_l - w_{\text{min}}).$$

Similarly, the optimal minimum wage

$$w^{***}_{\text{min}} = \phi_2.$$

Compare $\text{\widehat{SW}}(w^{***}_{\text{min}})$, $\text{\widehat{SW}}^*$ and $\text{SW}(w^*_1, w^*_2)$ with non-negative profits, it can yield the following proposition (corresponding to Proposition 7).

**Proposition 8'** (a) $\text{\widehat{SW}}(w^{***}_{\text{min}}) > \text{SW}(w^*_1, w^*_2)$ if $\phi_2 > \frac{m_2 u_l + \sqrt{3(\phi_1 - u_l)\phi_1}}{m_2}$, (b) $\text{\widehat{SW}}(w^{***}_{\text{min}}) > \text{\widehat{SW}}^* > 0$, if $\phi_2 > 2\phi_1 - u_l$ and $\phi_2 > \frac{m_2 u_l - (2\phi_1 - u_l)\phi_1}{m_2}$, and $\text{\widehat{SW}}(w^{***}_{\text{min}}) - \text{\widehat{SW}}^* < 0$, if $\phi_2 < 2\phi_1 - u_l$ and $\phi_2 < 2 \frac{\phi_1^2 - (\phi_1 - u_l)\phi_1}{m_2}$.

**Proof.**

$$\text{\widehat{SW}}(w^{***}_{\text{min}}) - \text{SW}(w^*_1, w^*_2) = \frac{(m_2 - 3m_1)u_l^2 + m_2 \phi_2^2 - 2m_2 u_l \phi_2 + 6m_1 u_l \phi_1 - 3m_1 \phi_1^2}{4k},$$

it can be solved to the result of part (a).

$$\text{\widehat{SW}}(w^{***}_{\text{min}}) - \text{\widehat{SW}}^* = \left\{ \begin{array}{ll} \frac{m_2(u_l^2 + \phi_2^2) - 2m_2 u_l \phi_2 - 4m_1 u_l^2 - 4m_1 \phi_1^2 + 8m_1 u_l \phi_1}{4k}, & \text{if } \phi_2 > 2\phi_1 - u_l, \\
\frac{m_2 \phi_2^2 - m_1 u_l^2 + (m_2 - m_1) \phi_1^2 + 2m_1 u_l \phi_1 - 2\phi_1 m_2}{k}, & \text{if } \phi_2 \leq 2\phi_1 - u_l, \end{array} \right.$$ 

it can be solved to the result of part (b). In words, when $\phi_2$ and $m_2$ are relatively large than $\phi_1$ and $m_1$, then $\widehat{SW}$ is higher than $\widehat{SW}^*$ and $\text{SW}(w^*_1, w^*_2)$. 

$\square$
9 The Oligopsony Model with Unemployment Subsidy

Instead of the minimum wage policy, suppose now the government can give a subsidy \( w_g \) to unemployed worker, and this subsidy can produce \( \phi_3 \) benefits (e.g. criminal rate reduced) in this section. Other assumptions are being equal and \( u_t \) should be replaced by \( u_g = w_g + u_t \). The same method can thus be applied to discuss the role of unemployment subsidy on this location-wage game.

Solving \( u_1 = u_g \) yields

\[
\begin{align*}
n^R_1 &= \frac{-w_g - u_t + w_1 + kx_1}{k}, & n^L_1 &= \frac{w_g + u_t - w_1 + kx_1}{k}.
\end{align*}
\]

The profit functions of firms and the government surplus are:

\[
\begin{align*}
\pi_1 &= (\phi_1 - w_1)N_1 = -\frac{2(-\phi_1 + w_1)(-w_g - u_t + w_1)}{k}, \\
\pi_2 &= (\phi_2 - w_2)N_2 = -\frac{2(-\phi_2 + w_2)(-w_g - u_t + w_2)}{k}, \\
G &= (\phi_3 - w_g)UN \\
&= -\frac{(-\phi_3 + w_g)(k + 2(m_1 + m_2)w_g + 2(m_1 + m_2)u_t - 2m_1w_1 - 2m_2w_2)}{k},
\end{align*}
\]

The equilibrium location is inside a range, instead of at a point. Solving \( \frac{\partial \pi_1}{\partial w_1} = 0 \), and \( \frac{\partial \pi_2}{\partial w_2} = 0 \) yield

\[
\begin{align*}
w^*_1 &= \frac{w_g + u_t - \phi_1}{2}, & w^*_2 &= \frac{w_g + u_t - \phi_2}{2}.
\end{align*}
\] (28)

Plugging \( w^*_1, w^*_2, \) and \( w^*_g \) into \( n^R_1, n^L_2 \) yield

\[
\begin{align*}
n^R_i &= x_i + \frac{w_g + u_t - \phi_j(i)}{2k}, \\
n^L_i &= x_i - \frac{w_g + u_t - \phi_j(i)}{2k}.
\end{align*}
\]

This solutions \( (n^R_i, n^L_i) \) are feasible only for the following conditions are satisfied

\[
\begin{align*}
n^L_{i+1} - n^R_i &\geq 0 \quad \Rightarrow \quad x_{i+1} - x_i \geq -\frac{\phi_j(i+1) + 2w_g + 2u_t - \phi_j(i)}{2k} \equiv x_{d(i+1)}, \\
n^L_1 &\geq 0 \quad \Rightarrow \quad x_1 \geq -\frac{\phi_1 + w_g + u_t}{2k} \equiv x_{1, min}, \\
n^R_M &\leq 1 \quad \Rightarrow \quad x_M \leq \frac{\phi_j(M) + w_g + u_t + 2k}{2k} \equiv x_{M, max}.
\end{align*}
\] (29-31)
Since $x_{i+1} > x_i$ by assumption, using the lower-bound of $x_1$ from (30) and the upper bound of $x_2$ from (31) yield:

$$x_M^{\max} - x_1^{\min} - \sum_{i=1}^{M-1} x_{d(i,i+1)}^{\min} = \frac{k - m_1 \phi_1 + 2w_g + 2u_l - m_2 \phi_2}{k} \geq 0.$$  

Therefore,

$$\frac{(m_1 + m_2)(m_1 \phi_1 + m_2 \phi_2) - k}{M} - u_l \equiv w_g^{\max}.$$  

Replacing the equilibrium wages into the market shares, then the firm’s employments and unemployment rate become:

$$N_1^{\ast} = \frac{-\phi_1 + w_g + u_l}{k}, \quad N_2^{\ast} = \frac{-\phi_2 + w_g + u_l}{k},$$  

$$UN^{\ast} = \frac{k - m_1 \phi_1 + (m_1 + m_2)w_g + (m_1 + m_2)u_l - m_2 \phi_2}{k}. \quad (32)$$  

Proposition 9’ When $M$ firms compete each other, the only possible equilibrium is that all firms occupy a range of market without overlapping and other areas are unemployed when $w_g < w_g^{\max}$. The equilibrium locations $(x_1^{\ast}, \ldots, x_M^{\ast})$ satisfy (29), (30), (31), the equilibrium wage is $w_1^{\ast} = \frac{w_g + u_l + \phi_1}{2}$ and $w_2^{\ast} = \frac{w_g + u_l + \phi_2}{2}$.

Proof. Plugging equilibrium wages into profit functions yields

$$\pi_1^{\ast} = \frac{(-\phi_1 + w_g + u_l)^2}{2k}, \quad \pi_2^{\ast} = \frac{(-\phi_2 + w_g + u_l)^2}{2k}.$$  

Since the above profit functions are independent to locations, a small change in locations does not alter the profit of firms. Moreover, undercutting is unprofitable as follows. For type 1 firms, given any $w_2$ and $w_g$, any undercutting price $w_1'$ can not yield a higher profit than $\pi_1^{\ast}$. Similarly for type 2 firms.  

Note that if $w_g > w_g^{\max}$, then the total employment falls and it makes no sense to do that.

To analyze a policy implication, we define a social welfare function as follows,

$$\max_{w_g} SW = CS_e + CS_u + \pi_1 + \pi_2 + G,$$

where the government can use $w_g$ to maximize social welfare.

$$SW = (3\phi_2^2 - 2w_g^2 + 4w_g u_l - 2w_g \phi_1 + 6u_l^2 - 6u_l \phi_1 + 3\phi_1^2 - 2w_g \phi_2 - 6u_l \phi_2$$  

$$+ 4k u_l + 4\phi_3 k - 4\phi_3 \phi_1 + 8\phi_3 w_g + 8\phi_3 u_l - 4\phi_3 \phi_2) / (4k).$$
Solving $\partial SW/\partial w_g = 0$ yields

$$w_g^{*s} = 2\phi_3 + u_l - \frac{\phi_1 + \phi_2}{M}. \tag{34}$$

We can discuss the optimal unemployment subsidy and unemployment rate as following propositions (corresponding Proposition 10).

**Proposition 10’** The optimal unemployment subsidy $w_g^{*s} = 2\phi_3 + u_l - \frac{m_1\phi_1 + m_2\phi_2}{M}$. When the economy is booming ($\phi_i$ increases, $i = 1, 2$), then the optimal unemployment subsidy must decrease. The number of firms ($m_1, m_2$) increases, then the optimum subsidy decreases.

**Proof.** It is clear from (34). □

**Proposition 11’** The optimal unemployment rate (under optimal unemployment subsidy) is

$$UN^{*s} = \frac{k - 2m_1\phi_1 + 2(m_1 + m_2)u_l - 2m_2\phi_2 + 2(m_1 + m_2)\phi_3}{k}.$$

The optimal unemployment rate is increasing in $k$ and $\phi_3$, while it is decreasing in $\phi_1, \phi_2$ and $k$. $\partial UN^{*s}/\partial m_1 = \frac{2(\phi_1 - \phi_1 + u_l)}{k}$ and $\partial UN^{*s}/\partial m_2 = \frac{2(\phi_1 + u_l - \phi_2)}{k}$.

**Proof.** It is clear form (33). □

The economic intuition is that the unemployment subsidy should be adjusted as per the situation of economic status: When the economy is booming (recession), the optimal subsidy should be decreased (increased); Similarly, the higher $\phi_3$, then the higher $w_g^{*s}$.

### 10 Conclusions

Using the setting of linear transportation costs and non-full employment in a duopsony-labor market, this paper analyzes the existence of pure strategy location-wage equilibrium and government unemployment policies. Precisely, it examines whether minimum wage is welfare improving or not. A higher minimum wage driving out low productivity firms may be most socially desirable when the dispersion of productivity among firms is relatively significant. Thus, the minimum wage may play a role to help the economy to drive out low productivity firms and concentrate on high productivity production. Moreover, it is also shown that the unemployment subsidy must be decreased (increased) when the economy is booming (transportation rate increases). Our results are also robust for considering a circular market and a Hotelling oligopsony.
Appendix

Proof of Proposition 4.

Proof. We solve the social optimal locations and market shares. The objective is the social welfare function:

$$\max_{N_1, N_2} \tilde{SW}(N_1, N_2) = CS_e + CS_u + \pi_1 + \pi_2$$

$$= \phi_1 N_1 + \phi_2 N_2 + (1 - N_1 - N_2) u_l - \int_{n_1^R}^{n_1^L} k|x - x_1| dx - \int_{n_2^R}^{x_2} k|x - x_2| dx$$

$$= \phi_1 N_1 + \phi_2 N_2 + (1 - N_1 - N_2) u_l - \frac{1}{2} kN_1^2 - \frac{1}{2} kN_2^2. \quad (35)$$

The first item in (35), $\phi_1 N_1 + \phi_2 N_2$ indicates the production of labor, and the second item $(1 - N_1 - N_2) u_l$ represents the utility of unemployed workers. The last part is the total transportation costs. The first order condition of (35) are

$$\frac{\partial \tilde{SW}(N_1, N_2)}{\partial N_i} = \phi_i - \frac{k}{2} N_i - u_l = 0, \quad i = 1, 2.$$

Thus, we have the optimal market shares

$$N_i^o = \frac{2(\phi_i - u_l)}{k} < N_i^*, \quad i = 1, 2, \quad (36)$$

and

$$UN^o = \frac{k - 2(\phi_1 + \phi_2) + 2u_l}{k} > UN^*.$$

Solving

$$N_i = \frac{2(w_i - u_l)}{k} = N_i^o \Rightarrow w_i^o = \phi_i, \quad i = 1, 2.$$

Plugging $w_i^o = \phi_i$, $i = 1, 2$ into $n_i^L$ and $n_i^R$ yield

$$n_i^{Lo} = x_i - \frac{\phi_i - u_l}{k}, \quad n_i^{Ro} = x_i + \frac{\phi_i - u_l}{k}, \quad i = 1, 2.$$

Therefore, similarly with (6), (7), and (8),

$$x_1^o \geq \frac{\phi_1 - u_l}{k},$$

$$x_2^o \leq 1 - \frac{\phi_2 + u_l}{k},$$

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and from $n_1^{Ro} - n_2^{Lo}$,

$$x_1^o + \frac{\phi_1 - u_l}{k} \leq x_2^o - \frac{\phi_2 - u_l}{k} \Rightarrow x_2^o - x_1^o \geq \frac{\phi_1 + \phi_2 - 2u_l}{k}. \quad \Box$$

**Proof of Proposition 5.**

**Proof.** Given that $\phi_1 < \frac{\phi_2 + u_l}{2}$ implies that $\phi_1 < w_2^*$, then for all $w_{min} \leq \phi_1$ are partially binding. Comparing (14) with (12) yields

$$SW_{min, w_2^*} - SW(w_1^*, w_2^*) = \frac{-3\phi_1 + u_l + 2w_{min}(\phi_1 + u_l - 2w_{min})}{4k}.$$ 

This is a concave function of $w_{min}$ which is equal to zero when $w_{min} = (\phi_1 + u_l)/2 = w_1^*$ or $w_{min} = (3\phi_1 - u_l)/2$. Since $(3\phi_1 - u_l)/2 > w_{min} = \phi_1 > (\phi_1 + u_l)/2$, therefore, $SW_{min, w_2^*} - SW(w_1^*, w_2^*)$ is positive if $(3\phi_1 - u_l)/2 > w_{min} > (\phi_1 + u_l)/2$. Since $\phi_1 < (3\phi_1 - u_l)/2$, thus (a) and (b) is valid. Comparing the highest social welfare in Case A and the case under laissez-faire:

$$SW^* - SW(w_1^*, w_2^*) = \frac{(\phi_1 - u_l)^2}{4k} > 0.$$ 

In the part (c), $w_{min} \geq w_2^*$ is the fully binding case. From (17) and (12),

$$SW_{min, w_{min}} - SW(w_1^*, w_2^*) \geq 0$$ 

iff $w_{min} \leq \frac{\phi_1 + \phi_2 - \sqrt{2\phi_1^2 - 2\phi_2^2 + 8\phi_1\phi_2 - 4u_l(\phi_1 + \phi_2) + 4u_l^2}}{4}$.

The maximal social welfare is at $w_{min}^* = w_{min}^* = \frac{\phi_1 + \phi_2}{2}$. Compare (a), (b), and (c), then from (16) and (18)

$$SW_{min, w_{min}} - SW^* = \frac{u_l^2 - 2\phi_1^2 + 4\phi_1\phi_2 - \phi_2^2 - 2u_l\phi_2}{4k}.$$ 

Since $\partial(SW_{min, w_{min}} - SW^*)/\partial u_l = \frac{u_l - \phi_2}{2k}$, it is a concave function in $u_l$ and the maximum value is at $u_l = \phi_2$. However, the necessary condition in this case is $u_l \leq \min\{\phi_1, \phi_2\}$. Thus, it is decreasing in $u_l$ in the feasible range. Thus, $SW_{min, w_{min}} - SW^* \geq 0$ iff $u_l < \phi_2 - \sqrt{2}(\phi_1 - u_l)$. That is, the optimal minimum wage is $w_{min} = \phi_1$ (given the social welfare $SW^*$) when $u_l \geq \phi_2 - \sqrt{2}(\phi_2 - \phi_1)$ and is $w_{min} = \frac{\phi_1 + \phi_2}{2}$ (given the social welfare $SW_{min, w_{min}}$), when $u_l \leq \phi_2 - \sqrt{2}(\phi_2 - \phi_1)$.

**Proof of Proposition 6.**

**Proof.** (a) When $\phi_1 \geq \frac{\phi_2 + u_l}{2}$ implies $\phi_1 \geq w_2^*$. If $w_{min} \leq w_2^*$, then it is a partial binding
case. Since $SW(w_{\text{min}}, w^*_2)$ is an increasing function in $w_{\text{min}}$. Therefore, $SW(w_{\text{min}}, w^*_2) > SW(w^*_1, w^*_2)$ for all $w_{\text{min}} < w^*_2$. If $w_{\text{min}} \geq w^*_2$, it is a fully binding case. Since $SW(w_{\text{min}}, w_{\text{min}})$ is increasing in $w_{\text{min}}$ when $w_{\text{min}} \leq \frac{\phi_1 + \phi_2}{2}$, therefore, it is always welfare improving for all $w_{\text{min}} \leq \phi_1$. Thus the optimal minimum wage is $w_{\text{min}} = \phi_1$. (b) When $\phi_1 \leq w_{\text{min}}$, it is the fully binding case.

$$SW(w_{\text{min}}, w_{\text{min}}) - SW(w^*_1, w^*_2) \geq 0$$

iff $w_{\text{min}} \leq \frac{\phi_1 + \phi_2}{2} + \sqrt{-2\phi_1^2 - 2\phi_2^2 + 8\phi_1\phi_2 - 4u_1(\phi_1 + \phi_2) + 4u_1^2}$.

The maximal social welfare is at $w_{\text{min}} = w^*_{\text{min}}$. (c) Since $SW(w_{\text{min}}, w_{\text{min}})$ is increasing in $w_{\text{min}}$ when $w_{\text{min}} \leq \phi_1 + \phi_2$ and $w_{\text{min}} = \phi_1$, gives the maximal social welfare in (a), which is the fully binding case since $\phi_1 > w^*_1$. Thus $w_{\text{min}} = \frac{\phi_1 + \phi_2}{2} = w^*_{\text{min}}$ is the optimal minimum wage.

\[\Box\]

**Proof of Proposition 8.**

**Proof.** Plugging $w^*_1$, $w^*_2$ into $n^R_1$, $n^L_2$ yield

$$n^R_1 = x_1 + \frac{w_g + u_t - \phi_1}{2k}, \quad n^L_2 = x_2 - \frac{w_g + u_t - \phi_2}{2k}.$$  \[(38)\]

This solutions $(n^R_1, n^L_2)$ are feasible only for the following conditions are satisfied

$$n^L_2 - n^R_1 \geq 0 \quad \Rightarrow \quad x_2 - x_1 \geq \frac{\phi_1 + \phi_2 - 2w_g + 2u_t}{2k} \equiv x^\text{min}_1,$$  \[(39)\]

$$n^L_1 \geq 0 \quad \Rightarrow \quad x_1 \geq \frac{\phi_1 - w_g - u_t}{2k} \equiv x^\text{min}_1,$$  \[(40)\]

$$n^R_2 \leq 1 \quad \Rightarrow \quad x_2 \leq 1 - \frac{\phi_2 - w_g - u_t}{2k} \equiv x^\text{max}_2.$$  \[(41)\]

Since $x_2 > x_1$ by assumption, using the lower-bound of $x_1$ and the upper bound of $x_2$ yield:

$$x^\text{max}_2 - x^\text{min}_1 - x^\text{min}_1 = \frac{k - \phi_1 - \phi_2 + 2w_g + 2u_t}{k} \geq 0.$$  

Therefore,

$$w_g \leq \frac{\phi_1 + \phi_2 - k}{2} - u_t \equiv w^\text{max}_g.$$  

Plugging $w^*_1$, $w^*_2$ into $n^R_1$, $n^L_2$ yield

$$n^R_1 = x_1 + \frac{w_g + u_t - \phi_1}{2k}, \quad n^L_2 = x_2 - \frac{w_g + u_t - \phi_2}{2k}.$$  \[(42)\]
This solutions \((n_1^{R*}, n_2^{L*})\) are feasible only for the following conditions are satisfied

\[
\begin{align*}
    n_2^{L*} - n_1^{R*} & \geq 0 \quad \Rightarrow \quad x_2 - x_1 \geq \frac{\phi_1 + \phi_2 - 2w_g + 2u_l}{2k} \equiv x_d^{\text{min}}, \quad (43) \\
    n_1^{L*} & \geq 0 \quad \Rightarrow \quad x_1 \geq \frac{\phi_1 - w_g - u_l}{2k} \equiv x_1^{\text{min}}, \quad (44) \\
    n_2^{R*} & \leq 1 \quad \Rightarrow \quad x_2 \leq 1 - \frac{\phi_2 - w_g - u_l}{2k} \equiv x_2^{\text{max}}. \quad (45)
\end{align*}
\]

Since \(x_2 > x_1\) by assumption, using the lower-bound of \(x_1\) and the upper bound of \(x_2\) yield:

\[
x_2^{\text{max}} - x_1^{\text{min}} - x_d^{\text{min}} = \frac{k - \phi_1 - \phi_2 + 2w_g + 2u_l}{2k} \geq 0.
\]

Therefore,

\[
w_g \leq \frac{\phi_1 + \phi_2 - k}{2} - u_l \equiv w_g^{\text{max}}.
\]

Plugging equilibrium wages into the profit function yields

\[
\pi_i^* = \left(\frac{-\phi_i + w_g + u_l}{2k}\right)^2, \quad i = 1, 2. \quad (46)
\]

Since the above profit functions are independent to locations, a small change in locations does not alter the profit of firms. Moreover, undercutting is unprofitable as follows. For firm 1, given any \(w_2\) and \(w_g\), any undercutting price \(w_1'\) can not yield a higher profit than \(\pi_1\) in (46). Similarly for firm 2.

**Proof of Proposition 13.**

**Proof.** The solving process are the same as in Section 3, and thus \(N_1, N_2, \pi_1, \pi_2\) are identical to the previous results. It can be got easily such that

\[
w_i^* = \frac{u_l + \phi_i}{2}, \quad i = 1, 2.
\]

and

\[
\begin{align*}
    n_1^{L*} & = \frac{2k + u_l - \phi_1}{2k}, \quad n_1^{R*} = \frac{\phi_1 - u_l}{2k}, \\
    n_2^{L*} & = x_2 - \frac{\phi_2 - u_l}{2k}, \quad n_2^{R*} = x_2 + \frac{\phi_2 - u_l}{2k}.
\end{align*}
\]

Therefore, the necessary conditions for location equilibrium are

\[
n_2^{L*} - n_1^{R*} \geq 0 \quad \Rightarrow \quad x_2 \geq \frac{\phi_1 + \phi_2 - 2u_l}{2k} \equiv x_2^{\text{min}}.
\]

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and

\[ n_1^L - n_2^R \geq 0 \Rightarrow x_2 \leq \frac{2k + 2u_1 - \phi_1 - \phi_2}{2k} \equiv x_2^{\text{max}}. \]

Since \( x_2^{\text{max}} > x_2^{\text{min}} \), then

\[ x_2^{\text{max}} - x_2^{\text{min}} = \frac{k + 2u_1 - \phi_1 - \phi_2}{k} > 0 \Rightarrow u_1 > \frac{\phi_1 + \phi_2 - k}{2} \equiv u_1^{\text{min}}. \]

To ensure each firm has a positive employment,

\[ N_1^* = \frac{\phi_1 - u_1}{k} > 0, \quad N_2^* = \frac{\phi_2 - u_1}{k} > 0. \]

Therefore, \( u_1 \) must satisfy

\[ u_1 < u_1^{\text{max}} = \min\{\phi_1, \phi_2\}. \]

\[ \square \]

Proof of Proposition 1’

Proof. The necessary condition for location equilibrium are

\[ n_{i+1}^L - n_i^R = \frac{2u_i - \phi_{j(i+1)} + 2k(x_{i+1} - x_i) - \phi_{j(i)} - 2k}{2k} \leq 0 \]

\[ \Rightarrow x_{i+1} - x_i^* \leq \frac{\phi_{j(i+1)} + \phi_{j(i)} - 2u_i}{2k} \equiv x_{d(i,i+1)}^{\text{min}}. \]

\[ n_i^L = x_1 - \frac{\phi_{j(1)} - u_1}{2k} \geq 0 \]

\[ \Rightarrow x_1 \leq \frac{\phi_{j(1)} - u_1}{2k} = x_1^{\text{min}}. \]

\[ n_M^R = x_M + \frac{\phi_{j(M)} - u_1}{2k} \leq 1 \]

\[ \Rightarrow x_M \leq \frac{u_1 - \phi_{j(M)} + 2k}{2k} \equiv x_M^{\text{max}}. \]

The above equations represent that the range of employed labors are disjointed.

Since \( x_{i+1} > x_i \), by assumption, using the lower bound of \( x_1 \) and the upper bound of \( x_M \),

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it is then
\[ x_M^{\text{max}} - x_1^{\text{min}} - \sum_{i=1}^{M-1} x_{d(i,i+1)}^{\text{min}} > 0, \]
\[ \Rightarrow 2k + u_l - \phi_i(M) - \phi_j(1) + u_l - \sum_{i=1}^{M-1} [\phi_j(j+1) - \phi_i(i)] + 2(M - 1)u_l > 0 \]
\[ \Rightarrow 2Mu_l + 2k - 2M\sum_j \phi_j(i) > 0 \]
\[ \Rightarrow 2Mu_l + 2(m_1\phi_1 + m_2\phi_2) > 0 \]
\[ \Rightarrow u_l > \frac{(m_1\phi_1 + m_2\phi_2) - k}{M}. \]

\[ \Box \]

**Proof of Proposition 5'**

**Proof.** (a) The proof is the same as the (a) part in the proof of Proposition 5. (b) Comparing \( SW, SW^*, \) and \( SW^{**} \) yield

\[ SW^* - SW(w_1^*, w_2^*) = \frac{m_1(\phi_1 - u_l)^2}{4k} > 0, \]

\[ SW^{**} - SW(w_1^*, w_2^*) = \frac{1}{4kM} \left[ M^2u_l^2 + u_l(2M(\phi_1 - \phi_2)m_2 - 2\phi_1)M^2 \right. \]
\[ \left. + 4(\phi_1 - \phi_2)^2m_2^2 - M(5\phi_1 - 3\phi_2)(\phi_1 - \phi_2)m_2 + \phi_1^2M^2 \right], \]

\[ SW^{**} - SW^* = \frac{1}{4kM} \left[ 4m_2^2(\phi_2 - \phi_1)^2 - 4Mm_2\phi_1^2 + m_2M(u_l^2 - 2\phi_2)u_l \right. \]
\[ \left. + 8\phi_1\phi_2 - 4\phi_1^2 - 3\phi_2^2 \right]. \]  \( (47) \)

Comparing (25) with (26) yields

\[ SW(w_{\text{min}}, w_2^*) - SW(w_1^*, w_2^*) = \frac{m_1(-3\phi_1 + u_l + 2w_{\text{min}})(\phi_1 + u_l - 2w_{\text{min}})}{4k}. \]

Other details are the same as the proof of Proposition 4. From (47), it is a concave function of \( u_l. \) Therefore, plugging \( u_l = \phi_1 = u_l^{\text{max}} \) into (47) yields

\[ SW^{**}\bigg|_{u_l=u_l^{\text{max}}} - SW(w_1^*, w_2^*) \bigg|_{u_l=u_l^{\text{max}}} = -\frac{3m_1m_2(\phi_1 - \phi_2)^2}{4kM} < 0. \] \( (48) \)

\[ \Box \]

**Proof of Proposition 5'**

**Proof.** (a) The proof is the same as the (a) part in the proof of Proposition 6. (b) \( SW^{**} - \)
$SW(w^*_1, w^*_2)$ is a concave function in $u_l$ and its minimum value is at $u_l = \phi_1$. Therefore, solving $SW^{**} - SW(w^*_1, w^*_2) > 0$ implying $u_l < \lambda_1$. Moreover,

\[
\frac{\partial \lambda_1}{\partial m_1} = \frac{m_2(\phi_2 - \phi_1)}{2M^2 \sqrt{m_1m_2}} \left( \sqrt{3}(m_2 - m_1) - 2\sqrt{m_1m_2} \right),
\]

$\partial \lambda_1 / \partial m_1 \geq 0$ whenever $\sqrt{3}(m_2 - m_1) - 2\sqrt{m_1m_2} \geq 0$, which means that $\sqrt{3}(m_2 - m_1) \geq 2\sqrt{m_1m_2}$, which implies $m_2 > m_1$ and thus

\[
3(m_2 - m_1)^2 \geq 4m_1m_2 \quad \Rightarrow \quad (3m_2 - m_1)(m_2 - 3m_1) \geq 0 \quad \Rightarrow \quad m_2 \geq 3m_1.
\]

$\partial \lambda_1 / \partial m_1 < 0$ when $m_2 \leq 3m_1$. Similarly,

\[
\frac{\partial \lambda_1}{\partial m_2} = \frac{-m_1(\phi_2 - \phi_1)}{2M^2 \sqrt{m_1m_2}} \cdot \left( \sqrt{3}(m_2 - m_1) - 2\sqrt{m_1m_2} \right),
\]

where $\partial \lambda_1 / \partial m_1 > 0$ when $m_2 \leq 3m_1$. \qed
References


