Skewness and Kurtosis Ratio Tests: With Applications to a Study of Multiperiod Tail Risk

Woon K. Wong*

Bristol Business School

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*Corresponding details: Frenchay Campus, Coldharbour Lane, Bristol, BS16 1QY, United Kingdom. Tel: +44 (0)117 32 82011; Fax: +44 (0)117 32 82810; Email: woon.wong@uwe.ac.uk
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Abstract

This article extends Lo and MacKinlay (1988)'s variance ratio test to that of skewness and kurtosis. In the proposed tests, overlapping observations are used and their dependencies under IID assumption are explicitly modelled so that more information from the data is incorporated, giving rise to relatively more powerful tests with good size properties. The proposed tests provide useful diagnostic check for GARCH-type time series models, in particular, where multiperiod forecasts of tail events are required. Empirical applications of the tests to world major equity markets show that failure to model higher order nonlinear dependencies is likely to have a negative effect on the accuracy of forecasts of multiperiod tail risk.

Keywords: Skewness, kurtosis, overlapping observations, multiperiod tail risk, Value-at-Risk

JEL Classification: C10, G11

1 Introduction

1.1 Motivation

The current financial crisis has highlighted the importance of having sufficient capital for banks’ trading activities. If an internal risk model is used, Basel II stipulates that the market risk capital for a bank’s trading portfolio to be determined using Value-at-Risk (VaR) over a 10-day horizon.\footnote{According to Basel II, the required market risk capital on trading day \( t \) is determined as the larger of 10-day VaR\(_{t-1}\) and \( M \times \) average of past 60 days of 10-day VaR’s, where \( M \) is the multiplication factor the value of which lies between 3 and 4.} Since VaR is estimated using daily returns, the Basle Committee sanctions the use of scaling law based on variance ratio relation, i.e. the 10-day VaR is approximated as \( \sqrt{10} \times 1 \)-day VaR. This scaling law is clearly far from perfect since Lo and MacKinlay (1988) show that the variance ratio relationship does not hold for stock market returns.

Existing time series risk models forecast VaR by focusing on two main components, which are the volatility process and the distribution function of the shocks; see Hsieh (1993). Popular diagnostic tests for the risk models ensure that the shocks and their squares are not autocorrelated. While such diagnostic tests seem sufficient for one-step-ahead forecasts (Giot and Laurent, 2003; Wong, 2010), it is desirable to check for higher order independence in the
case of multi-step forecasts of tail risks. This is because skewness and kurtosis measures the degree of asymmetry and tail fatness which associate closely with the severity of tail risks.

This article extends Lo and MacKinlay’s (1988) variance ratio test to that of skewness and kurtosis. If the returns are IID, similar intervaling effects on the skewness and kurtosis can be derived; see Hawawini (1980) and Lau and Wingender (1989). Specifically, the skewness and kurtosis of single-period returns are respectively $\sqrt{h}$ and $h$ times of those of $h$-period returns. Therefore, if for example the $h$-period skewness is significantly more negative than the value implied by the intervaling effects, multi-step forecasts of tail risk that incorrectly assume returns are IID would likely to be over optimistic.

In order to fully incorporate information from the data, this paper adopts the GMM approach used by Richardson and Smith (1991) in which overlapping observations are used and their dependencies under IID assumption are explicitly modelled. Such an approach not only provides more powerful tests, but also good size properties. The GMM tests and the weighting matrices are provided in the next section.

Since the asymptotic results of the proposed tests assume existence of moments up to eighth order, which is highly unlikely for financial returns, Section 3 investigates the robustness of the proposed ratio tests to moment condition failure. In comparison to the widely used nonlinearity test proposed by McLeod and Li (1983), simulation study shows that the proposed tests have relatively better size properties.

In Section 4, a study of multiperiod tail risks for world major equity markets is carried out. It is found that the scaling law underestimate multiperiod VaR for illiquid stocks and application of the proposed tests find signs of higher order nonlinear dependence present in the stock returns. Moreover, it is noted that in several cases the risk models studied pass the Ljung-Box and McLeod-Li tests but fail the proposed skewness-kurtosis ratio tests. Amongst the risk models considered in this paper, there is no single one that consistently remove higher order dependence from all stock market returns studied.

Finally, concluding remarks are provided in Section 5.

2 Higher order ratio tests

Let $r_t$ be a random return at time $t$ with mean $\mu$ and variance $\sigma^2$, and $\tilde{r}_t = r_{t-h+1} + \cdots + r_t$ be the associated $h$-period random return at $t$. Lo and MacKinlay (1988) made use of the fact that if $r_t$ is independent and identically distributed (IID), the stock price returns should pass the variance ratio test, i.e. the relationship

$$\text{var} (\tilde{r}_t) = h \text{var} (r_t)$$

(1)
holds. A few years later, Richardson and Smith (1991) proposed a GMM approach for the
variance ratio test, using (1) as a restriction in the sample moment conditions. A major con-
tribution by Richardson and Smith is the use of analytically derived weighting matrices in
the presence of overlapping returns for the GMM test. By explicitly modeling the dependen-
cies induced by the overlapping observations, the approach incorporates more information
from the data and thus enjoys better test powers and good size properties. This section
extends Richardson and Smith’s GMM approach to the skewness and kurtosis ratio tests.

2.1 Some preliminary notations and properties

In this paper, for reasons that will be given later, analyses and results are presented in
terms of cumulants. Formally, the $p$-th order joint cumulant of $p$-variate random variable
$(y_1, \ldots, y_p)$, denoted as $\text{cum}(y_1, \ldots, y_p)$, is defined as the coefficient of
$i^{\sum_{j=1}^p y_j t_j}$ in the Taylor series expansion of natural logarithm of $E\left[\exp \left(i \sum_{j=1}^p y_j t_j\right)\right]$. For the special case $y_j = y$, $j = 1, \ldots, p$, $\text{cum}(y_1, y_2, \ldots, y_p)$ is simply the $p$-th order cumulant of $y$. Note that $\text{cum}(y) = E(y)$. Let $\mu_p$ and $\kappa_p$ denote respectively the $p$-th order central moment and $p$-th order
 cumulant of $y$. It is well known that $\kappa_2 = \text{cum}(y, y) = \text{var}(y) = \mu_2$, $\kappa_3 = \mu_3$ and $\kappa_4 = \mu_4 - 3\mu_2$.
The appendix at the end of the paper provides further relations between higher order central
moments and cumulants.

Listed below are some properties that motivate the article to base its analyses in cumu-
lants.

**Lemma 1** Let $z_1$ and $y_1, \ldots, y_n$ be random variables whose joint cumulant exists. Then

1. $\text{cum}(y_1, \ldots, y_n)$ is symmetric in its argument.
2. $\text{cum}(y_1 + z_1, y_2, \ldots, y_n) = \text{cum}(y_1, y_2, \ldots, y_n) + \text{cum}(z_1, y_2, \ldots, y_n)$.
3. If any of $y_1, \ldots, y_n$ is independent of the remaining $y$’s, $\text{cum}(y_1, \ldots, y_n) = 0$.
4. If $a$ is a constant, $\text{cum}(a, y_1, \ldots, y_n) = 0$.

Let $x_t = r_t - \mu$ and $\bar{x}_t = \bar{r}_t - h\mu$ denote respectively the single- and $h$-period mean
adjusted random returns. If $x_t$ and $x_{t-l}$ are independent, by virtue of property 3, their joint
cumulant is zero. Indeed, under the assumption of IID $x_t$, application of properties 2, 3 and
4 give

\[ \bar{\kappa}_p = h\kappa_p, \]  

(2)

where $\bar{\kappa}_p$ is the $p$-th order cumulant of $\bar{x}_t$. The result in (2) not only forms the basis for the
higher order ratio tests proposed in this paper, but also is useful for the derivation of the
required theoretical weighting matrices for the GMM tests. Note that for \( p = 2 \), (2) reduces to (1), since the second order cumulant is simply the variance.

It is useful to consider the implications of (2) in terms of skewness and kurtosis as the two statistics are now widely used. Let \( \sigma^2, \rho_3 \) and \( \rho_4 \) be the variance, skewness and kurtosis of \( x_t \) respectively. From now onwards, as in \( \tilde{x}_t \) and \( \tilde{\kappa}_p \), we use ‘\( \sim \)’ to denote the corresponding \( h \)-period variable of interest. For example, \( \tilde{\sigma}^2 = \text{cum}(\tilde{x}_t, \tilde{x}_t) \). Then

\[
\tilde{\rho}_3 = \frac{\tilde{\kappa}_3}{\sigma^3} = \frac{h}{h^{3/2} \sigma^3} = \frac{1}{\sqrt{h}} \rho_3, \\
\tilde{\rho}_4 = \frac{\tilde{\kappa}_4}{\sigma^4} = \frac{h}{h^2 \sigma^4} = \frac{1}{h} \rho_4.
\]

(3) (4)

The above relations were studied by Hawawini (1980) and Lau and Wingender (1989) as the intervaling effect on skewness and kurtosis.

### 2.2 GMM test

To apply Hansen (1982)’s GMM test procedure, we construct at each time \( t \) an \( R \)-vector \( f_t (r_t, \tilde{r}_t, \theta) \) where \( \theta \) is a \( P \)-vector of unknown parameters, namely \( \mu, \sigma^2 \) and \( \kappa_j \), to be determined. Each element of \( f_t (\cdot) \) corresponds to a restriction, at least one of which is attributed to the higher order ratio relation given in (2). Given the time series \( \{r_t, \tilde{r}_t\}_{t=1}^T \),

\[
g_T (\theta) = \frac{1}{T} \sum_{t=1}^T f_t (r_t, \tilde{r}_t, \theta)
\]

(5)

tends to zero as \( T \) tends to infinity if the higher order ratio relation holds. The idea behind the GMM approach is to obtain the estimator \( \hat{\theta} \) such that it has a minimum variance-covariance matrix. Hansen (1982) showed that this can be achieved by solving the system of equations

\[
D_0^T S_0^{-1} g_T (\theta) = 0,
\]

(6)

where

\[
D_0 = E \left[ \frac{\partial g_0 (\theta)}{\partial \theta} \right],
\]

(7)

\[
S_0 = \sum_{l=-\infty}^{\infty} E \left[ f_t (\cdot) f_{t-l} (\cdot) \right].
\]

(8)

Of special interest to this paper is the following results:
\[
\sqrt{T} \left( \hat{\theta} - \theta \right) \rightarrow N \left( 0, [D'_0 S_0^{-1} D_0]^{-1} \right), \quad \tag{9}
\]
\[
T g_T \left( \hat{\theta} \right)' S_T^{-1} g_T \left( \hat{\theta} \right) \rightarrow \chi^2_{R-P}. \quad \tag{10}
\]

where \( R > P \). One reason for the popularity of the GMM approach lies in its validity when \( D_0 \) and \( S_0 \) are replaced by their consistent estimators, denoted \( D_T \) and \( S_T \). In particular, the \( S_T \) is often calculated by the two-step procedure of Hansen and Singleton (1982) or the Newey and West (1987) approach which guarantees a positive definite weighting matrix based on sample estimates of (5).

One objective of this article is to derive analytically under IID assumption the matrices \( D_0(\cdot) \) and \( S_0(\cdot) \) in the presence of overlapping observations. As Richardson and Smith (1991) have demonstrated for the variance ratio test, this approach incorporates more information from the data and yields desirable results in terms of higher test powers and better size properties. As can be seen in the following section, analytically deriving the variance-covariance matrix \( S_0 \) under the null hypothesis, reduces the problem to estimating only the required cumulants.

### 2.3 Skewness ratio test

For the skewness ratio test, \( f_t \) and \( D_0 \) are respectively

\[
f_t = \begin{bmatrix}
  r_t - \mu \\
  (r_t - \mu)^3 - \kappa_3 \\
  (\tilde{r}_t - h\mu)^3 - h\kappa_3
\end{bmatrix}, \quad D_0 = \begin{bmatrix}
  -1 & 0 \\
  -3\sigma^2 & -1 \\
  -3h^2\sigma^2 & -h
\end{bmatrix}, \quad \tag{11}
\]

with \( R = 3 \) and \( P = 2 \). Consider for example the covariance between the second and last elements of \( f_t \) in (11), i.e. \( \text{cov}((r_t - \mu)^3 - \kappa_3, (\tilde{r}_t - h\mu)^3 - h\kappa_3) \). Since \( \mu \), \( \sigma^2 \) and \( \kappa_3 \) are non-stochastic, by virtue of Property 4 in Lemma 1, the covariance is simply \( \text{cum}(x_t^3, \tilde{x}_t^3) \). So the elements of \( S_0 \) can be expressed in terms of cumulants of products of \( x_t \) and \( \tilde{x}_t \). In this case, the required covariance is \( \sum_{i=-\infty}^{\infty} \text{cum}(x_t^3, \tilde{x}_{t-1}^3) \), which can be denoted as \( s_{1,h}^{3,3} \) with the superscripts refer to the powers of random variables and subscripts the periods over which the returns are measured. Using the same notation, the required covariance matrix can be written as

\[
S_0 = \begin{bmatrix}
  s_{1,1}^{1,1} & s_{1,1}^{1,3} & s_{1,1}^{1,3} \\
  s_{1,1}^{3,1} & s_{1,1}^{3,3} & s_{1,1}^{3,3} \\
  s_{1,1}^{3,1} & s_{1,1}^{3,3} & s_{1,1}^{3,3} \\
  s_{h,1}^{3,1} & s_{h,1}^{3,3} & s_{h,1}^{3,3}
\end{bmatrix}.
\]
Exploiting the overlapping dependencies and the IID assumption, the elements of \( S_0 \) are derived analytically in Appendix as:

\[
\begin{align*}
{s_{1,1}}^{1,1} & = \sigma^2, \\
{s_{1,h}}^{1,3} & = h \left[ \kappa_4 + 3h\sigma^4 \right], \\
{s_{1,1}}^{3,3} & = h \left[ \kappa_6 + (3h + 12) \kappa_4\sigma^2 + 9\kappa_4^2 + (9h + 6)\sigma^6 \right], \\
{s_{h,h}}^{3,3} & = h^2\kappa_6 + [6h^3 + 9Ah] \kappa_4\sigma^2 + 9Ah\kappa_3^2 + [9h^4 + 6Bh] \sigma^6, 
\end{align*}
\]

where \( A_h = h (2\sigma^2 + 1)/3 \) and \( B_h = h^2 (h^2 + 1)/2 \). Setting \( h = 1 \), \( A_h = B_h = 1 \) and (13) reduces to \( s_{1,1}^{1,3} \) whereas both (14) and (15) simplify to \( s_{1,1}^{3,3} \).

### 2.4 Kurtosis ratio test

For the kurtosis ratio test, the corresponding \( f_t \) and \( D_0 \) are, respectively,

\[
\begin{align*}
&f_t = \begin{bmatrix}
- \mu & (r_t - \mu)^2 - \sigma^2 \\
(\sigma^2 - \mu)^2 - \sigma^2 & - \mu \\
(r_t - \mu)^4 - 3\sigma^4 \kappa_4 \\
(\sigma^4 - \mu)^4 - 3\sigma^4 \kappa_4
\end{bmatrix}, & D_0 &= \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
-4\kappa_3 & -6\sigma^2 & -1 \\
-4h^2\kappa_3 & -6h^2\sigma^2 - h
\end{bmatrix}
\end{align*}
\]

Using the same notation as in the skewness ratio test, the associated weighting matrix is given by

\[
S_0 = \begin{bmatrix}
{s_{1,1}}^{1,1} & s_{1,1}^{1,2} & s_{1,1}^{1,4} & s_{1,1}^{1,4} \\
{s_{1,1}}^{2,1} & s_{1,1}^{2,2} & s_{1,1}^{2,4} & s_{1,1}^{2,4} \\
{s_{1,1}}^{3,1} & s_{1,1}^{3,2} & s_{1,1}^{3,4} & s_{1,1}^{3,4} \\
{s_{1,1}}^{4,1} & s_{1,1}^{4,2} & s_{1,1}^{4,4} & s_{1,1}^{4,4} \\
s_{h,1} & s_{h,1} & s_{h,1} & s_{h,1}
\end{bmatrix}
\]
where the required covariances are derived in Appendix as

\[
\begin{align*}
s_{1,1}^{1,2} &= \kappa_3, \\
s_{1,1}^{2,2} &= \kappa_4 + 2\sigma^4, \\
s_{1,1}^{1,4} &= h \left[ \kappa_5 + 10h\kappa_3\sigma^2 \right], \\
s_{1,1}^{2,4} &= h \left[ \kappa_6 + (6h + 8)\kappa_4\sigma^2 + (4h + 6)\kappa_3^2 + 12h\sigma^6 \right], \\
s_{1,1}^{4,4} &= h \left[ \kappa_8 + (6h + 22)\kappa_6\sigma^2 + (4h + 52)\kappa_5\kappa_3 + 34\kappa_4^2 \\
&\quad + (84h + 120)\kappa_4\sigma^4 + (100h + 180)\kappa_3^2\sigma^2 + (72h + 24)\sigma^8 \right], \\
s_{h,h}^{4,4} &= h^2\kappa_8 + [12h^3 + 16A_h]\kappa_6\sigma^2 + [8h^3 + 48A_h]\kappa_5\kappa_3 + 34A_h\kappa_4^2 \\
&\quad + [36h^4 + 96hA_h + 72B_h]\kappa_4\sigma^4 + [64h^4 + 72hA_h + 144B_h]\kappa_3^2\sigma^2 \\
&\quad + [72h^2A_h + 24C_h]\sigma^8. \\
\end{align*}
\]

In (22), \( C_h = h (6h^4 + 10h^2 - 1) / 15 \). Similar to the case of skewness ratio test, when \( h = 1 \), \( C_h = 1 \); moreover, (19) yields \( s_{1,1}^{1,4} \), (20) \( s_{1,1}^{2,4} \) and both (21) and (22) simplify to \( s_{1,1}^{4,4} \).

### 2.5 Joint skewness and kurtosis ratio test

Skewness and kurtosis are often used together as a joint statistic; see for example the Jarque and Bera (1980) test for normality. For the case of joint skewness and kurtosis ratio test, we have

\[
f_t = \begin{bmatrix}
    r_t - \mu \\
    (r_t - \mu)^2 - \sigma^2 \\
    (r_t - \mu)^3 - \kappa_3 \\
    (r_t - \mu)^4 - 3\sigma^4 - \kappa_4 \\
    (\tilde{r}_t - h\mu)^3 - h\kappa_3 \\
    (\tilde{r}_t - h\mu)^4 - 3h^2\sigma^4 - h\kappa_4
\end{bmatrix}, \quad D_0 = \begin{bmatrix}
    -1 & 0 & 0 & 0 \\
    0 & -1 & 0 & 0 \\
    -3\sigma^2 & 0 & -1 & 0 \\
    -4\kappa_3 & -6\sigma^2 & 0 & -1 \\
    -3h^2\sigma^2 & 0 & -h & 0 \\
    -4h^2\kappa_3 & -6h^2\sigma^2 & 0 & -h
\end{bmatrix}
\]

The covariance matrix is

\[
S_0 = \begin{bmatrix}
    s_{1,1} & s_{1,2} & s_{1,3} & s_{1,4} & s_{1,h} & s_{1,h} \\
    s_{2,1} & s_{2,2} & s_{2,3} & s_{2,4} & s_{2,h} & s_{2,h} \\
    s_{3,1} & s_{3,2} & s_{3,3} & s_{3,4} & s_{3,h} & s_{3,h} \\
    s_{4,1} & s_{4,2} & s_{4,3} & s_{4,4} & s_{4,h} & s_{4,h} \\
    s_{h,1} & s_{h,1} & s_{h,1} & s_{h,1} & s_{h,h} & s_{h,h} \\
    s_{h,1} & s_{h,1} & s_{h,1} & s_{h,1} & s_{h,h} & s_{h,h}
\end{bmatrix}
\]
Most of the elements of $S_0$ have been provided in the above. The remaining required covariance elements are (see Appendix)

\[
\begin{align*}
\hat{s}^{2,3}_{1,h} & = h \left[ \kappa_5 + (3h + 6) \kappa_3 \sigma^2 \right], \\
\hat{s}^{4,3}_{1,h} & = h \left[ \kappa_7 + (3h + 18) \kappa_5 \sigma^2 + 34 \kappa_4 \kappa_3 + (30h + 72) \kappa_5 \sigma^4 \right], \\
\hat{s}^{3,4}_{1,h} & = h \left[ \kappa_7 + (6h + 15) \kappa_5 \sigma^2 + (4h + 30) \kappa_4 \kappa_3 + (66h + 36) \kappa_5 \sigma^4 \right], \\
\hat{s}^{3,4}_{h,h} & = h^2 \kappa_7 + \left[ 9h^3 + 12A_h \right] \kappa_5 \sigma^2 + \left[ 4h^3 + 30A_h \right] \kappa_4 \kappa_3 \\
& \quad + \left[ 30h^4 + 36hA_h + 36B_h \right] \kappa_3 \sigma^4.
\end{align*}
\]

Again, setting $h = 1$ reduces (24) to $\hat{s}^{2,3}_{1,1}$ whereas (25), (26) and (27) become $\hat{s}^{3,4}_{1,1}$.

### 3 A simulation study of size properties

For the analytical results in the above section to hold, moments of $r_t$ up to eighth order need to exist (sixth order for the skewness ratio test). The study by Loretan and Phillips (1994) suggests that while the second moments of financial time series seem to be finite, fourth-order moments do not. Moreover, based on simulation experiments, de Lima (1997) find that tests designed to have maximum power against misspecification of second or higher moments are sensitive to their nonexistence.

This section investigates using simulation experiments the robustness of the proposed ratio tests to moment condition failure. As a benchmark for comparison, the McLeod and Li (1983)’s test based on squared residual autocorrelations is also considered, for the test is not only widely used as diagnostic check for GARCH modelling, but also studied extensively for its robustness to moment condition failure in de Lima (1997).

Distributions that meet the moment condition of the proposed tests are first considered. Specifically, IID random data of size 250, 500 and 1000 that are distributed as standard normal and Student’s $t$ with 9 degrees of freedom are generated and subject to the skewness-kurtosis ratio tests as well as McLeod and Li’s test.\(^2\) Empirical test sizes are then calculated based on 5,000 replications of each simulation experiment. Table 1 provides the calculated test sizes at 10%, 5% and 1% levels for $h$ equals 5, 10 and 20 periods. McLi refers to the McLeod and Li’s test whereas Skew, Kurt and Joint are respectively the skewness, kurtosis and their joint ratio tests. Broadly speaking, the empirical test sizes are fairly close to their theoretical values. Closer observation finds the higher order ratio tests are generally under-sized at 10% and 5% levels, with noticeable improvements as the sample size increases.

\(^2\)Student’s $t$ with $v$ degrees of freedom has finite moments up to $(v - 1)$-th order.
At 1% level, the proposed tests tend to be mildly over-sized, with the joint skewness and kurtosis test most severe at 2.20% when $N = 500$ and $h = 20$. While the McLeod and Li’s test exhibits less under-sized tendencies at 10% and 5% levels, its over-sized problem at 1% level is more severe for all values of $N$ and $h$ when we move from normal distribution to Student distribution.

< Table 1 Empirical test sizes: all required moments exist >

Table 2 provides empirical test sizes when the moment condition fails. Specifically, IID random data are now generated from two Student’s $t$-distributions with 3 and 5 degrees of freedom, which correspond to the existence of moments up to the second and fourth orders, respectively. In the case of the skewness-kurtosis ratio tests, the under-sized tendency worsens noticeably at 10% and to a much smaller extent at 5% level. Interestingly at 1% level, the empirical test sizes remain broadly similar to those in Table 1 when the moment condition holds. For the McLeod and Li’s test, the over-sized problems are progressively more severe as the test size level moves from 5% to 1% and when a more heavy-tailed distribution is encountered.

< Table 2 Empirical test sizes: moment condition fails >

To sum up the above simulation study, the proposed higher order ratio tests are robust to violations of moment conditions. At conventional significance levels, while it is under-sized at 5% level but over-sized at 1% level, the deviations from theoretical values are small. This good size property and robustness is important as empirical evidence suggests the higher order moments of financial time series may not exist.

4 A study of multiperiod tail risk

This section illustrates the usefulness of the higher order ratio tests in a study of multiperiod tail risks for world major equity markets. Applications of the proposed tests confirm the presence of higher order dependence in stock market returns. While most of the nonlinear dependence can be removed using GARCH-type risk models, there is no single model that consistently passes the higher order tests for all stock markets studied. More importantly, there are several cases in which the GARCH-filtered returns pass the McLeod and Li’s test but fail to satisfy the skewness ratio relations. This illustrates not only the complementary role that can be played by the higher order ratio tests, but also indicates that the downside tail risks caused by asymmetry in stock returns distribution is a risk that should not be
overlooked. In this regard, the information provided by comparing the skewness of single- and multi-period stock returns can shed light on the nature of nonlinearities present in the financial returns. Finally, this section considers and discusses how the skewness-kurtosis ratio tests can be used as a guide to the task of multiperiod VaR forecasts.

4.1 Data and descriptive statistics

Large and small capitalization stock index log returns from US, UK, Germany and Japan are studied. Each time series of log returns comprises 4160 observations starting from 20 August 1991 to 22 August 2008. Table 3 provides the information on the standard deviation (sd), skewness (sk), kurtosis (ku) and 99% VaR (VaRL and VaRR are respectively VaRs on the left and right tails of the distribution) over the single- and multi-period horizons ($h = 5, 10$).

\begin{table}
\centering
\caption{Basic statistics}
\begin{tabular}{llll}
  \hline
  & sd & sk & ku & VaRL & VaRR \\
  \hline
  Large Cap & & & & & \\
  Small Cap & & & & & \\
  Japan & & & & & \\
  \hline
\end{tabular}
\end{table}

It is remarked that for multiperiod returns, all the figures presented in Table 3 are adjusted for intervaling effects so that they would remain constant for different values of $h$ if the ratio relations hold. This is achieved by setting $sd = h^{-1/2}\bar{\sigma}$, $sk = h^{1/2}\tilde{\rho}_3$, $ku = h\tilde{\rho}_4$, $VaRL = h^{-1/2}\bar{V}aRL$ and $VaRR = h^{-1/2}\bar{V}aRR$. It can be seen from the table that while the standard deviations are fairly stable, both large- and small-cap stock indexes are increasingly more left-skewed and leptokurtic (except for Japan’s skewness) as $h$ increases. As for the multiperiod VaR, only the small-cap indexes exhibit noticeable changes in magnitude as the return horizon increases. This should not be taken as evidence that higher order moments are not relevant in the determination of VaR for large-cap stock indexes, for as can be seen later this is due to decreasing standard deviations as return horizon increases. To illustrate, Figure 1 depicts for the single-period case the relationship between the skewness and VaRL divided by the associated standard deviation. The scatter plot shows that as skewness (measured by the horizontal axis) decreases, the magnitude of standardized VaRL increases. The only exception is the German small-cap stock index which has large negative VaRL but positive skewness at 1.68. The reason is that the German small-cap returns are extremely leptokurtic with kurtosis measure of 67.7 as compared with an average value 4.7 for the other stock indexes.

\begin{figure}[h]
\centering
\caption{Skewness and VaR}
\end{figure}

\footnote{99% VaR is obtained as the bottom 1 percentile of ranked (overlapping, if $h > 1$) returns in the full sample.}
4.2 Applying the skewness-kurtosis ratio tests

Before we apply the higher order ratio tests, it is worthwhile to consider the following example to illustrate why the higher order relations may not hold. Consider a stationary time series \( r_t \) and construct the associated two-period overlapping observations \( \tilde{r}_t = r_{t-1} + r_t \). By virtue of Lemma 1 in Section 2.1, the third order cumulant of \( \tilde{r}_t \) is

\[
\text{cum}(\tilde{r}_t, \tilde{r}_t, \tilde{r}_t) = \text{cum}(r_{t-1}, r_{t-1}, r_t) + \text{cum}(r_t, r_t, r_t) + 3\text{cum}(r_{t-1}, r_t, r_t) + 3\text{cum}(r_{t-1}, r_{t-1}, r_t) + 3\text{cum}(r_{t-1}, r_t, r_t).
\]

(28)

So testing \( \tilde{\kappa}_3 = 2\kappa_3 \) is equivalent to testing \( \text{cum}(r_{t-1}, r_{t-1}, r_t) + \text{cum}(r_{t-1}, r_t, r_t) = 0 \). That is, if higher order dependence exists between \( r_{t-1} \) and \( r_t \), skewness ratio relation does not hold.

Now, suppose \( r_t \) follows an AR(1) process: \( r_t = ar_{t-1} + e_t \) where the innovation \( e_t \) is IID random variable which has a finite non-zero third cumulant or moment. Then

\[
\text{cum}(r_{t-1}, r_{t-1}, r_t) = a \cdot \text{cum}(r_{t-1}, r_{t-1}, r_{t-1}) = a \cdot \kappa_3 \neq 0.
\]

(29)

Thus autocorrelation in \( r_t \) would also result in the rejection of the skewness ratio relation. Similar arguments also apply to the kurtosis ratio test, i.e., both linear and nonlinear dependence could cause the null hypothesis to be rejected.

For the reasons discussed above, the skewness-kurtosis ratio tests, McLeod and Li’s test as well as the test of linear correlation by Ljung and Box (1978) are applied to both raw and AR-filtered returns. The results are reported in Table 4 and 5. In the tables, LB is the Ljung-Box test whereas McLi, Skew, Kurt and Joint refer to the same tests reported in Table 1 and 2. The number of lags used in Ljung-Box and McLeod-Li tests is \( h \) and thus the associated test statistics are distributed as Chi-square with \( h \) degree of freedom under the null. The reported figures for Skew, Kurt and Joint are also Chi-square statistics with 1, 1, and 2 degree of freedom respectively. In the last three columns of the tables, \( m_2, k_3 \) and \( k_4 \) are sample estimates for \( \tilde{\sigma}^2/h, \tilde{\kappa}_3/(h\sigma^3/2) \) and \( \tilde{\kappa}_4/(h\sigma^4) \) respectively. If the skewness and kurtosis ratio relations hold, all three statistics would be independent of value of \( h \).

As can be seen from Table 4, for most stock indexes, magnitude of \( k_3 \) and \( k_4 \) rises as \( h \) increases, indicating that returns are increasingly more left-skewed and leptokurtic. The two exceptions are, firstly, the Japan’s large-cap stock index with relatively stable measures of skewness and kurtosis but in trends opposite to the others, and secondly, positive skewness measure of Japan’s small-cap stock index when \( h = 10 \). These observations are consistent
with the skewness-kurtosis tests which reject the skewness-kurtosis ratio relations for all stock indexes except Japan’s large-cap.

< Table 4 Tests on raw returns >

Table 5 reports results on returns filtered by AR(10) linear model. As expected, all stock indexes pass the Ljung-Box test. Moreover, the skewness and kurtosis statistics as well as the corresponding test statistics are drastically reduced for the small-cap stock indexes. For example, when \( h = 10 \), the values of \( k3, k4 \) and the associated Joint test statistic are respectively -10.71, 416.90 and 3168.8 for UK whereas the corresponding values in Table 4 are -1.90, 39.75 and 22.6, respectively. Similar results are observed for Germany and, to a much smaller extent, US and Japan. For the large-cap indexes, the opposite results are obtained although the differences are much smaller. Take UK again as an example. When \( h \) increases from 5 to 10, the kurtosis and the associated test statistic decrease respectively to 6.8 and 0.5 in Table 4 whereas the corresponding values in Table 5 are 24.92 and 22.9 respectively.

This discrepancy may be explained as follows. Large Ljung-Box test statistics and increasing sample variance (\( m2 \)) indicate that small-cap stocks, which are highly illiquid, exhibit high positive autocorrelation which exacerbate the deviation of \( \kappa_p \) from \( h \kappa_p \), a finding that is consistent with the above analysis described by (28) and (29). In the case of large-cap stock indexes, albeit with much smaller magnitude, significant Ljung-Box statistics and decreasing sample variance suggest they are negatively autocorrelated. The negative autocorrelation decreases the deviation of \( \kappa_p \) from \( h \kappa_p \), giving rise to several insignificant ratio tests for US and UK (see Table 4). Finally, these findings explain how the two opposing factors, namely decreasing sample variance and increasing skewness and kurtosis, render the scaling law seemingly good in estimating multiperiod VaR for large-cap stocks. For small-cap stocks with poor liquidity, scaling law clearly underestimates multiperiod VaR due to the relatively high positive correlation in their returns and a breakdown of the skewness-kurtosis ratio relations.

< Table 5 Tests on AR-filtered returns >

GARCH-type risk models have been shown successful in VaR forecasting; see for example Wong (2010). It will be interesting to see how well GARCH models can remove higher order dependencies in stock index returns. The standard GARCH-normal model

\[
\begin{align*}
    r_t &= a_0 + \sum_{j=1}^{p} a_j r_{t-j} + \varepsilon_t \\
    \varepsilon_t &= \sigma_t z_t
\end{align*}
\]
where \( z_t \sim \text{IID } N(0, 1) \) and
\[
\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2
\]
is first used to remove heteroscedastic structure from the stock index returns. The shocks \( z_t \) are then calculated based on the estimated GARCH model and subject to the same tests as above. The results are presented in Table 6. It can be seen that the GARCH model is successful in removing both linear and conditional variance dependencies from the stock returns and none of the Ljung-Box and McLeod-Li tests is significant. Also, except for Germany's large-cap stock index, all kurtosis ratio tests are insignificant, consistent with much less variable k4 statistics.

The skewness ratio relations, however, do not seem to hold for US and large-cap stock indexes of UK and Germany. Since the skewness measures of these stock indexes are progressively more negative as \( h \) increases, using the same analysis as described in (28), we can conjecture that most of \( \text{cum}(z_t, z_{t-i}, z_{t-j}) \)'s (where \( j \geq i > 0 \)) for these stock indexes are negative. Stock markets are famous for their dramatic crashes during crises which gives rise to negatively skewed distribution. While negative \( k3 \) when \( h = 1 \) means shocks are skewed to the left, increasingly more negative \( k3 \) for larger \( h \)'s suggest third order dependence is present in the shocks, possibly a reflection of the fact that during financial market crashes, negative returns tend to follow one after another, resulting in much larger skewness measure as otherwise would be if shocks are IID. In this case, multiperiod VaR forecasts would not provide sufficient risk coverage if they were made by (incorrectly) assuming the shocks \( z_t \) are IID.

\(< \text{Table 6 Tests on returns filtered by GARCH-Normal} > \)

Giot and Laurent (2003) and Wong (2010) provide evidence that APARCH models proposed by Ding, Granger and Engle (1993) provide accurate single-period VaR forecasts. In an APARCH model, the heteroscedasticity equation is given by
\[
\sigma_t^h = \alpha_0 + \alpha_1 (|\varepsilon_{t-1}| - \alpha_n \varepsilon_{t-1})^h + \beta \sigma_{t-1}^h.
\]

Assuming IID normal shocks driving the APARCH-normal process, the linear and nonlinearity tests described above are applied to the filtered returns, \( z_t \), and the results are provided in Table 7. One outcome that stands out for the APARCH-normal model is that it seems to capture the nonlinear dynamics of large-cap stock returns well and all tests except one fail to detect any dependence structure in \( z_t \). We offer two possible reasons for the performance of APARCH models. Firstly, the model allows a larger impact on volatility if there
is a negative shock. Secondly, as Ding et al points out, \(|r_t|^6\) has high autocorrelation for long lags and the model captures this property. This implies that when stock market falls, negative returns cause subsequent relatively higher and more persistent volatility clustering. This in turn implies the standardized shocks in APARCH models would be smaller than those in GARCH models, especially during market downturns. This is consistent with the k3 statistics of large-cap stock indexes, several of which actually become less negative as \(h\) increases. Indeed, in this case of Japan, the skewness ratio relation is rejected because the skewness turns from negative to positive as we move from single-period to multi-period.

There are, however, signs that the APARCH model may be less appropriate for small-cap stock indexes. For both UK and Japan, skewness measure turns from negative to positive as \(h\) increases to 10, rendering the associated skewness ratio tests significant.

< Table 7 Tests on returns filtered by APARCH-Normal >

### 4.3 Multiperiod VaR forecasts

Since GARCH models have been shown to be successful in VaR forecasting, the discussion below is based on forecasting multiperiod VaR using a GARCH-type model. Under general conditions, the VaR of a time series \(r_t\) may be expressed as

\[
VaR_{t+1} = \mu_{t+1} + \sigma_{t+1}q_{\alpha}
\]

where \(1-\alpha\) is the confidence level of VaR and \(q_{\alpha}\) is the \(\alpha\)-quintile of the distribution of shocks \(z_t\) to the GARCH-type process. According to Hsieh (1993), the impact of mean \(\mu_{t+1}\) is small and thus we focus on \(\sigma_{t+1}q_{\alpha}\) in the comparison below between the Basel II’s scaling law method and the GARCH approach in forecasting multiperiod VaR. The multiperiod VaR forecast by the \(\sqrt{h}\)-scaling law takes the form

\[
\widetilde{VaR}_{t+h} = \sqrt{h}\sigma_{t+1}q_{\alpha}
\]

whereas the GARCH modelling implies that

\[
\widetilde{VaR}_{t+h} = \tilde{\sigma}_{t+h}\tilde{q}_{\alpha}.
\]

Given the high volatility persistency in the heteroscedastic equation, \(\tilde{\sigma}_{t+h} \approx \sqrt{h}\sigma_{t+1}\). Hence, the performance of the scaling law depends on how well \(q_{\alpha}\) approximates \(\tilde{q}_{\alpha}\) as \(h\) increases. Now, if \(r_t\) is a normal process, then \(\tilde{q}_{\alpha} = q_{\alpha}\), since the distribution is closed under convolution. However, the relationship between \(\tilde{q}_{\alpha}\) and \(q_{\alpha}\) is much more complicated for financial returns.

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as they are highly non-normal and empirical evidence suggests that the property of closure under convolution does not apply to them. Nevertheless, the implication of intervaling effects suggests that as \( h \) increases, the distribution of \( \tilde{z}_t \) tends toward that of a normal distribution as its skewness and kurtosis are scaled by \( h^{-1/2} \) and \( h^{-1} \) respectively. Since stock returns are often left-skewed and leptokurtic, in most cases, \( q_\alpha < \tilde{q}_\alpha \) if the risk model is successful in yielding shocks \( z_t \) that pass the skewness-kurtosis ratio tests.

The above analysis predicts that the Basel’s \( \sqrt{h} \)-scaling law approach in comparison to the GARCH models would be over conservative in forecasting multiperiod VaR. This can be verified in a numerical exercise that bootstraps shocks \( z_t \) to simulate conditional distribution of \( \tilde{r}_{t+h} \) (this work is currently undertaken by the author).

5 Conclusion

Skewness and kurtosis ratio tests are developed using GMM technique in this paper. In particular, overlapping observations are used in order to incorporate more information into the proposed tests. This is achieved by by explicitly modelling the dependencies in the overlapping data under IID assumption. Simulation experiments demonstrate that the proposed tests are robust to moment condition failure and exhibit good size properties in comparison to other nonlinearity tests such as McLeod-Li’s.

Applications of the skewness-kurtosis ratio tests to world major equity markets illustrate their complementary role to existing linear and nonlinear diagnostic tests. Several cases are noted in which the Ljung-Box and McLeod-Li tests fail to detect presence of dependence structure but the proposed tests find violations of the skewness-kurtosis ratio relations. This ability of the higher order ratio tests is particularly useful where multiperiod forecasts of tail events are required, for tail risks are associated with measures of skewness and kurtosis which indicate the level of asymmetry and tail fatness of the distribution.

It is found that the \( \sqrt{h} \)-scaling law adopted by Basel II underestimates multiperiod VaR for illiquid stocks due to the fact that returns on illiquid stocks are positively autocorrelated. Generally speaking, in the context of heteroscedastic process, it is shown analytically that the scaling law is likely to yield over conservative multiperiod VaR forecasts since it ignores the intervaling effects on skewness and kurtosis of stock returns. In this regard, the proposed tests can be used to check if the higher order nonlinear dependencies have been modelled successfully, for failure to do so is likely to yield suboptimal forecasts for multiperiod tail risks.
A Appendix

Analytical proofs for the covariance matrices $S_0$ used in the kurtosis-skewness ratio tests are provided in the appendix here. The required covariances may be divided into three categories: covariance between products of single-period random returns (e.g. $S_{3,4}^{1}$), between products of single-period and $h$-period random returns (e.g. $S_{3,4}^{4,1}$), and between products of $h$-period random returns (e.g. $S_{3,4}^{h,h}$), with increasing degree of complexity.

In all three cases, the required covariances can be obtained using the indecomposable partition method stated in Lemma 2 below. However, in order to facilitate understanding (and cross verification) of the proofs, we first consider the results for the covariances between products of single-period random returns. These are provided in the next subsection, A.1, where relations between cumulants and moments are introduced. Subsection A.2 provides Lemma 2 which is required for the derivation of covariances of products of multiperiod random variables. Some examples are given to illustrate the Lemma can be applied. Subsection A.3 derives all the required covariances involving multiperiod random variances. Finally, this appendix ends with some remarks on the alternative approach of using multinomial theorem to derive the required covariance matrices for the GMM tests.

A.1 Proofs for $S_{1,1}^{n,q}$

To estimate the covariances which are expressed in cumulants, we can use the following formulae provided by Kendall and Stuart (1969, p.71) that show how we can use moments to obtain the required cumulants:

\[
\begin{align*}
\kappa_2 &= \mu_2 = \sigma^2, \\
\kappa_3 &= \mu_3, \\
\kappa_4 &= \mu_4 - 3\sigma^4, \\
\kappa_5 &= \mu_5 - 10\mu_3\sigma^2, \\
\kappa_6 &= \mu_6 - 15\mu_4\sigma^2 - 10\mu_3^2 + 30\sigma^6, \\
\kappa_7 &= \mu_7 - 21\mu_5\sigma^2 - 35\mu_4\mu_3 + 210\mu_3\sigma^4, \\
\kappa_8 &= \mu_8 - 28\mu_6\sigma^2 - 56\mu_5\mu_3 - 35\mu_4^2 + 420\mu_4\sigma^4 + 560\mu_3^2\sigma^2 - 630\sigma^8.
\end{align*}
\]

Kendall and Stuart (1969, p.70) also provides the formulae for expressing higher order
central moments in terms of cumulants (note from above that \( \mu_2 = \kappa_2, \mu_3 = \kappa_3 \) and):

\[
\begin{align*}
\mu_4 &= \kappa_4 + 3\sigma^4, \\
\mu_5 &= \kappa_5 + 10\kappa_3\sigma^2, \\
\mu_6 &= \kappa_6 + 15\kappa_4\sigma^2 + 10\kappa_3^2 + 15\sigma^6, \\
\mu_7 &= \kappa_7 + 21\kappa_5\sigma^2 + 35\kappa_4\kappa_3 + 105\kappa_3\sigma^4, \\
\mu_8 &= \kappa_8 + 28\kappa_6\sigma^2 + 56\kappa_5\kappa_3 + 35\kappa_4^2 + 210\kappa_4\sigma^4 + 280\kappa_3^2\sigma^2 + 105\sigma^8.
\end{align*}
\]

We shall now consider deriving expression of \( S_{1,1}^{p,q} (1 \leq p, q \leq 4) \) in terms of cumulants using the above formulae. Under the IID assumption, \( x_t \) and \( x_{t-l} \) are independent for \( l \neq 0 \). Thus by virtue of Property 3 in Lemma 1, \( \text{cum}(x_t^p, x_{t-l}^q) = 0 \) for \( l \neq 0 \). Using the above moment formulae and exploiting the fact that \( \text{E}(x_t) = 0 \), \( s_{1,1}^{p,q} = \sum \text{cum}(x_t^p, x_{t-l}^q) = \text{cov}(x_t^p, x_{t}^q) = \mu_{p+q} - \mu_p \mu_q \), it is straightforward that \( s_{1,1}^{1,1} = \sigma^2 \) and \( s_{1,1}^{1,2} = \kappa_3 \). For \( s_{1,1}^{2,2} \),

\[
s_{1,1}^{2,2} = \text{cov}(x_t^2, x_t^2) = \mu_4 - \mu_2 \mu_2.
\]

Substituting the above formulae that \( \mu_4 = \kappa_4 + 3\sigma^4 \) and \( \mu_2 = \kappa_2 \), we have

\[
s_{1,1}^{2,2} = \kappa_4 + 3\sigma^4 - \sigma^4 = \kappa_4 + 2\sigma^4.
\]

Using the same principles, the more complex ones are derived as follows.

\[
\begin{align*}
s_{1,1}^{1,3} &= \kappa_4 + 3\sigma^4, \\
s_{1,1}^{1,4} &= \kappa_5 + 10\kappa_3\sigma^2, \\
s_{1,1}^{2,3} &= \kappa_5 + 9\kappa_3\sigma^2, \\
s_{1,1}^{2,4} &= \kappa_6 + 14\kappa_4\sigma^2 + 10\kappa_3^2 + 12\sigma^6, \\
s_{1,1}^{3,3} &= \kappa_6 + 15\kappa_4\sigma^2 + 9\kappa_3^2 + 15\sigma^6, \\
s_{1,1}^{3,4} &= s_{1,1}^{1,1} = \kappa_7 + 21\kappa_5\sigma^2 + 34\kappa_4\kappa_3 + 102\kappa_3\sigma^4 \\
s_{1,1}^{4,4} &= \kappa_8 + 28\kappa_6\sigma^2 + 56\kappa_5\kappa_3 + 34\kappa_4^2 + 204\kappa_4\sigma^4 + 280\kappa_3^2\sigma^2 + 96\sigma^8.
\end{align*}
\]

Letting \( h = 1 \) in, for example, \((14-15)\) will give rise to the same formula for \( s_{1,1}^{3,3} \) above. One important observation to be made here is that \( s_{1,1}^{p,q} \) contains the basic structure for \( s_{1,h}^{p,q} \) and \( s_{h,h}^{p,q} \). Take the case \( p = q = 4 \) as another example, the right hand sides of \((21)\) and \((22)\) in the kurtosis ratio test share the same cumulant terms with \( s_{1,1}^{4,4} \) in \((49)\): \( \kappa_8, \kappa_6\sigma^2, \kappa_5\kappa_3, \kappa_4^2, \kappa_4\sigma^4, \kappa_3^2\sigma^2 \) and \( \sigma^8 \). Moreover, when \( h = 1 \), \( A_h = B_h = C_h = 1 \), yielding the same coefficients for all cumulant terms in \( s_{1,1}^{p,q}, s_{1,h}^{p,q} \) and \( s_{h,h}^{p,q} \), where \( 1 \leq p, q \leq 4 \). Therefore, as
can be seen below, $h^p$, $A_h$, $B_h$ and $C_h$ reflect the effects of having $h$-period returns in place of single-period returns under the null hypothesis of independent returns.

A.2 Cumulant of products of random variables

The above shows how $s_{1,1}^{p,q}$ can be obtained easily using the formulae of moments and cumulants provided by Kendall and Stuart (1969). However, things become complicated when multiperiod random variables are involved. Since the required covariances are essentially cumulants of products of random variables, we introduce here the concept of indecomposable partition used by Brillinger (1974, Section 2.3) to obtain cumulants of products of $x_t$.

Definition Consider a partition $P_1 \cup \cdots \cup P_M$ of the table of entries (not necessarily rectangular) given below

$$(1,1) \cdots (1,J_1)$$

$$\vdots$$

$$(I,1) \cdots (1,J_I)$$

Sets $P_{m'}$ and $P_{m''}$ are said to hook if there exists $(i_1,j_1) \in P_{m'}$ and $(i_2,j_2) \in P_{m''}$ such that $i_1 = i_2$; that is $(i_1,j_1)$ and $(i_2,j_2)$ are from the same row. $P_{m'}$ and $P_{m''}$ are said to communicate if there exists a sequence of sets $P_{m_1} = P_{m'}, P_{m_2}, \ldots, P_{m_N} = P_{m''}$ such that $P_{m_n}$ and $P_{m_{n+1}}$ hook for $n = 1, \ldots, N - 1$. A partition is said to be indecomposable if all its sets communicate.

Each row in the table above corresponds to a product of random returns in our paper. So, $I = 2$ as we need only covariances which equal to second order cumulants. Take the case of $\text{cum}(x_t^3, \tilde{x}_{t-1}^4)$ in $s_{1,h}^{3,4}$ for illustration, we can let the first row of entries in the above table correspond to $x_t^3$ whereas the second row to $\tilde{x}_{t-1}^4$, so that $J_1 = 3$ and $J_2 = 4$. An indecomposable partition as defined above is one that contains at least a set in which at least one element is from $x_t^3$ and the other from $\tilde{x}_{t-1}^4$.

The definition of indecomposable partition is used by Brillinger (1975) to obtain the joint cumulant of products of random variables, as presented in Lemma 2 below.

Lemma 2 Consider a two-way the $I$ random variables

$$Y_i = \prod_{j=1}^{J_i} X_{ij},$$

where $j = 1, \ldots, J_i$ and $i = 1, \ldots, I$. The joint cumulant $c(Y_1, \ldots, Y_I)$ is given by

$$\sum_{P} \text{cum} (X_{ij}; ij \in P_1) \cdots \text{cum} (X_{ij}; ij \in P_M)$$

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where the summation is over all indecomposable partitions $P = P_1 \cup \cdots \cup P_M$.

**Example 1** Consider the simple case of $c(x_t^2, x_{t-l}^2)$ in $s_{1,1}^{2,2}$. Then $Y_1 = X_{11}X_{12}$ and $Y_2 = X_{21}X_{22}$ would correspond to $x_t^2$ and $x_{t-l}^2$ respectively. Applying Lemma 1 and make use of the fact that $E(x_t) = E(\tilde{x}_{t-l}) = 0$,

$$
\text{cum}(Y_1, Y_2) = \text{cum}(X_{11}, X_{12}, X_{21}, X_{22}) + \text{cum}(X_{11}, X_{21}) \text{cum}(X_{12}, X_{22}) + \text{cum}(X_{11}, X_{22}) \text{cum}(X_{12}, X_{21}),
$$

which gives rise to

$$
\text{cum}(x_t^2, x_{t-l}^2) = \text{cum}(x_t, x_t, x_{t-l}, x_{t-l}) + 2\text{cum}(x_t, x_{t-l})^2. \quad (50)
$$

Note that $\text{cum}(x_t, x_t)\text{cum}(x_{t-l}, x_{t-l})$ is not an indecomposable partition because there is no cumulant term that links the $x_t^2$ and $x_{t-l}^2$ together.

### A.3  Proofs for $S_{1,h}^{p,q}$ and $S_{h,h}^{p,q}$

Covariances involving multiperiod random variables are derived in this subsection using Lemma 2.

#### A.3.1 Notation

First of all, to simplify the notation, the $j$-th order joint cumulant of random variables $y_1, \ldots, y_j$ is simply denoted as $\langle y_1 \cdots y_j \rangle$, that is

$$
\langle y_1 \cdots y_j \rangle = \text{cum}(y_1, \ldots, y_j).
$$

Note that $\langle \cdot \rangle$ cannot be used to represent cumulant of products of random variables. As (50) shows, $\text{cum}(x_t^2, x_t^2) \neq \langle x_t^4 \rangle$.

#### A.3.2 Preliminary results

There are two properties of $x_t$ which render the derivation of covariance matrices $S_0$ relatively straightforward. Firstly, $E(x_t) = 0$. Secondly, $x_t$ and $x_{t-l}$ are independent except for $l = 0$. The first property enables us to ignore all indecomposable partitions which result in $E(x_t)$ as a cumulant term. By virtue of Lemma 1, the second property implies that for $j$ random
variables $x$’s at time $t$ or $t - l$, we have

$$\langle x_t, \ldots, x_{t-l} \rangle = \begin{cases} \kappa_j & \text{if } l = 0, \\ 0 & \text{if } l \neq 0. \end{cases} \quad (51)$$

If the $j$ random variables are a mixture of $x$’s and $h$-period random returns $\bar{x}_{t-l}$’s,

$$\langle x_t, \ldots, \bar{x}_{t-l} \rangle = \begin{cases} \kappa_j & \text{for } 1 - h \leq l \leq 0, \\ 0 & \text{for } l > 0. \end{cases} \quad (52)$$

Finally, for $j h$-period random returns $\bar{x}$’s at time $t$ or $t - l$,

$$\langle \bar{x}_t, \ldots, \bar{x}_{t-l} \rangle = \begin{cases} (h - |l|) \kappa_j & \text{for } |l| < h, \\ 0 & \text{for } |l| \geq h. \end{cases} \quad (53)$$

### A.3.3 Covariances for skewness ratio test

The covariances between single-period returns are already provided in A.1. Now, we consider covariances that involve $h$-period random returns. First, consider the simple case of $s_{1,3}^{1,3} = \sum \text{cum}(x_t, \bar{x}_{t-l}^3)$. Applying Lemma 2,

$$\text{cum}(x_t, \bar{x}_{t-l}^3) = \langle x_t \bar{x}_{t-l}^3 \rangle + 3 \langle x_t \bar{x}_{t-l} \rangle \langle \bar{x}_{t-l}^2 \rangle$$

for $l = 1 - h, \ldots, -1, 0$, zero otherwise. Using results (52) and (53),

$$\text{cum}(x_t, \bar{x}_{t-l}^3) = \kappa_4 + 3h\sigma^4,$$

so $s_{1,3}^{1,3} = h[\kappa_4 + 3h\sigma^4]$. Similarly, for $1 - h \leq l \leq 0$,

$$\text{cum}(x_t^3, \bar{x}_{t-l}^3) = \langle x_t^3 \bar{x}_{t-l}^3 \rangle + 3 \langle x_t^3 \bar{x}_{t-l} \rangle \langle \bar{x}_{t-l}^2 \rangle + 3 \langle x_t \bar{x}_{t-l}^3 \rangle \langle x_{t-l}^2 \rangle + 9 \langle x_t^2 \bar{x}_{t-l}^2 \rangle \langle x_t \bar{x}_{t-l} \rangle + 9 \langle x_t^2 \bar{x}_{t-l} \rangle \langle x_t \bar{x}_{t-l}^2 \rangle + 9 \langle x_{t-l}^2 \rangle \langle x_t \bar{x}_{t-l} \rangle \langle \bar{x}_{t-l}^2 \rangle + 6 \langle x_t \bar{x}_{t-l} \rangle \langle x_t \bar{x}_{t-l} \rangle \langle x_t \bar{x}_{t-l} \rangle = \kappa_6 + (3h + 12) \kappa_4 \sigma^2 + 9\kappa_3^2 + (9h + 6) \sigma^6,$$

which if multiplied by $h$ gives rise to $s_{1,3}^{1,3}$. To see how the number of each type of indecomposable partitions is obtained in (54), take $\langle x_t^2 \bar{x}_{t-l} \rangle \langle x_t \bar{x}_{t-l} \rangle$ as an example: there are three combinations of choosing $x_t^2$ from $x_t^3$ and three combinations of $\bar{x}_{t-l}$ from $\bar{x}_{t-l}^3$ to yield $\langle x_t^2 \bar{x}_{t-l} \rangle$; it is left only one way for the remaining random variables to form $\langle x_t \bar{x}_{t-l} \rangle$. So, the
required number is \(3 \times 3 \times 1 = 9\).

Now in the case of \(\text{cum}(\bar{x}_t^3, \bar{x}_{t-l}^3)\) in \(s_{h,h}^{3,3}\), each term in the sum of products of cumulants will retain the same form as the right hand side of (54), and replacing \(x_t\) with \(\bar{x}_t\) yields the expression for \(\text{cum}(\bar{x}_t^3, \bar{x}_{t-l}^3)\). So, making use of (53),

\[
s_{h,h}^{3,3} = \sum (h - |l|) \kappa_6 + \left[ 6h \sum (h - |l|) + 9 \sum (h - |l|)^2 \right] \kappa_4 \sigma^2
+ 9 \sum (h - |l|)^2 \kappa_3^2 + \left[ 9h^2 \sum (h - |l|) + 6 \sum (h - |l|)^3 \right] \sigma^6,
\]

where the summation is from \(l = -h+1, \ldots, h-1\). Note that \(\sum (h - |l|) = h^2\), \(\sum (h - |l|)^2 = A_h\) and \(\sum (h - |l|)^3 = B_h\), and this completes the proof for the expression of \(s_{h,h}^{3,3}\) in (15).

### A.3.4 Covariances for kurtosis ratio test

From the above derivations for \(S_{1,h}^{1,3}\) and \(S_{1,h}^{3,3}\), we can see that covariances between products of single- and \(h\)-period random returns yield simple multiple of \(h\), and provide the basic form for more complex covariances between products of \(h\)-period random returns. These steps of proof are similar for covariances in the kurtosis ratio test. So we have

\[
s_{1,h}^{1,4} = \sum \text{cum}\left( x_t, \bar{x}_{t-l}^4 \right)
= \sum \left[ x_t \bar{x}_{t-l}^4 + 4 \left( x_t \bar{x}_{t-l} \right) \left( \bar{x}_{t-l}^3 \right) + 6 \left( x_t \bar{x}_{t-l} \right) \left( \bar{x}_{t-l}^2 \right) \right]
= h \left[ \kappa_5 + 10h \kappa_3 \sigma^2 \right].
\]

Also, multiplying by \(h\) the following cumulant

\[
\text{cum}\left( x_t^2, \bar{x}_{t-l}^4 \right) = \left( x_t^2 \bar{x}_{t-l}^4 \right) + 6 \left< x_t^2 \bar{x}_{t-l}^2 \right> \left< \bar{x}_{t-l}^2 \right> + 8 \left< x_t \bar{x}_{t-l} \right> \left< \bar{x}_{t-l}^3 \right>
+ 4 \left< x_t \bar{x}_{t-l} \right> \left< \bar{x}_{t-l}^3 \right> + 12 \left< x_t \bar{x}_{t-l} \right> \left< x_t \bar{x}_{t-l} \right> \left< \bar{x}_{t-l}^2 \right>
= \kappa_6 + (6h + 8) \kappa_4 \sigma^2 + (4h + 6) \kappa_3^2 + 12h \sigma^6
\]

yields \(s_{1,h}^{2,4}\). The case for \(s_{1,h}^{4,4}\) is more complex; the indecomposable partitions of \(\text{cum}\left( x_t^4, \bar{x}_{t-l}^4 \right)\)
are

\[
\langle x_t^4 \hat{x}_{t-l}^4 \rangle + 6 \langle x_t^1 \hat{x}_{t-l}^2 \rangle \langle \hat{x}_{t-l}^2 \rangle + 6 \langle x_t^2 \hat{x}_{t-l}^1 \rangle \langle \hat{x}_{t-l}^1 \rangle + 16 \langle x_t^3 \hat{x}_{t-l}^3 \rangle \langle \hat{x}_{t-l}^3 \rangle \\
+ 4 \langle x_t^4 \hat{x}_{t-l}^4 \rangle + 4 \langle x_t^1 \hat{x}_{t-l}^3 \rangle \langle \hat{x}_{t-l}^3 \rangle + 24 \langle x_t^2 \hat{x}_{t-l}^2 \rangle \langle \hat{x}_{t-l}^2 \rangle + 24 \langle x_t^3 \hat{x}_{t-l}^3 \rangle \langle \hat{x}_{t-l}^3 \rangle \\
+ 18 \langle x_t^4 \hat{x}_{t-l}^4 \rangle + 16 \langle x_t^3 \hat{x}_{t-l}^3 \rangle \\
+ 36 \langle x_t^2 \hat{x}_{t-l}^2 \rangle \langle \hat{x}_{t-l}^2 \rangle + 48 \langle x_t^3 \hat{x}_{t-l}^3 \rangle \langle \hat{x}_{t-l}^3 \rangle \\
+ 48 \langle x_t^4 \hat{x}_{t-l}^4 \rangle \langle x_t^2 \rangle + 72 \langle x_t^2 \hat{x}_{t-l}^2 \rangle \langle \hat{x}_{t-l}^2 \rangle + 144 \langle x_t^3 \hat{x}_{t-l}^3 \rangle \langle \hat{x}_{t-l}^3 \rangle \\
+ 36 \langle x_t^2 \hat{x}_{t-l}^2 \rangle \langle x_t^3 \hat{x}_{t-l}^3 \rangle + 36 \langle x_t^3 \hat{x}_{t-l}^3 \rangle \langle x_t^2 \hat{x}_{t-l}^2 \rangle + 144 \langle x_t^3 \hat{x}_{t-l}^3 \rangle \langle x_t^2 \hat{x}_{t-l}^2 \rangle \\
+ 72 \langle x_t^2 \rangle \langle x_t \hat{x}_{t-l} \rangle \langle \hat{x}_{t-l}^2 \rangle + 24 \langle x_t \hat{x}_{t-l} \rangle \langle x_t \hat{x}_{t-l} \rangle \langle x_t \hat{x}_{t-l} \rangle \langle \hat{x}_{t-l} \rangle.
\]

In the above, only \( \langle \hat{x}_{t-l}^2 \rangle \) and \( \langle \hat{x}_{t-l}^3 \rangle \) yield a factor \( h \). Thus

\[
\text{cum} (x_t^4, \hat{x}_{t-l}^4) = \kappa_8 + (6h + 22) \kappa_6 \sigma^2 + (4h + 52) \kappa_5 \kappa_3 + 34 \kappa_4^2 \\
+ (84h + 120) \kappa_4 \sigma^4 + (100h + 180) \kappa_3^2 \sigma^2 + (72h + 24) \sigma^6,
\]

for \( l = 1 - h \),.., 0. Thus multiplying the above by \( h \) yields \( s_t^{4,4} \). Replacing \( x_t^4 \) with \( \hat{x}_t^4 \) in (55) gives us \( \text{cum}(\hat{x}_t^4, \hat{x}_{t-l}^4) \) which, after applying the result of (53), yields

\[
(h - |l|) \kappa_8 + [12h (h - |l|) + 16 (h - |l|)^2] \kappa_6 \sigma^2 \\
+ [8h (h - |l|) + 48 (h - |l|)^2] \kappa_5 \kappa_3 + 34 (h - |l|)^2 \kappa_4^2 \\
+ [36h^2 (h - |l|) + 96 (h - |l|)^2 + 72 (h - |l|)^3] \kappa_4 \sigma^4 \\
+ [64h^2 (h - |l|) + 72h (h - |l|)^2 + 144 (h - |l|)^3] \kappa_3^2 \sigma^2 \\
+ [72h^2 (h - |l|)^2 + 24 (h - |l|)^4] \sigma^6.
\]

Summing the above from \( l = -h + 1 \) to \( h - 1 \) and noting \( \sum_{l=-h+1}^{h-1} (h - |l|)^4 = C(h) \), we have the required covariance.
A.3.5 Covariances for joint skewness and kurtosis ratio test

The remaining covariances to be derived for the joint skewness and kurtosis ratio test are $s_{1,h}^{2,3}$, $s_{1,h}^{4,3}$, $s_{1,h}^{3,4}$ and $s_{h,h}^{3,4}$. Using the same method as above,

$$s_{1,h}^{2,3} = \sum \text{cum} \left( x_t^2 \tilde{x}_{t-l}^3 \right)$$
$$= \sum \left[ \left( x_t^2 \tilde{x}_{t-l}^3 \right) + 6 \left( x_t \tilde{x}_{t-l}^2 \right) \left( x_t \tilde{x}_{t-l} \right) + 3 \left( x_t \tilde{x}_{t-l} \right) \left( \tilde{x}_{t-l} \right) \right]$$
$$= h \left[ \kappa_5 + (3h + 6) \kappa_3 \sigma^2 \right]$$

For $s_{1,h}^{4,3}$, applying the indecomposable partition method for $\text{cum} \left( x_t^4 \tilde{x}_{t-l}^3 \right)$ yields

$$\left( x_t^4 \tilde{x}_{t-l}^3 \right) + 3 \left( x_t^2 \tilde{x}_{t-l}^2 \right) \left( x_t \tilde{x}_{t-l} \right) + 6 \left( x_t^2 \tilde{x}_{t-l} \right) \left( \tilde{x}_{t-l} \right) + 12 \left( x_t \tilde{x}_{t-l} \right) \left( \tilde{x}_{t-l} \right)$$
$$+ 4 \left( x_t \tilde{x}_{t-l} \right) \left( \tilde{x}_{t-l} \right) + 12 \left( x_t^2 \tilde{x}_{t-l} \right) \left( x_t \tilde{x}_{t-l} \right) + 18 \left( x_t \tilde{x}_{t-l} \right) \left( \tilde{x}_{t-l} \right)$$
$$+ 12 \left( x_t^2 \tilde{x}_{t-l} \right) \left( \tilde{x}_{t-l} \right) + 18 \left( x_t^2 \tilde{x}_{t-l} \right) \left( x_t \tilde{x}_{t-l} \right)$$
$$+ 36 \left( x_t \tilde{x}_{t-l} \right) \left( \tilde{x}_{t-l} \right) + 36 \left( x_t \tilde{x}_{t-l} \right) \left( x_t \tilde{x}_{t-l} \right) \left( \tilde{x}_{t-l} \right)$$
$$= \kappa_7 + (3h + 18) \kappa_5 \sigma^2 + 34 \kappa_4 \kappa_3 + (30h + 72) \kappa_3 \sigma^4.$$ 

Multiplying the above result by a factor of $h$ gives rise to $s_{1,h}^{4,3}$. $s_{1,h}^{3,4}$ is a mirror image to $s_{1,h}^{4,3}$, so we have

$$s_{1,h}^{3,4} = \sum \left[ \left( x_t^2 \tilde{x}_{t-l}^4 \right) + 6 \left( x_t^2 \tilde{x}_{t-l}^3 \right) \left( \tilde{x}_{t-l} \right) + 3 \left( x_t \tilde{x}_{t-l} \right) \left( x_t \tilde{x}_{t-l} \right) + 12 \left( x_t \tilde{x}_{t-l} \right) \left( \tilde{x}_{t-l} \right) \right]$$
$$+ 4 \left( x_t \tilde{x}_{t-l} \right) \left( x_t \tilde{x}_{t-l} \right) + 12 \left( x_t \tilde{x}_{t-l} \right) \left( x_t \tilde{x}_{t-l} \right) + 18 \left( x_t \tilde{x}_{t-l} \right) \left( \tilde{x}_{t-l} \right)$$
$$+ 12 \left( x_t \tilde{x}_{t-l} \right) \left( \tilde{x}_{t-l} \right) + 18 \left( x_t \tilde{x}_{t-l} \right) \left( x_t \tilde{x}_{t-l} \right)$$
$$+ 36 \left( x_t \tilde{x}_{t-l} \right) \left( \tilde{x}_{t-l} \right) + 36 \left( x_t \tilde{x}_{t-l} \right) \left( x_t \tilde{x}_{t-l} \right) \left( \tilde{x}_{t-l} \right)$$
$$= h \left[ \kappa_7 + (6h + 15) \kappa_5 \sigma^2 + (4h + 30) \kappa_4 \kappa_3 + (66h + 36) \kappa_3 \sigma^4 \right]$$

Replacing the $x_t$ in the above with $\tilde{x}_t$ yields the required $s_{h,h}^{3,4}$:

$$s_{h,h}^{3,4} = \sum \left[ \left( h - |l| \right) \kappa_7 + (9h (h - |l|) + 12 (h - |l|)^2) \kappa_5 \sigma^2 \right.$$
$$+ (4h (h - |l|) + 30 (h - |l|)^2) \kappa_4 \kappa_3$$
$$+ (30h^2 (h - |l|) + 36h (h - |l|)^2 + 36 (h - |l|)^3) \kappa_3 \sigma^4 \right]$$
$$= h^2 \kappa_7 + \left[ 9h^3 + 12A_h \right] \kappa_5 \sigma^2 + \left[ 4h^3 + 30A_h \right] \kappa_4 \kappa_3$$
$$+ \left[ 30h^4 + 36hA_h + 36B_h \right] \kappa_3 \sigma^4,$$

and this completes the proofs.
A.4 The Multinomial Theorem

Finally, we briefly remark here that there is an alternative approach to deriving the required weighting matrices $(S_0)$. As noted before, $\text{cov}(\bar{x}_t^p, \bar{x}_t^q) = E(\bar{x}_t^{p+q}) - E(\bar{x}_t^p)E(\bar{x}_t^q)$, where $1 \leq p, q \leq 4$, we can use the Multinomial Theorem to evaluate

$$E(\bar{x}_t^{p+q}) = \sum_{k_1, \ldots, k_h} \frac{(p+q)!}{k_1! \cdots k_h!} E(\bar{x}_t^{k_1}) \cdots E(\bar{x}_t^{k_h}),$$

(56)

where the summation is taken over all sequences of nonnegative integer indices $k_1$ through $k_h$ such that $k_1 + \cdots + k_h = p + q$. Since $p + q$ can be as high as 8 for the higher order ratio tests, the task of writing (56) in terms of higher order moments of $x_t$ is also non-trivial.

References


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## Table 1. Empirical test sizes: all required moments exist

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5,000 relications are used to calculate the empirical test size. N, h and a(%) are the number of observations in a test, number of periods and theoretical test size in percentage respectively. Skew, Kurt and Joint are respectively the skewness ratio test, kurtosis ratio test and their joint test. McLi is the autocorrelation test of squares by McLeod and Li (1983).
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5,000 relications are used to calculate the empirical test size. N, h and a(%) are the number of observations in a test, number of periods and theoretical test size in percentage respectively. Skew, Kurt and Joint are respectively the skewness ratio test, kurtosis ratio test and their joint test. McLi is the autocorrelation test of squares by McLeod and Li (1983).
Table 3. Basic statistics

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sd, sk and ku are the standard deviation, skewness, excessive kurtosis respectively. VaRL and VaRR are respectively the 99% VaR on the left and right tails of the distribution. Note that all variables are scaled so that if the returns are IID, the figures should remain unchange with regard to h.
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Table 6. Tests on returns filtered by GARCH-Normal

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|      | UK   |      |      |      |       |     |     |     |
| 1    |      |      |      |      |       |     |     |     |
| 5    | 1.50 | 3.95 | 0.00 | 0.01 | 0.03  | 1.06| -1.04| 6.01 |
| 10   | 4.38 | 7.16 | 0.02 | 0.03 | 0.17  | 1.09| -0.98| 6.81 |
| Germany | |      |      |      |       |     |     |     |
| 1    |      |      |      |      |       |     |     |     |
| 5    | 5.62 | 3.63 | 1.18 | 0.85 | 1.18  | 1.05| 2.26 | 74.04|
| 10   | 14.09| 4.00 | 0.48 | 0.75 | 0.75  | 1.11| 2.26 | 82.67|
| Japan |      |      |      |      |       |     |     |     |
| 1    |      |      |      |      |       |     |     |     |
| 5    | 5.24 | 3.62 | 0.47 | 0.02 | 0.61  | 1.05| -0.67| 3.75 |
| 10   | 9.19 | 6.66 | 0.26 | 0.48 | 1.45  | 1.12| -0.36| 6.81 |

h refers to the number of periods. LB and McLi are respectively the Ljung-Box and McLeod-Li tests. Skew, Kurt and Joint are respectively the skewness ratio test, kurtosis ratio test and their joint test. m2, k3 and k4 are respectively variance, skewness and kurtosis. There are 4,160 observations in each time series and the reported test statistics are Chi square test statistics. Dark (light) shade indicates significance at 1% (5%) level.
### Table 7. Tests on returns filtered by APARCH-Normal

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h refers to the number of periods. LB and McLi are respectively the Ljung-Box and McLeod-Li tests. Skew, Kurt and Joint are respectively the skewness ratio test, kurtosis ratio test and their joint test. m2, k3 and k4 are respectively variance, skewness and kurtosis. There are 4,160 observations in each time series and the reported test statistics are Chi square test statistics. Dark (light) shade indicates significance at 1% (5%) level.
### Table 3. Basic statistics

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**Figure 1. A scatter plot of standardized VaRL against skewness**

Horizontal axis refers to the skewness and vertical axis refers to the VaR on the left tail of the distribution.
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