A Generalized Arbitrage-Free Nelson-Siegel Term Structure Model with Macroeconomic Fundamentals

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Abstract

This paper proposes a generalized arbitrage-free macro finance term structure model with both Nelson-Siegel latent yield factors and observable macro factors. Two subclasses are derived: spanned and unspanned models. In the spanned model, the yields are determined by both the Nelson-Siegel yield factors and macro variables. In the unspanned model, the yields are not spanned by macro variables but are only determined by the three Nelson-Siegel yield factors, although the Nelson-Siegel yield factors and macro variables interact in the state dynamics. Compared with existing models, the unspanned model is not only theoretically appealing but also proved to be empirically encouraging. We show that the new model improves predictive accuracy for long term yields than models with Nelson-Siegel yield factors only, with or without no-arbitrage restrictions; it can also improve forecast performance for nearly the whole yield curve than reduced-form dynamic Nelson-Siegel models, with or without macro factors. We also show the usefulness of this type of model in risk premia decomposition and explaining excess returns of yields.

Keywords: Yield curve, term structure of interest rates, macroeconomic fundamentals, factor models; state-space models

\textit{JEL Classification:} G1, E4, C5

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1 Introduction

Interest rate term structure or the yield curve is of fundamental importance to financial markets, monetary policy and fiscal policy. Substantial research efforts have been devoted to modelling the dynamics of the yield curve. While the goals of modelling are similar: goodness-of-fit, forecast accuracy; the approaches of modelling vary widely in the literature: reduced-form factor analysis versus no-arbitrage approach, and models with yield information versus models with macro information, etc.

Since the yield curve displays certain cross-sectional relationships along time and yields of different maturities comove closely from one another, it is both efficient and parsimonious to exploit the cross-sectional relationship and summarize the yield curve in a few factors. Litterman and Scheinkman (1991) find that three factors can explain up to 97% of all variations in the yield curve. In the literature, methods of extracting factors relying on yield curve information can be largely categorized into two ways: one is to impose no-arbitrage restrictions and exploit the cross-equation restrictions of bond pricing in an efficient market; the other is statistical interpolation of the cross-sectional relationship in the yield curve, such as principle component analysis, Nelson-Siegel (1987) interpolation, etc. In contrast to the first type of models which have theoretical restrictions, the second type can be regarded as reduced-form models.

The first approach of modelling the yield curve based on the assumption of no arbitrage has been pioneered by Vasicek (1977), Cox, Ingersoll and Ross (1985). Various specific models have since been developed in the no-arbitrage framework afterwards. For the ease of computation, the type of affine term structure models (denoted as ATSM) with closed-form bond pricing solutions have gained much popularity. Duffie and Kan (1996) derive a three factor affine term structure model which encompasses various previous affine term structure models. Dai and Singleton (2000) make a thorough specification analysis of three-factor affine term structure models. These models typically consider a handful of unobservable factors to explain the entire yield curve. However, estimation and identification of the risk price parameters are known to be particularly difficult without further restrictions. Many of these models exhibit poor empirical performance especially when forecasting future yields (Duffee, 2002). Among these models, the essentially affine class with constant volatility is proved to deliver better forecast than other types of models where stochastic volatility is considered. The models that we discuss later are all under constant volatility as well.

Among the second approach, Nelson-Siegel interpolation is extremely popular due to its goodness-of-fit, parsimony, and the implied conforming behavior of long-term yields. Diebold and Li (2006)
extend this model to dynamic Nelson-Siegel model (denoted as DNS) and find that (1) it is simple and stable to estimate, (2) it is quite flexible and fit well, and (3) it forecasts well. Effectively the DNS representation is a dynamic model with three Nelson-Siegel latent factors, where the extracted latent factors can be labeled as “level”, “slope” and “curvature” and are close to their empirical counterparts or the first three factors extracted from principle component analysis. However, the Nelson-Siegel model is lack of theoretical foundation to rule out riskless arbitrage opportunities which are important in efficient financial markets (Filipovic, 1999, Diebold, Piazzesi and Rudebusch, 2005).

Christensen, Diebold and Rudebusch (2007, 2009) derive an affine arbitrage-free class of Nelson-Siegel term structure models (denoted as AFNS) which address the theoretical weakness of the reduced-form Nelson-Siegel model and the empirical problem of affine term structure model. In this class of models, the three Nelson-Siegel yield factors and their traditional loadings are derived under risk-neutral dynamics of the underlying state vector. The arbitrage-free restrictions are imposed to the yield equations as additional constant terms which were absent from the reduced-form model. Models of this framework not only enjoys the parsimony and flexibility of the reduced-form Nelson-Siegel model, but also rules out arbitrage opportunity under the assumed setting. Further, the authors show that, compared to the DNS model, no-arbitrage restrictions help to improve significantly the goodness-of-fit and forecasting performance of long-term yields. Compared to the preferred specification of ATSM in Duffee (2002), their model performs better in general for yields with different maturities at various forecast horizons.

The above models discussed so far, ATSM, DNS and AFNS, all rely on yield curve information to extract factors that determine the dynamics and span the space of the yield curve. However, we know that yields at different maturities do not stand alone from the macroeconomy and its dynamics. Fisher equation (Fisher, 1930) states that in equilibrium nominal interest rate and inflation should have one-to-one relationship. In modern economies, central banks influence the short term interest rate through open market operations by reacting to economic conditions such as unemployment rate and inflation (Taylor, 1993) and even to a large amount of macroeconomic indicators (Bernanke and Boivin, 2003, Bernanke, Eliasz and Boivin, 2005). Since long-term interest rates are weighted expected short term interest rate adjusted for risks, the state and dynamics of the economy and the monetary policy transmission mechanism are useful in understanding the yield curve movements at proper frequencies. On the other hand, due to the limited information available for macroeconomic data, in order to investigate the dynamics and structure of the macroeconomy,
it would be very helpful to study the joint dynamics of the yield curve and the macroeconomy in an integrated framework, because rich data existing in the bond market can provide valuable information about market expectations of the underlying macroeconomy.

As a seminal work in the macro-finance literature, Ang and Piazzesi (2003) incorporate macroeconomic factors and yield factors in a vector autoregressive framework under no-arbitrage restrictions to study the interactions between the yield curve and the macroeconomy. They find that macro factors explain a bulk share of variations in the short to medium part of the yield curve, while latent yield factors account for most of movements for the long end. However, as the size of state vector increases, the curse of dimensionality imposes severe estimation challenge and results in some undesirable restrictions in the structure, such as unidirectional macro to yield linkage, have to be made. The identification of risk price is also problematic. Later works of such macro-finance affine term structure models (denoted as AFMA) suffers similar difficulties. Favero, Niu and Sala (2010) find that it is hard to combine latent yield factors and macro factors in such an affine term structure model to deliver better forecast than comparable models without macro factors or without the restrictions.

Closely related to Ang and Piazzesi (2003), Diebold, Rudebusch and Aruoba (2006) propose a reduced-form state-space framework to incorporate Nelson-Siegel yield factors and macroeconomic variables (denoted as NSMA)) in a standard VAR. The flexibility of the reduced-form framework enables mutual influences between the latent yield factors and macroeconomic variables in the VAR. Their paper finds strong evidence of the effects of macro variables on future movements in the yield curve and evidence for a reverse influence as well. However, they do not add no-arbitrage restrictions for yield curve, which are likely to hold at monthly frequency in an efficient financial market such as that in the US. This may cause theoretical inconsistency in the model and estimation.

To overcome the above stressed shortcomings of the macro-finance affine term structure model (AFMA) and the reduced-form Nelson-Siegel model with macro factors (NSMA), we propose a unified modelling framework to integrate the Nelson-Siegel yield factors and macro factors under the premise of absence of arbitrage. This can be regarded as an enlarged framework of the affine arbitrage-free class of Nelson-Siegel term structure models (AFNS) proposed by Christensen, Diebold and Rudebusch (2007, 2008). This enlarged class of models enjoy the features of AFNS models, i.e., parsimony and flexibility of Nelson-Siegel factors, and theoretical rigors of no-arbitrage conditions. More importantly, explicit macroeconomic fundamentals are no more absent. The joint dynamics of the yield curve and the macroeconomy can be analyzed within this framework under
no-arbitrage restrictions but without as much of the curse of dimensionality encountered in an affine term structure model.

This paper is organized as follows. Section 2 first reviews the various related modelling approaches of the yield curve. Section 3 proceeds to develop our model, which can be divided into two sub-classes: the unspanned versus spanned model, depending on whether the macro factors span the yield curve. The unspanned model will be focused as it enjoys more parsimony and is supported by empirical evidence. Section 4 summarizes the data. Section 5 discusses the estimation and forecast procedure and evaluate empirical results. Section 6 concludes.

2 A general framework of dynamic term structure models

The general framework of dynamic term structure models under constant volatility can be conveniently represented in a state-space form as follows,

\[ y_{t,n} = -\frac{1}{n} (A_n + B_n' X_t) + \varepsilon_{t,t+n} \quad \varepsilon_{t,t+n} \sim i.i.d.N(0, \sigma_n^2) \]

\[ X_t = \mu + \Phi X_{t-1} + v_t \quad v_t \sim i.i.d.N(0, \Omega) \]

Equation (1) is the measurement equation where \( y_{t,n} \) denotes the yield-to-maturity at time \( t \) of a zero-coupon bond maturing \( n \) periods ahead. At each point of time, there exists a set of yields with different maturities, and they are assumed to be determined by a set of state variables in the vector \( X_t \). The state equation (2) describes the dynamics of \( X_t \) which follows a VAR process.

Since the relationship between the continuously compounded yield-to-maturity and the price of a zero-coupon bond is

\[ y_{t,n} = -\frac{\log P_{t,t+n}}{n}, \]

equation (1) implies that the price of zero coupon bond is exponentially affine in the state vector \( X_t \),

\[ P_{t,n} = \exp(A_n + B_n' X_t). \]

Hence, the measurement equation (1) implicitly defines the pricing formula of zero-coupon bonds of different maturities with respect to factors.

As is discussed in Favero, Niu and Sala(2010), numerous dynamic yield curve models with constant volatility can be cast into the above state-space representation. Classifications of different models can be made along two dimensions: no-arbitrage models versus reduced form models with
cross-sectional interpolation; yield factor models versus mixed factor models or macro factor models. Distinction of models along the first dimension is reflected in the measurement equation on how the factor loadings are defined. To exploit cross-sectional relationship between yields, no-arbitrage restrictions and pure statistical interpolation are common methods to shrink the parameter space. Among the first group, affine term structure models are widely used where the factor loadings $A_n$ and $B_n$ are constrained by cross-equation restrictions to rule out arbitrage opportunities. Under no-arbitrage restrictions, the measurement equations not only achieve parsimony in the parameter setting of the measurement equation, but the model is also theoretically consistent. Among the second class, Nelson-Siegel factor interpolation is popular for its goodness of fit and parsimony. Difference in modelling along the second dimension is embodied in the information set contained in the state vector $X_t$. In the finance literature and professional applications, $X_t$ is often treated as latent factors extracted from the yield curve. In macroeconomic research and macro-finance literature, $X_t$ often contains macro factors in addition to yield factors to capture the joint relationship and dynamics between the macro economy and yield curve.

In the following, we first review some popular modelling approaches in this framework and the recent convergence along the first dimension, the affine arbitrage-free class of Nelson-Siegel model (Christensen, Diebold and Rudebusch, 2009, 2010). Then we propose a more generalized model to enable enlarged information set in the “converging” model, – the affine arbitrage-free Nelson-Siegel model with macro fundamentals.

### 2.1 The arbitrage-free affine term structure Model

In efficient financial market, no-arbitrage condition is crucial. The arbitrage-free term structure model is pioneered by Vasicek (1977). Then numerous models and specifications of the interest rate term structure under no-arbitrage conditions are proposed and studied. In order to facilitate analysis and ease the computational burden, the affine class of arbitrage-free term structure models that have closed-form solutions have gained much grounds. A discrete-time framework of affine term structure models with constant volatility has been popularized following Ang and Piazzesi (2003). In this framework, the state vector $X_t$ follows a VAR process as described in equation (2) \[ X_t = \mu + \Phi X_{t-1} + v_t, \quad v_t \sim i.i.d.N(0, \Omega). \]

The risk price $\Lambda_t$ is assumed to be associated with the state vector $X_t$ in a linear form \[ \Lambda_t = \lambda_0 + \lambda_1 X_t. \]
Given the short rate affine on the state vector,

\[ r_t = \delta_0 + \delta'_1 X_t \]  

the coefficients in the measurement equations of long term yields can be derived iteratively as functions of the basic parameters in the state dynamics, short rate equation and risk pricing equation. That is, for \( n \geq 1 \):

\[ A_{n+1} = A_n + B'_n (\mu - \Omega \lambda_0) + \frac{1}{2} B'_n \Omega B_n - \delta_0 \]  

\[ B'_{n+1} = B'_n (\phi - \Omega \lambda_1) - \delta'_1 \]

with \( A_1 = -\delta_0 \) and \( B_1 = -\delta_1 \).

One can show that under the risk-neutral measure, there exists a VAR of \( X_t \) such that

\[ X_t = \tilde{\mu} + \tilde{\Phi} X_{t-1} + \tilde{v}_t, \]

where the relationship between the parameters under the risk-neutral and physical processes take the following form

\[ \tilde{\mu} = \mu - \Omega \lambda_0 \]
\[ \tilde{\Phi} = \Phi - \Omega \lambda_1 \]

Under the risk-neutral measure, the pricing equation can be equivalently derived as

\[ A_{n+1} = A_n + B'_n \tilde{\mu} + \frac{1}{2} B'_n \Omega B_n - \delta_0 \]  

\[ B'_{n+1} = B'_n \tilde{\Phi} - \delta'_1 \]

With a three-factor specification, the estimated yield factors from the above model are fairly similar to the three Nelson-Siegel factors that we are going to discuss next. Compared with the dynamic Nelson-Siegel model, it rules out arbitrage opportunities in bond pricing. Parsimony of parameters in the measurement equations are ensured by the cross-equation restrictions. However, without further restrictions on the risk price \( \Lambda_t \), the model is likely over-parameterized and leads to poor fit and forecast for yields (Favero, Niu and Sala, 2010).

2.2 The dynamic Nelson-Siegel model in reduced-form

The dynamic Nelson-Siegel model (DNS) is proposed in Diebold and Li (2006). Three factors, extracted à la Nelson and Siegel (1987) are assumed to follow an unrestricted VAR.
We denote the three yield factors as \( L_t, S_t, \) and \( C_t \) such that \( X_t = [L_t \ S_t \ C_t]' \). Equation (1) in this model is specified as:

\[
y_{t,n} = L_t + S_t \left( \frac{1 - e^{-\gamma n}}{\gamma n} \right) + C_t \left( \frac{1 - e^{-\gamma n}}{\gamma n} - e^{-\gamma n} \right) + \varepsilon_{t,t+n}
\]  

(9)

Corresponding to the representation of equation (1), the coefficients take the following particular form:

\[
A_n = 0 \text{ and } B'_n = \left( -n, \ -\frac{1 - e^{-\gamma n}}{\gamma}, \ n e^{-\gamma n} - \frac{1 - e^{-\gamma n}}{\gamma} \right).
\]  

(10)

The dynamics of \( X_t \) is assumed to follow an unrestricted VAR(1) as specified in the state dynamic equation (2).

With the above imposed structure of factor loadings, the three yield factors \( L_t, S_t, \) and \( C_t \) have a natural interpretation as “level”, “slope” and “curvature”. As the yield maturity \( n \) varies, the loading on \( L_t \) is one for all yields, i.e. its impact on the whole yield curve is equal. Hence \( L_t \) can be interpreted as the long-term factor determining the level of the term structure. The empirical counterpart of the level is the long term yield of the yield curve. The loading on \( S_t \) is a monotone function that starts at one when maturity approaches zero, and decays to zero as \( n \) tends to infinity. Empirically, \( S_t \) correlates negatively with the spread between long and short yields, i.e. the slope of the yield curve. The third factor \( C_t \) has a hump-shaped loading which starts at zero, increases and then decays to zero, with the speed of decay and location of maximal hump determined by the parameter \( \gamma \). \( C_t \) is close to the empirical measure of curvature of the yield curve: \( (2 \times 2 \text{ year yield} - (10 \text{ year yield} + 3 \text{ month yield})) \).

This model enjoys the advantage of parsimony, goodness of fit, and superior forecast performance (Diebold and Li, 2006). However, DNS fails on an important theoretical dimension: it doesn’t impose the restrictions necessary to rule out opportunities for risk free arbitrage.

2.3 The arbitrage-free Nelson-Siegel Model

Affine arbitrage-free Nelson-Siegel (AFNS) class model is proposed in Christensen, Diebold and Rudebusch (2010). The model imposes the Nelson-Siegel functional coefficient structure on the canonical representation of affine models, which is derived under continuous-time framework. Zeng and Niu (2010) depict the model in discrete-time. We follow the discrete-time representation in this paper to make the discussion and notation consistent among different models. The main assumption in the affine arbitrage-free Nelson-Siegel model is that, under the risk-neutral measure,
the mean-reversion coefficient matrix of the state dynamics follows

\[
\tilde{\Phi} = \begin{bmatrix}
1 & 0 & 0 \\
0 & e^{-\gamma} & \gamma e^{-\gamma} \\
0 & 0 & e^{-\gamma}
\end{bmatrix}.
\]

This guarantees that the factor loadings of the three yield factors are the same as in the dynamic Nelson-Siegel model,

\[
B'_n = \left(-n, \frac{1-e^{-\gamma n}}{\gamma}, ne^{-\gamma n} - \frac{1-e^{-\gamma n}}{\gamma}\right).
\]

Further, in order to exclude arbitrage opportunities, the measurement equations of yields have to be adjusted by a constant term. Hence, different from the original Nelson-Siegel model, \( A_n \neq 0 \).

With \( A_1 = -\delta_0 \) and \( B_1 = -\delta_1 = -\left[1 \quad \frac{1-e^{-\gamma}}{\gamma} \quad \frac{1-e^{-\gamma}}{\gamma} - \gamma e^{-\gamma}\right]' \) in the short rate equation, and define \( \tilde{\mu} = 0 \) as its canonical form representation, \( A_n \) satisfies the following difference equation for \( n > 1 \)

\[
A_n = A_{n-1} + \frac{1}{2} B'_{n-1} \Omega B_{n-1} - \delta_0
\]  

Although the arbitrage free Nelson-Siegel (AFNS) model is appealing as a unifying framework that links the traditional Nelson-Siegel model to the affine no-arbitrage term structure model, it is significantly restrictive. Because it is only consistent with the presence of exactly three state variables and does not allow for the inclusion of state variables of different nature, such as observable macro variables.

### 2.4 An affine arbitrage-free macro finance model with latent yield factors and macro factors

Ang and Piazzesi (2003) propose a no-arbitrage macro-finance affine term structure model to capture the joint dynamics of three latent yield factors and two observable macro factors (inflation and real activity). With the risk price a linear function of state vector as in equation (3), the coefficients of the measurement equation \( A_n \) and \( B'_n \) satisfy the same recursive relationship as in equations (5) and (6).

Although this model is arbitrage free and contains macro fundamentals, the parsimony of no-arbitrage restrictions on the measurement equations is severely reduced by the increasing of dimensionality. In particular, the identification of risk price parameters is problematic, and estimation is technically challenging in order to find global optimum when implementing maximum likelihood estimation.
2.5 A reduced form macro finance model with Nelson-Siegel and macro factors

Diebold, Rudebusch, and Aruoba (2006) offer a reduced form joint vector autoregressive framework by combining popular Nelson-Siegel factors with observable macro factors. Its state vector is

\[ X_t = (L_t, S_t, C_t, CU_t, FFR_t, INFL_t)', \]

where the first three variables are Nelson Siegel factors and the other three are macro factors: capacity utilization, federal funds rate and inflation. We denote this model as NSMA model henceforth, where the measurement equations can be expressed as equation (1) and the corresponding coefficients for \( A_n \) and \( B_n' \) follow

\[ A_n = 0 \text{ and } B_n' = \begin{pmatrix} -n, & -\frac{1-e^{-\gamma n}}{\gamma}, & ne^{-\gamma n} - \frac{1-e^{-\gamma n}}{\gamma}, & 0, & 0, & 0 \end{pmatrix}. \]

This model not only enjoys the flexibility of jointly capturing the dynamics of yield curve and macro economy, but also maintains goodness-of-fit and a large extent of parsimony of the Nelson-Siegel model. However, as the authors recognize, it would be theoretically desirable to impose no-arbitrage restrictions for consistency of pricing.

We see from the above discussion that to study the joint dynamics of the yield curve and the macro economy, a desired model should combine the parsimony as in the Nelson-Siegel interpolation and the theoretical restrictions to rule out arbitrage opportunities. But the emerging affine arbitrage-free Nelson-Siegel class of models in Christensen, Diebold and Rudebusch (2010) have not opened up the space for observable macroeconomic variables. To overcome this limitation, we propose in the following section a generalized macro finance framework of term structure with Nelson-Siegel factors and observable macro factors under no-arbitrage restrictions. Figure 1 shows the classification of the above models along the two dimensions discussed. Our model is positioned as the encompassing model of the Nelson-Siegel yield factors, macro factors and no-arbitrage restrictions.

Figure 1: Model classification
3 Nelson-Siegel and macro factors in a generalized affine arbitrage-free macro finance framework

For mixed (i.e. latent and observable) factor ATSM under discrete time, Pericolo and Taboga (2009) discuss in details the specifications and classifications. Here, for the affine arbitrage-free Nelson-Siegel model with macro variables (AFNSMA henceforth), we distinguish two subclasses. The first class is unspanned model, where the macro variables do not span the yield curve such that yields are still determined only by the three Nelson-Siegel yield factors and a constant adjustment term. This class of model corresponds to the reduced form macro-finance model depicted by Diebold, Rudebusch and Aruoba (2006), but can exclude arbitrage opportunity. The second class is spanned model, where macro factors have non-zero factor loadings on the yield curve. In both
types of models, the yield factors and macro factors interact each other in the dynamic state equation under the physical measure.

3.1 Model derivation

3.1.1 Unspanned model

Proposition 1:
Assume that the one period risk-free rate is affine on a vector of variables \( X_t \), which contains three latent Nelson-Siegel factors \( X_t^u \) and observable variables \( X_t^o \).

\[
  r_t = \delta_0 + \delta_1^u X_t^u + \delta_1^o X_t^o
\]

where \( \delta_1^u = \begin{bmatrix} 1 & \frac{(1-e^{-\gamma})}{\gamma} & \frac{(1-e^{-\gamma})}{\gamma} - \gamma e^{-\gamma} \end{bmatrix} \), \( \delta_1^o = 0 \) and \( X_t^u = (X_1^u, X_2^u, X_3^u)' \). Define \( A_1 = -\delta_0 \) and \( B_1 = -\delta_1 \).

The state variables are described by the following VAR under the risk-neutral Q-measure

\[
  \begin{bmatrix} X_t^u \\ X_t^o \end{bmatrix} = \tilde{\mu} + \begin{bmatrix} \tilde{\Phi}^u \\ \tilde{\Phi}^{uo} \end{bmatrix} \begin{bmatrix} X_{t-1}^u \\ X_{t-1}^o \end{bmatrix} + \tilde{\nu}_t
\]

where

\[
  \tilde{\Phi}^u = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-\gamma} & \gamma e^{-\gamma} \\ 0 & 0 & e^{-\gamma} \end{bmatrix}
\]

Then the zero-coupon bond price is given by

\[
  P_{t,n} = \exp(A_n + B_n^u X_t^u)
\]

with \( B_n^u \) the Nelson-Siegel factor loadings

\[
  B_n^u = [-n, -\frac{1-e^{-\gamma n}}{\gamma}, ne^{-\gamma n} - \frac{1-e^{-\gamma n}}{\gamma}]'
\]

\[
  B_n^o = 0
\]

and \( A_n \) satisfies the difference equations:

\[
  A_{n+1} = A_n + B_n' \tilde{\mu} + \frac{1}{2} B_n^u \Omega^u B_n^u + A_1 \text{ for } n > 1.
\]

\( \Omega^u \) is the upper left block of the variance-covariance matrix corresponding to innovations to Nelson-Siegel factors in the state dynamics.

Proof:
By the ODE of $B_n$ as in equation (8), we have

$$B_{n+1} = \tilde{\Phi}' B_n - \delta_1,$$

that is

$$\begin{bmatrix} B_{n+1}^u \\ B_{n+1}^o \end{bmatrix} = \begin{bmatrix} \tilde{\Phi}' & \tilde{\Phi}'^o \\ 0 & \tilde{\Phi}'^o \end{bmatrix} \begin{bmatrix} B_n^u \\ B_n^o \end{bmatrix} - \begin{bmatrix} \delta_1^u \\ \delta_1^o \end{bmatrix}.$$

Because $B_1^o = -\delta_1^o = 0$,

$$B_{n+1}^o = \tilde{\Phi}'^o B_n^o = 0 \ \forall \ n > 0.$$

So we have

$$B_{n+1}^u = \tilde{\Phi}'^u B_n^u - \delta_1^u \ \forall \ n > 0.$$

Iteratively, this can be written as

$$B_n^u = B_1^u \sum_{k=0}^{n-1} \left( \tilde{\Phi}'^u \right)^k \ \forall \ n > 1.$$

Because

$$\left( \tilde{\Phi}'^u \right)^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-\gamma} & e^{-\gamma} \\ 0 & 0 & e^{-\gamma} \end{bmatrix}^k = e^{-k\gamma} \begin{bmatrix} e^\gamma & 0 & 0 \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix}^k = e^{-k\gamma} \begin{bmatrix} e^{k\gamma} & 0 & 0 \\ 0 & 1 & k\gamma \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-k\gamma} & k\gamma e^{-k\gamma} \\ 0 & 0 & e^{-k\gamma} \end{bmatrix},$$

$$\sum_{k=0}^{n-1} \left( \tilde{\Phi}'^u \right)^k = \begin{bmatrix} n & 0 & 0 \\ 0 & \sum_{k=0}^{n-1} e^{-k\gamma} & \sum_{k=0}^{n-1} k\gamma e^{-k\gamma} \\ 0 & 0 & \sum_{k=0}^{n-1} e^{-k\gamma} \end{bmatrix} = \begin{bmatrix} n & 0 & 0 \\ 0 & 1-e^{-n\gamma} & 1-e^{-n\gamma} \\ 0 & 0 & \frac{1-e^{-n\gamma}}{1-e^{-\gamma}} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{e^{-\gamma}-e^{-n\gamma}}{1-e^{-\gamma}} - \frac{(n-1)e^{-n\gamma}}{1-e^{-\gamma}} \\ \frac{1-e^{-n\gamma}}{1-e^{-\gamma}} \end{bmatrix}.$$
it follows that

\[
B^u_n = - \left\{ \sum_{k=0}^{n-1} \left( \tilde{\Phi}^u \right)^k \right\}' \delta_1
\]

\[
= - \begin{bmatrix}
  n & 0 \\
  0 & \frac{1-e^{-\gamma}}{1-e^{-\gamma}} \\
  0 & \gamma \left[ \frac{e^{-\gamma}-e^{-n\gamma}}{(1-e^{-\gamma})^{n-1}} - \frac{(n-1)e^{-n\gamma}}{1-e^{-\gamma}} \right] \\
\end{bmatrix}
\begin{bmatrix}
  1 \\
  \frac{1-e^{-\gamma}}{\gamma} - e^{-\gamma} \\
\end{bmatrix}
\]

\[
= \left[ -n, -\frac{1-e^{-\gamma n}}{\gamma}, ne^{-\gamma} - \frac{1-e^{-\gamma n}}{\gamma} \right]'.
\]

Since \( B^u_n = 0 \), we have

\[
B'_n \Omega B_n = B^u_n \Omega^u B^u_n
\]

where \( \Omega^u \) is the upper-left 3 \( \times \) 3 block of the variance-covariance matrix \( \Omega \) of the state innovation in the VAR.

Hence, the constant adjustment term becomes

\[
A_{n+1} = A_n + B^u_n \tilde{\mu}^u + \frac{1}{2} B^u_n \Omega^u B^u_n + A_1.
\]

### 3.1.2 Spanned model

**Proposition 2:**

Assume that the one period risk-free rate is affine on a vector of variables \( X_t \), which contains three latent Nelson-Siegel factors \( X^u_t \) and observable variables \( X^o_t \).

\[
r_t = \delta_0 + \delta^u_t \mu^u + \delta^o_t \mu^o
\]

where \( \delta^u = \begin{bmatrix}
  1 & \frac{1-e^{-\gamma}}{\gamma} & \frac{1-e^{-\gamma}}{\gamma} - \gamma e^{-\gamma} \end{bmatrix}' \) and \( X^u_t = (X^1_t, X^2_t, X^3_t)' \).

The state variables are described by the following VAR under the risk-neutral Q-measure

\[
\begin{bmatrix}
  X^u_t \\
  X^o_t \\
\end{bmatrix} = \tilde{\mu} + \begin{bmatrix}
  \tilde{\Phi}^u & \tilde{\Phi}^{uo} \\
  0 & \tilde{\Phi}^o \\
\end{bmatrix} \begin{bmatrix}
  X^u_{t-1} \\
  X^o_{t-1} \\
\end{bmatrix} + \tilde{v}_t,
\]

where

\[
\tilde{\Phi}^u = \begin{bmatrix}
  1 & 0 & 0 \\
  0 & e^{-\gamma} & \gamma e^{-\gamma} \\
  0 & 0 & e^{-\gamma} \\
\end{bmatrix}
\]
Then zero-coupon bond price is given by

$$P_{t,n} = \exp(A_n + B^u_n X^u_t + B^o_n X^o_t)$$

where $B^u_n$ is the Nelson-Siegel factor loadings

$$B^u_n = \left[-n, -\frac{1 - e^{-\gamma n}}{\gamma}, n e^{-\gamma n} - \frac{1 - e^{-\gamma n}}{\gamma}\right]'$$

and $A_n$ satisfies the difference equations:

$$A_{n+1} = A_n + B'_n \tilde{\mu} + \frac{1}{2} B'_n \Omega B_n + A_1 \text{ for } n > 0$$

with $A_1 = -\delta_0$.

**Proof:**

By the ODE of $B_n$, we have

$$B_{n+1} = \Phi' B_n - \delta_1,$$

that is

$$\begin{bmatrix} B^u_{n+1} \\ B^o_{n+1} \end{bmatrix} = \begin{bmatrix} \tilde{\Phi}^u & 0 \\ \tilde{\Phi}^o & \tilde{\Phi}^o \end{bmatrix} \begin{bmatrix} B^u_n \\ B^o_n \end{bmatrix} - \delta_1.$$

Then

$$B^u_{n+1} = \Phi^u B_n - \delta^u_1$$

Hence by the proof of proposition 1, we have that

$$B_n = \left[-n, -\frac{1 - e^{-\gamma n}}{\gamma}, n e^{-\gamma n} - \frac{1 - e^{-\gamma n}}{\gamma}\right]'$$

$$B^o_{n+1} = \tilde{\Phi}^o B^u_n + \tilde{\Phi}^o B^o_n - \delta^o_1,$$

and

$$A_{n+1} = A_n + B'_n \tilde{\mu} + \frac{1}{2} B'_n \Omega B_n + A_1 \text{ for } n > 0$$
3.2 Specification and Identification

For the identification of the latent yield factors with unique parametrization, we restrict $\bar{\mu}^{u} = 0$. One can show that there are other restrictions such as $\mu = 0$ or $\Omega^{u} = I$, which lead to observationally equivalent yields. However, for simplicity in the ODE of pricing coefficients $A_n$, $\bar{\mu}^{u} = 0$ is preferred.

As a special case of the essentially affine class of discrete-time term structure model, the AFNSMA model renders considerable parsimony by directly imposing restrictions on the risk-neutral dynamics, which in turn restrict the implied risk price.

For the unspanned model, the upper three rows of the risk-neutral dynamic coefficient $\tilde{\Phi}$ are mostly restricted to zeros and the only parameter involved is $\gamma$. Further, since the lower parts of $\tilde{\Phi}$ do not appear in the measurement equation coefficients, they are not identified, which also means that the time-varying risk price parameter $\gamma_1$ in equation (3) is not identified. However, this does not affect the identification of other parameters in this model, and inference on risk premia and excess returns can be easily made as will be discussed in the next section.

For the spanned model, the first three columns of $\tilde{\Phi}$ are restricted and $\gamma$ is the only parameter. Unlike the unspanned model, the right columns composing $\tilde{\Phi}^{uo}$ and $\tilde{\Phi}^{o}$ can be identified from the coefficients of $B_n^{o}$ in the measurement equations. Thus, the risk price parameters $\lambda_0$ and $\lambda_1$ can be fully identified from the relationship

\[
\bar{\mu} = \mu - \Omega \lambda_0 \\
\tilde{\Phi} = \Phi - \Omega \lambda_1
\]

such that

\[
\lambda_0 = \Omega^{-1}(\mu - \bar{\mu}) \\
\lambda_1 = \Omega^{-1}(\Phi - \tilde{\Phi})
\]

3.3 Risk premia and excess returns

With macro fundamentals included, we are not only interested in the model’s performance in in-sample fit and out-of-sample prediction, but also in the effects of macro factors on the risk premia and excess returns to the yield curve. To this end, whether the model is spanned or unspanned, we can infer the risk premia and excess returns from the following procedure without deriving the risk parameters $\lambda_0$ and $\lambda_1$.

Risk premium.
If we define the risk premium of bond price as the component to compensate risk, then it can be derived as the difference between the theoretical yield \( y_{t,t+\tau} \) and the yield \( \tilde{y}_{t,t+\tau} \) with zero risk compensation when setting \( \lambda_0 = 0 \) and \( \lambda_1 = 0 \). That is, to define risk premium as

\[
RP_n = y_{t,n} - \tilde{y}_{t,n}.
\]

For the yield without the premium

\[
\tilde{y}_{t,n} = -\frac{1}{n}(\tilde{A}_n + \tilde{B}_n'X_t)
\]

where

\[
\begin{align*}
\tilde{B}_{n+1}' &= \tilde{B}_n'\Phi - \delta_1' \\
\tilde{A}_{n+1} &= \tilde{A}_n + \tilde{B}_n'\mu + \frac{1}{2} \tilde{B}_n'\Omega \tilde{B}_n - \delta_0
\end{align*}
\]

Here, the coefficients can be derived solely based on the parameters of the state equation under the physical measure. Since there is no restrictions in the dynamic coefficient \( \Phi \), so that in general \( \tilde{B}_n^o \neq 0 \) and \( X^o_t \) will have non-zero loadings on \( \tilde{y}_{t,n} \).

Then the risk premium can be calculated as

\[
RP_{t,n} = y_{t,n} - \tilde{y}_{t,n}
\]

\[
= -\frac{1}{n}(A_n + B_n'X_t) + \frac{1}{n}(\tilde{A}_n + \tilde{B}_n'X_t)
\]

\[
= \frac{1}{n} \left[ (\tilde{A}_n - A_n) + (\tilde{B}_n' - B_n')X_t \right]
\]

Although \( B_n^o = 0 \), since \( \tilde{B}_n^o \neq 0 \), the loadings of \( X^o_t \) will be non-zero. This implies that even if the macro variables do not span the yield curve as in the unspanned model, they do have impact on the risk premia.

**Excess return**

The excess return of holding \( n \)-period bond for \( h \) periods relative to an \( h \)-period bond is

\[
x_{t+h}^{n,h} = ln(P_{t+h,n-h}) - ln(P_{t,n}) + lnP_{t,h}
\]

\[
= (A_{n-h} - A_n + A_h) + (B_{n-h})'X_{t+h} - (B_n - B_h)'X_t
\]

For the spanned model, it can be seen from above, both the current and future macro states affect the excess return.
For the unspanned model, as $B_n^u = 0$,

$$rx_{t+h}^n = (A_{n-h} - A_n + A_h) + (B_{n-h}^u)'X_{t+h}^u - (B_n^u - B_h^u)'X_t^u.$$ 

Although the macro variables at time $t$ do not have direct impact from the last term $(B_n^u - B_h^u)'X_t^u$, it affects excess return through its impact on future $X_{t+h}^u$ as

$$X_{t+h} = \sum_{i=0}^{h-1} \Phi^i \mu + \Phi^h X_t.$$ 

Since $\Phi$ is not restricted under the physical measure, $X_t^o$ has impact on $X_{t+h}^u$ if the upper right block of $\Phi^h$ is non zero. Similarly, the macro variables will influence the expected excess returns through its expected impact on $X_{t+h}^u$,

$$E_t(rx_{t+h}^n) = (A_{n-h} - A_n + A_h) + (B_{n-h}^u)'E_tX_{t+h}^u - (B_n^u - B_h^u)'X_t^u.$$ 

### 4 Data and macroeconomic factors

We use US Treasury zero-coupon equivalent yield data with maturities of 3, 6, 9, 12, 18, 24, 30, 36, 48, 60, 72, 84, 96, 108 and 120 months, a total of 15 yield series from August 1972 through September 2010. Among them, the 3 and 6 month yields are converted from the 3 and 6 month Treasury Bill rates of discounted basis, in Federal Reserve’s H.15 release of selected interest rates. The yields from 9 to 120 months are from research data of the Federal Reserve Board, released with the paper of Gurkaynak, Sack and Wright (2007). Both data are at daily frequency, updated constantly. Figure 2 shows the dynamics of this yield curve. For macro variables, we choose growth rate of Industrial Production Index ($IP_{gt}$) and inflation rate of consumer price index for all urban consumers (all items) ($INFL_t$). The IP growth and inflation are calculated as annual log difference $\log(X_t) - \log(X_{t-12})$. These two variables represent the level of real economic activity relative to potential and the inflation rate. They are widely considered to be the minimum set of fundamentals to capture basic macroeconomic information. The sample interval of macroeconomic variables is from August 1972 through September 2010, and are all seasonally adjusted. Figure 3 plots the time series of these three macro variables.

**Figure 2. Yield dynamics (1971:08 - 2010:09)


**Figure 3. Macroeconomic variables (1971:08 - 2010:09)**

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5 Estimation and forecast evaluation

Theoretically, a model combining the popular and parsimonious Nelson-Siegel yield factors, macroeconomic fundamentals and no-arbitrage restrictions is desirable. In odes to evaluate the empirical adequacy of our approach, we estimate and forecast with four different models under comparable
settings. Each of the models represents a modelling approach that we discuss in section 2. Here, we will use our unspanned no-arbitrage Nelson-Siegel macro finance model (AFNSMA) to compare with DNS model, AFNS model and the reduced-form Nelson-Siegel macro finance model (NSMA). We investigate whether there exists superior out-sample performance of our model compared to alternative models.

In choosing the unspanned specification instead of the spanned one, we have the following considerations. First, the unspanned model is more parsimonious compared to the spanned one and is likely to produce better forecast and to have less problem in the identification of the risk neutral dynamic coefficients. Second, this provides a good comparison to the NSMA model in Diebold, Rudebusch and Aruoba (2006), because their model is also an unspanned one in the reduced form. In fact, a spanned model in the reduced form will encounter severe curse of dimension without good restrictions on the factor loadings of macro variables in the measurement equations.

5.1 Estimation methods

As discussed above, each model has a state space representation. Hence, maximum likelihood estimation (MLE) with Kalman filter to extract the latent yield factors is an efficient and consistent estimation method for all models. We initialize the Kalman filter using the unconditional mean and unconditional variance matrix of the state vector, and then maximize the conditional likelihood by numerical algorithms.

5.2 Forecast procedure

We use the recursive estimation and forecasting procedure, and then calculate the forecast root mean squared errors (RMSE) to compare their out-of-sample forecast performance. We compute 1-, 6-, and 12-month-ahead forecast from these four models for the 15 yields. The recursive procedure is as follows: for the first set of forecast, the model is estimated from August 1971 to July 1986, a total of 180 months (15 years) and the out-of-sample 1-, 6-, and 12-month ahead forecast are obtained for August 1986, January 1987 and July 1987 respectively; we expand the previous estimation window to re-estimate the models when we move one-period forward, and another set of forecast is constructed; we repeat this procedure until the last estimation window for the 1-, 6- and 12-month ahead forecast ends in August 2010, March 2010, and September 2009, respectively. For the 1-, 6- and 12-month ahead forecast, we have 290, 285, and 279 forecasts respectively from each model.

For all models, when we have estimated the parameter $\Phi$ and $\mu$ from a sample that ends in
period $t$, the $h$-step ahead forecast for the state variable is \(^{1}\)

$$
\hat{X}_{t+h|t} = \sum_{i=0}^{h-1} \hat{\Phi}^i \hat{\mu} + \hat{\Phi}^h X_t
$$

(14)

For the DNS model and the NSMA model, we obtain $h$-step ahead forecasts for the $n$-maturity yield with the projected factors $h$-step ahead

$$
\hat{y}_{t+h|t} = \hat{L}_{t+h|t} + \hat{S}_{t+h|t} \left( \frac{1 - e^{-\gamma n}}{\gamma n} \right) + \hat{C}_{t+h|t} \left( \frac{1 - e^{-\gamma n}}{\gamma n} - e^{-\gamma n} \right).
$$

For AFNS model and the unspanned AFNSMA model, the $h$-step ahead forecast for yield is

$$
\hat{y}_{t+h,n|t} = \frac{-A_n(\hat{\Theta})}{n} + \hat{L}_{t+h|t} + \hat{S}_{t+h|t} \left( \frac{1 - e^{-\gamma n}}{\gamma n} \right) + \hat{C}_{t+h|t} \left( \frac{1 - e^{-\gamma n}}{\gamma n} - e^{-\gamma n} \right).
$$

where $A_n(\hat{\Theta})$ is the constant adjustment term as a function of the estimated underlying parameters.

5.3 Yield Forecast comparison

We evaluate the model’s adequacy by computing the forecast root mean squared errors (RMSE) of each model at each forecast horizon for each yield. The forecast RMSEs of these four models are shown in Table 1. Some observations from this table are in order.

First, under the same information set in the state vector, compare models with no-arbitrage restrictions to those without, i.e. the first column vs. the second column, and the third column vs. the fourth column. At the 1-month-ahead forecast horizon, the forecast performance is fairly similar between the two types of models and differences in the RMSEs are almost not distinguishable up to two decimal levels. At the 6-month-ahead forecast horizon, with Nelson-Siegel yield factors only, the AFNS model outperforms the DNS model for medium part of the yield curve; with mixed factors, the AFNSMA model beats the NSMA model beyond 1-year maturity. The advantage of imposing no-arbitrage restrictions becomes apparent when we move to longer forecast horizon at 12-month ahead, as in both cases, the no-arbitrage restricted models performs almost uniformly better than the reduced-form models.

Second, for each forecast horizon under the same modelling approach (no-arbitrage or reduced model), let us compare models with or without macro factors, i.e. the first column vs. the third column, and the second column vs. the fourth column. Again, at the 1-month-ahead forecast

\(^{1}\)The alternative would be to obtain forecasts by projecting $h$-step ahead: $\hat{X}_{t+h|t} = \hat{\mu}_h + \hat{\lambda}_h (X_t - \hat{\mu}_h)$. Given the nature of the no-arbitrage models, only iterated forecast can be computed for them. For this reason, we employ iterated forecasts for all models.
horizon, the results are similar. However, considering that the model with macro variables have significantly more parameters to estimation (for example, the NSMA model has 43 parameters in contrast to the DNS model with only 13 parameters) and hence higher parameter uncertainties which are typically disadvantage for forecasting, it is surprising that the models with macro variables do as good as the models without macro variables. When we move to longer horizons, the advantage of including macro information becomes more evident. At the 6-month-ahead forecast horizon, for no-arbitrage models, the AFNSMA model does better than the AFNS model for the whole yield curve and the advantage is more apparent with short-to-medium maturity yields; the same happens when comparing the NSMA model with the DNS model. At the 12-month-ahead forecast horizon, the AFNSMA model beats the AFNS model for the whole yield curve with a lower RMSE of 10 to 20 basis points; the NSMA model does better than the DNS model at short-to-medium maturities.

To summarize, adding macro information to the yield factor model helps to predict the medium-to-long segments of the yield curve. This is achieved even under the severe curse of dimensionality: as state factors increase from three to six, there are 30 more parameters to estimate simply from the VAR coefficients alone!

In Table 1, the best prediction among the four models for each horizon-maturity combination is denoted as bold numbers. If the no-arbitrage model does better than the corresponding reduced form model with the same information set, the result is underlined. From the patterns of the bold and underlined numbers, we see that overall, models with macro factors tend to do better for short-to-medium maturity yields, and models with no-arbitrage restrictions tend to outperform for medium-to-long maturity yields. These features become more evident when forecast horizon increases. At last, our unspanned AFNSMA model is able to profit from both features as it combines both no-arbitrage restrictions and macro factors together with the parsimonious Nelson-Siegel factor interpolation. At 6-month ahead forecast horizon, it provides the best forecast for the medium part of yields; for 12-month ahead forecast, it achieves the best results for all yields.

Table 1 about here.

5.4 Excess Return Forecast comparison

To be done soon
6 Conclusions

In this paper, we propose an arbitrage-free macro finance modelling framework where the state vector contains the popular Nelson-Siegel yield factors and observable macro factors. Our model can be regarded as an enlarged arbitrage-free Nelson-Siegel modelling framework of Christensen, Diebold and Rudebusch (2010) which does not open the space for observable macro factors. If their framework combines the best of two worlds: Nelson-Siegel factors and no-arbitrage, we combine the two elements in a larger world with macroeconomic fundamentals. The unspanned specification of our model is the no-arbitrage counterpart of the macro-finance model of Diebold, Rudebusch and Aruoba (2006) where the three Nelson-Siegel yield factors are combined with three macroeconomic variables in a reduced form state-space framework. Their model enjoys the parsimony of the Nelson-Siegel interpolation in the measurement equations and the flexibility of jointly studying the dynamics of yield curve factors with macroeconomic variables. Our model provides more rigor and structure without increasing the dimension of parameters, a problem often suffered by affine term structure models with macro factors which involve significant number of risk price parameters to estimate and identify.

We contrast our model theoretically and empirically with a selection of alternative models popularized in the literature. We show that our model provides a reliable and credible framework to analyze the joint dynamics of the yield curve and the macroeconomy. By comparing out-sample forecast performance, we find that combining Nelson-Siegel yield factors, no-arbitrage restrictions and macro factors together in a unified state-space framework is not only theoretically appealing but also empirically encouraging. Compared to the DRA model where Nelson-Siegel factors and macro variables are jointly modelled but no-arbitrage restrictions are absent, our model performs well at 1- and 6-month-ahead forecast and outperforms uniformly at 12-month ahead forecast. This demonstrates that our model is theoretically consistent and empirically adequate without sacrificing the parsimony and flexibility of the NSMA model. Compared to the DNS and AFNS models where macroeconomic factors are absent, our model fares better at medium- to long-term yields for forecast at all horizons. The superior performance than DNS and AFNS on long term yields show that macro factors are important for modelling long term yields. Ignoring macroeconomic information will result in large and systemic pricing errors in pricing and forecasting long term bonds.
7 References


8 Appendix

Here we provide the continuous-time counterpart of the model we derived in discrete-time. We first derive the spanned model, then we derive the unspanned model.

8.1 Spanned Model

Assume that the instantaneous risk-free rate is affine on a vector of variables $X_t$, which contains three latent Nelson-Siegel factors, denoted by $X^u_t = (X^1_t, X^2_t, X^3_t)'$, and observable variables, denoted by $X^o_t$, a total of $N$ factors.
\[ r_t = \delta' X_t = (\delta^u, \delta^o) \begin{pmatrix} X^u_t \\ X^o_t \end{pmatrix} = (1, 1, 0) \begin{pmatrix} X^1_t \\ X^2_t \\ X^3_t \end{pmatrix} + \delta^o X^o_t \] (15)

where \( \delta = \begin{pmatrix} \delta^u \\ \delta^o \end{pmatrix} \) and \( \delta^u = (1, 1, 0)' \).

The dynamics of the state vector \( X_t \) is described by the following system of SDEs under the risk-neutral Q-measure

\[ Q : \quad dX_t = K (\Theta - X_t) \, dt + \Sigma dW^Q; \] (16)

under the physical P-measure,

\[ P : \quad dX_t = k (\theta - X_t) \, dt + \Sigma dW^P, \] (17)

where \( \Sigma \) is a lower triangular and non-singular matrix

\[ \Sigma = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \\ \vdots & \vdots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \sigma_{N3} & \cdots & \sigma_{NN} \end{pmatrix}. \] (18)

The parameters of the physical dynamics are free in this model. Under the Q-measure, we need to make a few restrictions in order to obtain the Nelson-Siegel yield factor loadings in the measurement equations. Specifically, denote

\[ \Theta = \begin{pmatrix} \Theta^u \\ \Theta^o \end{pmatrix}, \]

the mean-reversion matrix \( K \) has to be block upper diagonal,

\[ K = \begin{pmatrix} K^u & K^{uo} \\ 0 & K^o \end{pmatrix}. \]

The Q-dynamics of equation (16) then becomes

\[ \begin{pmatrix} dX^u_t \\ dX^o_t \end{pmatrix} = \begin{pmatrix} K^u & K^{uo} \\ 0 & K^o \end{pmatrix} \left[ \begin{pmatrix} \Theta^u \\ \Theta^o \end{pmatrix} - \begin{pmatrix} X^u_t \\ X^o_t \end{pmatrix} \right] dt + \Sigma dW^Q. \] (19)

For a zero-coupon bond at time \( t \) which will mature at time \( T \), i.e., with maturity \( \tau = T - t \), the price is given by

\[
p(t, T) = E_t^Q [\exp(- \int_t^T r_u du)]
= \exp(A(t, T) + B^1(t, T)X^1_t + B^2(t, T)X^2_t + B^3(t, T)X^3_t + B^o(t, T)'X^o_t)
\]
where \( B^1(t,T), B^2(t,T), B^3(t,T), B^o(t,T) \) and \( A(t,T) \) are the unique solutions to the following system of ODEs:

\[
\begin{pmatrix}
\frac{dB^1(t,T)}{dt} \\
\frac{dB^2(t,T)}{dt} \\
\frac{dB^3(t,T)}{dt}
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
1 & \gamma & 0 \\
0 & -\gamma & \gamma
\end{pmatrix}
\begin{pmatrix}
B^1(t,T) \\
B^2(t,T) \\
B^3(t,T)
\end{pmatrix}
\]

\[
\frac{dB^o(t,T)}{dt} = \delta^o + K^{uo} B^u(t,T) + K^o B^o(t,T)
\]

and

\[
\frac{dA(t,T)}{dt} = B(t,T)'K\Theta - \frac{1}{2} \sum_{j=1}^{n} [\Sigma'B(t,T)B(t,T)']\Sigma_{jj}.
\]

With boundary conditions \( B^1(T,T) = B^2(T,T) = B^3(T,T) = B^o(T,T) = 0 \) and \( A(T,T) = 0 \), the unique solution to this system of ODEs is:

\[
B^1(t,T) = -(T - t)
\]

\[
B^2(t,T) = \frac{1 - e^{-\gamma(T-t)}}{\gamma}
\]

\[
B^3(t,T) = (T-t)e^{-\gamma(T-t)} - \frac{1 - e^{-\gamma(T-t)}}{\gamma}
\]

\[
B^o(t,T) = -\int_t^T e^{-K^o(s-t)}(\delta^o + K^{uo} B^u(s,T))ds
\]

(20)

\[
A(t,T) = \int_t^T (B(s,T)'K\Theta + \frac{1}{2} \sum_{j=1}^{n} [\Sigma'B(s,T)B(s,T)']\Sigma_{jj})ds
\]

(21)

Equivalently, the first three coefficients take the familiar form of Nelson-Siegel factor loadings as in expression (10), i.e.,

\[
B^1(\tau) = -\tau
\]

\[
B^2(\tau) = \frac{1 - e^{-\gamma\tau}}{\gamma}
\]

\[
B^3(\tau) = \tau e^{-\gamma \tau} - \frac{1 - e^{-\gamma \tau}}{\gamma}
\]

Finally, zero-coupon bond yields are given by

\[
y(\tau) = -\frac{A(\tau)}{\tau} + X_1^1 + \frac{1 - e^{-\gamma \tau}}{\gamma \tau} X_1^2 + \frac{1 - e^{-\gamma \tau}}{\gamma^3} X_1^3 - e^{-\gamma \tau} X_1^3 - \frac{1}{\tau} B^o(\tau)'X_1^o
\]

(22)
Proof:

For a general affine term structure model of diffusion process in its state dynamics and short rate as an affine function of the state vector

\[
\begin{align*}
\frac{dX_t}{dt} &= K[\Theta - X_t]dt + \Sigma D(X_t, t)dW^Q \\
rt &= \delta_0 + \delta'_1 X_t
\end{align*}
\]

where

\[
X_t = \begin{pmatrix} X_t^u \\ X_t^o \end{pmatrix}, \quad \delta_1 = \begin{pmatrix} \delta^u_1 \\ \delta^o_1 \end{pmatrix},
\]

and

\[
D(X_t, t) = \begin{pmatrix} \sqrt{\rho_{0,1} + \rho_{1,1}X_t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\rho_{0,n} + \rho_{1,n}X_t} \end{pmatrix}
\]

the ODE of the pricing equation is then

\[
\frac{dB(t, T)}{dt} = \delta_1 + K_t^iB(t, T) - \frac{1}{2} \sum_{j=1}^{n} [\Sigma_t^iB(t, T)B(t, T)^{\prime}\Sigma_t^i]_{jj}(\delta^j)^{\prime}
\]

and

\[
\frac{dA(t, T)}{dt} = -B(t, T)^{\prime}K_t\Theta_t - \frac{1}{2} \sum_{j=1}^{n} [\Sigma_t^iB(t, T)B(t, T)^{\prime}\Sigma_t^i]_{jj}\gamma^j.
\]

If we let \(\rho_{1,j} = 0\) for any \(j\), i.e. constant volatility in the diffusion process, then the first ODE becomes

\[
\begin{pmatrix} \frac{dB^u(t, T)}{dt} \\ \frac{dB^o(t, T)}{dt} \end{pmatrix} = \begin{pmatrix} \delta^u_1 \\ \delta^o_1 \end{pmatrix} + \begin{pmatrix} K^u^o & 0 \\ K^o^u & K^o^o \end{pmatrix} \begin{pmatrix} B^u(t, T) \\ B^o(t, T) \end{pmatrix}.
\]

Assume

\[
\delta^u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad K^u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & -\gamma & \gamma \end{pmatrix},
\]

we then have the first block of the ODE in \(B(t, T)\) as

\[
\frac{dB^u(t, T)}{dt} = \delta^u_1 + K^u B^u(t, T) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & -\gamma & \gamma \end{pmatrix} B^u(t, T).
\]

We can solve the above equation to obtain

\[
B^1(t, T) = -(T - t)
\]
\[ B^2(t, T) = -\frac{1 - e^{-\gamma(T-t)}}{\gamma} \]
\[ B^3(t, T) = (T-t)e^{-\gamma(T-t)} - \frac{1 - e^{-\gamma(T-t)}}{\gamma} \]

The second block of the ODE in \( B(t, T) \) is
\[ \frac{dB^o(t, T)}{dt} = \delta^o_1 + K^{uo}B^u(t, T) + K^{o}B^o(t, T), \]
and we can obtain its solution as
\[ B^o(t, T) = -\int_t^T e^{-K^{o}(s-t)}(\delta^o_1 + K^{uo}B^u(s, T))ds. \]

Assume and \( \rho_{0,1} = 1 \) for normalization, we have
\[ \frac{dA(t, T)}{dt} = -B(t, T)K\Theta - \frac{1}{2}\sum_{j=1}^{n}\left[\Sigma'B(t, T)B(t, T)\Sigma\right]_{jj}. \]

Hence
\[ A(t, T) = \int_t^T (B(s, T)K\Theta + \frac{1}{2}\sum_{j=1}^{n}\left[\Sigma'B(s, T)B(s, T)\Sigma\right]_{jj})dt. \]

### 8.2 Unspanned model

**Proposition 4:** When the parameters of Q-measure satisfy the conditions of proposition 3 without the restriction \( K^{ou} = 0 \); and also satisfy \( K^{uo} = 0 \) and \( \delta^o = 0 \), then we can obtain unspanned measurement equation where the yields are not spanned by the macro factors but just determined by the yield factors.

\[
\begin{pmatrix}
  y_t(\tau_1) \\
  y_t(\tau_2) \\
  \vdots \\
  y_t(\tau_2)
\end{pmatrix} = \begin{pmatrix}
  \frac{1}{\tau_1} & \frac{1-e^{-\tau_1\gamma}}{\tau_1\gamma} & \frac{1-e^{-\tau_1\gamma}}{\tau_1\gamma} & 0 & 0 \\
  \frac{1}{\tau_2} & \frac{1-e^{-\tau_2\gamma}}{\tau_2\gamma} & \frac{1-e^{-\tau_2\gamma}}{\tau_2\gamma} & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
  \frac{1}{\tau_n} & \frac{1-e^{-\tau_n\gamma}}{\tau_n\gamma} & \frac{1-e^{-\tau_n\gamma}}{\tau_n\gamma} & 0 & 0
\end{pmatrix} \begin{pmatrix}
  L_t \\
  S_t \\
  C_t \\
  IP_{gt} \\
  CPI_{gt}
\end{pmatrix} + \varepsilon_t
\]

**Proof:** Under Q measure
\[
\begin{pmatrix}
  dX^u_t \\
  dX^o_t
\end{pmatrix} = \begin{pmatrix}
  K^u & 0 \\
  K^{ou} & K^o
\end{pmatrix} \left[ \begin{pmatrix}
  \Theta^u \\
  \Theta^o
\end{pmatrix} - \begin{pmatrix}
  X^u_t \\
  X^o_t
\end{pmatrix} \right] dt + \Sigma dW^Q
\]

Then by the ODE,
\[
\frac{dB(t, T)}{dt} = \rho_1 + K^T B(t, T) - \frac{1}{2} \sum_{j=1}^{n} \Sigma^T B(t, T) B(t, T)^T \Sigma_{jj} (\delta^j)^T
\]

\[
\frac{dA(t, T)}{dt} = \rho_0 - (B(t, T))^T K \Theta - \frac{1}{2} \sum_{j=1}^{n} \Sigma^T B(t, T) B(t, T)^T \Sigma_{jj} \gamma^j
\]

With \( K^{ou} = 0 \) and \( \delta^o = 0 \), we have

\[
\left( \begin{array}{c}
\frac{dB^o(t, T)}{dt} \\
\frac{dB^u(t, T)}{dt}
\end{array} \right) = \left( \begin{array}{c}
\delta^o \\
0
\end{array} \right) + \left( \begin{array}{cc}
K^{ut} & K^{ou} \\
0 & K^{o}^{r}
\end{array} \right) \left( \begin{array}{c}
B^u(t, T) \\
B^o(t, T)
\end{array} \right)
\]

Thus with \( B(T, T) = 0 \), the solution of the loadings of macro factor is \( B^o(t, T) = 0 \) and \( B^u(t, T) \) is the Nelson Siegel loadings.

Compared with the model of Joslin, Priebsch and Singleton(2010), this proposition is a restricted version of their model.