Expansion of Poisson Process Functionals and Its Application in Econometric Estimation

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Abstract

In this paper, firstly both homogeneous and time–inhomogeneous functionals of Poisson process are expanded into orthogonal series in terms of the bases in respective Hilbert space. Approximations of truncation series to original one are given at the same time. Application of the expansion in econometrics is considered from practical standpoint, which includes three different sampling types, viz. on finite interval, on infinite interval and on a compact interval with its length approaching to infinity. In all the three cases, consistent estimators are obtained.

1 Introduction

The famous option pricing formula derived from the Black-Scholes model makes the following explicit assumptions:

(a) It is possible to borrow and lend cash at a known constant risk-free interest rate.

(b) The price follows a geometric Brownian motion with constant drift and volatility.

(c) There are no transaction costs.

(d) The stock does not pay a dividend.

(e) All securities are perfectly divisible.
(f) There are no restriction on short selling

In spite of worldwide recognition, this option-pricing model is inconsistent with option data. In Cont (2001), several imperfections of the option-pricing formula have been exposed which are summarized as follows. The datasets are drawn from S&P 500 (1970-2001), *S&P 500 (1970-2001 except for the crash of 19 October 1987), and S&P 500, Nasdaq-Composite, DAX, MSI, CAC-40 over the period 1997-1999. The daily log returns of the different indices shown some significant (negative) skewness, kurtosis bigger than 3, indicating that the tails of the Normal distribution go to zero much faster that the empirical data suggest and the empirical distributions are much more peaked than the Normal distribution.

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<th>Mean</th>
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<th>Skewness</th>
<th>Kurtosis</th>
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<tr>
<td>S&amp;P 500 (1970-2001)</td>
<td>0.0003</td>
<td>0.0099</td>
<td>-1.6663</td>
<td>43.36</td>
<td>0.0000</td>
</tr>
<tr>
<td>*S&amp;P 500 (1970-2001)</td>
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<td>0.0095</td>
<td>-0.1099</td>
<td>7.17</td>
<td>-</td>
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<td>S&amp;P 500 (1997-1999)</td>
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<td>0.0119</td>
<td>-0.4409</td>
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<td>0.0421</td>
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<tr>
<td>Nasdaq-Composite</td>
<td>0.0015</td>
<td>0.0154</td>
<td>-0.5439</td>
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<td>5.35</td>
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<td>-0.2116</td>
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</tr>
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In addition, comparing with the kernel density estimator $\hat{f}_h(x)$ for the estimation of density $f(x)$, the log density of the Normal distribution has a quadratic decay, whereas the empirical log density seems have a much more linear decay. This feature is typical for financial data and is often referred to as semi-heaviness of the tails. Moreover, $\chi^2$-test (which counts the number of sample points falling into certain intervals and compares them with the expected number of under the null hypothesis. The $P$-values of the $\chi^2$ statistics in the Table 1.1 show that the Normal distribution is always rejected. Basically, we can conclude that a two-parameter model, such as the Normal one, is not sufficient to capture all the features of the data.

Another important feature missing from the Black-Scholes model is the fact that volatility or, more generally, the environment is changing stochastically over time. This can be seen,
for example, by looking at historical volatilities. The historical volatility is a retrospective measure of volatility. It reflects how volatile the asset has been in the recent past. For the S&P index for every day from 1971-2001, the standard deviation of the daily log returns over a one-year period preceding the day, see Schoutens (2003), has been estimated. Then multiply the estimated standard deviation by the square root of the number of the trading days in one calendar year. Typically, there are 250 trading days in a year. This annualized standard deviation is called the historical volatility. Clearly, we see fluctuations in this historical volatility. Moreover, we see a kind of mean-reversion effect. The peak in the middle of the figure comes from the stock market crash on 19 October 1987; one-year windows including this day give rise to a very high volatilities.

However, to improve on the performance of Black-Sholes model, Lévy models were proposed in the late 1980s and early 1990s, since when they have been refined to take account of different stylized features of the markets. Lévy Processes enjoy the same properties of stationarity and independence of increments as Brownian motion, but they have more flexible distribution whose infinitely indivisibility enables them to represent skewness and excess kurtosis of the dataset in financial market. Examples of such distributions, which can take into account of skewness and kurtosis, are the Variance Gamma (VG), the Normal Inverse Gaussian (NIG), the CGMY (named after Carr, Geman, Madan and Yor), the Hyperbolic model and Mexiner distribution.

Researchers therefore supercede Brownian motion in conventional models with Lévy Process. A typical example is the HJM (Heath-Jarrow-Morton) model. With Lévy Process substitution, the zero-coupon price is modeled by an SDE of the form

$$dP_t(T) = P_t(T)(r_t dt + \sigma(t,T)dL_t), \quad P_0(T) > 0$$

where the typical choices of the Lévy Processes are the VG, the NIG, the Meixner or CGMY processes. The equation has a explicit solution and Eberlein and Raible (1999) reported that option prices as a function of the forward price/strike ratio are W-shaped: at-the-money prices are lower, while in-the-money and out-of-money prices are higher than in the case of Gaussian model. In this way, it is possible to capture the two key features of the empirical
behavior of the term structure: non-Gaussian behavior and stochastic volatility.

Financial mathematics has recently enjoyed considerable prestige as a result of its impact on financial industry. The theory of Lévy Processes has also see exciting development in recent years. The fusion of these two fields has provided new applied modeling with the context of finance and further stimulus for the study of problems within the ambit of Lévy Processes.

2 Levy processes

2.1 Definition and Infinite Divisibility

The term "Lévy process" honors the work of the French mathematician Paul Lévy who, although not alone in his contribution, played an instrumental role in bring together an understanding and characterization of processes with stationary independent increments. In earlier literature, Lévy processes can be found under a number of different names. In the 1940s, Lévy himself referred to them as a sub-class of processus additif (additive processes), that is processes with independent increment. For the most part however, research literature through 1960s to 1970s refers to Lévy processes simply as processes with stationary independent increments. One sees a change in language through the 1980s and by the 1990s the use of the term "Lévy process" had become standard.

**Definition 2.1** (Lévy process). A process $X = \{X_t, t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Lévy process if it possesses the following properties

(i) The paths of $X$ are $\mathbb{P}$-almost surely right continuous with left limits.

(ii) $\mathbb{P}(X_0 = 0) = 1$.

(iii) For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to $X_{t-s}$.

(iv) For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u, u \leq s\}$.

It is evident that both Brownian motion and Poisson process are Lévy process. From the definition alone it is difficult to see how rich a class of processes the class of Lévy processes
forms. Finetti (1929) introduced the notion of an infinitely divisible distribution and showed that they have intimate relationship with Lévy processes. This relationship gives a reasonably good impression of how varied the class of Lévy processes really is.

**Definition 2.2** (Infinitely divisible distribution). We say that a real-valued random variable $X$ has an infinitely divisible distribution if for each $n = 1, 2, \cdots$, there exists a sequence of i.i.d. random variables $X_{1,n}, \cdots, X_{n,n}$ such that

$$X \overset{d}{=} X_{1,n} + \cdots + X_{n,n}$$

where $\overset{d}{=}$ is equality in distribution.

In Kyprianou (2006), it has shown that any Lévy process has the property that for any $t > 0$

$$E\left[e^{iuX_t}\right] = e^{-t\psi(u)}$$

where $\psi(u) := \psi_1(u) = -\log E\left[e^{iuX_1}\right]$ is the characteristic exponent of $X_1$, which has an infinitely divisible distribution.

The following two theorems depict the relationship between Lévy processes and infinitely divisible distributions.

**Theorem 2.1** (Lévy-Khintchine formula). A probability law $\mu$ of a real-valued random variable is infinitely divisible with characteristic exponent $\psi(\theta)$

$$\int_{\mathbb{R}} e^{\theta x} \mu(dx) = e^{-\psi(\theta)}, \quad \text{for } \theta \in \mathbb{R},$$

if and only if there exists a triple $(a, \sigma, \Pi)$, where $a \in \mathbb{R}$, $\sigma \geq 0$ and $\Pi$ is a measure concentrated on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}} \max\{1, x^2\} \Pi(dx) < \infty$ such that

$$\psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x I_{|x|<1}) \Pi(dx)$$

for each $\theta \in \mathbb{R}$.

**Theorem 2.2** (Lévy-Khintchine formula for Lévy processes). Suppose for $a \in \mathbb{R}$, $\sigma \geq 0$ and $\Pi$ is a measure concentrated on $\mathbb{R} \setminus \{0\}$ such that $\int_{\mathbb{R}} \max\{1, x^2\} \Pi(dx) < \infty$. From this triple
define for each \( \theta \in \mathbb{R} \)

\[
\psi(\theta) = ia\theta + \frac{1}{2} \sigma^2 \theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x I_{(|x|<1)}) \Pi(dx).
\]

Then there exists a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) on which a Lévy process is defined having characteristic exponent \(\psi(\theta)\).

A process characterized by \((a, \sigma, \Pi)\) is a Brownian motion with drift if \(\Pi = 0\); while if \(\sigma = 0\) the process is a pure jump process.

### 2.2 Poisson process

Apart from Brownian motion, other processes which pertains to Lévy processes are Poisson process, Gamma process, Pascal process and Meixner process. Since we mainly focus on Poisson processes in this paper, the definition is given below.

**Definition 2.3** (Poisson distribution). A discrete random variable \(X\) with \(P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}, k = 0, 1, 2, \ldots, \) is said to have the Poisson distribution.

**Definition 2.4** (Poisson process). A process valued on the nonnegative integers \(N = \{N_t, t \geq 0\}\), defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), is said to be a Poisson process with intensity \(\lambda > 0\) if the following hold:

(i) The paths of \(N\) are \(\mathbb{P}\)-almost surely right continuous with left limits.

(ii) \(\mathbb{P}(N_0 = 0) = 1\).

(iii) For \(0 \leq s \leq t\), \(N_t - N_s\) is equal in distribution to \(N_{t-s}\).

(iv) For \(0 \leq s \leq t\), \(N_t - N_s\) is independent of \(\{N_u, u \leq s\}\).

(v) For each \(t > 0\), \(N_t\) is equal in distribution to a Poisson random variable with parameter \(\lambda t\).

It is easy to obtain the characteristic function of \(N_t\):

\[
E(e^{i\theta N_t}) = \sum_{k=0}^{\infty} e^{i\theta k} e^{-\lambda t} \frac{\lambda^k}{k!} = e^{-\lambda t(1-e^{i\theta})}
\]
3 Charlier orthogonal polynomials

First of all, two kinds of difference operation, backward and forward, are defined as:

\[ \nabla f(x) = f(x) - f(x - 1), \quad \triangle f(x) = f(x + 1) - f(x). \]

Because of the importance of them for the development of the following study, we consider a number of properties of the operators \( \nabla \) and \( \triangle \) here about.

(I) \( \triangle f(x) = \nabla f(x + 1); \)

(II) \( \triangle \nabla f(x) = \nabla \triangle f(x) = f(x + 1) - 2f(x) + f(x - 1); \)

(III) \( \triangle [f(x - 1)g(x)] = f(x)\triangle g(x) + g(x)\triangle f(x) - 1; \)

(IV) \( \nabla [f(x + 1)g(x)] = f(x)\nabla g(x) + g(x)\nabla f(x + 1); \)

(V) \( \triangle^n f(x) = \triangle(\triangle^{n-1} f(x)), \nabla^n f(x) = \nabla(\nabla^{n-1} f(x)), n \geq 2; \)

(VI) \( \nabla^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x - k); \)

(VII) \( \triangle^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x + k). \)

The monic Charlier polynomials \( C_i(\mu; x) \) are defined by generating function

\[ e^{-\mu w}(1 + w)^x = \sum_{i=0}^{\infty} C_i(\mu; x) \frac{w^i}{i!}, \quad \mu \neq 0. \]  

(3.1)

The explicit representation is

\[ C_i(\mu; x) = \sum_{k=0}^{i} \binom{i}{k} \binom{x}{k} \frac{k!(-\mu)^{i-k}}{i!}, \]  

(3.2)

and the orthogonality relation is

\[ \int_{0}^{\infty} C_m(\mu; x)C_i(\mu; x)d\varrho(\mu; x) = \mu^i!\delta_{mi}, \]

(3.3)

where \( \varrho(\mu; x) \) is a step function whose jumps are

\[ d\varrho(\mu; x) = \frac{e^{-\mu \mu^x}}{x!}, \quad \text{at } x = 0, 1, 2, \ldots \]  

(3.4)
Thus the positive-definite case occurs for $\mu > 0$ and in this case, $d\varrho(\mu; x)$ is the Poisson distribution. The Charlier polynomials can be expressed in terms of Laguerre polynomials

$$C_i(\mu; x) = i!L_i^{(x-i)}(\mu).$$  \hfill \text{(3.5)}$$

There is also the simple difference relation

$$\Delta C_i(\mu; x) = iC_{i-1}(\mu; x).$$  \hfill \text{(3.6)}$$

In fact, it follows from the following calculation:

$$\Delta C_i(\mu; x) = \sum_{k=0}^{i} \binom{i}{k}k!(-\mu)^{i-k}\Delta \left(\frac{x}{k}\right) = \sum_{k=0}^{i} \binom{i}{k}k!(-\mu)^{i-k} \left[ \frac{x+1}{k} - \frac{x}{k} \right]$$

$$= \sum_{k=1}^{i} \binom{i}{k}k!(-\mu)^{i-k} \left( \frac{x}{k} \right) \frac{k}{x+1-k}$$

$$= \sum_{k=0}^{i-1} \binom{i}{k+1}(k+1)!(-\mu)^{i-1-k} \left( \frac{x}{k+1} \right) \frac{k+1}{x-k}$$

$$= \sum_{k=0}^{i-1} \frac{i-1}{k+1} \binom{i}{k+1}(k+1)!(-\mu)^{i-1-k} \left( \frac{x}{k} \right) = i \sum_{k=0}^{i-1} \binom{i-1}{k}k!(-\mu)^{i-1-k} \left( \frac{x}{k} \right)$$

$$= iC_{i-1}(\mu; x).$$

Meanwhile, the finite difference Rodrigues’ type formula is given in Chihara (1978)

$$C_i(\mu; x) = (-1)^i\mu^{-x}\Gamma(x+1)\Delta^i \left( \frac{\mu^x}{\Gamma(x-i+1)} \right).$$  \hfill \text{(3.7)}$$

Denote $c_i(\mu; x) = (-1)^i\mu^{-i}C_i(\mu; x)$. Then we have the simple symmetry relation

$$c_i(\mu; x) = c_x(\mu; i)$$

and from this there follows the ”dual orthogonality” relation

$$\sum_{x=0}^{\infty} c_x(\mu; i)c_x(\mu; m)e^{-\mu^x}\frac{\mu^x}{x!} = \mu^{-i}i!\delta_{mi}.$$  \hfill \text{In (3.8)}, $c_i(\mu; x)$ is

$$c_i(\mu; x) = \frac{x!}{\mu^x}\nabla^i \left( \frac{\mu^x}{x!} \right).$$  \hfill \text{(3.8)}$$
The detailed discussion can be found in Nikiforov and Uvarov (1988).

A few Charlier polynomials are given below:

\begin{align*}
c_0(\mu; x) &= 1 \\
c_1(\mu; x) &= 1 - \frac{x}{\mu} \\
c_2(\mu; x) &= 1 - 2\frac{x}{\mu} + \frac{x(x-1)}{\mu^2} \\
c_3(\mu; x) &= 1 - 3\frac{x}{\mu} + 3\frac{x(x-1)}{\mu^2} - \frac{x(x-1)(x-2)}{\mu^3}
\end{align*}

4 Expansion of Poisson process functionals

4.1 Expansion of homogeneous functionals

1. Function expansion

Let

\[ \varrho(\mu; n) = e^{-\mu} \frac{\mu^n}{n!}, \quad n = 0, 1, \ldots \]

Suppose \( f(n) \) is defined on integers such that

\[ \sum_{n=0}^{\infty} f^2(n) \varrho(\mu; n) < \infty, \quad \text{for some } \mu > 0. \]  

(4.1)

All functions (actually sequences) satisfying (4.1) constitute a Hilbert space \( l^2(\mathbb{N}, \varrho(\mu; n)) \), in which the inner product is conventionally defined by

\[ (f(n), g(n)) = \sum_{n=0}^{\infty} f(n)g(n)\varrho(\mu; n), \]  

(4.2)

and a conventional norm is induced by \( \|f\| = (f, f)^{1/2} \). Moreover, Charlier orthogonal polynomials \( c_i(\mu; n) \) are in \( l^2(\mathbb{N}, \varrho(\mu; n)) \) and Charlier system is complete orthogonal in the space as well since it is a classical orthogonal polynomial system (see, P269 Example 2 of A.N.Shiryaev (1996)).

Observe that any polynomial is in \( l^2(\mathbb{N}, \varrho(\mu; x)) \). In fact, one first proves a slightly general statement, that is, for any \( f(n) = n^m, \) \( m \in \mathbb{N} \) fixed, \( \sum_{n=0}^{\infty} f(n)\varrho(\mu; n) < \infty \). Induction
is used. It is evidently right when \( f(n) = \text{constant}. \) For \( f(n) = n, \) \( \sum_{n=0}^{\infty} f(n) g(\mu; n) = e^{-\mu} \sum_{n=1}^{\infty} \frac{\mu^n}{(n-1)!} = \mu. \) Now suppose that \( f(n) = 1, n, \cdots, n^{m-1} \) have such property. Then

\[
\sum_{n=0}^{\infty} n^m g(\mu; n) = e^{-\mu} \sum_{n=1}^{\infty} n^{m-1} \frac{\mu^n}{(n-1)!} = \mu e^{-\mu} \sum_{n=0}^{\infty} (n+1)^{m-1} \frac{\mu^n}{n!} = \mu e^{-\mu} \sum_{k=0}^{\infty} C_k \sum_{n=0}^{\infty} n^k \frac{\mu^n}{n!} < \infty.
\]

This implies that all polynomials \( p_k(n) \in l^2(\mathbb{N}, g(\mu; n)) \).

In order to obtain an orthonormal basis, the polynomials are normalized:

\[
\mathcal{C}_i(\mu; n) = \sqrt{\frac{\mu^i}{i!}} c_i(\mu; n).
\]

Thus, \( \|\mathcal{C}_i(\mu; n)\| = 1 \) and \( \{\mathcal{C}_i(\mu; n)\}_{0}^{\infty} \) is an orthonormal basis in \( l^2(\mathbb{N}, g(\mu; n)) \). One hence has the following proposition.

**Proposition 4.1.** For any \( f(n) \in l^2(\mathbb{N}, g(\mu; n)) \), it can be expanded into an orthogonal series

\[
f(n) = \sum_{i=0}^{\infty} b_i \mathcal{C}_i(\mu; n),
\]

where \( b_i = b_i(\mu, f) = (f(n), \mathcal{C}_i(\mu; n)) = \sum_{n=0}^{\infty} f(n) \mathcal{C}_i(\mu; n) g(\mu; n). \)

**Proof.** By virtue of the Hilbert space, one has

\[
f(n) = \sum_{i=0}^{\infty} b_i \mathcal{C}_i(\mu; n).
\]

Multiple both sides of (4.4) by \( \mathcal{C}_j(\mu; n) \) and \( g(\mu; n) \), sum it up with respect to \( n \) from 0 to infinite, interchange the summations about \( i \) and \( n \) on the right hand side, invoke the orthogonality of \( \mathcal{C}_i(\mu; n), \)

\[
\sum_{n=0}^{\infty} f(n) \mathcal{C}_j(\mu; n) g(\mu; n) = \sum_{n=0}^{\infty} b_i \mathcal{C}_i(\mu; n) \mathcal{C}_j(\mu; n) g(\mu; n)
\]

\[
= \sum_{i=0}^{\infty} b_i \sum_{n=0}^{\infty} \mathcal{C}_i(\mu; n) \mathcal{C}_j(\mu; n) g(\mu; n)
\]

\[
= b_j.
\]

Then the assertion follows. \( \square \)
Given \( f(n) \in l^2(\mathbb{N}, g(\mu; n)) \) and (4.4) holds, then Parseval-Bessel equality is valid, viz.

\[
\|f(n)\|^2 = \sum_{i=0}^{\infty} b_i(\mu, f)^2. 
\]  

(4.5)

**Example 1** \( f(n) = n^2 \)

It is evident that \( n^2 = b_0 \mathcal{C}_0(\mu; n) + b_1 \mathcal{C}_1(\mu; n) + b_2 \mathcal{C}_2(\mu; n) \). In what follows, we only need to calculate the coefficients.

\[ b_0 = \langle n^2, \mathcal{C}_0(\mu; n) \rangle = e^{-\mu} \sum_{n=0}^{\infty} n^2 \frac{\mu^n}{n!} \]

\[ = e^{-\mu} \sum_{n=1}^{\infty} n^2 \frac{\mu^n}{n!} = e^{-\mu} \sum_{n=1}^{\infty} \frac{n^2 \mu^n}{(n-1)!} \]

\[ = \mu e^{-\mu} \sum_{n=0}^{\infty} (n+1) \frac{\mu^n}{n!} = \mu e^{-\mu} \sum_{n=0}^{\infty} \frac{n \mu^n}{n!} + \mu e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \]

\[ = \mu^2 + \mu \]

\[ b_1 = \langle n^2, \mathcal{C}_1(\mu; n) \rangle = e^{-\mu} \sum_{n=0}^{\infty} n^2 \mathcal{C}_1(\mu; n) \frac{\mu^n}{n!} \]

\[ = \sqrt{\mu} e^{-\mu} \sum_{n=0}^{\infty} n^2 \left( 1 - \frac{n}{\mu} \right) \frac{\mu^n}{n!} = \sqrt{\mu} e^{-\mu} \sum_{n=0}^{\infty} n^2 \frac{\mu^n}{n!} - \sqrt{\mu} e^{-\mu} \sum_{n=0}^{\infty} \frac{n \mu^n}{n!} \]

\[ = \sqrt{\mu}(\mu^2 + \mu) - \sqrt{\mu} e^{-\mu} \sum_{n=1}^{\infty} \frac{n^2 \mu^{n-1}}{(n-1)!} \]

\[ = \sqrt{\mu}(\mu^2 + \mu) - \sqrt{\mu} e^{-\mu} \sum_{n=0}^{\infty} (n+1) \frac{\mu^n}{n!} \]

\[ = \sqrt{\mu}(\mu^2 + \mu) - \sqrt{\mu} e^{-\mu} \sum_{n=0}^{\infty} n^2 \frac{\mu^n}{n!} + 2 \sqrt{\mu} e^{-\mu} \sum_{n=0}^{\infty} n \frac{\mu^n}{n!} - \sqrt{\mu} e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \]

\[ = -\sqrt{\mu}(2\mu + 1) \]

\[ b_2 = \langle n^2, \mathcal{C}_2(\mu; n) \rangle = e^{-\mu} \sum_{n=0}^{\infty} n^2 \mathcal{C}_2(\mu; n) \frac{\mu^n}{n!} \]

\[ = \frac{\mu}{\sqrt{2}} e^{-\mu} \sum_{n=0}^{\infty} n^2 \left( 1 - 2 \frac{n}{\mu} + \frac{n(n-1)}{\mu^2} \right) \frac{\mu^n}{n!} \]

\[ = \frac{\mu}{\sqrt{2}} e^{-\mu} \sum_{n=0}^{\infty} n^2 \frac{\mu^n}{n!} - 2 \frac{\mu}{\sqrt{2}} e^{-\mu} \sum_{n=0}^{\infty} n^2 \frac{\mu^n}{n!} + \frac{\mu}{\sqrt{2}} e^{-\mu} \sum_{n=0}^{\infty} \frac{n^3(n-1) \mu^n}{\mu^2 (n-1)!} \]

\[ = \frac{\mu}{\sqrt{2}} (\mu^2 + \mu) - 2 \frac{\mu}{\sqrt{2}} (\mu^2 + 3\mu + 1) + \frac{\mu}{\sqrt{2}} e^{-\mu} \sum_{n=1}^{\infty} \frac{n^2(n-1) \mu^n}{\mu^2 (n-1)!} \]
\[
\begin{align*}
\frac{\mu}{\sqrt{2}}(\mu^2 + \mu) - 2\frac{\mu}{\sqrt{2}}(\mu^2 + 3\mu + 1) + \frac{\mu}{\sqrt{2}} e^{-\mu} \sum_{n=2}^{\infty} \frac{n^2 \mu^n}{n!} &= \mu^2 \mu^2 + \mu^2 \mu^2 + \mu^2 \\
= -\frac{\mu}{\sqrt{2}}(\mu^2 + 5\mu + 2) + \frac{\mu}{\sqrt{2}} e^{-\mu} \sum_{n=0}^{\infty} (n + 2) \frac{\mu^n}{n!} &= -\frac{\mu}{\sqrt{2}}(\mu^2 + 3\mu + 1) + 4 \frac{\mu}{\sqrt{2}} e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \\
= -\frac{\mu}{\sqrt{2}}(\mu^2 + 5\mu + 2) + \frac{\mu}{\sqrt{2}} (\mu^2 + \mu) + 4 \frac{\mu}{\sqrt{2}} + 4 \frac{\mu}{\sqrt{2}} &= \sqrt{2}\mu.
\end{align*}
\]

Therefore, \(n^2 = (\mu^2 + \mu)\mathcal{C}_0(\mu; n) - \sqrt{\mu}(2\mu + 1)\mathcal{C}_1(\mu; n) + \sqrt{2}\nu\mathcal{C}_2(\mu; n)\).

**Example 2** Expansions of \(2^n\) and \(2^{-n}\).

In generating function letting \(w = 1\) and \(w = -1/2\) yields
\[
\begin{align*}
2^n &= e^{\mu} \sum_{i=0}^{\infty} (-1)^i \sqrt{\frac{\mu^i}{i!}} \mathcal{C}_i(\mu; n) \\
2^{-n} &= e^{-\mu/2} \sum_{i=0}^{\infty} \frac{1}{2^i} \sqrt{\frac{\mu^i}{i!}} \mathcal{C}_i(\mu; n)
\end{align*}
\]

Given a truncation parameter \(k\), one has the truncation series for the expansion (4.4):
\[
f_k(n) = \sum_{i=0}^{k} b_i \mathcal{C}_i(\mu; n).
\]

(4.6)

Now let us investigate the degree of the approximation \(f_k(n)\) to \(f(n)\). For the sake of convenience of operation on \(f\) and other functions involved, we may extend the domains of those functions to negative integers assuming values of zero. It is evident that the extension does not affect anything at all.

**Theorem 4.1.** Suppose that \(f(n) \in l^2(\mathbb{N}, \varrho(\mu; n))\) for some \(\mu > 0\). Suppose further that there is a positive integer \(r\) such that \(\nabla^r f(n + r) \in l^2(\mathbb{N}, \varrho(\mu; n))\). Then we have for large \(k\)
\[
\|f(n) - f_k(n)\|^2 \leq \frac{\mu^r(k + 1 - r)!}{(k + 1)!} R^2(k),
\]
where \(R^2(k) = \sum_{i=k+1}^{\infty} b_i^2 (\mu; \nabla^r f(n + r))\) which will converge to zero as \(k \to \infty\).
Proof. Invoking the Rodrigues formula and the property of difference yields

\[ b_i(\mu; f) = \sum_{n=0}^{\infty} f(n)c_i(\mu; n)g(\mu; n) = \sqrt{\frac{\mu^n}{i!}} \sum_{n=0}^{\infty} f(n)c_i(\mu; n)g(\mu; n) \]

\[ = e^{-\mu} \sqrt{\frac{\mu^n}{i!}} \sum_{n=0}^{\infty} f(n)c_i(\mu; n)\frac{\mu^n}{n!} \]

\[ = e^{-\mu} \sqrt{\frac{\mu^n}{i!}} \sum_{n=0}^{\infty} f(n)\nabla \left( \frac{\mu^n}{n!} \right) \]

\[ = e^{-\mu} \sqrt{\frac{\mu^n}{i!}} \sum_{n=0}^{\infty} \left\{ \nabla \left[ f(n+1)\nabla^{i-1} \left( \frac{\mu^n}{n!} \right) \right] - \nabla f(n+1)\nabla^{i-1} \left( \frac{\mu^n}{n!} \right) \right\} \]

\[ = e^{-\mu} \sqrt{\frac{\mu^n}{i!}} \sum_{n=0}^{\infty} \nabla \left[ f(n+1)\nabla^{i-1} \left( \frac{\mu^n}{n!} \right) \right] - e^{-\mu} \sqrt{\frac{\mu^n}{i!}} \sum_{n=0}^{\infty} \nabla f(n+1)\nabla^{i-1} \left( \frac{\mu^n}{n!} \right) \]

Notice that

\[ \sum_{n=0}^{\infty} \nabla \left[ f(n+1)\nabla^{i-1} \left( \frac{\mu^n}{n!} \right) \right] \]

\[ = \lim_{n \to \infty} f(n+1)\nabla^{i-1} \left( \frac{\mu^n}{n!} \right) - f(n+1)\nabla^{i-1} \left( \frac{\mu^n}{n!} \right) \bigg|_{n=1} \]

\[ = \lim_{n \to \infty} f(n+1)\nabla^{i-1} \left( \frac{\mu^n}{n!} \right) = e^{-\mu} \lim_{n \to \infty} f(n+1) \left[ \frac{n!}{\mu^n} \nabla^{i-1} \left( \frac{\mu^n}{n!} \right) \right] g(\mu; n) \]

\[ = e^{-\mu} \lim_{n \to \infty} f(n+1)c_{i-1}(\mu; n)g(\mu; n), \]

using again the Rodrigues formula. However,

\[ \sum_{n=0}^{\infty} f(n+1)c_{i-1}(\mu; n)g(\mu; n) = \frac{1}{\mu} \sum_{n=1}^{\infty} n f(n)c_{i-1}(\mu; n-1)g(\mu; n) \]

\[ = \frac{1}{\mu} (f(n), nc_{i-1}(\mu; n-1)) < \infty, \]

since \( nc_{i-1}(\mu; n-1) \) is a polynomial which belongs to the space \( l^2(\mathbb{N}, g(\mu; n)) \). Therefore, the limit is zero. It follows that the calculation of \( b_i \) is simplified as

\[ b_i(\mu; f) = -e^{-\mu} \sqrt{\frac{\mu^n}{i!}} \sum_{n=0}^{\infty} \nabla f(n+1)\nabla^{i-1} \left( \frac{\mu^n}{n!} \right). \]

Repeat the above process \( r \) times, we eventually have

\[ b_i(\mu; f) = (-1)^r e^{-\mu} \sqrt{\frac{\mu^n}{i!}} \sum_{n=0}^{\infty} \nabla^r f(n+r)\nabla^{i-r} \left( \frac{\mu^n}{n!} \right). \]
\[ r \mu_i \left( (i - r) \pi \right) \]

\[ i \pi \pi \mu_i - i \pi \nu \]

\[ \sum_{n=0}^{\infty} \nabla^r f(n + r) \nabla^{i - r} \left( \frac{\mu^n}{n!} \right) \]

\[ (-1)^r \sqrt{\frac{\mu^r (i - r)!}{i!}} \left( \frac{\mu^r}{(i - r)!} \right) \sum_{n=0}^{\infty} \nabla^r f(n + r) \nabla^{i - r} \left( \frac{\mu^n}{n!} \right) \]

\[ = (-1)^r \sqrt{\frac{\mu^r (i - r)!}{i!}} b_{i - r}(\mu; \nabla^r f(n + r)). \] (4.8)

Finally, by virtue of orthogonality of \( C_i(\mu; n) \)

\[ \| f(x) - f_k(x) \|^2 = \left\| \sum_{i=k+1}^{\infty} b_i C_i(\mu; n) \right\|^2 = \sum_{n=0}^{\infty} \left( \sum_{i=k+1}^{\infty} b_i C_i(\mu; n) \right)^2 \rho(\mu; n) \]

\[ = \sum_{i=k+1}^{\infty} b_i^2 = \sum_{i=k+1}^{\infty} \frac{\mu^r (i - r)!}{i!} b_{i - r}(\mu; \nabla^r f(n + r)) \]

\[ \leq \frac{\mu^r (k + 1 - r)!}{(k + 1)!} \sum_{i=k+1}^{\infty} b_{i - r}(\mu; \nabla^r f(x + r)) \]

\[ := \frac{\mu^r (k + 1 - r)!}{(k + 1)!} R^2(k), \]

where \( R^2(k) = \sum_{i=k+1}^{\infty} b_{i - r}(\mu; \nabla^r f(n + r)) \) which will converge to zero as \( k \to \infty \) due to \( \nabla^r f(n + r) \in l^2(N, \rho(\mu; n)) \) and Parseval-Bessel equality.

2. Expansion in stochastic space

A functional of Poisson process in the form of \( f(N_t) \) is called of homogeneity where \( N_t \) is a Poisson process with intensity \( \mu \) meaning that \( N_t \sim Pois(\mu t) \). Suppose that for a fixed \( t > 0, f(n) \in l^2(N, \rho(\mu t; n)) \), viz.

\[ \sum_{n=0}^{\infty} f^2(n) \rho(\mu t; n) < \infty, \] (4.9)

where \( \rho(\mu t; n) = e^{-\mu t} \frac{(\mu t)^n}{n!} \). It is apparent that (4.9) implies that \( EF^2(N_t) < \infty \). This enables us to construct a mapping between \( l^2(N, \rho(\mu t; n)) \) and \( L^2(\Omega) \):

\[ T : f \mapsto f(N_t). \] (4.10)

Lemma 4.1. The mapping \( T \) has the following properties

1. \( T \) is linear;

2. \( T \) is one-one;

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3. \( T \) is isomorphism.

Proof. (1) For any \( f, g \in l^2(N, \varrho(\mu_t; n)) \), \( T(af + bg) = (af + bg)(N_t) = af(N_t) + bg(N_t) = aT(f) + bT(g) \).

(2) For any \( f, g \in l^2(N, \varrho(\mu_t; n)) \),

\[
\langle T(f), T(g) \rangle_{L^2(\Omega)} = \langle f(N_t), g(N_t) \rangle = E[f(N_t)g(N_t)] = \sum_{n=0}^{\infty} f(n)g(n)\varrho(\mu_t; n) = (f(n), g(n))_{l^2(N, \varrho(\mu_t; n))}.
\]

Thus, \( T \) is inner product preserving and hence \( \|T(f) - T(g)\|_{L^2(\Omega)} = \|T(f - g)\| = \|f - g\|_{l^2(N, \varrho(\mu_t; n))} \). We therefore can assert that \( f \neq g \Leftrightarrow T(f) \neq T(g) \), which indicates that \( T \) is one-one.

(3) In view of linearity and one-one mapping of \( T \), it is isomorphism. \( \square \)

Denote by \( \Xi \) the image of \( T \) so that \( \Xi \) is a subset of \( L^2(\Omega) \). The following lemma shows the further picture of \( \Xi \).

**Lemma 4.2.** \( \Xi \) is a closed linear subspace of \( L^2(\Omega) \), hence it is a Hilbert space as well.

Proof. The linearity of \( \Xi \) is induced by that of \( T \). Next, suppose that \( \{\xi_k\} \) is a Cauchy sequence in \( \Xi \). Since \( T \) is one-one, there is a sequence \( f_k \in l^2(N, \varrho(\mu_t; n)) \) such that \( T(f_k) = \xi_k \). Meanwhile, because \( T \) is inner product preserving, \( \{f_k\} \) is a Cauchy sequence as well. Therefore there is an element \( f \in l^2(N, \varrho(\mu_t; n)) \) such that \( \lim_{k \to \infty} f_k = f \) in the sense of mean square due to completeness of Hilbert space \( l^2(N, \varrho(\mu_t; n)) \). It follows that

\[
\|\xi_k - T(f)\| = \|T(f_k) - T(f)\| = \|f_k - f\| \to 0,
\]

which means that \( \lim \xi_k \to T(f) \in \Xi \) and \( \Xi \) is closed. Hence, it is a Hilbert space. \( \square \)

**Lemma 4.3.** \( \{c_i(\mu_t, N_t)\} \) is an orthonormal basis in \( \Xi \).
Proof. Because \( \{ \mathcal{C}_i(\mu t, n) \} \) is an orthonormal basis in \( l^2(\mathbb{N}, g(\mu t; n)) \) and \( T \) is one-one and inner-product preserving, the orthogonality and completeness of \( \{ \mathcal{C}_i(\mu t, n) \} \) are transited into \( \Xi \). The assertion follows.

Now that \( \Xi \) is a Hilbert space with orthonormal basis \( \{ \mathcal{C}_i(\mu t, N_t) \} \), we have the following theorem.

**Theorem 4.2.** For any \( f(N_t) \in \Xi \), it admits a Fourier series expansion

\[
f(N_t) = \sum_{i=0}^{\infty} c_i \mathcal{C}_i(\mu t, N_t),
\]

(4.11)

where \( c_i = c_i(t, f) = \langle f(N_t), \mathcal{C}_i(\mu t, N_t) \rangle \).

**Proof.** In view of that \( \Xi \) is a Hilbert space with orthonormal basis \( \{ \mathcal{C}_i(\mu t, N_t) \} \), it is valid. \( \square \)

**Example 1**

As \( \langle f(N_t), g(N_t) \rangle_\Xi = (f(n), g(n))_{l^2(\mathbb{N})} \), we have the following expansions from preceding examples

\[
N_t^2 = \mu(1 + \mu)\mathcal{C}_0(\mu t; N_t) - \sqrt{\mu t(2\mu t + 1)}\mathcal{C}_1(\mu t; N_t) + \sqrt{2\mu}\mathcal{C}_2(\mu t; N_t)
\]

\[
2^{N_t} = e^{\mu t} \sum_{i=0}^{\infty} (-1)^i \sqrt{\frac{(\mu t)^i}{i!}} \mathcal{C}_i(\mu t; N_t)
\]

\[
2^{-N_t} = e^{-\mu t/2} \sum_{i=0}^{\infty} \frac{1}{2^i} \sqrt{\frac{(\mu t)^i}{i!}} \mathcal{C}_i(\mu t; N_t)
\]

**Example 2**

Assuming \( w = e^i - 1 \) and \( w = e^{-i} - 1 \) \((i = \sqrt{-1})\) in exponential generating function (3.1) with \( \mu := \mu t \) yields

\[
e^{ix} = e^{i(t)(e^i-1)} \sum_{k=0}^{\infty} C_k(\mu t; x) \frac{(e^{i} - 1)^k}{k!}
\]

\[
e^{ix} = e^{i(t)(e^i-1)} \sum_{k=0}^{\infty} (-1)^k (\mu t)^k c_k(\mu t; x) \frac{(e^{i} - 1)^k}{k!}
\]

\[
e^{ix} = e^{i(t)(e^i-1)} \sum_{k=0}^{\infty} (-1)^k \sqrt{\frac{(\mu t)^k}{\sqrt{k!}}} \mathcal{C}_k(\mu t; x),
\]

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\[ e^{-ix} = e^{\mu t(e^{-i}-1)} \sum_{k=0}^{\infty} C_k(\mu t; x) \frac{(e^{-i}-1)^k}{k!} \]

\[ = e^{\mu t(e^{-i}-1)} \sum_{k=0}^{\infty} (-1)^k(\mu t)_k c_k(\mu t; x) \frac{(e^{-i}-1)^k}{k!} \]

\[ = e^{\mu t(e^{-i}-1)} \sum_{k=0}^{\infty} (-1)^k \sqrt{(\mu t)_k} \frac{(e^{-i}-1)^k}{k!} \mathcal{C}_k(\mu t; x) \]

Hence we have

\[ \cos x = \frac{1}{2} \left( e^{ix} + e^{-ix} \right) \]

\[ = \sum_{k=0}^{\infty} \frac{1}{2}(-1)^k \sqrt{(\mu t)_k} \left[ e^{\mu t(e^{-i}-1)}(e^{i}-1)^k + e^{\mu t(e^{-i}-1)}(e^{-i}-1)^k \right] \mathcal{C}_k(\mu t; x) \]

\[ = \sum_{k=0}^{\infty} \frac{1}{2}(-1)^k \sqrt{(\mu t)_k} e^{-\mu t(1-\cos 1)} \sqrt{2 - 2 \cos 1}^k \left[ e^{\mu t} \cos(\alpha k + \mu t \sin 1) + e^{-\mu t} \cos(\alpha k - \mu t \sin 1) \right] \mathcal{C}_k(\mu t; x), \]

where \( \alpha = \arctan \left( \frac{\sin 1}{\cos 1 - 1} \right) \). Meanwhile,

\[ \sin x = \frac{1}{2i} \left( e^{ix} - e^{-ix} \right) \]

\[ = \sum_{k=0}^{\infty} \frac{1}{2i}(-1)^k \sqrt{(\mu t)_k} \left[ e^{\mu t(e^{-i}-1)}(e^{i}-1)^k - e^{\mu t(e^{-i}-1)}(e^{-i}-1)^k \right] \mathcal{C}_k(\mu t; x) \]

\[ = \sum_{k=0}^{\infty} (-1)^k \sqrt{(\mu t)_k} \frac{k!}{k!} e^{-\mu t(1-\cos 1)} \sqrt{2 - 2 \cos 1}^k \sin(\alpha k + \mu t \sin 1) \mathcal{C}_k(\mu t; x), \]

where \( \alpha \) retains the same meaning.

In view of the relation between \( l^2(\mathbb{N}, g(t; n)) \) and \( \Xi \), we obtain

\[ \cos N_t = \sum_{k=0}^{\infty} (-1)^k \sqrt{(\mu t)_k} e^{-\mu t(1-\cos 1)} \sqrt{2 - 2 \cos 1}^k \cos(\alpha k + \mu t \sin 1) \mathcal{C}_k(\mu t; N_t), \]

\[ \sin N_t = \sum_{k=0}^{\infty} (-1)^k \sqrt{(\mu t)_k} e^{-\mu t(1-\cos 1)} \sqrt{2 - 2 \cos 1}^k \sin(\alpha k + \mu t \sin 1) \mathcal{C}_k(\mu t; N_t). \]

Given a truncation parameter \( p \) for \( i \), the truncation series to (4.11) is defined as

\[ f_p(N_i) = \sum_{i=0}^{p} c_i \mathcal{C}_i(\mu t, N_i). \]
Theorem 4.3. Suppose that \( f(n) \in l^2(\mathbb{N}, \varrho(\mu t; n)) \) for some \( \mu, t > 0 \). Suppose further that there is a positive integer \( r \) such that \( \nabla^r f(n+r) \in l^2(\mathbb{N}, \varrho(\mu t; n)) \). Meanwhile, \( N_t \) is a Poisson process with intensity \( \mu \). Then we have for large \( p \)

\[
\| f(N_t) - f_p(N_t) \|_{L^2(\Omega)}^2 \leq \frac{(\mu t)^r (p+1-r)!}{(p+1)!} R^2(k),
\]

(4.13)

where \( R^2(p) = \sum_{i=p+1}^{\infty} c_{i-r}^2(\mu t; \nabla^r f(n+r)) \) which will converge to zero as \( p \to \infty \).

Proof. It follows from the orthogonality of \( \mathcal{C}_i(\mu t, N_t) \) that

\[
\| f(N_t) - f_p(N_t) \|_{L^2(\Omega)}^2 = \left\| \sum_{i=p+1}^{\infty} c_i(t, f) \mathcal{C}_i(\mu t, N_t) \right\|^2 = \sum_{i=p+1}^{\infty} c_i(\mu t, f)^2.
\]

However, since \( f(n) \) satisfies all the conditions in Theorem 4.1 with \( \mu \) being substituted by \( \mu t \), we can derive the following relation by (4.8)

\[
c_i(\mu t, f) = (-1)^r \sqrt{\frac{(\mu t)^r (i-r)!}{i!} c_{i-r}(\mu t; \nabla^r f(n+r))}.
\]

Therefore,

\[
\sum_{i=p+1}^{\infty} c_i(\mu t, f)^2 = \sum_{i=p+1}^{\infty} \frac{(\mu t)^r (i-r)!}{i!} c_{i-r}^2(\mu t; \nabla^r f(n+r)) \leq (\mu t)^r \frac{(p+1-r)!}{(p+1)!} \sum_{i=p+1}^{\infty} c_{i-r}^2(\mu t; \nabla^r f(n+r)) = (\mu t)^r \frac{(p+1-r)!}{(p+1)!} R^2(p)
\]

where \( R^2(p) = \sum_{i=p+1}^{\infty} c_{i-r}^2(\mu t; \nabla^r f(n+r)) \) which will converge to zero as \( p \to \infty \). \( \square \)

4.2 Expansion of Time-Inhomogeneous Functionals of Poisson Processes

4.2.1 Expansion on \( \mathbb{R}^+ \times \mathbb{N} \)

1. Function expansion

Suppose that \( f(t, n) \) is defined on \( \mathbb{R}^+ \times \mathbb{N} \) where \( \mathbb{R}^+ \) denotes the set of positive reals and \( \mathbb{N} \) signifies the set of nonnegative integers. Let \( \nu = \varrho(\mu t; n) \times \lambda \) be the product measure of
$\rho(\mu; n)$ and $\lambda$ (Lebesgue measure) on $\mathbb{R}^+ \otimes \mathbb{N}$. Therefore, $(\mathbb{R}^+ \times \mathbb{N}, \nu)$ becomes a measure space.

Define

$$L^2(\mathbb{R}^+ \times \mathbb{N}, \rho(\mu; n) \times \lambda) = \left\{ f(t, n) : \int_0^\infty \sum_{n=0}^\infty f^2(t, n) \rho(\mu; n) dt < \infty \right\},$$

where $\rho(\mu; n) = e^{-\mu t}(\mu)^n/n!$, $\mu > 0$ and $\lambda$ is the Lebesgue measure. Since as a measure $\rho(\mu; n)$ is finite, and Lebesgue measure is $\sigma$-finite, $L^2(\mathbb{R}^+ \times \mathbb{N}, \rho(\mu; x) \times \lambda)$ becomes a Hilbert space (see P162 of Dudley (2003)). We shall simplify the notation $L^2(\mathbb{R}^+ \times \mathbb{N}, \rho(\mu; n) \times \lambda) = L^2(\mathbb{R}^+ \times \mathbb{N})$ for brevity.

If $f(t, n) \in L^2(\mathbb{R}^+ \times \mathbb{N})$, $\sum_{n=0}^\infty f(t, n) \rho(\mu; n)$ is square integrable on $\mathbb{R}^+$ due to the following inequality

$$\int_0^\infty \left( \sum_{n=0}^\infty f(t, n) \rho(\mu; n) \right)^2 dt \leq \int_0^\infty \sum_{n=0}^\infty f^2(t, n) \rho(\mu; n) dt < \infty.$$

This fact allows us to be able to define the following operation.

For any two elements $f(t, n) \in L^2(\mathbb{R}^+ \times \mathbb{N})$ and $g(t, n) \in L^2(\mathbb{R}^+ \times \mathbb{N})$ define a conventional inner product

$$(f, g) = \int_0^\infty \sum_{n=0}^\infty f(t, n) g(t, n) \rho(\mu; n) dt$$

on the space and the induced norm follows. Observe that the reason that we say this definition is a conventional inner product is due to the consideration that if we regard $\rho$ as a measure on all subsets of $\mathbb{N}$ such that at a single point $n$, the measure is $\rho(\mu; n)$, the summation can be viewed as an integral:

$$\sum_{n=0}^\infty f(t, n) \rho(\mu; n) = \int_\mathbb{N} f(t, n) d\rho(\mu, n).$$

Equipped with this inner product and the induced norm, the space $L^2(\mathbb{R}^+ \times \mathbb{N}, \nu)$ admits one complete orthogonal sequence which is the product of orthonormal bases $\mathcal{L}_j(t)$ in $l^2(\mathbb{N}, \rho(\mu; n))$ and $\mathcal{C}_i(\mu, n)$ in $L^2(\mathbb{R}^+, \lambda)$ respectively. This is because the space $L^2(\mathbb{R}^+ \times \mathbb{N}, \rho(\mu; n) \times \lambda)$ can be treated as the product space of $l^2(\mathbb{N}, \rho(\mu; n))$ and $L^2(\mathbb{R}^+, \lambda)$. The detailed discussion is analogous to the part in Chapter one. We therefore have a complete
orthogonal sequence for the space: \( \{ \mathcal{C}_i(\mu; n)L_j(t) \}_{i,j=0}^{\infty} \). The orthogonality of the sequence is verified as follows

\[
\langle \mathcal{C}_i(\mu; n)L_j(t), \mathcal{C}_m(\mu; n)L_l(t) \rangle = \int_0^{\infty} \sum_{n=0}^{\infty} \mathcal{C}_i(\mu; n)L_j(t) \mathcal{C}_m(\mu; n) L_l(t) \varrho(\mu; n) dt
\]

\[
= \int_0^{\infty} L_j(t)L_l(t) \left[ \sum_{n=0}^{\infty} \mathcal{C}_i(\mu; n)n \mathcal{C}_m(\mu; n) \right] dt
\]

\[
= \begin{cases} 
\int_0^{\infty} L_j(t)L_l(t) dt, & \text{if } i = m; \\
0, & \text{if } i \neq m.
\end{cases}
\]

\[
= \begin{cases} 
1, & \text{if } i = m \text{ and } j = l; \\
0, & \text{if } i \neq m \text{ or } j \neq l.
\end{cases}
\]

By virtue of Hilbert space and orthonormal basis, the following proposition is natural.

**Proposition 4.2.** For any \( f(t, n) \in L^2(\mathbb{R}^+ \times \mathbb{N}, \varrho(\mu; n) \times \lambda) \), it admits an orthogonal expansion

\[
f(t, n) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij} L_j(t) \mathcal{C}_i(\mu; n),
\]

(4.14)

where \( b_{ij} = (f(t, n), L_j(t) \mathcal{C}_i(\mu; n)) \).

For the sake of convenient discussion, denote \( b_i = b_i(\mu, f) = \sum_{n=0}^{\infty} f(t, n) \mathcal{C}_i(\mu; n) \varrho(\mu; n) \), which is actually the coefficient of expansion of \( f(t, n) \) in terms of \( \mathcal{C}_i(\mu; n) \) in the space \( l^2(\mathbb{N}, \varrho(\mu; n)) \). Notice that \( b_{ij} = \int_0^{\infty} b_i(\mu, f) L_j(t) dt \) indicating that the expansion (4.14) can be regarded as a two-step expansion, firstly in \( l^2(\mathbb{N}, \varrho(\mu; n)) \), then in \( L^2(\mathbb{R}^+, \lambda) \). In addition, by Parseval-Bessel equality,

\[
\|f(t, n)\|_{L^2(\mathbb{R}^+ \times \mathbb{N})}^2 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij}^2 = \sum_{i=0}^{\infty} \|b_i(\mu, f)\|_{L^2(\mathbb{R}^+)}^2.
\]

(4.15)

Given a bundle of truncation parameters \( k \) for \( i \) and \( p_i \) for \( j \)s, one has a truncation series for (4.14)

\[
f_{k,p}(t, n) = \sum_{i=0}^{k} \sum_{j=0}^{p_i} b_{ij} L_j(t) \mathcal{C}_i(\mu; n).
\]

(4.16)
Next theorem will show the degree of approximation $f_{k,p}(t, n)$ to $f(t, n)$. We coin two notations $\nabla_n$ and $\triangle_n$ to signify that the differences of two types are operated with respect to $n$ respectively.

**Theorem 4.4.** Suppose $f(t, n) \in L^2(\mathbb{R}^+ \times \mathbb{N}, \sigma(\mu t; n) \times \lambda)$ for some $\mu > 0$. Moreover, $t^{r_1/2}\nabla_n^r f(t, n + r_1) \in L^2(\mathbb{R}^+ \times \mathbb{N}, \sigma(\mu t; n) \times \lambda)$ for some positive integer $r_1 > 1$, and $b_i = b_i(\mu, f) = \sum_{n=0}^{\infty} f(t, n) \mathcal{C}_i(\mu t; n) \sigma(\mu t; n)$ has derivative until $r_2$-th order such that $t^{r_2/2}D^{r_2}b_i \in L^2(\mathbb{R}^+)$ for all sufficient large $i$. Then

$$\|f(t, n) - f_{k,p}(t, n)\|_{L^2(\mathbb{R}^+ \times \mathbb{N})}^2 \leq \frac{\mu^{r_1}}{k^{r_1}} R_1^2(k) + \frac{k}{p^{r_2}} R_2^2(p_{min}),$$

(4.17)

where $R_1^2(k) = (1+o(1))\sum_{i=k+1}^{\infty} \|b_i - r_i(\mu t; t^{r_1/2}\nabla_n^r f(t, n + r_1))\|^2$, $p_{min} = \min\{p_0, p_1, \cdots, p_k\}$ and $R_2^2(p_{min}) = \sum_{j=p_{min}+1}^{\infty} \|b_i^o(\mu t, \tilde{b}_i(t))\|^2$ is an infinitesimal with $p_{min} \to \infty$ in which $\tilde{b}_i(t) = t^{r_2/2}e^{t/2}[b_i(\mu, f)e^{t/2}]^{(r_2)}$.

**Proof.** It follows from the orthogonality that

$$\|f(t, n) - f_{k,p}(t, n)\|_{L^2(\mathbb{R}^+ \times \mathbb{N})}^2 = \sum_{i=0}^{k} \sum_{j=p_{i+1}}^{\infty} b_{ij} \mathcal{L}_j(t) \mathcal{C}_i(\mu t; n) + \sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} b_{ij} \mathcal{L}_j(t) \mathcal{C}_i(\mu t; n)$$

$$= \sum_{i=0}^{k} \sum_{j=p_{i+1}}^{\infty} b_{ij}^2 + \sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} b_{ij}^2$$

$$= \sum_{i=0}^{k} \left\|b_i - \sum_{j=0}^{p_i} b_{ij} \mathcal{L}_j(t)\right\|_{L^2(\mathbb{R}^+)}^2 + \sum_{i=k+1}^{\infty} \left\|b_i\right\|_{L^2(\mathbb{R}^+)}^2.$$

According to Theorem 4.1, $b_i = (-1)^{r_1} \sqrt{\left(\mu t)^{r_1}(i-r_1)\right) b_{i-r_1}(\mu t; \nabla_n^{r_1} f(t, n + r_1))}$,

$$\sum_{i=k+1}^{\infty} \left\|b_i\right\|_{L^2(\mathbb{R}^+)}^2 = \sum_{i=k+1}^{\infty} \left\|(-1)^{r_1} \sqrt{\left(\mu t)^{r_1}(i-r_1)\right) b_{i-r_1}(\mu t; \nabla_n^{r_1} f(t, n + r_1))\right\|^2$$

$$= \sum_{i=k+1}^{\infty} \frac{\mu^{r_1}}{i!} \left\|t^{r_1/2}b_{i-r_1}(\mu t; \nabla_n^{r_1} f(t, n + r_1))\right\|^2$$

$$= \sum_{i=k+1}^{\infty} \frac{\mu^{r_1}}{i!} \left\|b_{i-r_1}(\mu t; t^{r_1/2}\nabla_n^{r_1} f(t, n + r_1))\right\|^2$$

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\[ \leq \frac{\mu^r_i (k + 1 - r_1)!}{(k + 1)!} \sum_{i=k+1}^{\infty} \left\| b_{i-r_1}(\mu t; t^{r_1/2} \nabla_{\mathcal{R}_i} f(t, n + r_1)) \right\|^2 \]

\[ = \frac{\mu^r_i}{k^{r_1}} R_i^2(k), \]

where \( R_i^2(k) = (1 + o(1)) \sum_{i=k+1}^{\infty} \left\| b_{i-r_1}(\mu t; t^{r_1/2} \nabla_{\mathcal{R}_i} f(t, n + r_1)) \right\|^2 \) which will converge to zero as \( k \to \infty \) since \( t^{r_1/2} \nabla_{\mathcal{R}_i} f(t, n + r_1) \in L^2(\mathbb{R}^+ \times \mathbb{N}). \)

In addition, it follows from the result in Chapter one that \( \left\| b_i - \sum_{j=0}^{p_i} b_{ij} \mathcal{L}_j(t) \right\|^2 \leq \frac{(p_i + 1 - r_2)!}{(p_i + 1)!} R_2^2(p_i) \), where \( R_2^2(p_i) = \sum_{j=p_i+1}^{\infty} \left| b_{i-r_2}(\mu t, \tilde{b}_i(t)) \right|^2 \) is an infinitesimal with \( p_i \to \infty \) in which \( \tilde{b}_i(t) = t^{r_2/2} e^{-t/2} [b_1(\mu t, f) e^{t/2}]^{(r_2)} \). Note that \( b_{i-r_2}^{(r_2)} \) is not the \( r_2 \)-th derivative but the coefficient of expansion of the function involved with respect to \( \mathcal{L}_j^{(r_2)}(t) \).

In conclusion,

\[ \| f(t, x) - f_{k,p}(t, x) \|_{L^2(\mathbb{R}^+ \times \mathbb{N})}^2 = \sum_{i=k+1}^{\infty} \| b_i \|_{L^2(\mathbb{R}^+)}^2 + \sum_{i=0}^{k} \left\| b_i - \sum_{j=0}^{p_i} b_{ij} \mathcal{L}_j(t) \right\|_{L^2(\mathbb{R}^+)}^2 \]

\[ \leq \frac{\mu^r_i}{k^{r_1}} R_i^2(k) + \sum_{i=0}^{k} \frac{(p_i + 1 - r_2)!}{(p_i + 1)!} R_2^2(p_i) \]

\[ \leq \frac{\mu^r_i}{k^{r_1}} R_i^2(k) + \frac{k(p_{\min} + 1 - r_2)!}{(p_{\min} + 1)!} R_2^2(p_{\min}) \]

\[ \leq \frac{\mu^r_i}{k^{r_1}} R_i^2(k) + \frac{k}{p_{\min}} R_2^2(p_{\min}). \]

\[ \square \]

2. Stochastic process expansion

Let \( N = (N_t, t \geq 0) \) be a Poisson process with intensity parameter \( \mu > 0 \). As indicated by the definition, \( N_t \) follows a Poisson distribution \( \text{Pio}(\mu t) \), i.e., its density is \( g(\mu t; n) \) at nonnegative integer point \( n \).

For \( t > 0 \) fixed and a function \( f(t, n) \in l^2(\mathbb{N}, g(\mu t; n)) \), \( E f^2(t, N_t) = \sum_{n=0}^{\infty} f^2(t, n) g(\mu t; n) \) \( < \infty \). That enables us to consider a map between \( l^2(\mathbb{N}, g(\mu t; n)) \) and \( L^2(\Omega) \):

\[ \mathcal{T}_t : f(t, n) \mapsto f(t, N_t). \] (4.18)

Since \( \langle f(t, N_t), g(t, N_t) \rangle = E[f(t, N_t)g(t, N_t)] = (f(t, n), g(t, n)) \) for any two functions \( f(t, n), g(t, n) \in l^2(\mathbb{N}, g(\mu t; n)) \), the inner product is preserved by the mapping \( \mathcal{T}_t \). Therefore
many properties are retained by the transformation as well. For example, the orthonormal basis \( C_i(\mu t; n) \) has been converted as an orthogonal system \( C_i(\mu t; N_t) \) in \( L^2(\Omega) \). Meanwhile, in view of completeness of \( l^2(\mathbb{N}, \varrho(\mu t; n)) \), the image of the mapping

\[
\Theta_t = \{ f(t, N_t) : f(t, n) \in l^2(\mathbb{N}, \varrho(\mu t; n)) \}
\]

forms a closed subspace in \( L^2(\Omega) \), hence it is a Hilbert space as well. Accordingly, \( C_i(\mu t; N_t) \) becomes an orthonormal basis in \( \Theta_t \).

**Proposition 4.3.** For any \( f(t, N_t) \in \Theta_t \), it admits an orthogonal expansion

\[
f(t, N_t) = \sum_{i=0}^{\infty} b_i(t) C_i(\mu t; N_t)
\]

where \( b_i(t) = b_i(\mu t, f) = \langle f(t, N_t), C_i(\mu t; N_t) \rangle_{\Theta_t} \).

**Proof.** By virtue of Hilbert space, it follows.

Our next step is to relax \( t \) being any real number on \( \mathbb{R}^+ \). The relaxation of \( t \) makes all elements \( f(t, N_t) \in \Theta_t \) become stochastic processes. However, the coefficient \( b_i(t) \) therefore turns into a function of \( t \). It is reasonable to require that \( b_i(t) \) be in \( L^2(\mathbb{R}^+) \) since in which the functions are tractable.

Consider a mapping from \( L^2(\mathbb{R}^+ \times \mathbb{N}, \lambda \times \varrho(\mu t; n)) \) into random space

\[
\mathcal{T} : f(t, n) \mapsto f(t, N_t).
\]

It is tantamount to saying that \( \mathcal{T} \) maps a family of functions with respect to \( t \) defined on \( \mathbb{N} \) into a stochastic process. Let \( \Theta \) be the image of \( \mathcal{T} \),

\[
\Theta = \{ f(t, N_t) : f(t, n) \in L^2(\mathbb{R}^+ \times \mathbb{N}, \lambda \times \varrho(\mu t; n)) \}.
\]

Note that if \( f(t, n) \in L^2(\mathbb{R}^+ \times \mathbb{N}, \lambda \times \varrho(\mu t; n)) \), \( \int_{\mathbb{R}^+} \sum_{n=0}^{\infty} f(t, n) \varrho(\mu t; n)dt < \infty \); that amounts to say that for almost every \( t \), \( Ef^2(t, N_t) = \sum_{n=0}^{\infty} f(t, n)^2 \varrho(\mu t; n) < \infty \). If it happens that for some fixed \( t = t_0 > 0 \), \( Ef^2(t, N_t) \) does not exist, we fix the values of \( f(t_0, n) \) such that the expectation is finite. Without loss of generality, in the sequel we assume
\(Ef^2(t, N_t) < \infty\) for every \(t > 0\). Therefore, from every single \(t\) point of view, \(\Theta\) is a subset of \(L^2(\Omega)\).

Introduce an operation in \(\Theta \times \Theta\):

\[
\langle f(t, N_t), g(t, N_t) \rangle = \int_0^\infty E[f(t, N_t)g(t, N_t)]dt. \tag{4.23}
\]

First of all, this operation makes sense because the conditions on \(f\) and \(g\) ensure the existence of the integral. In effect,

\[
\left| \langle f(t, N_t), g(t, N_t) \rangle \right| = \left| \int_0^\infty E[f(t, N_t)g(t, N_t)]dt \right|
\leq \int_0^\infty \sqrt{E[f^2(t, N_t)]} \sqrt{E[g^2(t, N_t)]}dt
\leq \int_0^\infty E[f^2(t, N_t)]dt \int_0^\infty E[g^2(t, N_t)]dt
= \sum_{n=0}^\infty f^2(t, n)g(\mu t; n)dt \sum_{n=0}^\infty g^2(t, n)g(\mu t; n)dt
= \|f(t, n)\|_{L^2(\mathbb{R}^+ \times N)}^2 \|g(t, n)\|_{L^2(\mathbb{R}^+ \times N)}^2 < \infty.
\]

Secondly, since \(\langle f(t, N_t), g(t, N_t) \rangle = (f(t, n), g(t, n))\), this operation possesses all properties of inner product, hence it is an inner product in \(\Theta\).

**Lemma 4.4.** The mapping \(\mathcal{T}\) defined by (4.21), has the following properties.

(1) \(\mathcal{T}\) is linear;

(2) \(\mathcal{T}\) is a one-one mapping from \(L^2(\mathbb{R}, \phi_t(x))\) to \(\Theta\);

(3) \(\mathcal{T}\) is an isomorphism.

**Lemma 4.5.** \(\Theta\), defined by (4.22), is a closed subspace of \(L^2(\Omega)\), hence it is a Hilbert space with inner product \(\langle \cdot, \cdot \rangle_\Theta\) and induced norm \(\| \cdot \|_\Theta\). In addition, \(\{\mathcal{C}_i(\mu t, N_t)\mathcal{L}_j(t)\}_{i,j=0}^\infty\) is an orthonormal basis in \(\Theta\).
Therefore, we settle down on the following theorem.

**Theorem 4.5.** Let \( N = (N_t, t \geq 0) \) be a Poisson process with intensity parameter \( \mu > 0 \). Suppose that \( f(t, n) \in L^2(\mathbb{R}^+ \times \mathbb{N}, \lambda \times \varrho(\mu; n)) \). Then \( f(t, N_t) \) admits an orthogonal expansion

\[
f(t, N_t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij}(t) \mathcal{E}_i(\mu t; N_t),
\]

(4.24)

where \( b_{ij} = \langle f(t, N_t), \mathcal{E}_i(\mu t; N_t) \rangle_\Theta \).

For the sake of convenience, denote \( b_i(t) = \sum_{n=0}^{\infty} f(t, n) \mathcal{E}_i(\mu t; n) \varrho(\mu, n) \). Then \( b_{ij} = \int_{\mathbb{R}^+} b_i(t) \mathcal{E}_j(t) dt \). It can be shown that \( b_i(t) \) for all \( i \) are square integrable on \( \mathbb{R}^+ \). In fact,

\[
\int_0^{\infty} (b_i(t))^2 dt = \int_0^{\infty} (\langle f(t, N_t), \mathcal{E}_i(\mu t; N_t) \rangle)^2 dt
\]

\[
\leq \int_0^{\infty} \| f(t, N_t) \|^2 \| \mathcal{E}_i(\mu t; N_t) \|^2 dt = \int_0^{\infty} \| f(t, N_t) \|^2 dt
\]

\[
= \int_0^{\infty} \sum_{n=0}^{\infty} f^2(t, n) \varrho(\mu, n) dt < \infty,
\]

i.e. \( b_i(t) \in L^2(\mathbb{R}^+) \). Thus, \( b_{ij} \) can be viewed as the coefficients of the expansion of \( f(t, N_t) \) in terms of the orthonormal basis \( \mathcal{E}_i(\mu t; n) \) in \( L^2(\mathbb{N}, \varrho(\mu, n)) \).

Given a bundle of truncation parameters \( k \) for \( i \) and \( p_i \) for \( j \)'s, one has a truncation series for (4.24)

\[
f_{k,p}(t, N_t) = \sum_{i=0}^{k} \sum_{j=0}^{p_i} b_{ij}(t) \mathcal{E}_i(\mu t; N_t).
\]

(4.25)

**Theorem 4.6.** Let \( N = (N_t, t \geq 0) \) be a Poisson process with intensity parameter \( \mu > 0 \). Suppose that \( f(t, N_t) \in \Theta \). Moreover, there is a positive integer \( r_1 \) such that \( \nabla_{n_1}^r f(t, n+r_1) \in l^2(\mathbb{N}, \varrho(\mu t; n)) \) for each \( t > 0 \). Furthermore, \( b_i = b_i(\mu t, f) = \sum_{n=0}^{\infty} f(t, n) \mathcal{E}_i(\mu t; n) \varrho(\mu t; n) \) has derivatives up to \( r_2 \)-th order \( (r_2 > 1) \) such that \( t^{r_2/2} D^{r_2} b_i \in L^2(\mathbb{R}^+) \) for all sufficient large \( i \).

Then

\[
\| f(t, N_t) - f_{k,p}(t, N_t) \|_{L^2}^2 \leq \frac{\mu r_1}{k r_1} R_1^2(k) + \frac{k}{r_2!} R_2^2(p_{\min}),
\]

(4.26a)

\[
E(f(t, N_t) - f_{k,p}(t, N_t))^2 \leq \mu r_1 A_k(t) \frac{(k + 1 - r_1)!}{(k + 1)!} + B \frac{k}{\sqrt{t} r_2 - 1/2} R_2^2(p_{\min})
\]

(4.26b)
where $R_2^2(k) = (1 + o(1)) \sum_{i=k+1}^{\infty} \left\| b_{i-r_1}(\mu; t; \nu_1/2 \nabla_{\nu_1} f(t, n + r_1)) \right\|^2$, $A_k(t)$ is a quantity related with $t, k$ such that for each fixed $t$, when $k \to \infty$, $A_k(t) \to 0$; $B$ is a constant, $p_{\min} = \min\{p_0, \cdots, p_k\}$, $R_2^2(p_i) = \sum_{j=p_i+1}^{\infty} [b_{i-j-r_2}^r(\tilde{b}_i)]^2$ is an infinitesimal with $p_i \to \infty$ in which $\tilde{b}_i(t) = t^{r_2/2} e^{-t_2/2} [b_i(t)e^{t/2}]^{r_2}$.

Proof. It follows from the orthogonality of $\mathcal{C}_i(\mu; N_i)$ that

$$E(f(t, N_i) - f_{k,p}(t, N_i))^2$$

$$= E \left( \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} b_{ij} \mathcal{L}_j(t) \mathcal{C}_i(\mu; N_i) + \sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} b_{ij} \mathcal{L}_j(t) \mathcal{C}_i(\mu; N_i) \right)^2$$

$$= \sum_{i=0}^{k} \left( \sum_{j=p_i+1}^{\infty} b_{ij} \mathcal{L}_j(t) \right)^2 + \sum_{i=k+1}^{\infty} \left( \sum_{j=0}^{\infty} b_{ij} \mathcal{L}_j(t) \right)^2$$

$$= \sum_{i=0}^{k} \left( \sum_{j=p_i+1}^{\infty} b_{ij} \mathcal{L}_j(t) \right)^2 + \sum_{i=k+1}^{\infty} b_i^2(t).$$

In view of (4.8), $b_i(t) = b_i(\mu; f(t, x)) = (-1)^{r_1} \sqrt{\frac{(\mu)^{r_1}(i-r_1)!}{i!}} b_{i-r_1}(\mu; \nabla_{\nu_1} f(t, x + r_1))$

$$\leq (\mu)^{r_1} (k + 1 - r_1)! \sum_{i=k+1}^{\infty} b_{i-r_1}^2(\mu; \nabla_{\nu_1} f(t, x + r_1)).$$

Since $\sum_{i=k+1}^{\infty} b_{i-r_1}^2(\mu; \nabla_{\nu_1} f(t, x + r_1)) = \|\nabla_{\nu_1} f(t, x + r_1)\|^2_{H(\nabla_{\nu_1}(\mu, x))}$, for any fixed $t$, $A_t(k) := t^{r_1} \sum_{i=k+1}^{\infty} b_{i-r_1}^2(\mu; \nabla_{\nu_1} f(t, x + r_1))$ will converge to zero as $k \to \infty$.

On the other hand, it follows from Chapter one that

$$\left( \sum_{j=p_i+1}^{\infty} b_{ij} \mathcal{L}_j(t) \right)^2 \leq \left( \frac{1}{r_2 - 1} \frac{1}{(p_i - r_2 + 1)^{r_2 - 1}} \right) \left( \sup_{j \geq p_i+1} |\mathcal{L}_j(t)| \right) R_2^2(p_i)$$

where $R_2^2(p_i) = \sum_{j=p_i+1}^{\infty} [b_{i-j-r_2}(\tilde{b}_i)]^2$ is an infinitesimal with $p_i \to \infty$ in which $\tilde{b}_i(t) = t^{r_2/2} e^{-t_2/2} [b_i(t)e^{t/2}]^{r_2}$. Thus, invoking the bound $\sup_{j \geq p_i+1} |\mathcal{L}_j(t)| \leq B(t_2)^{-1/4}$ yields

$$\sum_{i=0}^{k} \left( \sum_{j=p_i+1}^{\infty} b_{ij} \mathcal{L}_j(t) \right)^2$$

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\[ \leq \sum_{i=0}^{k} \left( \frac{1}{r_2 - 1} \left( p_i - r_2 + 1 \right)^{r_2 - 1} \right) \left( \sup_{j \geq p_{i+1}} |\mathcal{L}_j(t)| \right)^2 R^2(p_i) \]
\[ \leq B \frac{k}{p_{\min}^{r_2 - 1}} \left( \sup_{j \geq p_{\min} + 1} |\mathcal{L}_j(t)| \right)^2 R^2(p_{\min}) \leq B \frac{k}{\sqrt{t} p_{\min}^{r_2 - 1/2}} R^2(p_{\min}), \]

where \( B \) is a constant which may vary in the derivation.

Eventually,

\[ E (f(t, N_t) - f_{k,p}(t, N_t))^2 = \sum_{i=0}^{k} \sum_{j=p_{i+1}}^{\infty} b_{ij}^2 \phi_{jT}^2(t) \]
\[ \leq \mu r_1 A_k(t) \frac{(k+1 - r_1)!}{(k+1)!} + B \frac{k}{\sqrt{t} p_{\min}^{r_2 - 1/2}} R^2(p_{\min}). \]

\[ \square \]

4.2.2 Expansion on \([0, T] \times \mathbb{N}\]

1. Function expansion

Let \( f(t, n) \) be defined on \([0, T] \times \mathbb{N}\). Let us construct a space of functions as follows.

For a parameter \( \mu > 0 \), we introduce the product measure \( \nu \) of Lebesgue measure \( \lambda \) on \([0, T]\) and \( \varrho(\mu t, n) \) on \( \mathbb{N} \). Since both \( \lambda \) and \( \varrho(\mu t, n) \) are finite measure, \( \nu := \lambda \times \varrho(\mu t, n) \) is finite. Hence, it follows from the conventional \( L^2 \) theory that \( L^2([0, T] \times \mathbb{N}, \nu) = \{ f(t, n) : \int_{[0, T] \times \mathbb{N}} f^2(t, n) d\nu < \infty \} \) is a Hilbert space with inner product

\[ (f, g) = \int_{[0, T] \times \mathbb{N}} f(t, n) g(t, n) d\nu = \int_0^T \sum_{n=0}^{\infty} f(t, n) g(t, n) \varrho(\mu t, n) dt \]

and induced norm \( ||f|| = \sqrt{(f, f)} \).

In addition, as we know that \( \varphi_jT(t) \) and \( \mathcal{C}_i(\mu t, n) \) are orthonormal bases for \( L^2([0, T], \lambda) \) and \( L^2(\mathbb{N}, \varrho(\mu t, n)) \) respectively, it can be shown that \( \varphi_jT(t) \mathcal{C}_i(\mu t, n) \) is an orthonormal basis for \( L^2([0, T] \times \mathbb{N}, \nu) \).

Suppose now that \( f(t, n) \in L^2([0, T] \times \mathbb{N}, \nu) \). By virtue of Hilbert space, we have the following expansion for \( f(t, n) \):

\[ f(t, n) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij} \varphi_jT(t) \mathcal{C}_i(\mu t, n), \quad (4.27) \]
Accordingly, it follows from Parseval-Bessel equality that

\[ \sum_{n=0}^{\infty} f(t, n) \mathcal{C}_i(\mu t, n) \phi(\mu t, n) dt, \]

we denote \( b_i(t, f) = \sum_{n=0}^{\infty} f(t, n) \mathcal{C}_i(\mu t, n) \phi(\mu t, n) \) and it is known that \( b_i(t, f) \) is finite almost everywhere [\( \lambda \)]. If necessary, we modify the values of \( b_i(t, f) \) such that it is finite at every \( t \).

Thus, \( b_i(t, f) \) can be viewed as the coefficient function of the expansion of \( f(t, n) \) by means of \( \mathcal{C}_i(\mu t, n) \) in the space \( L^2(N, \phi(\mu t, n)) \). Also, due to the relation between \( b_{ij} \) and \( b_i(t, f) \) and the square integrability of \( b_i(t, f) \) on \([0, T]\) implied by Cauchy-Schwarz inequality

\[ \int_0^T b_i^2(t, f) dt = \int_0^T (f(t, n), \mathcal{C}_i(\mu t, n))^2 dt \leq \int_0^T \|f(t, n)\|^2 dt < \infty, \]

where \( b_{ij} \) is the coefficients of the expansion of \( b_i(t, f) \) in terms of \( \varphi_{jT}(t) \) in the space \( L^2[0, T] \).

Accordingly, it follows from Parseval-Bessel equality that

\[ \|f(t, n)\|_{L^2([0, T] \times N)}^2 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij}^2 = \sum_{i=0}^{\infty} \|b_i(t, f)\|_{L^2(N)}^2. \]

The expansion (4.27) therefore is regarded as a two-step expansion, firstly in \( L^2(N, \phi(\mu t, n)) \), then in \( L^2[0, T] \).

Given a bundle of truncation parameters \( k \) for \( i \) and \( p_i \) for \( j \)’s (all truncation parameters will diverge in some kind of styles which will be specified when it is necessary), the truncation series to (4.27) is defined as

\[ f_{k,p}(t, n) = \sum_{i=0}^{k} \sum_{j=0}^{p_i} b_{ij} \varphi_{jT}(t) \mathcal{C}_i(\mu t, n). \] (4.28)

**Theorem 4.7.** Suppose that \( f(t, n) \in L^2([0, T] \times N, \nu) \). Moreover, there is a positive integer \( r \) such that \( \nabla^r_n f(t, n + r) \in L^2(N, \phi(\mu t; n)) \) for each \( t \in [0, T] \), and \( r^{\nu/2} \nabla^r_n f(t, n + r) \in L^2(\mathbb{R}^+ \times N) \). Furthermore, \( b_i = b_i(t, f) = \sum_{n=0}^{\infty} f(t, n) \mathcal{C}_i(\mu t; n) \phi(\mu t; n) \) has derivatives up to of second order, and \( \sup_i \max(|b'_i(0)|, |b'_i(T)|) \leq B \). Denote \( p_{\min} = \min(p_0, p_1, \ldots, p_k) \). Suppose that \( k/p_{\min}^3 \to 0 \). Then

\[ \|f(t, n) - f_{k,p}(t, n)\|_{L^2([0, T] \times N)}^2 \leq C B^2 T^3 \frac{k}{p_{\min}^3} + \frac{\mu r}{k^p} R^2(k) \]

where \( C \) is a pure constant; \( R^2(k) = (1 + o(1)) \sum_{i=k+1}^{\infty} \|b_i-r(\mu t, t^{\nu/2} \nabla^r_n f(t, n + r))\|^2 \) which will converge to zero as \( k \to \infty \).
Proof.

\[
\| f(t, n) - f_{k,p}(t, n) \|_{L^2([0,T] \times \mathbb{N})}^2 = \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} b_{ij} \varphi_j(t) \| \mathcal{E}_i(\mu, n) + \sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} b_{ij} \varphi_j(t) \| \mathcal{E}_i(\mu, n) \|^{2} \\
= \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} b_{ij}^2 + \sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} b_{ij}^2 \\
= \sum_{i=0}^{k} \left| b_i(t, f) - \sum_{j=0}^{p_i} b_{ij} \varphi_j(t) \right| \| \mathcal{E}_i(\mu, n) \|^{2} + \sum_{i=k+1}^{\infty} \left| b_i(t, f) \right| \| \mathcal{E}_i(\mu, n) \|^{2} \\
= \sum_{i=k+1}^{\infty} \| b_i(t, f) \|_{L^2([0,T])}^2.
\]

Since \( f(t, n) \) satisfies all the conditions about variable \( n \) in Theorem 4.1, according to (4.8),

\[
b_i(t, f) = (-1)^r \sqrt{\frac{(\mu)^r (i-r)}{i!}} b_{i-r} (\mu; \nabla_n f(t, n + r)).
\]

Thus,

\[
\sum_{i=k+1}^{\infty} \| b_i(t, f) \|_{L^2(\mathbb{R}^+)}^2 = \sum_{i=k+1}^{\infty} \left| (-1)^r \sqrt{\frac{(\mu)^r (i-r)}{i!}} b_{i-r} (\mu; \nabla_n f(t, n + r)) \right|^{2} \\
= \sum_{i=k+1}^{\infty} \frac{\mu^r (i-r)!}{i!} \left| t^{r/2} b_{i-r} (\mu; \nabla_n f(t, n + r)) \right|^{2} \\
= \sum_{i=k+1}^{\infty} \frac{\mu^r (i-r)!}{i!} \left| b_{i-r} (\mu; t^{r/2} \nabla_n f(t, n + r)) \right|^{2} \\
\leq \frac{\mu^r (k + 1 - r)!}{(k + 1)!} \sum_{i=k+1}^{\infty} \left| b_{i-r} (t; t^{r/2} \nabla_n f(t, n + r)) \right|^{2} \\
= \frac{\mu^r}{k^r} R^2(k),
\]

where \( R^2(k) = (1 + o(1)) \sum_{i=k+1}^{\infty} \left| b_{i-r} (\mu; t^{r/2} \nabla_n f(t, n + r)) \right|^{2} \) which will converge to zero as \( k \to \infty \) since \( t^{r/2} \nabla_n f(t, n + r) \in L^2(\mathbb{R}^+ \times \mathbb{N}) \).

For another part, since \( b_i(t, f) \) is twice differentiable, the result in the first chapter gives

\[
\left\| b_i(t, f) - \sum_{j=0}^{p_i} b_{ij} \varphi_j(T)(t) \right\|_{L^2([0,T])}^2 \leq C_1 \frac{B^2 T^3}{p_i^4} + C_2 \frac{BT^{3.5}}{p_i^4} R_2(p_i) + C_3 \frac{T^4}{p_i^4} R_2^2(p_i) \\
= C_1 (1 + o(1)) \frac{B^2 T^3}{p_i^4}.
\]

where \( C_j (j = 1, 2, 3) \) are absolutely constants; \( R_2^2(p_i) = \sum_{j=p_i+1}^{\infty} \left| b_j (g^r) \right|^{2} \) which converge to
zero when $N \to \infty$. Thus,
\[
\sum_{i=0}^{k} \left\| b_i(t, f) - \sum_{j=0}^{p_i} b_{ij} \varphi_j(t) \right\|_{L^2[0,T]}^2 \leq \sum_{i=0}^{k} C_1(1 + o(1)) \frac{B^2 T^3}{p_i^3} \leq C_1(1 + o(1)) \frac{B^2 T^3 k}{p_{\min}^3}
\]

Consequently, the assertion follows.

\[\square\]

Remark 4.1. There are a variety of functions satisfying the conditions of the theorem. Take $f_1(t, n) = t \sin(n)$ and $f_2(t, n) = t \cos(n)$. Let us calculate the differences:

\[
\nabla \sin(n + 1) = \sin(n + 1) - \sin(n) = \sin(n) \cos 1 + \sin 1 \cos(n) - \sin(n) = - (1 - \cos 1) \sin(n) + \sin 1 \cos(n);
\]

\[
\nabla^2 \sin(n + 2) = \nabla(\sin(n + 2) - \sin(n + 1)) = \nabla \sin(n + 2) - \nabla \sin(n + 1) = [\sin(n + 2) - \sin(n + 1)] - [\sin(n + 1) - \sin(n)] = 2 \sin \frac{1}{2} \cos \frac{2n + 3}{2} - 2 \sin \frac{1}{2} \cos \frac{2n + 1}{2} = 2 \sin \frac{1}{2} \left[ \cos \frac{2n + 3}{2} - \cos \frac{2n + 1}{2} \right] = -4 \sin^2 \frac{1}{2} \sin(n + 1) = -4 \sin^2 \frac{1}{2} \cos 1 \sin(n) - 4 \sin^2 \frac{1}{2} \sin 1 \cos(n);
\]

\[
\nabla^3 \sin(n + 3) = \nabla(\nabla^2 \sin(n + 3)) = -4 \sin^2 \frac{1}{2} \nabla \sin(n + 2) = -4 \sin^2 \frac{1}{2} \cos 2 \sin(n) - 4 \sin^2 \frac{1}{2} \sin 2 \cos(n).
\]

Suppose that $\nabla^{r-1} \sin(n + r - 1) = A_{r-1} \sin(n) + B_{r-1} \cos(n)$, then

\[
\nabla^r \sin(n + r) = \nabla(\nabla^{r-1} \sin(n + r)) = \nabla[\nabla^{r-1} \sin(n + 1 + r - 1)] = \nabla[A_{r-1} \sin(n + 1) + B_{r-1} \cos(n + 1)] = A_{r-1}[\sin(n + 1) - \sin(n)] + B_{r-1}[\cos(n + 1) - \cos(n)] = -A_{r-1}(1 - \cos 1) \sin(n) + A_{r-1} \sin 1 \cos(n)
\]
where $A_r = -B_{r-1} \sin 1 - A_{r-1}(1 - \cos 1)$, and $B_r = A_{r-1} \sin 1 - B_{r-1}(1 - \cos 1)$.

We therefore assert that for any $r \geq 1$, $\nabla^r \sin(n + r) = A_r \sin(n) + B_r \cos(n)$ where $A_r$ and $B_r$ are constants.

It follows from Example 2 that $t^{r/2} \nabla^r f(t, n+r) \in L^2(\mathbb{R}^+ \times \mathbb{N})$. Meanwhile, the derivatives of $b_i(t, f_1)$ and $b_i(t, f_2)$ contain a factor $t^{i/2} / \sqrt{i!}$ which converges to zero uniformly on $[0, T]$, and the rest are continuous function on the interval. Hence, $b_i(t, f_1)$ and $b_i(t, f_2)$ satisfy the conditions.

It should be pointed out that any function in the form of either $t^q \sin x$ or $t^q \cos x$ where $q(t)$ is smooth enough satisfies the theorem. In addition, the class of functions in the form $q(t)p_k(x)$ where $p_k(x)$ is a polynomial of degree $k < \infty$ and $q(t)$ smooth enough satisfies the conditions, since $b_i(t, q(t)p_k(x)) = 0$ if $i > k$.

2. Stochastic process expansion

Suppose that $f(t, n) \in L^2([0, T] \times \mathbb{N}, \nu)$, namely, $\int_0^T \sum_{n=0}^{\infty} f^2(t, n) g(\mu t, n) dt < \infty$ for some $\mu > 0$. We now construct a mapping between $L^2([0, T] \times \mathbb{N})$ and $L^2([0, T] \times \Omega)$:

$$T : f(t, n) \mapsto f(t, N_t),$$

where $N_t$ is a Poisson process on $[0, T]$ with intensity $\mu$. Denote by $\Xi$ the image of $T$. It follows from the definition of $T$ that $\Xi$ is a linear vector space. Let us define an operation on $\Xi \times \Xi$:

$$\langle f(t, N_t), g(t, N_t) \rangle_\Xi = \int_0^T E[f(t, N_t)g(t, N_t)] dt.$$

Since $\langle f(t, N_t), g(t, N_t) \rangle_\Xi = (f(t, n), g(t, n))_{L^2([0,T] \times \mathbb{N})}$, it is evident that $\langle \cdot, \cdot \rangle_\Xi$ is an inner product in $\Xi$. Meanwhile, it is easy to verify that $T$ is linear, one-one mapping and isomorphism, and $\Xi$ is a closed subspace of $L^2([0, T] \times \Omega)$. Hence, $\Xi$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_\Xi$ and induced norm. In view of that $T$ is inner product preserving, $T$ maps the orthonormal basis $\varphi_j(t) e_i(\mu t, n)$ of $L^2([0, T] \times \mathbb{N}, \nu)$ into $\Xi$ as its counterpart. Then, we have the following theorem.
Theorem 4.8. For any process \( f(t, N_i) \in \mathcal{X} \), it admits an Fourier expansion

\[
f(t, N_i) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} \varphi_{jT}(t) \mathcal{G}_i(\mu t, N_i)
\]

for \( t \in [0, T] \), where \( c_{ij} = \langle f(t, N_i), \varphi_{jT}(t) \mathcal{G}_i(\mu t, N_i) \rangle_\mathcal{X} \).

Proof. It follows because \( \mathcal{X} \) is a Hilbert space with orthonormal basis \( \varphi_{jT}(t) \mathcal{G}_i(\mu t, N_i) \).

In view of \( c_{ij} = \int_{0}^{T} \varphi_{jT}(t) E[f(t, N_i) \mathcal{G}_i(\mu t, N_i)] dt \), define \( c_i(t, f) = E[f(t, N_i) \mathcal{G}_i(\mu t, N_i)] \) for the sake of convenience. Hence, \( c_{ij} = \int_{0}^{T} \varphi_{jT}(t) c_i(t, f) dt \). \( c_i(t, f) \) is a function on \([0, T]\) which belongs to \( L^2[0, T] \) as indicated by Cauchy-Schwarz inequality. Since \( \int_{0}^{T} E[f^2(t, N_i)] dt = \int_{0}^{T} \sum_{n=0}^{\infty} f^2(t, n) g(\mu t, n) dt < \infty \), \( E[f^2(t, N_i)] < \infty \) almost everywhere, that is tantamount to saying that except perhaps on a \( t \)-negligible set, \( E[f^2(t, N_i)] \) is finite. However, on the possible negligible set, we would fix the value of \( f(t, N_i) \) to be zero such that \( E[f^2(t, N_i)] \) is finite for every \( t \). Thus, \( f(t, N_i) \in L^2(\mathbb{N}, g(\mu t, n)) \). It therefore follows from (4.11) that \( c_i(t, f) = E[f(t, N_i) \mathcal{G}_i(\mu t, N_i)] \) is a coefficient of the expansion of \( f(t, N_i) \) in terms of \( \mathcal{G}_i(\mu t, N_i) \). Moreover, the expansion (4.30) can be regarded as a two-step expansion, first in \( L^2(\mathbb{N}, g(\mu t, n)) \), then in \( L^2[0, T] \). Observe that Parseval-Bessel equality gives

\[
\|f(t, N_i)\|_\mathcal{X}^2 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij}^2 = \sum_{i=0}^{\infty} \|c_i(t, f)\|_{L^2[0, T]}^2.
\]

Given a bundle of truncation parameters \( k \) for \( i \) and \( p_i \) for \( j \)'s (all truncation parameters will diverge in some kind of styles which will be specified when it is necessary), the truncation series to (4.30) is defined as

\[
f_{k,p}(t, N_i) = \sum_{i=0}^{k} \sum_{j=0}^{p_i} c_{ij} \varphi_{jT}(t) \mathcal{G}_i(\mu t, N_i).
\]

Theorem 4.9. Suppose \( f(t, n) \in L^2([0, T] \times \mathbb{N}, \nu) \). Moreover, there is an integer \( r \) such that \( \nabla_n^r f(t, n + r) \in L^2(\mathbb{N}, g(\mu t, n)) \). Furthermore, \( c_i(t, f) = E[f(t, N_i) \mathcal{G}_i(\mu t, N_i)] \) is twice differentiable for every \( i \) on \([0, T]\) and \( \sup_i \max(|c_i'(0)|, |c_i''(T)|) < \infty \). Then

\[
E(f(t, N_i) - f_{k,p}(t, N_i))^2 \leq M(T) \frac{k}{p_{\min}^2} + \frac{(\mu T)^r (k + 1 - r)!}{k!} R^2(t, k),
\]

where \( M(T) \) is a constant depending on \( T \); for fixed \( t \), \( R^2(t, k) = \sum_{i=k+1}^{\infty} c_{i-r}^2(\mu t; \nabla_n^r f(t, n + r)) \) will converge to zero when \( k \to \infty \).
Proof. By the orthogonality of \( \mathcal{C}_i(\mu t, N_t) \) we have

\[
E(f(t, N_t) - f_{k,p}(t, N_t))^2 = \left( \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} c_{ij} \varphi_{T,t}(t) \mathcal{C}_i(\mu t, N_t) + \sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} c_{ij} \varphi_{T,t}(t) \mathcal{C}_i(\mu t, N_t) \right)^2
\]

\[
= \left( \sum_{i=0}^{k} \left( \sum_{j=p_i+1}^{\infty} c_{ij} \varphi_{T,t}(t) \right)^2 + \sum_{i=k+1}^{\infty} \left( \sum_{j=0}^{\infty} c_{ij} \varphi_{T,t}(t) \right)^2 \right)^2
\]

\[
= \sum_{i=0}^{k} \left( c_i(t, f) - \sum_{j=+1}^{p_i} c_{ij} \varphi_{T,t}(t) \right)^2 + \sum_{i=k+1}^{\infty} c_i(t, f)^2.
\]

It follows from Chapter one that

\[
\left( c_i(t, f) - \sum_{j=+1}^{p_i} c_{ij} \varphi_{T,t}(t) \right)^2 \leq C T^2 a_i^2 f_i^2 (1 + o(1))
\]

where \( a_i = |c_i'(T, f)| + |c_i'(0, f)| \). Thus,

\[
\sum_{i=0}^{k} \left( c_i(t, f) - \sum_{j=+1}^{p_i} c_{ij} \varphi_{T,t}(t) \right)^2 \leq C T^2 \sum_{i=0}^{k} a_i^2 f_i^2 \leq M(T) \frac{k}{P_{\min}},
\]

where \( M(T) = C T^2 \sup_{t} \max(|c_i'(0, f)|, |c_i'(T, f)|)^2 \) in which \( C \) is a pure constant.

Meanwhile, according to (4.8), \( c_i(t, f) = (-1)^r \sqrt{\frac{(\mu t)^r (i-r)!}{n^r}} c_{i-r}(\mu t; \nabla f(t, n + r)) \). Thus,

\[
\sum_{i=k+1}^{\infty} c_i(t, f)^2 = \sum_{i=k+1}^{\infty} \frac{(\mu t)^r (i-r)!}{i!} c_{i-r}^2(\mu t; \nabla f(t, n + r))
\]

\[
\leq \frac{(\mu T)^r (k + 1 - r)!}{(k + 1)!} \sum_{i=k+1}^{\infty} c_{i-r}^2(\mu t; \nabla f(t, n + r)) = \frac{(\mu T)^r (k + 1 - r)!}{(k + 1)!} R^2(t, k)
\]

where for fixed \( t \), \( R^2(t, k) = \sum_{i=k+1}^{\infty} c_{i-r}^2(\mu t; \nabla f(t, n + r)) \) will converge to zero when \( k \to \infty \).

\[
\square
\]

5 Applications in econometric estimation

Let \( N_t \) be a Poisson process with intensity \( \mu \), which implies that for every \( t > 0 \), \( N_t \sim \text{Poi}(\mu t) \).

Since the effect of \( \mu \) can be scaled by variable \( t \), we simply assume in what follows that \( \mu = 1 \).
The econometric model we are going to study is as follows

\[ Y_t = f(t, N_t) + \varepsilon(t), \quad t \in [0, T] \quad (5.1) \]

where \( f \) is unknown functional, \( N_t \) is a Poisson process with intensity 1, \( \varepsilon(t) \) is an error process with zero mean and finite variance.

We shall consider three types of sampling, that is, observations equally spaced on \([0, T]\) with fixed \( T \), observations at \( s = 1, 2, \cdots, m \) on \((0, \infty)\) and observations uniformly distributed on \([0, T_m]\) with \( T_m \to \infty \) with \( m \to \infty \). Accordingly, we divide the section into three subsections.

5.1 Finite horizon \([0, T]\) with fixed \( T \)

First of all, we propose some assumptions on the unknown functional in the model.

**Assumption A.1**

(a) Suppose that \( f(t, n) \in L^2([0, T] \times \mathbb{N}, \lambda \times g) \) and \( \nabla^2 f(t, n + 2) \in L^2([0, T] \times \mathbb{N}, \lambda \times g) \). Moreover, suppose \( f(t, n) \) as a function of \( t \) is continuous on \([0, T]\). Let \( g(n) = \max_{0 \leq t \leq T} f(t, n) \). Suppose that \( E[g(N_T)]^2 < \infty \).

(b) \( c_i(t, f) = E[f(t, N_t)\xi_i(t, N_t)] \) is twice differentiable for every \( i \geq 0 \). In addition, all these functions, \( c_i(t, f) \) and its first two derivatives, belong to \( L^2[0, T] \).

(c) Moreover, \( \sup_{i \geq 0} (|c_i'(0)| + |c_i'(T)|) < C_T < \infty \).

**Remark 5.1.** All conditions are rather weak, so that there are many functions satisfying them.

(1) \( f_1(t, n) = t^n q(t)p_k(n) \) where \( \eta \geq 2 \), \( q(t) \in C^2[0, T] \) and \( p_k(\cdot) \) is a polynomial with fixed order \( k \). Since \( \nabla f_1(t, n + 1) = t^n q(t)\nabla p_k(n + 1) \), \( \nabla^2 f_1(t, n + 2) = t^n q(t)\nabla^2 p_k(n + 2) \) and the difference of any polynomial is still a polynomial with a lower order, it is evident that the condition (a) is fulfilled.

In addition, because \( c_i(t, f_1) = t^n q(t)c_i(t, p_k) \) and \( c_i(t, p_k) \) is a power function of \( t \) with power greater than or equal to 0.5, \( c_i(t, f_1) \) is a product of \( q(t) \) and a power function whose power greater than or equal to 2. On account of supposition on \( q(t) \), Condition (b) is satisfied.
The function suits (c) because $c_i(t, f_1) = 0$ when $i > k$.

(2) $f_2(t, x) = t^{2^n}$ and $f_3(t, n) = t^{2^n} - n$ with $\eta \geq 2$.

Let us focus on $f_2(t, x)$. Since $E[f_2(t, N_t)]^2 = t^{2^n}E[2^{2N_t}] = t^{2^\eta}e^{3t} < \infty$ for any $t > 0$ and $\nabla f_2(t, n + 1) = t^{2^n} = f_2(t, n)$ and $\nabla^2 f_2(t, n + 2) = f_2(t, n)$, Condition (a) is satisfied.

Apropos of Condition (b), it follows from Example 1 in Section 4 that $c_i(t, f_2) = t^{\eta}c_i(t, 2^n) = (-1)^i t^{\eta}e^t\sqrt{\frac{\nu}{i!}}$. Hence, $c_i$ is twice differentiable and both it and its derivatives pertain to $L^2[0, T]$.

As for Condition (c), note that

$$\frac{d}{dt} c_i(t, f_2) = \frac{(-1)^i}{\sqrt{i!}} t^{\eta + \frac{i}{2}} e^t + \frac{(-1)^i}{\sqrt{i!}} t^{\eta + \frac{i}{2}} e^t,$$

for $i \geq 0$. Clearly $\frac{d}{dt} c_i(t, f_2)|_{t=0} = 0$ and

$$\left| \frac{d}{dt} c_i(t, f_2)|_{t=T} \right| = \frac{\eta + \frac{i}{2}}{\sqrt{i!}} T^{\eta - 1 + \frac{i}{2}} e^T + \frac{1}{\sqrt{i!}} T^{\eta + \frac{i}{2}} e^T$$

$$= (\eta T^{\eta - 1} + T^{\eta}) e^T \sqrt{\frac{T}{i!}} + \frac{1}{2} T^{\eta} e^T \sqrt{\frac{i/2}{i!}} < M(T)$$

a constant depending on $T$, for both $\frac{T}{i!}$ and $i^2\frac{T}{i!}$ converge to zero. Therefore condition (c) is fulfilled.

In a very similar way, $f_3(t, n)$ satisfies (a), (b) and (c) too.

(3) $f_4(t, n - t) = t^\beta \sin(n)$ and $f_5(t, n) = t^\beta \cos(n)$.

Note that $f_4(t, n - t) = t^\beta \cos t \sin n - t^\beta \sin t \cos n$. It follows Example 2 in last section that

$$c_i(t, f_4) = (-1)^i t^\beta \sqrt{\frac{i}{\pi}} e^{-t(1 - \cos 1)} \frac{2}{2(1 - \cos 1)} \cos t \sin(\alpha i + t \sin 1) - \sin t \cos(\alpha i + t \sin 1)$$

$$= (-1)^i t^\beta \sqrt{\frac{\nu}{i!}} e^{-t(1 - \cos 1)} \left[2(1 - \cos 1)\right] \sin[\alpha i - t(1 - \sin 1)]$$

$$= \frac{(-1)^i}{\sqrt{i!}} \sqrt{2(1 - \cos 1)} t^{\beta + i/2} e^{-t(1 - \cos 1)} \sin[\alpha i - t(1 - \sin 1)]$$

where $\alpha = \arctan \frac{\sin 1}{\cos 1 - 1}$. 

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Apparently, if \( \eta \geq 2 \), all \( c_i(t, f_4) \) are twice differentiable on \([0, T]\) and
\[
c_i'(t, f_4) = \left( -\frac{1}{i} \right) \sqrt{2(1 - \cos 1)} \left\{ (\beta + i/2)t^{\beta - 1 + i/2}e^{-t(1 - \cos 1)} \sin[\alpha i - t(1 - \sin 1)] \\
- (1 - \cos 1)t^{\beta + i/2}e^{-t(1 - \cos 1)} \sin[\alpha i - t(1 - \sin 1)] \\
- (1 - \sin 1)t^{\beta + i/2}e^{-t(1 - \cos 1)} \cos[\alpha i - t(1 - \sin 1)] \right\}.
\]
It is clear that \( c_i'(0, f_4) = 0 \) and \( |c_i'(T, f_4)| \) are bounded uniformly in \( i \) since
\[
i \sqrt{\frac{2(1 - \cos 1)T}{i!}} \to 0.
\]
as \( i \to \infty \) for any fixed \( T \).

We have observations \( Y_{t, m} \) at \( t_{s,m} = T \frac{s}{m} \) for \( s = 1, \ldots, m \) which are uniformly distributed on \([0, T]\). On account of Assumption A.1, \( f(t, N_t) \) can be expanded at observation points into orthogonal series as follows
\[
f(t_{s,m}, N_{t_{s,m}}) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} \varphi_j T(t_{s,m}) \mathcal{C}_i(t_{s,m}, N_{t_{s,m}})
\]
\[
= \sum_{i=0}^{k} \sum_{j=0}^{p_i} c_{ij} \varphi_j T(t_{s,m}) \mathcal{C}_i(t_{s,m}, N_{t_{s,m}}) + \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} c_{ij} \varphi_j T(t_{s,m}) \mathcal{C}_i(t_{s,m}, N_{t_{s,m}})
\]
\[
+ \sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} c_{ij} \varphi_j T(t_{s,m}) \mathcal{C}_i(t_{s,m}, N_{t_{s,m}})
\]
\[
:= x'_s \theta' + \delta_s + \gamma_s, \quad \text{for} \ s = 1, \ldots, m,
\]
where
\[
\theta' = (c_{00}, \ldots, c_{0p_0}, \ldots, c_{k0}, \ldots, c_{kp_k}),
\]
\[
x'_s = (\varphi_0 T(t_{s,m}) \mathcal{C}_0(t_{s,m}, N_{t_{s,m}}), \ldots, \varphi_{p_0} T(t_{s,m}) \mathcal{C}_0(t_{s,m}, N_{t_{s,m}}),
\]
\[
\ldots, \ldots,
\]
\[
\varphi_{p_0} T(t_{s,m}) \mathcal{C}_k(t_{s,m}, N_{t_{s,m}}), \ldots, \varphi_{p_k} T(t_{s,m}) \mathcal{C}_k(t_{s,m}, N_{t_{s,m}}))
\]
\[
\delta_s = \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} c_{ij} \varphi_j T(t_{s,m}) \mathcal{C}_i(t_{s,m}, N_{t_{s,m}})
\]
\[
\gamma_s = \sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} c_{ij} \varphi_j T(t_{s,m}) \mathcal{C}_i(t_{s,m}, N_{t_{s,m}}) = \sum_{i=k+1}^{\infty} c_i(t_{s,m}, f) \mathcal{C}_i(t_{s,m}, N_{t_{s,m}}).
\]
Therefore, we obtain a system of \( m \) equations
\[
Y_{t_{s,m}} = x_s'\theta + \delta_s + \gamma_s + u_s, \quad \text{for } s = 1, \ldots, m, \tag{5.2}
\]
where \( u_s = \varepsilon(T \frac{s}{m}) \) are the error term at observation points.

We further denote \( Y' = (Y_{t_{1,m}}, \ldots, Y_{t_{m,m}}) \) and \( u' = (u_1, \ldots, u_m) \). In addition, signify \( X = (x_1, \ldots, x_m)' \) as a \( m \times (p_0 + \cdots + p_k) \) matrix and \( \delta' = (\delta_1, \ldots, \delta_m) \) and \( \gamma' = (\gamma_1, \ldots, \gamma_m) \) as \( m \)-dimension column vectors. Therefore, we can write the system of equations (5.3) in a matrix form
\[
Y = X\theta + \delta + \gamma + u. \tag{5.3}
\]

The Ordinary Least Squares (OLS) estimator of \( \theta \) is given by
\[
\hat{\theta} = (X'X)^{-1}X'Y. \tag{5.4}
\]

The following assumptions are proposed to tackle the so-called dimensionality problem.

**Assumption A.2**

(a) Let \( S = \{a_0, a_1, a_2, \ldots\} \), where \( a_i = \{a_{ij}\}_{j=0}^{\infty} \) is a sequence such that \( \sum_{j=1}^{\infty} j|a_{ij}| < \infty \) for \( i = 0, 1, 2, \ldots \).

(b) Suppose further that \( \sum_{i=0}^{\infty} i \left( \sum_{j=0}^{\infty} |a_{ij}| \right)^2 < \infty \).

Under the conditions of Assumption A.2 the following proposition is valid. Basically it is because of the theorem in the section of converse questions of expansion and Riesz-Fischer theorem.

**Proposition 5.1.** Let Assumption A.2 hold. We have

(1) For every \( i \) function \( a_i(t) = \sum_{j=0}^{\infty} a_{ij}\varphi_j(t) \) exists and is differentiable on \( [0, T] \). Moreover, \( a_i'(t) \in L^2[0, T] \) for \( i \geq 0 \).

(2) There exists a function \( F(t, n) \in L^2(\mathbb{N}, q(t, n)) \) for \( 0 < t \leq T \) such that
\[
F(t, N_t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij}\varphi_j(t)\varepsilon_i(t, N_t), \tag{5.5}
\]
where convergence is in the sense of mean square. Moreover, for every \( t \in (0, T) \), \( \nabla F(t, n + 1) \in L^2(\mathbb{N}, \varrho(t, n)) \).

**Proof.** (1). The assertion has been proved in the first part of the paper. (2) Because the second part of Assumption A.2 is much stronger than the requirement of Riesz-Fischer theorem, the existence follows. Meanwhile, the following derivation gives the finiteness of the second moment of the functional for any \( t > 0 \),

\[
E[F^2(t, N_t)] = \sum_{i=0}^{\infty} \left( \sum_{j=0}^{\infty} a_{ij} \varphi_j T(t) \right)^2 \leq \frac{2}{T} \sum_{i=0}^{\infty} \left( \sum_{j=0}^{\infty} |a_{ij}| \right)^2 < \infty.
\]

Moreover, recall that \( C_i(t; n) = \sqrt{\frac{t}{i!}} c_i(t; n) = \left( -1 \right)^i \sqrt{\frac{t}{i!}} C_i(t; n) \) and \( \nabla C_i(t; n) = iC_{i-1}(t; n) \) from (3.6). Accordingly, we have

\[
\nabla C_i(t; n + 1) = \nabla C_i(t; n) = \left( -1 \right)^i \frac{1}{\sqrt{i! \sqrt{T}}} \nabla C_i(t; n) = \left( -1 \right)^i \frac{1}{\sqrt{i! \sqrt{T}}} \nabla C_i(t; n) = - \frac{i}{\sqrt{i! \sqrt{T}}} C_i(t; n).
\]

It follows that

\[
\nabla F(t, n + 1) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} \varphi_j T(t) \nabla C_i(t; n + 1) = \sum_{i=0}^{\infty} a_i(t) \nabla C_i(t, n + 1)
\]

\[
= - \sum_{i=1}^{\infty} a_i(t) \frac{1}{\sqrt{T}} C_i(t; n) = - \frac{1}{\sqrt{T}} \sum_{i=1}^{\infty} a_{i+1}(t) \sqrt{i} C_i(t; n)
\]

Thus, for any \( t > 0 \),

\[
E \left[ \nabla F(t, N_t + 1) \right]^2 = E \left[ \left( - \frac{1}{\sqrt{T}} \sum_{i=0}^{\infty} a_{i+1}(t) \sqrt{i} C_i(t; N_t) \right)^2 \right]
\]

\[
= \frac{1}{t} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} i a_{i+1}(t) \left( \sum_{j=0}^{\infty} a_{i+1,j} \varphi_j T(t) \right)^2 \leq \frac{2}{t T} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( a_{i+1,j} \right)^2 \leq \frac{2}{t T} \sum_{i=0}^{\infty} \left( \sum_{j=0}^{\infty} a_{i+1,j} \right)^2 < \infty.
\]
Given a bundle of truncation parameters $k$ and $(p_0, \ldots, p_k)$, we can obtain a truncated sequence $a$ from $S$ in Assumption A.2, viz. $a = (a_{00}, \ldots, a_{0p_0}, \ldots, a_{k0}, \ldots, a_{kp_k})$. In the sequel, it is said that $a$, instead of $S$, satisfies Assumption A.2 for the sake of simplicity.

Suppose now that vector $a$ satisfies Assumption A.2. Then we can apply a transformation for $\hat{\theta}$ and using (5.5) obtains

$$aX'(\hat{\theta} - \theta) = aX'(\delta + \gamma + u) = (F' - A' - B')(\delta + \gamma + u)$$

where $F' = (F(t_{1,m}, N_{t_1,m}), \ldots, F(t_{m,m}, N_{t_m,m}))$, and

$$A' = (\alpha_1, \ldots, \alpha_m) \quad \text{with} \quad \alpha_s = \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} a_{ij} \varphi_j T(t_{s,m}) C_i(t_{s,m}, N_{t_s,m}), \quad s = 1, \ldots, m$$

$$B' = (\beta_1, \ldots, \beta_m) \quad \text{with} \quad \beta_s = \sum_{i=k+1}^{\infty} a_i(t_{s,m}) C_i(t_{s,m}, N_{t_s,m}), \quad s = 1, \ldots, m.$$

Observe that at the observation points the process can be rephrased and derived that

$$N_{t_s,m} = N_{t_s,m} - t_{s,m} + t_{s,m}$$

$$= \sum_{l=1}^{s} [(N_{t_l,m} - t_{l,m}) - (N_{t_{l-1},m} - t_{l-1,m})] + t_{s,m}$$

$$= \sqrt{T} \frac{1}{\sqrt{m}} \sum_{l=1}^{s} w_l + T \frac{s}{m} = \sqrt{T} \frac{1}{\sqrt{m}} \sum_{l=1}^{[mr]} w_l + T \frac{[mr]}{m}$$

$$= \sqrt{T} x_{[mr],m} + T \frac{[mr]}{m} \to a.s. \ W_r + Tr,$$

as $m \to \infty$, where $x_{s,m} = \frac{1}{\sqrt{m}} \sum_{l=1}^{s} w_l$, $w_l = \sqrt{\varphi} \left[(N_{t_l,m} - t_{l,m}) - (N_{t_{l-1},m} - t_{l-1,m})\right]$ form an i.i.d. $(0,1)$ sequence due to the property of Poisson process, $0 \leq r \leq 1$, and $W_r$ is a standard Brownian motion on $[0,1]$. It follows from the functional central limit theorem that $x_{[mr],m}$ converges almost surely to a standard Brownian motion on $r \in [0,1]$. Palpably, $x_{s,m}$ satisfies Assumption 4.2, as intimated earlier.

**Theorem 5.1.** Let Assumption A.1 and A.2 hold. Suppose $x_{s,m}$ and $u_s$ satisfy Assumption 4.3. Assume that the truncation parameters satisfy Assumption 5.6. Moreover, suppose that

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function $F(t, x)$ generated by vector $a$ satisfying Assumption A.2 is continuous in both $t$ and $x$.

\[
\frac{1}{\sqrt{m}} a X' X \widehat{\theta} \rightarrow_p \int_0^1 F(rT, \sqrt{T}W_r + Tr) dU_r \tag{5.7}
\]

where $(W, U)$ is a vector of Brownian motion processes on $[0, 1]$ specified in Assumption 4.3.

**Proof.** As $\frac{1}{\sqrt{m}} a X' X (\widehat{\theta} - \theta) = \frac{1}{\sqrt{m}} (F' - A') (\delta + \gamma + u)$, we start with the convergence of $F' u$. Observe that

\[
\frac{1}{\sqrt{m}} F' u = \frac{1}{\sqrt{m}} \sum_{s=1}^{m} F(t_{s,m}, N_{t_{s,m}}) u_s = \frac{1}{\sqrt{m}} \sum_{s=1}^{m} F \left( \frac{T}{m}, N_{t_{s,m}} - t_{s,m} + T \frac{s}{m} \right) u_s
\]

\[
= \sum_{s=1}^{m} F \left( \frac{T}{m} \frac{s}{m} + \sqrt{T}x_{s,m} + T \frac{s}{m} \right) u_s
\]

\[
= \sum_{s=1}^{m} \int_{\frac{s-1}{m}}^{\frac{s}{m}} F \left( T \left[ \frac{mr}{m} + o(1), \sqrt{T}W_m(r) + T \frac{m}{m} + o_p(1) \right] \right) dU_m(r)
\]

\[
= \int_{0}^{1} F \left( T \left[ \frac{mr}{m} + o(1), \sqrt{T}W_m(r) + T \frac{m}{m} + o_p(1) \right] \right) dU_m(r).
\]

Because of continuity of $F$ and convergence of $(W_m(r), U_m(r)) \rightarrow_a.s. (W, U)$, using Theorem 2.2 in Kurtz and Protter (1991) yields

\[
\int_{0}^{1} F \left( T \left[ \frac{mr}{m} + o(1), \sqrt{T}W_m(r) + T \frac{m}{m} + o_p(1) \right] \right) dU_m(r) \rightarrow_p \int_{0}^{1} F(rT, W_r + Tr)dU_r
\]

as $m \rightarrow \infty$.

Apropos of the other terms, Cauchy-Schwarz inequality gives

\[
\frac{1}{\sqrt{m}} |F' \delta| = \frac{1}{\sqrt{m}} \left| \sum_{s=1}^{m} F(t_{s,m}, N_{t_{s,m}}) \delta_s \right|
\]

\[
\leq \left( \frac{1}{m} \sum_{s=1}^{m} F^2(t_{s,m}, N_{t_{s,m}}) \right)^{1/2} \left( \sum_{s=1}^{m} \delta_s^2 \right)^{1/2}
\]

\[
\frac{1}{\sqrt{m}} |F' \gamma| \leq \left( \frac{1}{m} \sum_{s=1}^{m} F^2(t_{s,m}, N_{t_{s,m}}) \right)^{1/2} \left( \sum_{s=1}^{m} \gamma_s^2 \right)^{1/2}
\]

\[
\frac{1}{\sqrt{m}} |A' \delta| \leq \left( \frac{1}{m} \sum_{s=1}^{m} \alpha_s^2 \right)^{1/2} \left( \sum_{s=1}^{m} \delta_s^2 \right)^{1/2}
\]

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\[
\frac{1}{\sqrt{m}} |A'\gamma| \leq \left( \frac{1}{m} \sum_{s=1}^{m} \alpha_s^2 \right)^{1/2} \left( \sum_{s=1}^{m} \gamma_s^2 \right)^{1/2}
\]

\[
\frac{1}{\sqrt{m}} |B'\delta| \leq \left( \frac{1}{m} \sum_{s=1}^{m} \beta_s^2 \right)^{1/2} \left( \sum_{s=1}^{m} \delta_s^2 \right)^{1/2}
\]

\[
\frac{1}{\sqrt{m}} |B'\gamma| \leq \left( \frac{1}{m} \sum_{s=1}^{m} \beta_s^2 \right)^{1/2} \left( \sum_{s=1}^{m} \gamma_s^2 \right)^{1/2}
\]

and invoking that \( \{ u_s \} \) is a martingale difference and \( x_{s,m} \) is adapted with \( F_{s-1,m} \) yields

\[
E \left( \frac{1}{\sqrt{m}} A'u \right)^2 = \frac{1}{m} E \left( \sum_{s=1}^{m} \alpha_s u_s \right)^2
\]

\[
= \frac{1}{m} \sum_{s=1}^{m} E[\alpha_s^2 u_s^2] + \frac{1}{m} \sum_{s_1=1}^{m-1} \sum_{s_2=s_1+1}^{m} E[\alpha_{s_1} \alpha_{s_2} u_{s_1} u_{s_2}]
\]

\[
= \frac{1}{m} \sum_{s=1}^{m} E[\alpha_s^2 E(u_s^2|F_{s-1,m})] + \frac{1}{m} \sum_{s_1=1}^{m-1} \sum_{s_2=s_1+1}^{m} E[\alpha_{s_1} \alpha_{s_2} u_{s_1} E(u_{s_2}|F_{s_2-1,m})]
\]

\[
= \frac{\sigma^2}{m} \sum_{s=1}^{m} E[\alpha_s^2],
\]

similarly,

\[
E \left( \frac{1}{\sqrt{m}} B'u \right)^2 = \frac{\sigma^2}{m} \sum_{s=1}^{m} E[\beta_s^2].
\]

Therefore, in order to complete the proof of the theorem, it suffices to show that

\[
\sum_{s=1}^{m} \delta_s^2 \to_P 0, \quad \sum_{s=1}^{m} \gamma_s^2 \to_P 0, \quad (5.8a)
\]

\[
\frac{1}{m} \sum_{s=1}^{m} E[\alpha_s^2] \to_P 0, \quad \frac{1}{m} \sum_{s=1}^{m} E[\beta_s^2] \to_P 0, \quad (5.8b)
\]

because we can demonstrate that

\[
\frac{1}{m} \sum_{s=1}^{m} F^2(t_{s,m}, N_{t_{s,m}}) = \frac{1}{m} \sum_{s=1}^{m} F^2 \left( T \frac{s}{m}, \sqrt{T} x_{s,m} + T \frac{mr}{m} \right)
\]

\[
= \sum_{s=1}^{m} \int_{\frac{s-1}{m}}^{\frac{s}{m}} F^2 \left( T \frac{r}{m}, \sqrt{T} W_{r,m} + T \frac{mr}{m} \right) dr
\]

\[
- \frac{1}{m} F^2(0, 0) + \frac{1}{m} F^2(T, \sqrt{T} x_{m,m} + T)
\]

\[
= \int_{0}^{1} F^2 \left( T \frac{r}{m}, \sqrt{T} W_{r,m} + T \frac{mr}{m} \right) dr
\]
\[- \frac{1}{m} F^2(0,0) + \frac{1}{m} F^2(T, N_T) \]
\[- \to P \int_0^1 F^2(Tr, \sqrt{T} W_r + Tr) dr, \]

as \( m \to \infty \), due to the convergence of \( W_m(r) \) and finiteness of \( E[F^2(T, N_T)] \) by Proposition 5.1.

To begin with (5.8a), observe that

\[
\sum_{s=1}^{m} E[\delta_s^2] = \sum_{s=1}^{m} E \left( \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} c_{ij} \varphi_j T(t_s,m) \mathcal{C}_i(t_s,m, N_{t_s,m}) \right)^2
\]
\[= \sum_{s=1}^{m} \sum_{i=0}^{k} \left( \sum_{j=p_i+1}^{\infty} c_{ij} \varphi_j T(t_s,m) \right)^2. \]

It follows from Assumption A.1 that \( c_i(t, f) \) is twice differentiable on \([0,T]\). Thus, a theorem in Chapter one reads

\[
\left| \sum_{i=0}^{p_i} c_{ij} \varphi_j T(t_s,m) \right| \leq C(1 + o(1)) Ta_i T/p_i, \]

where \( C \) is a pure constant and \( a_{iT} = |c'_i(0)| + |c'_i(T)| \). Therefore,

\[
\sum_{s=1}^{m} E[\delta_s^2] = \sum_{s=1}^{m} \sum_{i=0}^{k} \left( \sum_{j=p_i+1}^{\infty} c_{ij} \varphi_j T(t_s,m) \right)^2 \leq C(1 + o(1)) T^2 \sum_{s=1}^{m} \sum_{i=0}^{p_i} \frac{a_{iT}^2}{p_i^2} \leq C(1 + o(1)) T^2 \left( \sup_{i \geq 0} a_{iT}^2 \right) m k m^{1+\kappa_1-2\kappa_2} \to 0, \]

as \( m \to \infty \).

Additionally, we also have

\[
\sum_{s=1}^{m} E[\gamma_s^2] = \sum_{s=1}^{m} E \left( \sum_{i=k+1}^{\infty} c_i(t_s,m, f) \mathcal{C}_i(t_s,m, N_{t_s,m}) \right)^2
\]
\[= \sum_{s=1}^{m} \sum_{i=k+1}^{\infty} c_i(t_s,m, f)^2. \]

However, invoking the relation in (4.8) with \( r = 2 \) by virtue of Assumption A.1, we obtain

\[
c_i(t_{s,m}, f) = \sqrt{\frac{(t_{s,m})^2(i-2)!}{i!}} c_{i-2}(t_{s,m}; \nabla^2 f(t, n + 2)). \]
We therefore have
\[
\sum_{s=1}^{m} E[\gamma_s^2] = \sum_{s=1}^{m} \sum_{i=k+1}^{\infty} c_i(t_{s,m}, f)^2 = \sum_{s=1}^{m} \sum_{i=k+1}^{\infty} \frac{(t_{s,m})^2}{i(i-1)} c_{i-2}(t_{s,m}; \nabla^2 f(t, n + 2)) \\
\leq \frac{1}{k(k + 1)} \sum_{s=1}^{m} (t_{s,m})^2 \sum_{i=k+1}^{\infty} c_{i-2}(t_{s,m}; \nabla^2 f(t, n + 2)) \\
\leq \frac{T^2 m}{k(k + 1)} \sup_{t \in [0, T]} E(\nabla^2 f(t, N_t + 2))^2,
\]
where we have used the fact that \( \sum_{i=2}^{\infty} c_{i-2}(t; \nabla^2 f(t, n + 2)) = E[f(t, N_t + 2)^2] \).

Note that
\[
E[f^2(t, N_t)] = \sum_{n=0}^{\infty} f^2(t, n) e^{-t} \frac{t^n}{n!} \leq \sum_{n=0}^{\infty} \max_{0 \leq t \leq T} f^2(t, n) \frac{T^n}{n!} e^T \sum_{n=0}^{\infty} \frac{g(n)}{n!} e^{-t} \frac{T^n}{n!}
\]
which implies that the convergence of the series to \( E[f^2(t, N_t)] \) is uniformly (by Weierstrass M-Test). On account of the continuity of \( f(t, n) \) in \( t \) on \( [0, T] \), it follows that \( E[f^2(t, N_t)] \) is continuous in \( t \in [0, T] \). A very similar derivation indicates that \( E(\nabla^2 f(t, N_t + 2))^2 \) is continuous in \( t \) as well, which ushers to the finiteness of \( \sup_{t \in [0, T]} E(\nabla^2 f(t, N_t + 2))^2 \).

Consequently, that \( \sum_{s=1}^{m} \gamma_s^2 \rightarrow_P 0 \) follows from the limit that \( \frac{m}{k(k+1)} = m^{\kappa_1 - 2\kappa_2}(1 + o(1)) \rightarrow 0 \) as \( m \rightarrow \infty \).

Now, we are in a position to prove (5.8b). Firstly
\[
\frac{1}{m} \sum_{s=1}^{m} E[\alpha_s^2] = \frac{1}{m} \sum_{s=1}^{m} E \left( \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} a_{ij} \varphi_j(T(s,m)) \xi_i(t(s,m), N_i, m) \right)^2 \\
= \frac{1}{m} \sum_{s=1}^{m} \sum_{i=0}^{k} \left( \sum_{j=p_i+1}^{\infty} a_{ij} \varphi_j(T(s,m)) \right)^2 \leq \frac{2}{Tm} \sum_{s=1}^{m} \sum_{i=0}^{k} \left( \sum_{j=p_i+1}^{\infty} |a_{ij}| \right)^2 \\
\leq \frac{2}{Tm} \sum_{s=1}^{m} \sum_{i=0}^{k} \frac{1}{p_i^2} \left( \sum_{j=p_i+1}^{\infty} j|a_{ij}| \right)^2 \leq \frac{2}{T} \sum_{i=0}^{k} \frac{1}{p_i^{2\min}} \left( \sum_{j=p_i+1}^{\infty} j|a_{ij}| \right)^2 \\
\leq \frac{o(1)}{T} \frac{k}{p_i^{2\min}} = \frac{o(1)}{T} m^{\kappa_1 - 2\kappa_2} \rightarrow 0,
\]
as \( m \rightarrow \infty \), in which the implication of Assumption A.2 that \( \sum_{j=p_i+1}^{\infty} j|a_{ij}| = o(1) \) has been engaged.
Secondly, for the second part,

\[
\frac{1}{m} \sum_{s=1}^{m} E[\beta_2^2] = \frac{1}{m} \sum_{s=1}^{m} E \left( \sum_{i=k+1}^{\infty} a_i(t_{s,m}) \varphi_i(t_{s,m}, N_{t_{s,m}}) \right)^2
\]

\[
= \frac{1}{m} \sum_{s=1}^{m} \sum_{i=k+1}^{\infty} a_i(t_{s,m})^2 = \frac{1}{m} \sum_{s=1}^{m} \sum_{i=k+1}^{\infty} \left( \sum_{j=0}^{\infty} a_{ij} \varphi_j(t_{s,m}) \right)^2
\]

\[
\leq \frac{2}{Tm} \sum_{s=1}^{m} \sum_{i=k+1}^{\infty} \left( \sum_{j=0}^{\infty} |a_{ij}| \right)^2
\]

\[
\leq \frac{2}{Tk} \sum_{i=k+1}^{\infty} i \left( \sum_{j=0}^{\infty} |a_{ij}| \right)^2 = o(1) m^{-\kappa_1} \rightarrow 0,
\]

as \( m \to \infty \), where \( \sum_{i=k+1}^{\infty} i \left( \sum_{j=0}^{\infty} |a_{ij}| \right)^2 = o(1) \) on account of the second part of Assumption A.2. This finishes the proof. \( \square \)

### 5.2 Infinite horizon

Suppose there is an econometric model as follows

\[ Y_t = f(t, N_t) + \varepsilon(t), \quad t > 0, \quad (5.9) \]

where \( f \) is unknown functional, \( N_t \) is a Poisson process with intensity 1, \( \varepsilon(t) \) is an error process with zero mean and finite variance.

The following assumptions are assumed for functional \( f(t, N_t) \).

**Assumption B.1**

(a) For every \( t > 0, f(t, n) \) and its difference up to third order with respect to \( n \) all belong to \( L^2((0, \infty) \times \mathbb{N}, \lambda \times \varrho) \).

(b) \( c_i(t, f) = E[f(t, N_t) \varphi_i(t, N_t)] \) is twice differentiable for every \( i \geq 0 \). In addition, all these functions belong to \( L^2(0, \infty) \).

(c) Moreover, \( c_i(t, \nabla^3 f(t, n + 3)) \) satisfy that \( t^3 [c_i(t, \nabla^3 f(t, n + 3))]^2 \) are bounded on \( (0, \infty) \) uniformly in \( i \) for large \( i \).
Remark 5.2. The conditions are quit weak. There are sizeable variety of functions satisfying them, for example,

(1) All functions $f_1(t, n) = t^3 e^{-\xi t} p_k(n)$ on $(0, \infty) \times \mathbb{N}$ where $p_k(n)$ is a polynomial with finite degree $k$ and $\xi > 0$ satisfy all conditions (a), (b) and (c). To begin with, $f_1(t, n)$ and all differences $\nabla f_1(t, n + 1) = t^3 e^{-\xi t} \nabla p_k(n + 1)$, $\nabla^2 f_1(t, n + 2) = t^3 e^{-\xi t} \nabla^2 p_k(n + 2)$, $\nabla^3 f_1(t, n + 3) = t^3 e^{-\xi t} \nabla^3 p_k(n + 3)$ are in the form of $t^3 e^{-\xi t} p_l(n)$ where $l \leq k$. Hence, for any fixed $t > 0$, they are all in $L^2((0, \infty) \times \mathbb{N}, \lambda \times \varrho)$.

Secondly, since $c_i(t, f_1) = E[e^{-\xi t} p_k(N_l) \mathcal{C}_l(t; N_l)] = e^{-\xi t} E[p_k(N_l) \mathcal{C}_l(t; N_l)] = e^{-\xi t} c_i(t, p_k)$, while if $i \leq k$, $c_i(t, p_k)$ unambiguously is a superposition of power functions of $t$ (in view of Example 1 in Section 4) with power less than or equal to $k$, and $c_i(t, p_k) = 0$ if $i > k$. Palpably, $c_i(t, f_1)$ are twice differentiable and belong to $L^2((0, \infty)$. Apparently, this statement is also suitable for the differences of $f_1(t, n)$.

Thirdly, the condition (c) is fulfilled because there are sorely finite number of coefficients for $f_1$ and its differences which are not zero and each of them is bounded on $(0, \infty)$.

(2) $f_2(t, n) = t^\xi 2^{-n}$ where $\xi \geq 2$. It is easy to see that $\nabla f_2(t, n + 1) = -t^\xi 2^{-(n+1)} = -2^{-1} f_2(t, n)$, $\nabla^2 f_2(t, n + 2) = t^\xi 2^{-(n+2)} = 2^{-2} f_2(t, n)$, and $\nabla^3 f_2(t, n + 3) = -t^\xi 2^{-(n+3)} = -2^{-3} f_2(t, n)$. Since for every $t > 0$, $E[f_2(t, N_l)]^2 = t^{2\xi} e^{-3t/4}$, $f_2(t, n) \in L^2((0, \infty) \times \mathbb{N}, \lambda \times \varrho)$; so do all the three differences.

In addition, it follows from Example 1 in Section 4 that $c_i(t, f_2) = t^\xi e^{-t/2} (1 - \frac{t}{4})$. We therefore assert that $c_i(t, f_2)$ are all in $L^2((0, \infty)$.

Furthermore, observe that

$$t^3[c_i(t, \nabla^3 f_2(t, n + 3))]^2 = \frac{1}{64} t^{3+2\xi} e^{-t} \frac{1}{2^i} \frac{t^i}{i!}$$

$$\leq \frac{1}{64} \frac{1}{2^i} \frac{t^{3+2\xi + i} e^{-t}(3 + 2\xi + i)^{3+2\xi + i} e^{-(3+2\xi + i)}}{(3 + 2\xi + i)!}$$

$$\leq \frac{1}{64 \sqrt{2\pi i}} \frac{1}{2^i t^i} (3 + 2\xi + i)^{3+2\xi + i} e^{-(3+2\xi + i)}$$

$$\leq \frac{1}{64 \sqrt{2\pi i}} \frac{1}{2^i t^i} (3 + 2\xi + i)^{3+2\xi + i} e^{-(3+2\xi + i)}$$

$$\leq \frac{1}{64 \sqrt{2\pi i}} \left(1 + \frac{3 + 2\xi}{t}ight)^i e^{-(3+2\xi) \frac{3 + 2\xi + i)^{3+2\xi}}{2^i}$$
\[ \leq \frac{C}{64\sqrt{2\pi i}} \leq \frac{C}{64\sqrt{2\pi}}, \]
because \((1 + \frac{3+2\xi}{i})^i \uparrow e^{(3+2\xi)}\) and \(\frac{(3+2\xi+1)^{3+2\xi}}{2\pi i} \rightarrow 0\) with \(i\) increasing, where \(C\) is some constant and we have exploited the fact that \(i! > \sqrt{2\pi i} (\frac{i}{e})^i\).

(3) \(f_3(t, n) = \frac{\xi}{t+\eta} \sin n\) and \(f_4(t, n) = \frac{\xi}{t+\eta} \cos n\) with \(\xi \geq 1\) and \(\eta \geq \xi + 1.25\). Obviously, \(\nabla f_3(t, n+1) = \sin 1 f_3 - (1 - \cos 1) f_3, \nabla^2 f_3(t, n+2) = (1 - \cos 1 + \cos 2) f_3 - 2 \sin 1 f_4\) and \(\nabla^3 f_3(t, n+3) = (\cos 3 - 3 \cos 2 + 3 \cos 1 - 1) f_3 + (\sin 3 - 3 \sin 2 + 3 \sin 1) f_4\). They are all in \(L^2((0, \infty) \times N, \lambda \times \varrho)\).

Additionally, it follows from Example 2 in Section 4,
\[ c_i(t, f_3) = (-1)^i t^\xi \frac{\xi}{1+t+\eta} \sqrt{\frac{t^{i+1}}{i!} e^{-t(1-\cos 1)}} \sqrt{2(1-\cos 1)^i} \sin(\alpha i + t \sin 1) \]
\[ c_i(t, f_4) = (-1)^i t^\xi \frac{\xi}{1+t+\eta} \sqrt{\frac{t^{i+1}}{i!} e^{-t(1-\cos 1)}} \sqrt{2(1-\cos 1)^i} \cos(\alpha i + t \sin 1) \]
where \(\alpha\) is a constant.

To verify the condition (c), we estimate that
\[ t^3 [c_i(t, f_3)]^2 = \frac{t^{3+2\xi+i}}{i!(1+t+\eta)^2} e^{-2t(1-\cos 1)} [2(1 - \cos 1)]^i \sin^2(\alpha i + t \sin 1) \]
\[ \leq \frac{1}{i!} (0.5 + i)^{0.5+i} \left( 2(1 - \cos 1) \right)^i \]
\[ \leq \frac{1}{i!} \left( \frac{0.5 + i}{2(1 - \cos 1)} \right)^{0.5+i} \]
\[ = \frac{(0.5 + i)^i}{1} \sqrt{2(1 - \cos 1)} e^{-(0.5+i)} \]
\[ \leq \frac{0.5 + i}{\sqrt{2\pi i}} \frac{1}{\sqrt{2(1 - \cos 1)}} \left( 1 + \frac{0.5}{i} \right)^i e^{-(0.5+i)} \]
\[ \leq \frac{1}{\sqrt{2\pi i}} \frac{1}{\sqrt{2(1 - \cos 1)}} \]

Similarly \(c_i(t, f_4)\) satisfies the corresponding condition as well.

In view of that the differences of \(f_4\) have the same structure as that of \(f_3, f_4\) fulfills Assumption B.1 too.

Suppose we have \(m\) observations \((s, Y_s)\) at points \(s = 1, 2, \ldots, m\). Because of Assumption
B.1, we can expand the unknown functional in the model (5.9) as follows

\[ f(t, N_t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij} L_j(t) C_i(t; N_t). \]  

(5.10)

Given a bundle of truncation parameters \( k \) for \( i \) and \((p_0, \cdots, p_k)\) for \( j \)'s which are all relevant with sample size \( m \), the expansion (5.10) is separated into three parts, so that the model at observation points becomes

\[
Y_s = \sum_{i=0}^{k} \sum_{j=0}^{p_i} b_{ij} L_j(s) C_i(s; X_s) + \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} b_{ij} L_j(s) C_i(s; X_s)
+ \sum_{i=k+1}^{\infty} b_i(s) C_i(s; X_s) + u_s, \quad s = 1, \cdots, m, \tag{5.11}
\]

where \( X_s = N_s \) signifies the Poisson process at point \( s \), \( b_i(s) = \sum_{j=0}^{\infty} b_{ij} L_j(s) \), and \( u_s = \varepsilon(s) \) constitute an error sequence with mean zero and finite variance.

We can rewrite the \( m \) equations in (5.13) into a matrix form:

\[
Y = X\theta + \delta + \gamma + u, \tag{5.12}
\]

where all notations possess similar meanings as in the case of finite horizon. We leave them out of account for brevity.

The Ordinary Least Squares (OLS) estimator of \( \theta \) is given by

\[
\hat{\theta} = (X'X)^{-1}X'Y. \tag{5.13}
\]

Let \( a \) be a row vector satisfying Assumption A.2. A transformation may be applied to \( \hat{\theta} \),

\[
aX'X(\hat{\theta} - \theta) = aX'(\delta + \gamma + u) \tag{5.14}
\]

Meanwhile, vector \( a \) can generate a functional of \( t \) and \( N_t \) when it meets with the basis \( L_j(t)C_i(t; X_t) \), as the following proposition stated.

**Proposition 5.2.** Let Assumption A.2 hold.

1. Functions \( a_i(t) = \sum_{j=0}^{\infty} a_{ij} L_j(t) \) exist for each \( i \geq 0 \) and are twice differentiable on \((0, \infty)\).
2. **Function**

\[
\bar{F}(t, N_t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} L_j(t) C_i(t; N_t) \quad (5.15)
\]

exists and for each \( t > 0 \), both \( \bar{F}(t, N_t) \) and \( \nabla \bar{F}(t, N_t) \) are in \( L^2(N, \varrho(t; n)) \).

**Proof.**

1. The assertion is valid because the condition (a) in Assumption A.2 is stronger than that in the theorem of converse question.

2. Since condition (b) in Assumption A.2 is stronger than Riesz-Fischer theorem \( F \) exists. In addition, because of boundedness of \( L_j(t) \), denoted by \( C \) for the time being,

\[
E[\bar{F}(t, N_t)]^2 = \sum_{i=0}^{\infty} a_i(t)^2 \leq C^2 \sum_{i=0}^{\infty} \left( \sum_{i=0}^{\infty} |a_{ij}|^2 \right) < \infty
\]

which implies for each \( t > 0 \), \( \bar{F}(t, N_t) \in L^2(N, \varrho(t; n)) \). Moreover, making use of (5.6), we have

\[
E[\nabla \bar{F}(t, N_t)]^2 = \frac{1}{t} \sum_{i=1}^{\infty} i a_i(t)^2 \leq \frac{C^2}{t} \sum_{i=1}^{\infty} \left( \sum_{i=0}^{\infty} |a_{ij}|^2 \right) < \infty
\]

which indicates the inclusion. \( \square \)

In view of the representation (5.15), we can write

\[
a X' = \bar{F}' - \alpha' - \beta', \quad (5.16)
\]

where \( \bar{F}' = (\bar{F}(1, N_1), \cdots, \bar{F}(m, N_n)) \), and

\[
\alpha' = (\alpha_1, \cdots, \alpha_m) \quad \text{with} \quad \alpha_s = \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} a_{ij} L_j(s) C_i(s, N_s), \quad s = 1, \cdots, m,
\]

\[
\beta' = (\beta_1, \cdots, \beta_m) \quad \text{with} \quad \beta_s = \sum_{i=k+1}^{\infty} a_i(s) C_i(s, N_s), \quad s = 1, \cdots, m.
\]

Let \( \bar{F}(t, N_t) = \bar{F}(t, N_t - t + t) = H(t, N_t - t) \). Notice that at each observation point, \( N_s - s = \sum_{l=0}^{s} [(N_l - l) - (N_{l-1} - (l - 1))] = \sum_{l=0}^{t} w_l \), in which \( w_l \) is an iid(0,1) sequence. Thus, \( x_{s,m} = \frac{1}{\sqrt{m}} (N_s - s) \) satisfies Assumption 4.2.
Theorem 5.2. Let Assumption B.1, 5.2 and 5.3 hold. Suppose that $H(t, x)$ is homogeneous with respect to $t$ with functions $F(t, x)$ and $v(m)$ defined in Assumption 5.4. Moreover, $x_{s,m}$ and $u_s$ satisfy Assumption 4.3. Then

$$\frac{1}{\sqrt{mv(m)}} aX'X(\hat{\vartheta} - \vartheta) \to_D \left( \int_0^1 G_3(t) dL_W(t, 0) \right)^{1/2} B(1), \quad (5.17)$$

where $G_3(t) = \int F^2(t, x) dx$, $L_W(t, 0)$ is the local–time process of $W_r$ and $B(1)$ signifies a standard normal variable independent of $W_r$.

Proof. In view of equations (5.14) and (5.16) we have

$$aX'X(\hat{\vartheta} - \vartheta) = (\bar{F}' - \alpha' - \beta') (\delta + \gamma + u). \quad (5.18)$$

We start with investigation of the convergence of $\bar{F}'u$. Observe that

$$\frac{1}{\sqrt{mv(m)}} \bar{F}'u = \frac{1}{\sqrt{mv(m)}} \sum_{s=1}^{m} H(s, N_s - s)u_s$$

$$= \frac{1}{\sqrt{m}} \sum_{s=1}^{m} F \left( \frac{s}{m}, \sqrt{mx_{s,m}} \right) u_s + \frac{1}{\sqrt{mv(m)}} \sum_{s=1}^{m} R_m \left( \frac{s}{m}, \sqrt{mx_{s,m}} \right) u_s.$$

By virtue of Theorem 4.2,

$$\frac{1}{\sqrt{m}} \sum_{s=1}^{m} F \left( \frac{s}{m}, \sqrt{mx_{s,m}} \right) u_s \to \left( \int_0^1 G_3(t) dL_W(t, 0) \right)^{1/2} B(1) \quad (5.19)$$

$m \to \infty$. Next we are going to show that

$$\frac{1}{\sqrt{mv(m)}} \sum_{s=1}^{m} R_m \left( \frac{s}{m}, \sqrt{mx_{s,m}} \right) u_s \to 0 \quad (5.20)$$

as $m \to \infty$.

Invoking Assumption 4.3,

$$E \left[ \frac{1}{\sqrt{mv(m)}} \sum_{s=1}^{m} R_m \left( \frac{s}{m}, \sqrt{mx_{s,m}} \right) u_s \right]^2$$

$$= \frac{1}{\sqrt{mv(m)^2}} \sum_{s_1=1}^{m} \sum_{s_2=1}^{m} E \left[ R_m \left( \frac{s_1}{m}, \sqrt{mx_{s_1,m}} \right) u_{s_1} R_m \left( \frac{s_2}{m}, \sqrt{mx_{s_2,m}} \right) u_{s_2} \right]$$

$$= \frac{1}{\sqrt{mv(m)^2}} \sum_{s=1}^{m} E \left[ R_m^2 \left( \frac{s}{m}, \sqrt{mx_{s,m}} \right) u_s^2 \right]$$

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where \( \varphi \) and \( \varpi \) are the density functions of \( \sqrt{m}x_{s,m} = N_s - s \) and \( \frac{1}{\sqrt{s}}(N_s - s) \) respectively. It is clear that \( g_s(x) = \frac{1}{\sqrt{s}} \varpi_s\left( \frac{x}{\sqrt{s}} \right) \). The following result is useful which can be found in Chow and Teicher (1988),

\[
\sup_x |\varpi_s(x) - \varphi(x)| = o(1)
\]  

(5.21)

where \( \varphi(x) \) is the density of standard normal distribution. Hence, when \( s \) is large enough, \( \varpi_s\left( \frac{x}{\sqrt{s}} \right) = \varphi(0)(1 + o(1)) = \frac{1}{\sqrt{2\pi}}(1 + o(1)) \).

If \( H(t,x) \in T(H_1) \), then \( |R_m\left( \frac{s}{m}, x \right)| \leq a_m\left( \frac{s}{m} \right) P(x) \). Therefore,

\[
\frac{1}{\sqrt{m}v(m)^2} \sum_{s=1}^{m-1} \sum_{s_1 \neq s_2} E\left[ R_m\left( \frac{s_1}{m}, \sqrt{m}x_{s_1,m} \right) R_m\left( \frac{s_2}{m}, \sqrt{m}x_{s_2,m} \right) u_{s_1} u_{s_2} \right]
\]

\[
= \frac{1}{\sqrt{m}v(m)^2} \sum_{s=1}^{m} E\left[ R_m^2\left( \frac{s}{m}, \sqrt{m}x_{s,m} \right) | F_{s-1,m} \right]
\]

\[
+ \frac{1}{\sqrt{m}v(m)^2} \sum_{s_1 \neq s_2} E\left[ R_m\left( \frac{s_1}{m}, \sqrt{m}x_{s_1,m} \right) R_m\left( \frac{s_2}{m}, \sqrt{m}x_{s_2,m} \right) u_{s_1} E\left( u_{s_2} | F_{s_2-1,m} \right) \right]
\]

\[
= \frac{\sigma^2}{\sqrt{m}v(m)^2} \sum_{s=1}^{m} E\left[ R_m^2\left( \frac{s}{m}, \sqrt{m}x_{s,m} \right) \right].
\]

It follows from the classical central limit theorem that \( \frac{1}{\sqrt{s}}(N_s - s) = \frac{1}{\sqrt{s}} \sum_{t=0}^{s} w_t \) converges to \( N(0,1) \) as \( s \to \infty \). Suppose that \( g_s(x) \) and \( \varpi_s(x) \) are the density functions of \( \sqrt{m}x_{s,m} = N_s - s \) and \( \frac{1}{\sqrt{s}}(N_s - s) \) respectively. It is clear that \( g_s(x) = \frac{1}{\sqrt{s}} \varpi_s\left( \frac{x}{\sqrt{s}} \right) \). The following result is useful which can be found in Chow and Teicher (1988).

\[
\sup_x |\varpi_s(x) - \varphi(x)| = o(1)
\]  

(5.21)

where \( \varphi(x) \) is the density of standard normal distribution. Hence, when \( s \) is large enough, \( \varpi_s\left( \frac{x}{\sqrt{s}} \right) = \varphi(0)(1 + o(1)) = \frac{1}{\sqrt{2\pi}}(1 + o(1)) \).

If \( H(t,x) \in T(H_1) \), then \( |R_m\left( \frac{s}{m}, x \right)| \leq a_m\left( \frac{s}{m} \right) P(x) \). Therefore,

\[
\frac{1}{\sqrt{m}v(m)^2} \sum_{s=1}^{m} E\left[ R_m^2\left( \frac{s}{m}, \sqrt{m}x_{s,m} \right) \right] = \frac{1}{\sqrt{m}v(m)^2} \sum_{s=1}^{m} \int_{-\infty}^{\infty} R_m^2\left( \frac{s}{m}, x \right) g_s(x)dx
\]

\[
\leq \frac{1}{\sqrt{m}v(m)^2} \sum_{s=1}^{m} \int_{-\infty}^{\infty} a_m^2\left( \frac{s}{m} \right) P^2(x)g_s(x)dx
\]

\[
= \frac{1}{\sqrt{m}v(m)^2} \left( \sum_{s=1}^{\Gamma_m} + \sum_{s=\Gamma_m+1}^{m} \right) a_m^2\left( \frac{s}{m} \right) \int_{-\infty}^{\infty} P^2(x)g_s(x)dx
\]

where \( \Gamma_m = o(\sqrt{m}) \to \infty \) as \( m \to \infty \).

Because \( P^2(\cdot) \) is integrable, without loss of generality we can assume that \( P(x)^2 < C \) for some constant \( C > 0 \). In addition, in the case that \( a_m(t)/v(m) \to 0 \) uniformly in \( t \) as \( m \to \infty \), for any \( \epsilon > 0 \), when \( m \) is large, we have \( |a_m(t)/v(m)| \leq \epsilon \). Thus,

\[
\frac{1}{\sqrt{m}v(m)^2} \left( \sum_{s=1}^{\Gamma_m} + \sum_{s=\Gamma_m+1}^{m} \right) a_m^2\left( \frac{s}{m} \right) \int_{-\infty}^{\infty} P^2(x)g_s(x)dx
\]
\[
\leq \frac{e^2}{\sqrt{m}} \sum_{s=1}^{\Gamma_m} \int_{-\infty}^{\infty} P^2(x)(g_s(x)dx + \frac{e^2}{\sqrt{m}} \sum_{s=1}^{m} \int_{-\infty}^{\infty} P^2(x)g_s(x)dx
\]
\[
\leq \frac{e^2C^2}{\sqrt{m}} \Gamma_m + \frac{e^2}{\sqrt{m}} \sum_{s=1}^{m} \int_{-\infty}^{\infty} P^2(x) \frac{1}{\sqrt{s}} \varpi_s \left( \frac{x}{\sqrt{s}} \right) dx
\]
\[
+ \frac{1 + o(1)}{\sqrt{2\pi}mv(m)^{2}} \sum_{s=1}^{m} a^2 \left( \frac{s}{m} \right) \int_{-\infty}^{\infty} P^2(x) dx
\]
\[
= \frac{e^2C^2}{\sqrt{m}} \Gamma_m + \frac{e^2}{\sqrt{m}} \sum_{s=1}^{m} \int_{-\infty}^{\infty} P^2(x) \frac{1}{\sqrt{s}} \varpi_s \left( \frac{x}{\sqrt{s}} \right) dx
\]
which can be as small as we wish.

In the case that \(a_n(t) = a(t)\) which is square integrable on \([0,1]\) and \(v(m) \to \infty\) as \(m \to \infty\). Reset \(\Gamma_m\) such that \(\Gamma_m < m\), \(\Gamma_m = o(v(m)^{4})\) and \(\Gamma_m \to \infty\). This purpose can be fulfilled if \(\Gamma_m = \min\{m^\rho, v(m)^{4\rho}\}\) where \(0 < \rho < 1\). It follows that

\[
\frac{1}{\sqrt{mv(m)^2}} \left( \sum_{s=1}^{\Gamma_m} \sum_{s=1}^{m} a^2 \left( \frac{s}{m} \right) \frac{1}{\sqrt{s/m}} \right) \int_{-\infty}^{\infty} P^2(x)g_s(x)dx
\]
\[
= \frac{1}{\sqrt{mv(m)^2}} \left( \sum_{s=1}^{\Gamma_m} \sum_{s=1}^{m} a^2 \left( \frac{s}{m} \right) \frac{1}{\sqrt{s/m}} \right) \int_{-\infty}^{\infty} P^2(x)g_s(x)dx
\]
\[
\leq \frac{C^2}{\sqrt{mv(m)^2}} \sum_{s=1}^{\Gamma_m} \sum_{s=1}^{m} a^2 \left( \frac{s}{m} \right) \frac{1}{\sqrt{s/m}} \int_{-\infty}^{\infty} P^2(x) dx
\]
\[
= \frac{C^2}{\sqrt{mv(m)^2}} \sum_{s=1}^{\Gamma_m} \sum_{s=1}^{m} a^2 \left( \frac{s}{m} \right) \frac{1}{\sqrt{s/m}} \int_{-\infty}^{\infty} P^2(x) dx
\]
\[
\leq \frac{1 + o(1)}{\sqrt{2\pi}mv(m)^{2}} \sum_{s=1}^{m} a^2 \left( \frac{s}{m} \right) \frac{1}{\sqrt{s/m}} \int_{-\infty}^{\infty} P^2(x) dx
\]
\[
+ \frac{1 + o(1)}{\sqrt{2\pi}mv(m)^{2}} \sum_{s=1}^{m} a^2 \left( \frac{s}{m} \right) \frac{1}{\sqrt{s/m}} \int_{-\infty}^{\infty} P^2(x) dx
\]
Since \(\frac{1}{m} \sum_{s=1}^{m} a^2 \left( \frac{s}{m} \right) \frac{1}{\sqrt{s/m}} \to f_0^1 a^2(t) \frac{1}{\sqrt{t}} dt\) when \(m \to \infty\), the last term can be as small as we wish as well.

If \(H(t,x) \in T(H_2)\), then \(|R_m \left( \frac{s}{m}, x \right) | \leq b_m \left( \frac{s}{m} \right) Q(s)P(x)\). Hence,

\[
\frac{1}{\sqrt{mv(m)^2}} \sum_{s=1}^{m} E \left[ R_m \left( \frac{s}{m}, \sqrt{m}x_s, m \right) \right] = \frac{1}{\sqrt{mv(m)^2}} \sum_{s=1}^{m} \int_{-\infty}^{\infty} R_m \left( \frac{s}{m}, x \right) g_s(x)dx
\]
\[
\leq \frac{1}{\sqrt{mv(m)^2}} \sum_{s=1}^{m} \int_{-\infty}^{\infty} b_m^2 \left( \frac{s}{m} \right) Q^2(s) P^2(x) g_s(x) \, dx \\
= \frac{1}{\sqrt{mv(m)^2}} \left( \sum_{s=1}^{\Gamma_m} + \sum_{s=\Gamma_m+1}^{m} \right) b_m^2 \left( \frac{s}{m} \right) Q^2(s) \int_{-\infty}^{\infty} P^2(x) g_s(x) \, dx
\]

where \( \Gamma_m = o(\sqrt{m}) \to \infty \) as \( m \to \infty \).

As \( \lim_{m \to \infty} b_m(t)/v(m) = l(t) \in C[0,1] \), we can write \( b_m(t)/v(m) = l(t)(1 + o(1)) \) when \( m \) is large. Meanwhile, for \( Q(y) \to 0 \) as \( y \to \infty \), we have \( |Q(y)| < \epsilon \) as \( y \) large enough for any given \( \epsilon > 0 \). Accordingly,

\[
\frac{1}{\sqrt{mv(m)^2}} \left( \sum_{s=1}^{\Gamma_m} + \sum_{s=\Gamma_m+1}^{m} \right) b_m^2 \left( \frac{s}{m} \right) Q^2(s) \int_{-\infty}^{\infty} P^2(x) g_s(x) \, dx
\]

\[
= \frac{1}{\sqrt{m}} \sum_{s=1}^{\Gamma_m} l^2 \left( \frac{s}{m} \right) (1 + o(1)) Q^2(s) \int_{-\infty}^{\infty} P^2(x) g_s(x) \, dx
\]

\[
+ \frac{1}{\sqrt{m}} \sum_{s=\Gamma_m+1}^{m} l^2 \left( \frac{s}{m} \right) (1 + o(1)) Q^2(s) \int_{-\infty}^{\infty} P^2(x) g_s(x) \, dx
\]

\[
\leq \frac{C^2(1 + o(1))}{\sqrt{m}} \Gamma_m \max_{t \in [0,1]} l^2(t) \max_{1 \leq s \leq \Gamma_m} Q^2(s)
\]

\[
+ \frac{\epsilon^2(1 + o(1))}{\sqrt{2\pi}} \sum_{s=\Gamma_m+1}^{m} l^2 \left( \frac{s}{m} \right) \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} P^2(x) \, dx
\]

\[
= \frac{C^2(1 + o(1))}{\sqrt{m}} \Gamma_m \max_{t \in [0,1]} l^2(t) \max_{1 \leq s \leq \Gamma_m} Q^2(s)
\]

\[
+ \frac{\epsilon^2(1 + o(1))}{\sqrt{2\pi}} \sum_{s=\Gamma_m+1}^{m} l^2 \left( \frac{s}{m} \right) \frac{1}{\sqrt{s/m}} \int_{-\infty}^{\infty} P^2(x) \, dx
\]

Notice that in view of Assumption 5.4 \( \max_{1 \leq s \leq \Gamma_m} Q^2(s) \) is a bounded variable since \( Q \) is bounded on any compact interval and vanishes at infinity. Also observe that \( \frac{1}{m} \sum_{s=1}^{m} l^2 \left( \frac{s}{m} \right) \frac{1}{\sqrt{s/m}} \)

\[
\text{converges to } \int_{0}^{1} l^2(t) \frac{1}{\sqrt{t}} \, dt. \text{ It therefore follows that the quantity will converges to zero as } m \to \infty.
\]

We eventually can assert that (5.20) is valid on all accounts.
We are now in a position to prove all the rest terms in (5.18) are convergent to zero in probability. Due to Cauchy-Schwarz inequality,

$$\frac{1}{\sqrt{mv(m)}} |F'\delta| \leq \left( \frac{1}{\sqrt{mv(m)^2}} \sum_{s=1}^{m} H^2(s, N_s - s) \right)^{1/2} \|\delta\|$$

$$= \left[ \frac{1}{\sqrt{mv(m)^2}} \sum_{s=1}^{m} \left( v(m)E \left( \frac{s}{m}, N_s - s \right) + R_m \left( \frac{s}{m}, N_s - s \right) \right)^2 \right]^{1/2} \|\delta\|$$

$$\leq \left[ \frac{2}{m} \sum_{s=1}^{m} E^2 \left( \frac{s}{m}, N_s - s \right) + \frac{2}{\sqrt{mv(m)^2}} \sum_{s=1}^{m} R^2_m \left( \frac{s}{m}, N_s - s \right) \right]^{1/2} \|\delta\|,$$

and similarly,

$$\frac{1}{\sqrt{mv(m)^2}} |F'\gamma| \leq \left( \frac{1}{\sqrt{mv(m)^2}} \sum_{s=1}^{m} H^2(s, N_s - s) \right)^{1/2} \|\gamma\|$$

$$\leq \left[ \frac{2}{m} \sum_{s=1}^{m} E^2 \left( \frac{s}{m}, N_s - s \right) + \frac{2}{\sqrt{mv(m)^2}} \sum_{s=1}^{m} R^2_m \left( \frac{s}{m}, N_s - s \right) \right]^{1/2} \|\gamma\|$$

where $\| \cdot \|$ signifies the Euclidean norm. Meanwhile,

$$\frac{1}{\sqrt{mv(m)^2}} |\alpha'\delta| \leq \left( \frac{1}{\sqrt{mv(m)^2}} \sum_{s=1}^{m} \alpha_s^2 \right)^{1/2} \|\delta\|,$$

$$\frac{1}{\sqrt{mv(m)^2}} |\alpha'\gamma| \leq \left( \frac{1}{\sqrt{mv(m)^2}} \sum_{s=1}^{m} \alpha_s^2 \right)^{1/2} \|\gamma\|$$

$$\frac{1}{\sqrt{mv(m)^2}} |\beta'\delta| \leq \left( \frac{1}{\sqrt{mv(m)^2}} \sum_{s=1}^{m} \beta_s^2 \right)^{1/2} \|\delta\|,$$

$$\frac{1}{\sqrt{mv(m)^2}} |\beta'\gamma| \leq \left( \frac{1}{\sqrt{mv(m)^2}} \sum_{s=1}^{m} \beta_s^2 \right)^{1/2} \|\gamma\|.$$
By virtue of the above, what we need to do is to prove that

\[
\sum_{s=1}^{m} \delta_s^2 \rightarrow_{p} 0, \quad \sum_{s=1}^{m} \gamma_s^2 \rightarrow_{p} 0 \quad (5.22a)
\]

\[
\frac{1}{\sqrt{m}v(m)^2} \sum_{s=1}^{m} E[\alpha_s^2] \rightarrow 0, \quad \frac{1}{\sqrt{m}v(m)^2} \sum_{s=1}^{m} E[\beta_s^2] \rightarrow 0, \quad (5.22b)
\]

because we have shown (5.20) and it follows from Theorem 4.1 that

\[
\frac{1}{m} \sum_{s=1}^{m} F^2 \left( \frac{s}{m}, N_s - s \right) \rightarrow P_0 \int_0^1 G_3(t) dL_W(t,0)
\]

where \(G_3(t) = \int F^2(t,x) dx\) and \(L_W(t,0)\) is the local time of Brownian motion spent at zero over time interval \([0,t]\).

Firstly, apropos of the first part of (5.22a) it follows that

\[
E \left[ \sum_{s=1}^{m} \delta_s^2 \right] = \sum_{s=1}^{m} \left( \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} c_{ij} \mathcal{L}_j(s) \mathcal{E}_i(s;N_s) \right)^2 \leq \sum_{s=1}^{m} \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} o(1) \frac{1}{p_i} \frac{1}{\sqrt{s}} \leq \sum_{s=1}^{m} \frac{1}{\sqrt{s}} = o(1) m^{1/2 + \kappa_1 - \frac{3}{2} \kappa_2} \rightarrow 0,
\]

as \(m \rightarrow \infty\), by virtue of the estimate of truncation series of \(c_i(s)\) and Assumption 5.3. We obtain that \(\sum_{s=1}^{m} \delta_s^2 \rightarrow_{p} 0\).

As for the second part of (5.22a), we have

\[
E \left[ \sum_{s=1}^{m} \gamma_s^2 \right] = \sum_{s=1}^{m} \left( \sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} c_{ij} \mathcal{L}_j(s) \mathcal{E}_i(s;N_s) \right)^2 \leq \sum_{s=1}^{m} \sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} o(1) k \frac{1}{p_{\min}} \frac{1}{\sqrt{s}} \leq \sum_{s=1}^{m} \frac{1}{\sqrt{s}} = o(1) m^{1/2 + \kappa_1 - \frac{3}{2} \kappa_2} \rightarrow 0.
\]

However, invoking the relation (4.8) with \(r = 3\) by virtue of Assumption B.1, we obtain

\[
c_i(s,f) = \sqrt{s^4(i-3)!} \frac{1}{i!} c_{i-3}(t_{s,m}; \nabla^3 f(t,n+3)).
\]
As $m \to \infty$, where $A$ is the uniform bound of $s^3[c_{i-3}(t_{s,m}; \nabla^3 f(t, n + 3))]^2$ postulated in Assumption B.1.

Let us turn to prove (5.22b). For the first part,

$$
\frac{1}{\sqrt{mv(m)^2}} \mathbb{E} \left[ \sum_{s=1}^{m} \alpha_s^2 \right] = \frac{1}{\sqrt{mv(m)^2}} \mathbb{E} \left[ \sum_{s=1}^{m} \sum_{i=0}^{k} \sum_{j=p_{i+1}}^{k} a_{ij} \mathcal{L}_j(s) \mathcal{E}_i(s; N_s) \right]^2
$$

$$
= \frac{1}{\sqrt{mv(m)^2}} \mathbb{E} \left[ \sum_{s=1}^{m} \sum_{i=0}^{k} \sum_{j=p_{i+1}}^{k} a_{ij} \mathcal{L}_j(s) \right]^2
$$

$$
\leq \frac{C^2}{\sqrt{mv(m)^2}} \sum_{s=1}^{m} \sum_{i=0}^{k} \left( \sum_{j=p_{i+1}}^{k} |a_{ij}| \right)^2 \leq \frac{C^2 \sqrt{m}}{v(m)^2} \sum_{i=0}^{k} \sum_{j=p_{i+1}}^{k} \left( \sum_{j=0}^{k} |a_{ij}| \right)^2
$$

$$
\leq \frac{o(1)}{v(m)^2} \frac{k \sqrt{m}}{p_{\min}^2} = \frac{o(1)}{v(m)^2} m^\kappa_1 + \frac{1}{2} - \kappa_2 \to 0
$$

using Assumption 5.2 and 5.3 as $m \to \infty$ where $C$ is the bound of $\mathcal{L}_j(s)$.

Meanwhile,

$$
\frac{1}{\sqrt{mv(m)^2}} \mathbb{E} \left[ \sum_{s=1}^{m} \beta_s^2 \right] = \frac{1}{\sqrt{mv(m)^2}} \mathbb{E} \left[ \sum_{s=1}^{m} \sum_{i=k+1}^{\infty} \left( \sum_{j=0}^{\infty} a_{ij} \mathcal{L}_j(s) \right)^2 \right]
$$

$$
= \frac{1}{\sqrt{mv(m)^2}} \mathbb{E} \left[ \sum_{s=1}^{m} \sum_{i=k+1}^{\infty} \left( \sum_{j=0}^{\infty} a_{ij} \mathcal{L}_j(s) \right)^2 \right]
$$

$$
\leq \frac{C^2}{\sqrt{mv(m)^2}} \sum_{s=1}^{m} \sum_{i=k+1}^{\infty} \left( \sum_{j=0}^{\infty} |a_{ij}| \right)^2 = \frac{C^2 \sqrt{m}}{v(m)^2} \sum_{i=k+1}^{\infty} \left( \sum_{j=0}^{\infty} |a_{ij}| \right)^2
$$

$$
\leq \frac{C^2 \sqrt{m}}{v(m)^2} \frac{1}{k} \sum_{i=k+1}^{\infty} \left( \sum_{j=0}^{\infty} |a_{ij}| \right)^2 = \frac{o(1)}{v(m)^2} m^{\frac{1}{2} - \kappa_1} \to 0,
$$

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by Assumption 5.2 and 5.3 as \( m \to \infty \). This finishes the proof. \( \square \)

5.3 On \([0, T]\) with \( T \) approaching infinity

Suppose that the sampling interval is \([0, T_m]\) where \( T_m \) diverges to infinity with sample size increasing. Let \( t_{s,m} = T_m \frac{s}{m} \), \( s = 1, 2, \cdots, m \), be the sample points. First and foremost, we impose some assumptions for \( f(t, n) \) function in the model (5.1).

**Assumption C.1**

a) For every \( t > 0 \), \( f(t, n) \) and its difference up to the third order are all in \( L^2(\mathbb{N}, g(t, n)) \).

b) For each \( i \), \( b_i(t, f) = E[f(t, N_t)\zeta_i(t, N_t)] \) is differentiable up to the second order and they belong to \([0, T]\) for any \( T > 0 \).

c) For \( i \) large, the coefficient function \( b_i(t, \nabla^3 f(t, n + 3)) \) are such that

d) The derivative of \( b_i(t, f) \) are uniformly bounded by \( M > 0 \) on \((0, \infty)\).

**Remark 5.3.** There are a variety of functions satisfying all the conditions.

On account of Assumption C.1, \( f(t, N_t) \) can be expanded into orthogonal series in terms of orthonormal basis \( \{ \varphi_{jT_m}(t)\zeta_i(t, N_t) \} \). According to the truncation parameters, the expansion of \( f(t, N_t) \) at sample points is expressed as the sum of the truncation series and the residuals,

\[
\begin{align*}
    f(t_{s,m}, N_{s,m}) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij} \varphi_{jT_m}(t)\zeta_i(t, N_t) \\
    &= \sum_{i=0}^{k} \sum_{j=0}^{p_i} b_{ij} \varphi_{jT_m}(t)\zeta_i(t, N_t) + \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} b_{ij} \varphi_{jT_m}(t)\zeta_i(t, N_t) \\
    &\quad + \sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} b_{ij} \varphi_{jT_m}(t)\zeta_i(t, N_t) \\
    &= x'_s \beta + \delta_s + \gamma_s, \quad s = 1, \cdots, m,
\end{align*}
\]

where all notations remain the similar meanings as usual; we omit to recite them for brevity. Thus the model at sample points has the corresponding expression,

\[
Y_{s,m} = x'_s \beta + \delta_s + \gamma_s + u_s, \quad s = 1, \cdots, m,
\]
which hence can be rephrased in matrix form

\[ Y = X\beta + \delta + \gamma + \varepsilon, \tag{5.23} \]

where

\[ Y' = (Y_{t_1,m}, \ldots, Y_{t_m,m}), \quad X = (x_1, \ldots, x_m)', \]

\[ \delta = (\delta_1, \ldots, \delta_m)', \quad \gamma = (\gamma_1, \ldots, \gamma_m)', \]

\[ \varepsilon = (u_1, \ldots, u_m)'. \]

The Ordinary Least Squares (OLS) estimate of \( \beta \) is

\[ \hat{\beta} = (X'X)^{-1}X'Y. \tag{5.24} \]

In order to delve into the consistent issue of \( \hat{\beta} \), we introduce a transformation to tackle the curse of dimensionality. Let \( a \) be the column vector from Assumption 5.2; in other words, \( a \) is truncated from \( S \) according to the truncation parameters.

In view of Proposition 5.1, there exists a function \( a_i(t) = \sum_{j=0}^{\infty} a_{ij}\varphi_j T_m(t) \) for each \( i \) on \([0, T_m]\) and such defined \( a_i(t) \) is differentiable such that \( a_i'(t) \in L^2[0, T_m] \). Meanwhile, there exists a functional \( \bar{F}(t, N_t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij}\varphi_j T_m(t)C_i(t, N_t) \) for \( t \in [0, T_m] \) and its difference for every for \( t \in [0, T_m] \) is in \( L^2(\mathbb{N}, g(t, n)) \). By virtue of the elements of \( X \), we have

\[ a'X' = (\bar{F}(t_1,m, N_{t_1,m}), \ldots, \bar{F}(t_{1,m}, N_{t_{1,m}})) - (\bar{\delta}_1, \ldots, \bar{\delta}_m) - (\bar{\gamma}_1, \ldots, \bar{\gamma}_m) \]

\[ := \bar{F}' - \bar{\delta}' - \bar{\gamma}' \tag{5.25} \]

where for \( s = 1, \ldots, m, \bar{\delta}_s = \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} a_{ij}\varphi_j T_m(t_{s,m})C_i(t_{s,m}, N_{t_{s,m}}) \) and \( \bar{\gamma}_s = \sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} a_{ij}\varphi_j T_m(t_{s,m})C_i(t_{s,m}, N_{t_{s,m}}) \).

By the help of (5.25), we apply a transformation for \( \hat{\beta} \),

\[ a'X'(\hat{\beta} - \beta) = a'X'(\delta + \gamma + \varepsilon) \]

\[ = (\bar{F} - \bar{\delta} - \bar{\gamma})(\delta + \gamma + \varepsilon) \tag{5.26} \]

which facilitates the later use.
Let us rephrase $\tilde{F}(t,N_t) = \tilde{F}(t,N_t - t + t) = H(t, N_t - t)$. Observe that at each sample point, $N_{t,s,m} - t_{s,m} = \sum_{l=0}^{s}\left[(N_{t,l,m} - t_{l,m}) - (N_{t-1,l,m} - t_{l-1,m})\right] = \sqrt{T_m} \frac{1}{\sqrt{m}} \sum_{l=1}^{s} w_l$ in which $w_l = \sqrt{\frac{m}{T_m}} \left[(N_{t,l,m} - t_{l,m}) - (N_{t-1,l,m} - t_{l-1,m})\right]$ form an i.i.d.(0,1) sequence. Signify that $x_{s,m} = \frac{1}{\sqrt{m}} \sum_{l=1}^{s} w_l$. Palpably, $\{x_{s,m}\}$ satisfies Assumption 4.2.

References


