Government and Private Employment, Wages, and Social Welfare in a Hotelling Oligopsony

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Introduction: Minimum Wage

- Competitive market (simple supply-demand model): Impose a minimum wage leads to an excess supply of labor
- Monosony: A minimum wage makes the employer a price-taker, up to the level of employment.
- Minimum wage in a Hotelling model
  - Firm choose locations as non-wage characteristics, such as actual job specification, hours of work, distance of the firm from the worker’s home and social environment (*horizontal job differentiation*).
    - Delfgaauw (2007) provides empirical evidence by using survey data.
    - A teenager might prefer working at the local McDonalds if some friends also work there (Bhaskar and To (1999, EJ)).
- Heterogeneous Firms with different productivities
  - Impose a minimum wage reduce the duopsony power over workers.
- Basic Assumption: full employment is assumed (the effect on employment is not considered).
Introduction: Main Research Interests in This Study

▸ Present a revised Oligopsony-labor model with multiple private firms and one government intervene (minimum wage or unemployment subsidy).
▸ There exists only one type of “pure strategy location-wage equilibrium” such that both firms occupy disjointed areas and all remaining markets are unemployed.
▸ The socially optimal minimum wage as well as the unemployment subsidy is solved analytically.
▸ Imposing a minimum wage without driving out lower productivity firms is welfare improving (because reduce the oligopsony power of firms over workers, and increase employment).
▸ Imposing a minimum wage will increase the differentiation between firms.
▸ Imposing a higher minimum wage driving out lower productivity firms can be more welfare improving than lower minimum wages.
▸ The optimal unemployment subsidy is positively related to the transportation rate and negative to the productivity and the leisure utility from unemployment.
Recent Literature

繁忙和To (1999, EJ) first consider a circular space of workers and uniformly distributed oligopsonic firms who offer jobs to the workers nearby, and suggest that a rise in the minimum wage increases social welfare.

繁忙和To (2003, EER) further develop the implications on wage distribution, with the assumption of the uniform distribution of firms.

繁忙While the above two papers treat locations as exogenous, Kaas and Madden (2010, ET) solve the location-wage game of the duopsony labor market and suggest that imposing minimum wage is always welfare-improving.

繁忙Parallel with Search Models of the Labor Market, such as Acemoglu and Shimer (2000, RES) and Postel-Vinay and Robin (2002, Econometrica)
Main Comparisons Between This Study and The Previous Studies

_solve a location-wage game with pure strategies:_

- Bhaskar and To (1999) and Bhaskar and To (2003): the assumption of the uniform distribution of firms
- Kaas and Madden (2010) and this paper: solve the location-wage game

*Oligopsony Labor Market without Full Employment:*

- Bhaskar and To (2003) and Kaas and Madden (2010): assumption of full employment

_linear transportation costs:_

- Bhaskar and To (2003) and this paper: linear in distance, with exogenous location
- Kaas and Madden (2010): quadratic transportation costs

_Labor Productivity:_

- Bhaskar and To (1999): constant return to scale, with endogenous capital.
- Bhaskar and To (2003): decreasing return to scale in capital and labor
- Kaas and Madden (2010) and this paper: constant return to scale, ignoring capital
The Model (with Duopsony)-1

Start to present a duopsony model and then it is extend to an oligopsony model.

Suppose all labors are uniformly distributed on a linear market with a unit distance.

Each labor sells his labor either to one of the two private firms (1,2) who are located at $x_1 \in [0,1], x_2 \in [0,1]$, respectively, with $x_1 \leq x_2$, or stays unemployment.

Suppose the wage offered by firm $i$ is $w_i$, $i = 1, 2$. The utility of a labor located at $x$ who are hired by firm $1, 2$ respectively are:

$$u_i = w_i - k |x - x_i|, \quad i = 1, 2. \quad (1)$$

where $k$ is the unit transportation cost. Moreover, $u_i$ is the utility of any unemployed worker when enjoys leisure.

The employment of firm $i$ is $N_i = (n_i^r - n_i^l) = 2(w_i - u_i)/k$, $i = 1, 2$.

The unemployment rate is $UN = 1 - N_1 - N_2$. 

The Model-2

The profit functions of firms are:

\[ \pi_i(w_1, w_2) = (\phi_i - w_i)N_i = \frac{2(\phi_i - w_i)(w_i - u_i)}{k}, \quad i = 1, 2, \]

where \( \phi_1 \) and \( \phi_2 \) are firms’ productivity for firm 1 and firm 2, respectively.

Figure 1: The equilibrium locations and labors’ utilities
Location-wage equilibrium -1

Solving subgame perfection equilibrium under *laissez-faire*:

- Second stage: wage competition
- First stage: location choice

**Proposition 1.** Given \( u_l^{\min} < u_l < u_l^{\max} \), when two firms compete each other, the only possible equilibrium is that both firms occupy a disjointed range of labor market and other areas are unemployed. The equilibrium locations \((x_1^*, x_2^*)\) satisfy

\[ x_1^* \geq (\phi_1 + \phi_2 - 2u_l)/(2k), \quad x_2^* \leq 1 - [(\phi_2 - u_l)/(2k)], \quad \text{and} \quad x_2^* - x_1^* \geq (\phi_1 + \phi_2)/(2k), \]

and the equilibrium wages are \( W_1 = \frac{u_l + \phi_1}{2} \) and \( W_2 = \frac{u_l + \phi_2}{2} \).

\[ u_l^{\min} = \frac{(\phi_1 + \phi_2 - k)}{2} \quad \text{and} \quad u_l^{\max} = \min\{\phi_1, \phi_2\} \]
 Equilibrium wages

\[ w_i^* = \frac{u_i}{2} + \frac{\phi_i}{2}, \quad i = 1, 2. \tag{2} \]

The equilibrium firm’s employments and unemployment rate yield:

\[ N_i^* = \frac{\phi_i - u_i}{k}, \quad i = 1, 2, \tag{3} \]

\[ UN^* = \frac{k + 2u_i - \phi_1 - \phi_2}{k}. \tag{4} \]
Proof to Proposition 1 - 1

Plugging $w_1^*$, $w_2^*$ into $n_1^R$, $n_2^L$ yield

\[ n_1^{R*} = x_1 + \frac{\phi_1 - u_l}{2k}, \quad n_2^{L*} = x_2 - \frac{\phi_2 - u_l}{2k}. \quad (5) \]

This solutions $(n_1^{R*}, n_2^{L*})$ are feasible only for the following conditions are satisfied

\[ n_2^{L*} - n_1^{R*} = -\frac{-2u_l + \phi_2 - 2kx_2 + \phi_1 + 2kx_1}{2k} \geq 0 \quad \Rightarrow \quad x_2^* - x_1^* \geq \frac{\phi_1 + \phi_2 - 2u_l}{2k} \equiv x_d^\min, \quad (6) \]

\[ n_1^{L*} = x_1 - \frac{\phi_1 - u_l}{2k} \geq 0 \quad \Rightarrow \quad x_1 \geq \frac{\phi_1 - u_l}{2k} \equiv x_1^\min, \quad (7) \]

\[ n_2^{R*} = x_2 + \frac{\phi_2 - u_l}{2k} \leq 1 \quad \Rightarrow \quad x_2 \leq 1 - \frac{\phi_2 - u_l}{2k} \equiv x_2^\max. \quad (8) \]
Proof to Proposition 1 - 2

Equation (6) represents the ranges of employed labors are disjointed. Equation (7) means that the left boundary of employed labors for firm 1 does not touch the left limit of labor. Equation (8) represents that the right boundary of employed labors for firm 2 does not over the right limit of labor.

Since \( x_2 > x_1 \) by assumption, using the above three inequities yield:

\[
x_2^{\text{max}} - x_1^{\text{min}} - x_d^{\text{min}} = \frac{k - \phi_1 + 2u_l - \phi_2}{k} \geq 0,
\]

which implies,

\[
u_l \geq \frac{\phi_1 + \phi_2 - k}{2} \equiv u_l^{\text{min}}. \quad \text{(9)}
\]

Undercutting is not profitable.
Location Differentiation

**Corollary 1.** The equilibrium locations of firms are not maximal differentiation when unemployment is considered.

- The additional implications of Proposition 1 suggest that maximum location differentiation in duopsony labor market (as Kaas and Madden, 2010) is no longer hold. This result shows that the unemployment can play a role on the location game. Allowing unemployment eliminates the incentive of both firms to move further apart to reduce wage competition.
- Each firm enjoys the monopsony position on a local labor market near its location.
Wage Dispersion

Proposition 2. The equilibrium wage \( w_1^* \) (\( w_2^* \)) is increasing in \( u_i \) and \( \phi_1 \) (\( \phi_2 \)) with \( \frac{\partial w_i}{\partial \phi_i} = 1/2 \), \( i = 1, 2 \). Moreover, \( w_1^* - w_2^* = \frac{\phi_1 - \phi_2}{2} \), therefore, \( w_1^* < w_2^* \) if and only if \( \phi_1 < \phi_2 \).

- The intuition of Proposition 2 is that the wage rate is positive related to firm’s own productivity and the utility of unemployment.
- The change in wages reflects one half of the changes of firm’s own productivities.
- The firm with a higher productivity will pay a higher wage rate in equilibrium.
Equilibrium Unemployment Rate

**Proposition 3.** With a minimum wage $w_{min}$, the unemployment rate is

$$UN^* = 1 - (\phi_1 + \phi_2 - 2u_i)/k.$$  

Moreover, the unemployment rate is increasing in $u_i$ and $k$, while it is decreasing in $\phi_i$, $i = 1, 2$.

Proposition 3 suggests that the unemployment rate is negatively related to the productivities, but positive related to the unit transportation cost and the utility of unemployment.
An Example of Location-Wage Equilibrium

Example 1:
A numerical examples is given by the following values of parameters: $k = 0.64$, $u_1 = 1.8$, $\phi_1 = 2$, $\phi_2 = 2.2$. The equilibrium locations, wages, market shares, and unemployment rate are:

- $x_1^* \geq x_{1\text{min}}^* = 0.15625$
- $x_2^* \leq x_{2\text{max}}^* = 0.6875$
- $x_2^* - x_1^* \geq x_{d\text{min}}^* = 0.46875$
- $w_1^* = 1.9$
- $w_2^* = 2.0$
- $N_1^* = 0.3125$
- $N_2^* = 0.625$
- $UN^* = 0.0625$. 


Social Welfare - Definition

Consumer surplus of workers is defined as
\[
CS_e(w_1, w_2) = \int_{n_1^L}^{n_1^R} (w_1 - k \mid x - x_1 \mid) dx + \int_{n_2^L}^{n_2^R} (w_2 - k \mid x - x_2 \mid) dx
= \frac{w_1^2 + w_2^2 - 2u_i^2}{k}.
\]

Consumer surplus of unemployed people is defined as
\[
CS_u(w_1, w_2) = (1 - N_1 - N_2)u_i
= \frac{u_i(k + 4u_i - 2(w_1 + w_2))}{k}.
\]

Social welfare is then defined as
\[
SW(w_1, w_2) = CS_e(w_1, w_2) + CS_u(w_1, w_2) + \pi_1(w_1, w_2) + \pi_2(w_1, w_2)
= \frac{2u_i^2 - w_1^2 - w_2^2 + 2(\phi_1 w_1 + \phi_2 w_2) + u_i k - 2u_i(\phi_1 + \phi_2)}{k}.
\]
Proposition 4. The first-best locations, wages and unemployment rate are: \( x_1^o \geq \frac{\phi_i u_l}{k} \), \( x_2^o \leq 1 - \frac{\phi_2 + u_l}{k} \), and \( x_2^o - x_1^o \geq \frac{\phi_1 + \phi_2 - 2u_l}{k} \); \( w_1^o = \phi_1 \), \( w_2^o = \phi_2 \); \( UN^o = \frac{k - 2(\phi_1 + \phi_2) + 2u_l}{k} \).

The objective is the social welfare function:

\[
\max_{N_1, N_2} SW(N_1, N_2) = CS_e + CS_u + \pi_1 + \pi_2
\]

\[
= \phi_1 N_1 + \phi_2 N_2 + (1 - N_1 - N_2)u_l - \int_{n_1}^{n_1^R} k |x - x_1| \, dx - \int_{n_2}^{x_2^R} k |x - x_2| \, dx
\]

\[
= \phi_1 N_1 + \phi_2 N_2 + (1 - N_1 - N_2)u_l - \frac{1}{2} kN_1^2 - \frac{1}{2} kN_2^2. \quad (11)
\]
The Best-best market shares

\[ N_i^o = \frac{2(\phi_i - u_i)}{k} > N_i^*, \quad i = 1, 2, \quad (12) \]

The First-best Unemployment

\[ UN^o = \frac{k - 2(\phi_1 + \phi_2) + 2u_i}{k} > UN^*. \quad (13) \]

The first-best solution provides a lower unemployment rate that the equilibrium under *laissez-faire*. 
Minimum Wage: Redundant, Partial Binding and Full Binding

If \( w_{\text{min}} \leq \min(w_1^*, w_2^*) = (u_1 + \min(\phi_1, \phi_2))/2 \), the minimum wage is not binding and thus the minimum wage plays no role in the labor market.

Partial binding minimum wage: \( w_{\text{min}} \geq w_1^* = (\phi_1 + u_1)/2 \), and \( w_{\text{min}} < w_2^* = (\phi_2 + u_1)/2 \). Denote the optimal social welfare as \( SW^* \)

Full binding minimum wage: \( w_{\text{min}} \geq \max\{w_1^*, w_2^*\} \)
Denote the optimal social welfare as \( SW^{**} = SW(w_{\text{min}}^*, w_{\text{min}}^*) \).
Minimum Wage: Three Cases

4.3 Minimum Wage Policy when the Productivity Difference is Relatively Large such that $\phi_2 - \phi_1 > u_1 - u$, (without considering driving out the low productivity firm)

4.4 Minimum Wage Policy when the Productivity Difference is Relatively Small such that $\phi_2 - \phi_1 < u_1 - u$, (without considering driving out the low productivity firm)

4.5 The Social Welfare with a Higher Minimum Wage Driving Out the Low Productivity Firm
4.3 Minimum Wage Policy when the Productivity Difference is Relatively Large

**Proposition 5.** Given \( \phi_2 - \phi_1 > \phi_1 - u_i \), then (a) when \( w_{\text{min}} \leq \phi_1 \), then it is always welfare improving, and the maximal welfare improving is at \( w_{\text{min}} = \phi_1 \). (b) when \( \phi_1 \leq w_{\text{min}} < \frac{\phi_1 + u_i}{2} (= w_2^*) \), then it is social welfare improving iff \( (3\phi_1 - u_i)/2 > w_{\text{min}} > \frac{\phi_1 + u_i}{2} \). However, the maximal social welfare is at \( w_{\text{min}} = \phi_1 \). (c) when \( \frac{\phi_1 + u_i}{2} \leq w_{\text{min}} < \frac{\phi_1 + u_i}{2} - u_i \), then it is social welfare improving iff \( w_{\text{min}} \leq \frac{\phi_2 + \phi_1}{2} + \sqrt{-2\phi_1^2 - 2\phi_2^2 + 8\phi_1\phi_2 - 4\phi_1 u_i (\phi_2 + \phi_1) + 4u_i^2} \). The maximal social welfare is at \( w_{\text{min}} = \frac{\phi_1 + \phi_2}{2} \). (d) compare the maximal social welfare in (a), (b) and (c). The optimal minimum wage is \( w_{\text{min}} = \phi_1 \) if \( u_i \geq \phi_2 - \sqrt{2}(\phi_2 - \phi_1) \) and \( w_{\text{min}} = \frac{\phi_1 + \phi_2}{2} \) if \( u_i \leq \phi_2 - \sqrt{2}(\phi_2 - \phi_1) \).

![Figure 2: The position of \( w_{\text{min}} \) in Proposition 5](image)
4.3 Minimum Wage Policy when the Productivity Difference is Relatively Large

Proposition 5 states that imposing an optimal minimum wage \( w^*_\text{min} \) is always social welfare improving. However, for any minimum wage \( w^\text{min} \), there is not always welfare improving when

\[
\min w > \frac{\phi_1 + \phi_2}{2} + \frac{\sqrt{-2\phi_2^2 - 2\phi_1^2 + 8\phi_1\phi_2 - 4u_1(\phi_1 + \phi_2) + 4u_2^2}}{4}.
\]

**Example 2:** A numerical examples is given by the following values of parameters:

\( k = 0.64 \), \( \phi_1 = 2 \), \( \phi_2 = 2.2 \). \( SW^* - SW(w_1^*, w_2^*) \) is always positive, measure that imposing the optimal minimum wage \( w^*_\text{min} \) is always social welfare improving as shown in the second part of Proposition 5.
Figure 3: The social welfare with minimum wage when the productivity difference is relatively large such that $3\phi_1 - \phi_2 \geq 2u_l$. 

$SW(w_{\text{min}}, w_{\text{min}})$
(when $u_i < \phi_2 - \sqrt{3}(\phi_2 - \phi_1)$)

$SW(w_{\text{min}}, w^*_2)$

$SW(w^*_1, w^*_2)$

$w^*_1$
$(u_1 + \phi_1)/2$

$w^*_{\text{min}}$
$\phi_1$

$w^*_2$
$(u_1 + \phi_2)/2$

$w^{**}_{\text{min}}$
$(\phi_1 + \phi_2)/2$

$w_{\text{min}}$
Figure 4: The social welfare with minimum wage when the productivity difference is relatively large such that $3\phi_1 - \phi_2 < 2u_l$ and $u_l < (1 + \sqrt{3})\phi_2 + (1 - \sqrt{3})\phi_1$
Figure 5: The social welfare with minimum wage when the productivity difference is relatively large such that $3\phi_1 - \phi_2 < 2u_l$ and $u_l > (1 + \sqrt{3})\phi_2 + (1 - \sqrt{3})\phi_1$
Minimum Wage and Social Welfare: A Example

- $SW(w_2^*, w_2^*) - SW(w_1^*, w_2^*)$ may be less than zero when $u_l$ is relatively large ($u_l > 1.9$).

- As shown in Figure 4, imposing an arbitrary minimum wage $w_{\min}$ with one firm binding may be social welfare worsening.

- Third, for $SW^{**} - SW(w_1^*, w_2^*)$, it is social welfare improving (Figure 3) when $u_l$ is relatively small ($u_l < 1.927$), while it is welfare worsening (Figure 4) when $u_l$ is large ($u_l > 1.927$) as shown in Proposition 5.

- Finally, comparing $SW^{**} - SW(w_1^*, w_2^*)$ and $SW^* - SW(w_1^*, w_2^*)$, we have $SW^{**} - SW^* \geq 0$ when $u_l$ is relatively small ($u_l < 1.917$) as shown in Proposition 5.
Figure 5: The comparisons among $SW^*$, $SW^{**}$ and $SW(w_1^*, w_2^*)$
4.4 Minimum Wage Policy when the Productivity Difference is Relatively Small

Proposition 6. Given \( \phi_2 - \phi_1 < u_1 - u_2 \), if a government imposes a minimum wage such that (a) when \( w_{\text{min}} \leq \phi_1 \), it is always welfare improving, and the maximal welfare is at \( w_{\text{min}} = \phi_1 \). (b) when \( \phi_1 \leq w_{\text{min}} \), then it is social welfare improving iff \( w_{\text{min}} < \frac{\phi_1 + \phi_2}{2} + \frac{\sqrt{-2\phi_1^2 - 2\phi_2^2 + 8\phi_1\phi_2 - 4u_1(\phi_1 + \phi_2) + 4u_2^2}}{4} \), and the maximal welfare is at \( w_{\text{min}} = \frac{\phi_1 + \phi_2}{2} \). (c) compare the maximal welfare in (a) and (b), we find that the global welfare maximization is at \( w_{\text{min}} = \frac{\phi_1 + \phi_2}{2} \).

**Figure 6**: The comparisons among \( SW^* \), \( SW^{**} \) and \( SW(w_1^*, w_2^*) \)

**Figure 7**: The position of \( w_{\text{min}} \) in Proposition 6
Figure 8: The social welfare with minimum wage when the productivity difference is relatively small such that $\phi_2 - \phi_1 < \phi_1 - w_1$

if $\phi_2 > 2\phi_1 - w_1$. 
4.5 A Higher Minimum wage Driving out low productivity firms

A government can enact a high minimum wage to drive out low productivity firms and whether the social welfare increases or decreases. Define

\[ \widehat{SW} = \phi_2 N_2 + (1 - N_2)u_i - \int_{n_2^l}^{n_2^u} k |x - x_2| \, dx. \]

represent the case that \( w_{\text{min}} > \phi_1 \) and thus firm 1 is forced out of the industry.

\[ \widehat{SW}(w_{\text{min}}) = \frac{u_i^2 - w_{\text{min}}^2 + u_i k - 2\phi_2 u_i + 2\phi_2 w_{\text{min}}}{k}. \]

Differentiate \( \widehat{SW} \) with respect to \( w_{\text{min}} \) yield

\[ w_{***}^* = \phi_2. \]

\( w_{\text{min}} = \phi_1 \) if the non-negative profit condition is considered. Then
4.5 A Higher Minimum wage Driving out low productivity firms

\[
\widehat{SW}^* = \begin{cases} 
SW(\phi_1, w_2^*) & \text{when } \phi_1 < \frac{u_i + \phi_2}{2}, \quad (21) \\
SW(\phi_1, \phi_2) & \text{when } \phi_1 > \frac{u_i + \phi_2}{2}, \quad (22)
\end{cases}
\]

Compare \(\widehat{SW}(w_{\min}^*)\), \(SW^*\), and \(SW(w_1^*, w_2^*)\) with non-negative profit can yield the following proposition.

**Proposition 8.** (a) \(\widehat{SW}(w_{\min}^*) > SW(x_1^*, w_2^*)\) if \(\phi_2 > u_i + \sqrt{3}(\phi_1 - u_i)\), (b) \(\widehat{SW}(w_{\min}^*) > \widehat{SW}^*\) if \(\phi_2 > 2\phi_1 - u_i\).

The above proposition suggests that imposing minimum wage driving out the low productivity firm is social welfare improving when \(\phi_2\) is relatively large, and can be better than the case with the minimum wage with partial binding (\(w_{\min}^* > \phi_1\)).
Figure 6: The comparisons among $\widehat{SW}^+$, $SW(w_1^*, w_2^*)$ and $SW^{***}$.
The Model with unemployment subsidy

Instead of the minimum wage policy, suppose now the government can give a subsidy ($w_g$) to unemployed worker, and this subsidy can produce $\phi_3$ benefits (e.g. criminal rate reduced) in this section. Other assumptions are being equal and thus equations (1) are still valid, and $u_l$ should be replaced by $u_g = w_g + u_l$.

Figure 7: The equilibrium locations and labors’ utilities with unemployment subsidy
Equilibrium Wage, Location, and Unemployment

**Proposition 9.** When two firms compete each other, the only possible equilibrium is that both firms occupy a range of market without overlapping and other areas are unemployed when \( w_g < w_g^{\text{max}} \). The equilibrium locations \((x_1^*, x_2^*)\) satisfy \( x_1^* \geq \frac{\phi_1 - w_g - u_l}{2k} \), \( x_2^* \leq 1 - \frac{\phi_2 + w_g - u_l}{2k} \), and \( x_2^* - x_1^* \geq \frac{\phi_1 + \phi_2 - 2w_g - 2u_l}{2k} \), the equilibrium wage is \( w_1 = \frac{w_g + u_l + \phi_1}{2} \) and \( w_2 = \frac{w_g + u_l + \phi_2}{2} \).

The firm’s employments and unemployment rate become:

\[
N_1^* = \frac{\phi_1 - w_g - u_l}{k}, \quad N_2^* = \frac{\phi_2 - w_g - u_l}{k},
\]

\[
UN = \frac{k - \phi_1 - \phi_2 + 2w_g + 2u_l}{k}.
\]
Comparative Statics and Social Welfare

To ensure the existence of a reasonable equilibrium, \( w_g < w_g^{\text{max}} \) is hereafter assumed.

**Proposition 10.** Under a government subsidy, \( w_1^* \) and \( w_2^* \) are increasing in \( k \).
Moreover, \( w_1^* - w_2^* = \frac{\phi_1 - \phi_2}{2} \), therefore, \( w_1^* < w_2^* \) if and only if \( \phi_1 < \phi_2 \). Finally, \( \partial w_i / \partial \phi_i = 1/2 \), which means that \( w_i^* \) is increasing (half) in \( w_g \), and decreasing (half) in \( t \).

**Proposition 11.** Given a wage subsidy \( w_g \), the unemployment rate is

\[
UN^* = \frac{k - \phi_1 + 2w_g + 2u_i - \phi_2}{k}.
\]
Moreover, the unemployment rate is increasing in \( w_g \), \( u_i \) and \( k \), while it is decreasing in \( \phi_i \), \( i = 1, 2 \).
Social Welfare under Government Subsidy-1

Social welfare function as follows,

$$\max_{w_g} SW = CS_e + CS_u + \pi_1 + \pi_2 + G,$$

where $G = (\phi_3 - w_g) \cdot UN$ is the government surplus and the government can use $w_g$ to maximize social welfare.

$$SW = (3\phi_2^2 - 2w_g^2 + 4w_g u_l - 2w_g \phi + 6u_l^2 - 6u_1 \phi_1 + 3\phi_1^2 - 2w_g \phi_2 - 6u_1 \phi_2$$

$$+ 4ku_l + 4\phi_3 k - 4\phi_3 \phi_1 + 8\phi_3 w_g + 8\phi_3 u_l - 4\phi_3 \phi_2) / (4k). \quad (34)$$

**Proposition 12.** The optimal unemployment subsidy $w_g^* = 2\phi_3 + u_l - \frac{\phi_1 + \phi_2}{2}$. When the economy is booming ($\phi_i$ increases, $i = 1, 2$), then the optimal unemployment subsidy must decrease. Similarly, $\partial w_g / \partial u_l > 0$, and $\partial w_g / \partial k = 0$. 
Social Welfare under Government Subsidy-2

Proposition 13. The optimal unemployment rate (under optimal unemployment subsidy) as the second-best solution is

\[ UN = \frac{k - 2\phi_1 + 4u_i - 2\phi_2 + 4\phi_3}{k}. \]

The optimal unemployment rate is increasing in \( u_i, k \) and \( \phi_3 \), while it is decreasing in \( \phi_1, \phi_2 \).

The economic intuition is that the unemployment subsidy should adjust as per the situation of economic status: When the economy is booming (recession), the optimal subsidy should be decreased (increased); Similarly, the higher \( \phi_3 \), then the higher \( w_g^* \).
Circular Market

Without loss of generality, assume $x_1 = 0$, and $x_2 \in [0, y_2]$.

**Proposition 14.** Consider a duopsony-labor market in circular market, the location-wage equilibrium is $x_1 = 0$, $x_2 \in [x_2^{\min}, x_2^{\max}]$. The equilibrium wages, profits, and market shares are all identical to the linear market.
Oligopsony Labor Markets-1

The duopsony model can be easily extended to oligopsony-labor market. In order to simplify our model, suppose there are $m_1$ firms with productivity $\phi_1$, and $m_2$ firms with productivity $\phi_2$, and $m_1 + m_2 = M$. Suppose firm $i \in [1,2,\ldots,M]$ locate at $x_1 \leq x_2 \leq \ldots \leq x_M$, respectively, and the utility function are

$$u_i = \begin{cases} 
  u^L_i = w_i - k(x_i - x), & \forall \ 0 \leq x \leq x_i, \ i = 1,2,\ldots,M, \\
  u^R_i = w_i - k(x - x_i), & \forall \ x_i < x \leq 1, \ i = 1,2,\ldots,M.
\end{cases}$$

Figure 8: The equilibrium locations and labors utilities in a circular market.
Minimum Wage under A Oligoposony Labor Market -1

Proposition 1’. Given \( \frac{(m_1\phi_1 + m_2\phi_2) - k}{M} < u_l < \min\{\phi_1, \phi_2\} \), when there are \( M \) oligoposony firms compete each other, the only possible equilibrium is that each firm occupies a disjointed range of labor market, and other areas are unemployed. The equilibrium locations \( (x_1^*, x_2^*, x_3^*, \ldots, x_M^*) \) satisfy \( x_1^* \geq x_1^{\min}, \quad x_M^* \leq x_M^{\max}, \) and

\[
x_{i+1}^* - x_i^* \geq x_{d(i,i+1)}^{\min},
\]

the equilibrium wages are \( W_i = \frac{u_l + \phi_j(i)}{2} \).

Proposition 5’. (a) Given \( \phi_1 < \frac{\phi_2 + u_l}{2} \), if a government imposes a minimum wage such that \( w_{\min} \leq \phi_1 \) (all firms get non-negative profits), then it is always welfare improving. (b) If negative profits are allowable, then imposing a minimum wage \( w_{\min} \) with low productivity firms binding such that \( w_2^* > w_{\min} \geq w_1^* \) is social welfare improving if and only if \( (3\phi_1 - u_l)/2 > w_{\min} > (\phi_1 + u_l)/2 \). (c) Specifically, imposing the optimal minimum wage \( w_{\min}^* = \phi_1 \) is always a social welfare improving.
Proposition 17. (a) Given \( \phi_1 \geq \frac{\phi_2 + u_l}{2} \), if a government imposes a minimum wage such that \( w_{\min} < \phi_1 \) (all firms get non-negative profits), then it is always welfare improving. The maximum welfare improving is at \( w_{\min} = \phi_1 \). (b) If negative profits are allowable, imposing the minimum wage (\( w_{\min}^{**} \geq \max \{w^*, w_2^*\} \)) is (not) a social welfare improving if only if \( u_l < (>) \frac{\phi_1 m_1 + \phi_2 m_2 - \sqrt{3} (\phi_2 - \phi_1) \sqrt{m_1 m_2}}{m_1 + m_2} \equiv \lambda_1 \), and \( \partial \lambda_1 / \partial m_1 \leq 0 \), \( \partial \lambda_1 / \partial m_2 \geq 0 \) when \( m_2 \leq 3m_1 \).
Proposition 18. \( SW^{**} - SW^* \geq 0 \) iff 
\[ u_l \leq \frac{m_1\phi_1 + m_2\phi_2 - 2(\phi_2 - \phi_1)\sqrt{m_1(m_1 + m_2)}}{m_1 + m_2} \equiv \lambda_2, \] and 
\[ \partial \lambda_2 / \partial m_1 < 0, \; \partial \lambda_2 / \partial m_2 > 0. \]

Proposition 19. (a) \( \widehat{SW}(w_{\text{min}}^{**}) > SW(w_1^*, w_2^*) \) if 
\[ \phi_2 > \frac{m_2u_l + \sqrt{3}(\phi_1 - u_l)\sqrt{m_1m_2}}{m_2}, \] (b) \( \widehat{SW}(w_{\text{min}}^{**}) > \widehat{SW}^* > 0 \), if \( \phi_2 > 2\phi_1 - u_l \) and 
\[ \phi_2 > \frac{m_2u_l - (2\phi_1 - u_l)\sqrt{m_1m_2}}{m_2}, \] and 
\[ \widehat{SW}(w_{\text{min}}^{**}) - \widehat{SW}^* < 0, \]
if \( \phi_2 < 2\phi_1 - u_l \) and 
\[ \phi_2 < \frac{m_2\phi_1 - (\phi_1 - u_l)\sqrt{m_1m_2}}{m_2}. \]
Proposition 20. When two firms compete each other, the only possible equilibrium is that both firms occupy a range of market without overlapping and other areas are unemployed when \( w_g < w_g^{\text{max}} \). The equilibrium locations \((x_1^*, x_2^*)\) satisfy (43), (44), (45), the equilibrium wage is \( w_1 = \frac{w_g + u_l + \phi_1}{2} \) and \( w_2 = \frac{w_g + u_l + \phi_2}{2} \).

Proposition 21. The optimal unemployment subsidy \( w_g^* = 2\phi_3 + u_l - \frac{m_1\phi_1 + m_2\phi_2}{M} \). When the economy is booming (\( \phi_i \) increases, \( i = 1, 2 \)), then the optimal unemployment subsidy must decrease. The number of firms \( (m_1, m_2) \) increases, then the optimum subsidy decreases.
Proposition 22. The optimal unemployment rate (under optimal unemployment subsidy) is

\[ UN^o = \frac{k - 2m_1 \phi_1 + 2(m_1 + m_2)u_l - 2m_2 \phi_2 + 2(m_1 + m_2)\phi_3}{k} . \]

The optimal unemployment rate is increasing in \( k \) and \( \phi_3 \), while it is decreasing in \( \phi_1, \phi_2 \) and \( k \). \( \frac{\partial UN}{\partial m_1} = \frac{2(\phi_3 - \phi_1 + u_l)}{k} \) and \( \frac{\partial UN}{\partial m_2} = \frac{2(\phi_3 + u_l - \phi_2)}{k} \).
Conclusions

We show duopsony firms are located without maximal differentiation. Instead, the “pure strategy location-wage equilibrium” is that both firms occupy disjointed areas and all remaining markets are unemployed.

Imposing the optimal minimum wage will increase the differentiation between firms. However, imposing a higher minimum wage may be welfare worsening under some conditions.

Imposing a minimum wage without driving out lower productivity firms is welfare improving (because reduce the oligopsony power of firms over workers, and increase employment).

Imposing a minimum wage will increase the differentiation between firms.

The optimal unemployment subsidy is positively related to the transportation rate and negative to the productivity and the leisure utility from unemployment.

The unemployment rate will decreases when a minimum wage is imposed without driving out effects. However, the unemployment rate may increase when low productivity firms are drived out, even it is social welfare improving.