Lattices and Lotteries

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Introduction
Comparative Statics: Two Approaches

- Topology, implicit function theorem
- Ordered space/lattice, lattice programming
Both goods are normal, in the sense that there exists bundle with higher consumption of each good at the higher income.

Differentiability, convexity fail; implicit function theorem does not apply, but lattice programming method does.
Introduction
Why Lattice Programming?

- Remove many common, technical, restrictions:
  - Differentiability, (Quasi-)concavity
  - Continuous choice
  - Local analysis

- Exploit the inherent order structure that models have:
  - Use order-based properties
  - Sufficient conditions involve observable choice behavior
Introduction

Background: Lattice Programming (LP) Foundations

- Veinott, A., 1992. Lattice Programming: Qualitative Optimization and Equilibria. (manuscript)
Introduction

Background: LP application to Problems with budgetary trade-offs

Generalize work on certainty choice \((u(x, y))\) to choice when one good is risky \((U(x, \tilde{y}) = E_{\tilde{y}} u(x, \tilde{y}))\).

Much is built on \(Eu(x + \tilde{y} - p(\tilde{y}))\) model with restrictions on \(u\).

- Keep expected utility hypothesis
- Relax restrictions on \(u\) (e.g. differentiability which is crucial in existing work)
- Relax restriction on preferences (perfect substitutes)
- Expand lottery choice; encompass discrete lottery choice
Literature dealing with multivariate measures of risk aversion, and risk aversion comparability and to a lesser extent with comparative statics of change in risk aversion and income

- Kihlstrom and Mirman 1974, 1981
- Diamond and Stiglitz 1974
- Richard 1975
- Duncan 1977
- Karni 1979, 1982
- Demers and Demers 1991
- Levy and Levy 1990
- Schlee 1990
- Athey 2002
- and a number of others...
For Leontief $U(x, \tilde{y}) = \int (\min \{x, qs\})^\alpha dF_y$ slope of $(x, q)$ IC is
\[
\frac{dq}{dx} = - \left( \frac{x}{q} \right)^{\alpha - 1} \Pr\left\{ \frac{\tilde{y}}{q} \geq \frac{x}{q} \right\} \frac{E \tilde{y} | \tilde{y} \leq \frac{x}{q}}{E \tilde{y} | \tilde{y} \leq \frac{x}{q}}.
\]

Homothetic; convex, for $\alpha \leq 1$ ... but for $\alpha > 1$?
A partially ordered set (Poset) \((\mathcal{X}, \leq_{\mathcal{X}})\) is a set \(\mathcal{X}\) with a partial order \(\leq_{\mathcal{X}}\) (a binary relation that is reflexive, antisymmetric, and transitive).

For \(X, X' \in \mathcal{X}\), \(X \lor X'\) and \(X \land X'\) denote the join (least upper bound) and the meet (greatest lower bound) respectively, when these exist.

Poset \((\mathcal{X}, \leq_{\mathcal{X}})\) is a lattice if the meet and join exist in \(\mathcal{X}\) for all \(X, X'\) in \(\mathcal{X}\).
\( X \lor X' = U \) and \( X \land X' = W \).

\((\mathbb{R}^2_+, \leq_\varepsilon)\) is a lattice.

\((A, \leq_\varepsilon)\) with \( A = \{X, X', W, Z\} \) is a lattice, although not a sublattice of \((\mathbb{R}^2_+, \leq_\varepsilon)\).
Space as a lattice

Set orders to compare feasible sets and optimal sets

Superextremal, SE, (or quasisupermodular) properties on objective function to give sufficient (and possibly necessary) conditions for comparative statics

Main (ordinal) Theorem (Veinott, Veinott and Li Calzi, Milgrom and Shannon), relates SE properties of objective to set ordering of optimal sets:

Strong set order increases in constraint sets are associated with set order increases in optimal sets if and only if objective function has relevant SE property.

How strong are the results depends on the lattice and on the ability to appropriately order feasible sets.
LP and Comparative Statics
Set Orders

Definition
Let \((\mathcal{X}, \leq)\) be a lattice and \(A, B\) be two sets in \(\mathcal{X}\).

- **Strong set order:** \(A \leq_a B\) iff for all \(X \in A, X' \in B\),
  \[ X \land X' \in A \text{ and } X \lor X' \in B. \]

- **Strongly lower than:** \(A \leq_s B\) iff for all \(X \in A, X' \in B\),
  \[ X \leq X \]

- **Chain-lower-than:** \(A \leq_c B\) iff for all \(X \in A, X' \in B\),
  \[ X \text{ and } X' \text{ are comparable with } X \land X' \in A \text{ and } X \lor X' \in B. \]
Lattices and Comparative Statics

Superextremal Properties

**Definition**
Let \((\mathcal{X}, \leq_\mathcal{X})\) be a lattice and consider \(f : \mathcal{X} \rightarrow \mathbb{R}\).

- **Supermodular (SM):** \(f\) is SM iff, for all \(X, X' \in \mathcal{X}\),
  \[
  f(X \lor X') + f(X \land X') \geq f(X) + f(X').
  \]

- **Lattice-superextremal (LSE):** \(f\) is LSE iff, for all \(X, X' \in \mathcal{X}\),
  \[
  f(X) \geq (>) f(X \land X') \Rightarrow f(X \lor X') \geq (>) f(X').
  \]

- **Strictly superextremal (SSE):** \(f\) is LSE iff, for all incomparable \(X, X' \in \mathcal{X}\),
  \[
  f(X) \geq f(X \land X') \Rightarrow f(X \lor X') > f(X').
  \]
Theorem (Veinott, Milgrom and Shannon)
Let \((\mathcal{X}, \leq_{\mathcal{X}})\) be a lattice and consider \(f : \mathcal{X} \rightarrow \mathbb{R}\). Then,

\[
f \text{ is LSE iff } \arg \max_A f \leq_a \arg \max_B f \quad \forall A \leq_a B.
\]

\[
f \text{ is SSE iff } \arg \max_A f \leq_c \arg \max_B f \quad \forall A \leq_a B.
\]

Often, economic applications involve only some of the strongly ranked sets in \((\mathcal{X}, \leq_{\mathcal{X}})\) so the necessity part of the theorem cannot be used, and only sufficient conditions can be had.
LP and Comparative Statics

Euclidean lattice: inappropriate for value problems

- An increasing $f$ is always LSE/SSE: $f \left( X \lor X' \right) > f(X), f(X')$ for all incomparable $X, X'$.
- Feasible sets with budgetary trade-offs are not strong set comparable in the Euclidean lattice, $X \in A, X' \in B$, but $X \lor X' \notin B$.

- **Lattices and LP are not just about Euclidean order and positive cross partials.**
Consumer problem under certainty

Economic Problem

Consumer problem with two goods

\[
\max_{(x, y) \in \mathbb{R}_+^2} u(x, y) \quad s.t. \quad (x, y) \in B(I)
\]

where \( B(I) = \{(x, y) \in \mathbb{R}_+^2, \ p_x x + p_y y \leq I\} \) under linear pricing, and \( B(I) = \{(x, y) \in \mathbb{R}_+^2, \ p_x x + p(y) \leq I\} \) when \( y \) is subject to a non-linear price function \( p(y) \) (strictly increasing).

- Good \( y \) is \textit{strongly normal} if every optimal choice at a higher income is at least as large as every optimal choice at a lower income.
- Good \( y \) is \textit{pathwise normal} if for each optimal choice at lower income, there is an optimal choice at high income that is at least as large (and conversely).
Define lattice \((\mathbb{R}^2_+, \leq)\) so that:

- Partial order \(\leq\) reflects comparative statics on \(y\)
  \[ X \leq X' \Rightarrow y \leq y' \]

- Partial order \(\leq\) is consistent with increases in income
  \[ X \leq X' \Rightarrow p_x x + p_y y \leq p_x x' + p_y y' \]

- Or, if instead \(y\) is subject to a non-linear pricing function \(p(y)\)
  \[ X \leq X' \Rightarrow p_x x + p(y) \leq p_x x' + p(y') \]

- These conditions define the class of partial orders called value orders.
Certainty Case
Examples of Value orders

Direct Value Order (Antoniadou 1996, 2007)

- Budget set comparability assured under linear pricing. Hence SSE (LSE) property in this lattice is sufficient for good $y$ to be strongly (pathwise) normal - works with non perfectly divisible $y$ too.
Certainty case
LSE property in direct value lattice

\( u \) is LSE if, for \( X, X' \) incomparable with \( y > y' \) and \( p \cdot X < p \cdot X' \):

\[
\begin{align*}
u(x, y) \geq (>) u(x + \Delta, y') &\Rightarrow u(x' - \Delta, y) \geq (>) u(x', y')
\end{align*}
\]

where \( \Delta = \frac{p_y}{p_x} (y - y') \).

- This has the form of a revealed preference condition:
  \( U(X) \geq U(W) \Rightarrow U(X') \not> U(Z) \)

- If a function satisfies this LSE condition for every price vector, then it satisfies the Quah (2007) condition, and vice versa.
Certainty Case
Examples of Value orders

The Radial Value Order (Ruble and Mirman 2008)

Budget set comparability assured under linear pricing. Hence SSE (LSE) property in this lattice is sufficient for good $y$ to be strongly (pathwise) normal - satisfied by non-convex preferences.
The Expenditure Value Order (AMR 2009)

\[ X \leq_{ev(p,y)} X' \iff p(y) \leq p(y') \text{ and } p_x x + p(y) \leq p_x x' + p(y') \]

- \( B(I) \leq_{a} B(I') \) and comparative statics can be carried out.
  Constraints needn’t be linear for value order to work. There could be quantity discounts or premia or a combination.
Uncertainty and Consumer Choice

Objective Function with a Risky Good

Consumer preferences with one safe, one risky good

\[ U(x, \tilde{y}) = \int u(x, s) dF_y \]

- Consumption set \( \mathbb{R}_+ \times \mathcal{F} \), where \( \mathcal{F} = \) distribution functions with non-negative support.
- What consumption lattice?
- What comparative statics?
- What relationship between lattice properties of \( u \) and \( U \)?

Antoniadou, Mirman and Ruble ()
Lattices and Lotteries
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Uncertainty and Product Lattice

Product Lattice with FOSD

- FOSD is a good place to start to capture idea of "increase" for the risky good (most of what we say can be replicated for other orders such as SOSD)
- With one distribution function, an increase in the quantity of risky good corresponds to an increase wrt to FOSD

\[ F_y \leq_{FOSD} F_{y'} \iff F_y(s) \geq F_{y'}(s) \text{ for all } s \]

- The obvious consumption lattice is the product lattice

\[ X \leq_{\epsilon \times FOSD} X' \iff x \leq x' \text{ and } F_y \leq_{FOSD} F_{y'} \]

- \((\mathbb{R}_+ \times \mathcal{F}, \leq_{\epsilon \times FOSD})\) is a lattice. But this will not work much better than the Euclidean lattice!
In the product lattice \((\mathbb{R}_+ \times \mathcal{F}, \leq_{\epsilon \times FOSD})\):

- \(U\) is SM in \((\mathbb{R}_+ \times \mathcal{F}, \leq_{\epsilon \times FOSD})\) if and only if \(u\) is SM in \((\mathbb{R}_+^2, \leq_{\epsilon \times FOSD})\).
- \(u\) increasing on \(\mathbb{R}_+^2 \Rightarrow U\) SSE in \((\mathbb{R}_+ \times \mathcal{F}, \leq_{\epsilon \times FOSD})\).

In the product lattice \((\mathbb{R}_+ \times \mathcal{F}, \leq_{\epsilon \times SOSD})\):

- \(u\) increasing on \(\mathbb{R}_+^2\) and strictly concave in \(y\) \(\Rightarrow U\) LSE in \((\mathbb{R}_+ \times \mathcal{F}, \leq_{\epsilon \times SOSD})\).
Consumer problem with one certain and one uncertain good

$$\max_{(x, \tilde{y}) \in \mathbb{R}_+ \times \mathcal{F}} U(x, \tilde{y}) = \int u(x, s) \, dF_y \quad \text{s.t.} \quad (x, \tilde{y}) \in B(l)$$

where $B(l) = \{(x, \tilde{y}) \in \mathbb{R}_+ \times \mathcal{F}, p_x x + p(\tilde{y}) \leq l\}$, $p(\tilde{y})$ being the price function for lotteries.

- Aim: Embed problem into appropriate lattice $(\mathbb{R}_+ \times \mathcal{F}, \leq ?)$ where budget sets increase in $l$, then if $U$ is LSE, optimal sets increase in $l$ also, in a way that is meaningful in terms of the consumption of the risky good.
Define lattice \((\mathbb{R}_+ \times \mathcal{F}, \leq)\) so that:

- Partial order \(\leq\) reflects comparative statics on \(\tilde{y}\)
  \[ X \leq Y \implies \tilde{y} \leq_{FOSD} \tilde{y}' \]

- Partial order \(\leq\) is consistent with increases in income
  \[ X \leq Y \implies p_x x + p(\tilde{y}) \leq p_x x' + p(\tilde{y}') \]

- This class of partial orders are described as *stochastic value orders*.
- Arguably the most natural extension of earlier certainty case method.
First Order Stochastic Value Order

Let $X, X' \in \mathbb{R}_+ \times \mathcal{F}$. Then,

$$X \leq_{\text{FOSV}(\hat{y}, p)} X' \text{ iff } \hat{y} \leq_{\text{FOSD}} \hat{y}' \text{ and } p_{x} x + p(\hat{y}) \leq p_{x} x' + p(\hat{y}') .$$

We need some assumptions on the price function:

$$p(y_0) = 0 \quad \text{(A2)}$$

$$F_y \leq_{\text{FOSD}} (<_{\text{FOSD}}) F_{y'} \Rightarrow p(\hat{y}) \leq (<) p(\hat{y}') \quad \text{(A3)}$$

($y_0$ is degenerate lottery at 0)

- For example, could have a “fair” pricing function, $p(\hat{y}) = \int a(y) dF_y$. 
Suppose \( p(\tilde{y}) \) satisfies (A2) and (A3). Then \( \left( \mathbb{R}_+ \times \mathcal{F}, \leq_{FOSV(\tilde{y}, p)} \right) \) is a lattice.

Let \( X, X' \) be incomparable with \( p_x x + p(\tilde{y}) \leq p_x x' + p(\tilde{y}') \). Then, have \( X \lor X' = (x^\lor, \tilde{y}^\lor) \) and \( X \lor X' = (x^\land, \tilde{y}^\land) \) with

\[
x^\lor = \max \left\{ x' - \frac{p(\tilde{y}^\lor) - p(\tilde{y}')}{p_x}, 0 \right\}, \quad F^\lor(s) = \min \{ F_y(s), F_{y'}(s) \}
\]

and

\[
x^\land = \max \left\{ x + \frac{p(\tilde{y}) - p(\tilde{y}^\land)}{p_x}, 0 \right\}, \quad F^\land(s) = \max \{ F_y(s), F_{y'}(s) \}
\]
The consumption set is in a sense too rich, with all possible lotteries on the non-negative line. The join of two incomparable lotteries wrt FOSD can be prohibitively expensive, so that budget sets may fail strong set comparability.

Example:

Let \( X = (0.5, y_1) \) and \( X' = (0, \tilde{y}') \) with \( \tilde{y}' = \{0.5, 0.5; 0, 4\} \), so \( X \in B(1.5) \) and \( X' \in B(2) \).

But \( X \lor X' = (0, \tilde{y}^{\lor}) \) with \( \tilde{y}^{\lor} = \{0.5, 0.5; 1, 4\} \) and \( p(\tilde{y}^{\lor}) = 2.5 \).

So \( X \lor X' \notin B(2) \), therefore \( B(1.5) \not\subseteq_a B(2) \).

Hence, we proceed by either restricting the support of the distribution functions, or the allowable distribution functions.

A third possible way that may work is to further restrict the price function by linking it to properties of preferences.
Bounded Support

- Problem arises because “bigger” (join) lottery \( \tilde{y}^\vee \) might not be affordable.
- If the support is bounded, \( p(\tilde{y}^\vee) \) is also.
- We know (from certainty case) we can restrict attention to a sublattice
  - For income high enough, “biggest” lottery is always affordable.
- Support of \( \tilde{y} \subseteq [0, k] \).
- Then \( p(\tilde{y}) \leq p(y_k) \) where \( y_k \) is degenerate lottery at \( k \).
- If \( l \geq p(y_k) \), then \( B(l) \) has a maximum, \( \bar{X} = \left( \frac{l-p(y_k)}{p_x}, y_k \right) \) and the problem with the unaffordability of the join is avoided.
Proposition
Suppose the consumption set is a sublattice \( A \subseteq \mathbb{R}_+ \times \mathcal{F} \) with \( X \in A \Rightarrow p_x \times + p(\tilde{y}) \geq p(y_k) \), and \( p(\tilde{y}) \) satisfies (A2) and (A3). Then, \( B(I) \leq_a B(I') \) for \( I' \geq I \), and therefore:

- if \( U \) is LSE in \( (\mathbb{R}_+ \times \mathcal{F}, \leq_{FOSV(\tilde{y}, p)}) \), the lottery \( \tilde{y} \) is pathwise increasing wrt FOSD (hence pathwise "normal")
- if \( U \) is SSE in \( (\mathbb{R}_+ \times \mathcal{F}, \leq_{FOSV(\tilde{y}, p)}) \), the lottery \( \tilde{y} \) is strongly increasing wrt FOSD
Consumption set is $\mathbb{R}_+ \times \mathcal{F}^C$ where $\mathcal{F}^C \subseteq \mathcal{F}$ is a chain with respect to FOSD.

Then $\tilde{y}^\vee, \tilde{y}^\wedge \in \{\tilde{y}, \tilde{y}'\}$. Thus the join is trivially affordable.

**Proposition**

Suppose the consumption set is a sublattice $\mathbb{R}_+ \times \mathcal{F}^C \subseteq \mathbb{R}_+ \times \mathcal{F}$ with $\mathcal{F}^C$ a chain with respect to $\leq_{FOSD}$, and $p(\tilde{y})$ satisfies (A2) and (A3). Then, $B(I) \leq_a B(I')$ for $I' \geq I$, and therefore:

- if $U$ is LSE in $\left(\mathbb{R}_+ \times \mathcal{F}^C, \leq_{FOSV}(\tilde{y}, p)\right)$, the lottery $\tilde{y}$ is pathwise increasing wrt FOSD
- if $U$ is SSE in $\left(\mathbb{R}_+ \times \mathcal{F}^C, \leq_{FOSV}(\tilde{y}, p)\right)$, the lottery $\tilde{y}$ is strongly increasing wrt FOSD
A Bridge to the Certainty Case

The Expenditure Value Order on Consumption set with uncertainty

- Suppose (and this is not a minor assumption) the price function satisfies:
  \[ p(\tilde{y}) = p(\tilde{y}') \iff \tilde{y} = \tilde{y}' \] (A1)

- Let \( X, X' \in \mathbb{R}_+ \times \mathcal{F} \). Then, \( X \leq_{ev(p(\tilde{y}))} X' \) if and only if \( p(\tilde{y}) \leq p(\tilde{y}') \) and \( p_x x + p(\tilde{y}) \leq p_x x' + p(\tilde{y}') \).

Proposition

If \( p(\tilde{y}) \) satisfies (A1), then \( \left( \mathbb{R}_+ \times \mathcal{F}, \leq_{ev(p(\tilde{y}))} \right) \) is a lattice.

- If \( U \) is LSE in \( \left( \mathbb{R}_+ \times \mathcal{F}, \leq_{ev(p(\tilde{y}))} \right) \), then expenditure on the risky good is pathwise increasing with income.

- If \( U \) is SSE in \( \left( \mathbb{R}_+ \times \mathcal{F}, \leq_{ev(p(\tilde{y}))} \right) \) and the constraint is binding, then expenditure on the risky good is strongly increasing with income.
FOSV and EVO

Is there a relation?

- Relationship between \( \mathbb{R}_+ \times \mathcal{F}, \leq_{eV(p(\tilde{y}))} \) and \( \left( \mathbb{R}_+ \times \mathcal{F}, \leq_{FOSV(\tilde{y},p)} \right) \) lattices:
  
  Suppose the price function satisfies (A1)-(A3). Then the FOSV order is finer than the EVO
  
  \[ X \leq_{FOSV(\tilde{y},p)} X' \implies X \leq_{eV(p(\tilde{y}))} X' \]
  
  They are equivalent on subsets of the consumption set \( \mathbb{R}_+ \times \mathcal{F}^C \), i.e. when the allowable lotteries form a chain.

- Unless in the special case of a chain, the lattice properties LSE or SSE on the two lattices cannot imply one another. They are distinct.

- The expenditure monotonicity implied by the LSE/SSE property in \( \left( \mathbb{R}_+ \times \mathcal{F}, \leq_{eV(p(\tilde{y}))} \right) \) does not in general imply monotonicity wrt FOSD.
Application: Discrete Lottery Choice
An intuitive demonstration of SE Conditions

A simple binary choice problem:

\[
\max_{(x, \tilde{y}) \in \mathbb{R}_+ \times \{y_0, \tilde{y}\}} U(x, \tilde{y}) = \int u(x, s) \, dF_y \quad \text{s.t.} \quad (x, \tilde{y}) \in B(I)
\]

- Discrete choice \{y_0, \tilde{y}\} outside scope of usual method. But \((\mathbb{R}_+ \times \{y_0, \tilde{y}\}, \leq_{FOSV(\tilde{y}, p)})\) is a sublattice of \((\mathbb{R}_+ \times \mathcal{F}, \leq_{FOSV(\tilde{y}, p)})\) and the LP apparatus can be used.

**Proposition**

\(U\) is LSE (SSE) if and only if \(\tilde{y}\) is normal.

- Restricted framework yields a complete characterization.
Application: Discrete Choice

LSE/SSE Condition

$U$ is LSE if, for $l' \geq l$ (with $X, X'$ so that $p_x x + p(\bar{y}) \leq p_x x'$):

$$U(x, \bar{y}) \geq (>) U \left(x + \frac{p(\bar{y})}{p_x}, 0\right) \Rightarrow U \left(x' - \frac{p(\bar{y})}{p_x}, \bar{y}\right) \geq (>) U(x', 0)$$

- Cobb-Douglas $u$ is trivially LSE.
- $u(x, y) = (x + a)^\alpha (y + b)^\beta$ is also LSE.
The LSE Condition can be rewritten as:

\[ U(x, \tilde{y}) - u(x, 0) \geq (> ) u\left(x + \frac{p(\tilde{y})}{p_x}, 0\right) - u(x, 0) \Rightarrow \]

\[ U\left(x' - \frac{p(\tilde{y})}{p_x}, \tilde{y}\right) - u\left(x' - \frac{p(\tilde{y})}{p_x}, 0\right) \geq (> ) U(x', 0) - u\left(x' - \frac{p(\tilde{y})}{p_x}, 0\right) \]

- LHS is risk attitude term, RHS is opportunity cost term.
Ys represent risk effect.

Risk effect term decreases relative to opportunity cost term (a kind of substitution).

$U$ is not LSE, $\tilde{y}$ is not normal.
• Risk effect term increases relative to opportunity cost term (a kind of complementarity).
• $U$ is LSE, $\tilde{y}$ is normal.
Recall the LSE Condition:

\[ U(x, \tilde{y}) \geq (>) u \left( x + \frac{p(\tilde{y})}{p_x}, 0 \right) \Rightarrow U \left( x' - \frac{p(\tilde{y})}{p_x}, \tilde{y} \right) \geq (>) u \left( x', 0 \right) \]

- Let \( C^e_x \) denote the certainty equivalent in additional units of good \( x \) at \((x, \tilde{y})\). That is, \( C^e_x \) satisfies \( u(x + C^e_x, 0) = U(x, \tilde{y}) \) and suppose that this is well-defined. Then, the LSE condition can be reformulated as:

  \[ p_x C^e_x \geq (>) p(\tilde{y}) \Rightarrow p_x C^e_{x'} \geq (>) p(\tilde{y}) \quad \forall x, x' \text{ with } x' > x. \]

- \( C^e_x \) must be a nondecreasing function of \( x \).