Estimation of confidence intervals for the mean of heavy tailed loss distributions
ESAM, 2009

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   Estimating the mean of a heavy tailed distribution
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Introduction: Operational Risk losses

The main feature of loss distributions - heavy right tails
Measuring Operational Risks

The capital to cover unexpected losses

The Basel Committee: the capital = unexpected losses

- Expected loss = average of the observed data values
- Unexpected loss = 99.9% quantile - expected loss

Severity of Operational Risk (OR) losses

2008 was worst on record in terms of severity of OR (Algorithmics Gr.)

- in 2008 Société Générale lost €4.9bn on unauthorised trading
Motivation: bias in Loss Distribution Approach (LDA)

Bias of the estimates

- The mean computed as a sample average is greatly affected by extreme values

$\mu$ as a sample average, quantile is read from ordered data

$\mu = 2126.564$

$\mu$ and quantile computed using specific methods

$\mu = 1632.033$ that is the difference of more than 494mln US$ (20%)
Objectives: propose improvements to the LDA

1. Select methods that are specifically designed for heavy tailed distributions:
   - Estimate the mean
   - To reflect accuracy of the point estimates construct confidence intervals for the mean

2. Compare the performances of the methods in terms of coverage probabilities, expected width, and rates of missing left/right endpoints of confidence intervals

3. Based on the results of the simulation studies
   Recommend the best method that will be useful in practice
Methodology

Estimate the mean by

- Method specifically developed for heavy tailed distributions (Peng, 2001)

Estimate confidence intervals for the mean by:

- \( m \) out of \( n \) with replacement bootstrap (Hall and LePage, 1996)
- \( m \) out of \( n \) without replacement bootstrap (Politis et al., 1999)
- Empirical likelihood method (Peng, 2004)
Evaluate performance of estimation methods depending on

**Sample size for all methods**

Evaluate the accuracy of the confidence interval estimates across various sample sizes

**Resampling size for bootstrap methods**

Evaluate the accuracy of the bootstrap confidence interval estimates across various values for $m$
Estimating the mean of a heavy tailed distribution

Define $X_{n,1}, \ldots, X_{n,n}$ as order statistics of a sample $X_1, \ldots, X_n$

Use the threshold to split the ordered data to the nontail and tail regions

the mean for the nontail region

$$
\mu_n^{(1)} = \frac{1}{n} \sum_{i=1}^{n-k} X_i,
$$

the mean for the tail region

$$
\hat{\mu}_n^{(2)} = \frac{k}{n} X_{n,n-k+1} \frac{\hat{\gamma}}{\hat{\gamma} - 1}
$$

where $\hat{\gamma}$ is the Hill’s estimator

Then the mean is the sum of $\mu_n^{(1)}$ and $\mu_n^{(2)}$
Selecting the optimal threshold parameter

For every $k$, we obtain another $\hat{\gamma}$: for small $k$ bias is small, but variance is large

$$AVar(H_{k,n}) \sim \frac{\gamma^2}{k}$$
Double bootstrap method (Danielsson, 2001)

Optimal $k$ - achieves optimal balance between bias and variance

- Use $m$ out of $n$ bootstrap and draw samples of size $m_1 = n^{(1-\epsilon)}$
- For a range of $k_1$ (eg. from $0.1m_1$ to $0.9m_1$) compute AMSE
- Find $k_{m_1}$ where the AMSE is minimal
- Repeat first three steps for $m_2 = \frac{m_1^2}{n}$ to find $k_{m_2}$
- Use this formula to calculate optimal $k$

$$\hat{k}_n^{opt} = f(\hat{k}_{m_1}, \hat{k}_{m_2})$$
Bootstrap based confidence interval for the mean

Define $X_1^*, \ldots, X_m^*$ iid rv drawn randomly from $X_1, \ldots, X_n$

- $m$ out $n$ with replacement bootstrap
- $m$ out $n$ without replacement bootstrap

To ensure consistency of the bootstrap estimates

- $m < n$, where $m = n^{1-\epsilon}$ for $0 < \epsilon < 0.5$
- for example for $n=1000$, $32 < m < 933$
Asymptotic bootstrap confidence intervals for the mean

Centred studentised means for both bootstrap methods

\[
\bar{X}_m^* = \frac{1}{m} \sum_{i=1}^{m} X_i^* \quad (3)
\]

\[
S_m^* = \frac{1}{m} \sum \left( X_i^* - \bar{X}_m^* \right)^2 \quad (4)
\]

\[
T_m^* = \sqrt{m} \frac{\bar{X}_m^* - \bar{X}}{S_m^*} \quad (5)
\]
Sampling distribution of $T_m^*$

100(1 – $\alpha$)% confidence intervals for $\mu$

$$I_{1-\alpha}^* = (\bar{X}_n - \hat{x}_{1-\alpha}S_n / \sqrt{n}, \bar{X}_n + \hat{x}_{1-\alpha}S_n / \sqrt{n})$$.

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Empirical likelihood method

Let $X_1, \ldots, X_n$ be i.i.d. observations with a common distribution function $F$ and $X_{n,1}, \ldots X_{n,n}$ are the order statistics.

Approximate the distribution in the right tail of $F$ as follows:

$$F(x) = 1 - cx^{-\gamma} \text{ for } x > X_{n,n-k}$$

where $\gamma$ is an unknown tail index and $X_{n,n-k}$ is the threshold value.

The log-likelihood function

$$l_0(c, \gamma, p_i) = l_1(c, \gamma) + l_2(p_i)$$

estimates nontail region nonparametrically and tail region parametrically using Extreme Value Theory (Qin and Wong, 1996).
$H_0$: under constrains (9)-(10), $H_1$: under all constrains

the distribution is heavy tailed

$$\gamma > 1, \ c > 0, \ p_i > 0$$ (9)

the nontail region is treated nonparametrically

$$\sum_{\delta_i=0} p_i = 1 - cX_{n,n-k+1}^{-\gamma}$$ (10)

the mean is computed as a sum of $\mu_n^{(1)}$ and $\mu_n^{(2)}$

$$\sum_{\delta_i=0} p_i X_i = \mu - c \frac{\gamma}{\gamma - 1} X_{n,n-k+1}^{1-\gamma}$$ (11)
Empirical likelihood confidence interval for the mean

The semiparametric likelihood ratio statistic is defined as

\[ l(\mu) = -2 \left\{ l_0 (\hat{c}, \hat{\gamma}, \hat{p}_i) - l_0 (\bar{c}, \bar{\gamma}, \bar{p}_i) \right\}. \] (12)

\( l_0 (\hat{c}, \hat{\gamma}, \hat{p}_i) \) is estimated under constraints (9) and (10).
\( l_0 (\bar{c}, \bar{\gamma}, \bar{p}_i) \) is estimated under constraints (9)-(11).
Asymptotic empirical likelihood interval for the mean

$l(\mu)$ at the true mean has $\chi^2_{(1)}$ distribution (Peng, 2004)
Simulation design: Fréchet($\gamma$) distribution

Parameter $\gamma$ determines the shape of the tail: the tail is less heavy for large $\gamma$. 
Simulation design

500 samples from the Fréchet(1.5) distribution

\( n = 100, 300, 500, 1000 \)

Confidence intervals were computed at two significance levels

\( \alpha = 0.1 \text{ and } 0.05 \)

Bootstrap methods

Size \( m = n^{1-\epsilon} \) for any \( \epsilon \) such that \( 0 < \epsilon < 0.5 \) (Danielsson, 2001)

We use \( \epsilon = 0.1, 0.3, 0.35, 0.4 \)

Number of iterations \( b = 50000 \)
<table>
<thead>
<tr>
<th>Definitions of accuracy measures for confidence intervals</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Coverage probability</strong></td>
</tr>
<tr>
<td>proportion of intervals covering $\mu$</td>
</tr>
<tr>
<td><strong>Expected width</strong></td>
</tr>
<tr>
<td>average length of estimated intervals covering $\mu$</td>
</tr>
<tr>
<td><strong>Rate of missing left endpoint</strong></td>
</tr>
<tr>
<td>proportion of intervals with left endpoints exceeding $\mu$</td>
</tr>
<tr>
<td><strong>Rate of missing right endpoint</strong></td>
</tr>
<tr>
<td>proportion of intervals with right endpoints below $\mu$</td>
</tr>
</tbody>
</table>
### Simulation results for Fréchet(1.5)-95% confidence interval

<table>
<thead>
<tr>
<th>Method</th>
<th>Normal Approxim</th>
<th>$m$ out of $n$ with repl</th>
<th>$m$ out of $n$ without repl</th>
<th>Emp Likelihood</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>n=1000</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Obs cov-ge</td>
<td>0.828</td>
<td>0.964</td>
<td>0.970</td>
<td>0.996</td>
</tr>
<tr>
<td>Exp width</td>
<td>1.554</td>
<td>2.508</td>
<td>2.638</td>
<td>6.783</td>
</tr>
<tr>
<td>Miss left</td>
<td>0</td>
<td>0.004</td>
<td>0.004</td>
<td>0.004</td>
</tr>
<tr>
<td>Miss right</td>
<td>0.172</td>
<td>0.032</td>
<td>0.026</td>
<td>0</td>
</tr>
<tr>
<td><strong>n=500</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Obs cov-ge</td>
<td>0.786</td>
<td>0.976</td>
<td>0.986</td>
<td></td>
</tr>
<tr>
<td>Exp width</td>
<td>1.800</td>
<td>2.712</td>
<td>2.893</td>
<td></td>
</tr>
<tr>
<td>Miss left</td>
<td>0</td>
<td>0.006</td>
<td>0.006</td>
<td></td>
</tr>
<tr>
<td>Miss right</td>
<td>0.214</td>
<td>0.018</td>
<td>0.008</td>
<td></td>
</tr>
</tbody>
</table>
Conclusions

**Confidence interval for the mean of a heavy tailed distribution**

- Empirical likelihood works well in sample of size 1000
- Both $m$ out of $n$ bootstrap methods work well in samples of various sizes, but only with resampling size $m = n^{1-\epsilon}$, where $\epsilon = (0.35, 0.4)$
  Theoretically recommended range of values is $\epsilon = (0, 0.5)$
Future directions

Calibrated empirical likelihood
Calibrate the critical values (to be used instead of $\chi^2_1$) using $m$ out of $n$ bootstrap to obtain calibrated empirical likelihood confidence intervals in samples of smaller than or equal to 500

Refined bootstrap
Adapt refined bootstrap (Cornea and Davidson, 2008) proposed for hypothesis testing to confidence interval estimation for heavy tailed distributions