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1 James Cooper provided many helpful comments on the underlying mathematics of this paper. All errors are mine.
**Introduction:** In his Nobel lecture Samuelson [1972] gave a simple geometric characterization of profit maximization. This characterization can be expressed in analytic terms, see Cooper, Russell, Samuelson [2001], Cooper and Russell [2006], but differential equations do not provide any insight into the reason why constrained optimization processes should be captured by an area condition. In this paper we use the tools of differential geometry to try to make this connection clearer.

Samuelson’s condition is shown in Fig 1. A firm (which at this point may or may not be profit maximizing) hires two inputs $q_1, q_2$, at prices $p^1, p^2$, to produce output $q_0$ which it sells at some market price. The diagram shows families of own price demand curves for $q_1$ in two different constraint regimes. In the steeper family, labeled $V_2$, the quantity of $q_2$ is fixed. In the other family labeled $P_2$, the price $p^2$ is fixed. The Le Chatelier principle guarantees that this family will be flatter.

Samuelson’s (S) test for profit maximization is as follows. Calculate the areas $a, b, c, d$ of any four ‘quadrilaterals’ cut out by the “leaves” of the demand systems as shown in the Figure. Then if $ad=cb$, the firm is maximizing profits.

Where does such a result come from? More generally, what is the relationship between textbook duality theory, which captures the partial differential equations of constrained maximization systems and Samuelson’s area condition? To answer these questions we need some concepts from area (symplectic/contact) geometry. In this framework economics shares a common mathematical basis with mechanics and more especially thermodynamics.
Economic Phase Space:
We assume that the economic agent (individual or firm) can be represented by a point in $\mathbb{R}^{2n+1}$, Euclidean $2n+1$ dimensional space. Points in this space have coordinates $q_i \ i=1,n,$ (quantities) and $p_j \ j=1,n,$ (prices). The $(2n+1)st$ coordinate is given by the value of some
objective function. For the firm, this objective function is profits $\prod$. Profits are generated by selling output $q_0$ produced according to some production function $q_0 = F(q_1, \ldots, q_n)$ into a demand function which is not upward sloping.

When we come to the consumer we will think of the $(2n+1)st$ coordinate as $M$, income, in the context of the compensation function of McKenzie (1957). We stress that to this point we are imposing no optimizing conditions so that the $q$s, and $p$s and together with the relevant objective function all are to be thought of as independent variables.

Samuelson’s condition is a test for the maximization of profits. Now when profits are maximized, we know from standard duality theory that there exists a profit function $\prod(p^1 \ldots p^n)$ such that $d\prod = q_i dp^i$. Here we use the convention that lowered and raised indices are to be summed.

Contact geometry provides tools which enable us to reverse these steps. We start with a differential form $\theta = d\prod - q_i dp^i$ and look for a subset of $\mathbb{R}^{2n+1}$ over which $\theta = 0$. It is a theorem that such spaces exist, that their maximum dimension is $n$, and that they are defined by a function $\prod$ with $\frac{\partial \prod}{\partial p^i} = q_i$ so that on these $n$ dimensional submanifolds we are indeed assured that profits are maximized.²

More precisely we have,

Definition: Let $(\mathbb{R}^{2n+1}, \theta)$ be a contact manifold. A Legendrian submanifold³ is an $n$ dimensional submanifold on which $\theta = 0$.

A local description of Legendre submanifolds may be given in terms of a generating function by the following theorem, Arnold [1978].

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² Strictly speaking the firm is extremizing profits. To go further and say that the firm is maximizing profits requires the introduction of second order convexity assumptions. These will simply be assumed in this paper.

³ There are now many books dealing with the mathematics of Legendrian submanifolds of contact manifolds, but Arnold [1978] is the clearest and the deepest. The role of contact geometry in maximizing systems was first noted by Hermann [1973] in his treatment of thermodynamics. In that same respect, Arnold [1989] used contact geometry to derive exactly the equivalent of profit functions. Indeed many of the ideas of this paper are drawn from the modern geometric treatment of thermodynamics, see Lichnerowicz[1970]. This is not too surprising since Samuelson developed his area condition, “while reading Clerk Maxwell’s charming introduction to thermodynamics” Samuelson [op.cit.]
Theorem (Arnold): For any partition \( I \cup J \) of the set of indices \( \{1, 2, \ldots, n\} \) into two distinct subsets \( I \) and \( J \) and for a function \( \varphi (q_i, p^j) \), of \( n \) variables \( q_i, i \in I, p^j, j \in J \), the \((n+1)\) equations

\[
q_i = \frac{\partial \varphi}{\partial p^j}, \quad p^j = \frac{\partial \varphi}{\partial q_i}, \quad s = \varphi - q_i \frac{\partial \varphi}{\partial q_i}
\]

define a Legendrian submanifold of \( \mathbb{R}^{(2n+1)} \). Conversely, every Legendrian submanifold is locally defined by such functions for some at least one of the \( 2^n \) choices of the subset \( I \).

Clearly one possible choice for the generating function \( \varphi \) is a profit function \( \prod (p^1 \ldots p^n) \) but, as the theorem shows there are many other such “constrained profit functions” with any \( q, p \) mixture of \( n \) variables, the only restriction being that a constrained profit function cannot contain both an input and that input’s price.

We have therefore characterized economic processes as follows.

**Proposition 1:** Profit maximizing equilibrium processes are flows on \( n \) dimensional Legendrian sub-manifolds of a \((2n+1)\) dimensional contact manifold with contact form

\[
\theta = d[\prod q_i dp^i]
\]

Remarks:

1. The pair \((\mathbb{R}^{(2n+1)}, \theta)\) is known in general as a contact manifold. In our particular circumstances we may call it Economic Phase Space (EPS).
2. For an economist the easiest entrée to contact geometry is through the profit function and its differential. The Legendrian submanifold, however, is defined by the condition that \( \theta = 0 \), so it is defined on the dual space to the space of differentials. This \( 2n \) dimensional vector space, a subset of the tangent space to \( \mathbb{R}^{(2n+1)} \), is known as called the contact structure generated by \( \theta = d[\prod qa_i dp^i] \) and is written \( \text{ker} (\theta) \). Once we have \( \text{ker} (\theta) \) it is obvious that if \( \tau \) is a non-vanishing function on \( M \), then the 1-form \( \tau \theta \) has the same kernel and thus generates the same contact structure.
3. By a direct calculation it is straightforward to see that the form \( \theta = d[\prod q_i dp^i] \) has the feature that it is as far away as possible from being integrable. More precisely, \( \theta \wedge (d\theta)^n = 0 \). Here \( \wedge \) denotes the exterior product and
\((d\theta)^n = d\theta \wedge d\theta\) \((n\ times)\). Geometrically this means that the field of hyperplanes given by \(\ker(\theta)\) will nowhere line up to form a surface in \(2n\) dimensional space.

This is precisely what we need to describe general (i.e. pre maximizing) economic behavior. Even before we introduce maximization into economic behavior, we have a geometric structure given by the fact that for example \(\text{Cost} = q_ip_i\). Usually this structure is captured by an isocost line in output space. Contact geometry is the geometry which preserves this structure. Because the contact one-form is not integrable, we can generate all possible isocost lines as a projection of the contact structure to input quantity space. If the one form were to be integrable most isocost (budget) lines would be lost in this projection.

**The Samuelson Area Condition: Lagrangian Submanifolds.**

In some sense Legendrian submanifolds contain too much information. Recall that the level of profits is one of the coordinates of \((2n+1)\) dimensional space. But to determine whether or not profits are being maximized, it is not strictly necessary to know the value of the maximand. This suggests that we might confine attention to the lower dimensions of \(\mathbb{R}^{2n}\), an even dimensional space. Projected to an even dimensional space, the contact form \(\theta\) has an exterior derivative, a symplectic two-form \(d\theta\). Together \((\mathbb{R}^{2n}, d\theta)\) form a symplectic space. Submanifolds on which \(d\theta = 0\) are called Lagrangian submanifolds.

We obtain a Lagrangian submanifold from a Legendrian submanifold in the obvious way. Consider the projection \(P: \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}\) along the \(\prod\) direction, \(P(q_i, p_j, \prod) := (q_i, p_j)\), then \((\mathbb{R}^{2n}, d\theta)\) is a symplectic manifold. This gives a natural relationship between Legendrian submanifolds in \(\mathbb{R}^{2n+1}\) and Lagrangian submanifolds in \(\mathbb{R}^{2n}\). In particular if \(S\) is a Legendrian manifold in \(\mathbb{R}^{2n+1}\), \(P(S)\) is a Lagrangian manifold in \(\mathbb{R}^{2n}\). This leads to a Lagrangian counterpart to Proposition 1.
**Proposition 2:** Cost minimizing equilibrium processes on a $2n$ dimensional quantity price space are Lagrangian submanifolds of a $(2n)$ dimensional symplectic manifold with symplectic form $d\theta = dq^i \wedge dp_i$.

To apply this proposition to Samuelson’s diagram, we note first that Economic Phase Space now has 4 dimensions, $q_1, q_2, p^1, p^2$. On this space, a Lagrangian submanifold has 2 dimensions. A 2 dimensional space is defined on $\mathbb{R}^4$ without any symplectic structure by 2 equations $q_2 = q(q_1, p^1), p^2 = p(q_1, p^1)$.

The Samuelson diagram shows the level curves of two such functions. What guarantees that these two functions can be used to define a symplectic form on $\mathbb{R}^4$ and therefore a Lagrangian submanifold? The answer to this question is given by the following version of a theorem of Tabatchnikov (1993).

**Theorem:** Let $M^2$ be a symplectic manifold with area form $\omega$. A foliation $F$ on $M$ is said to be a Lagrangian foliation if its leaves are Lagrangian submanifolds, i.e. each leaf $N$ has $\dim N = 1$ and $\omega (X, Y) = 0$, for every $X, Y$ tangent to $N$. Then there is a natural symplectic torsion free connection on $TM$. The connection in question measures the failure of the equality $ab = cd$, where $a, b, c, d$ are as before. When the connection is flat, $ab = cd$, and $\omega$ can be extended to a symplectic form $\Omega$ on $M^4$ with the property that the restriction of $\Omega$ to $M^2$ is $\omega$.

**Proof:** Tabatchnikov shows that the condition $ab = cd$ means that there are coordinate transformations in which the foiations become parallel vertical and horizontal straight lines with the area form $\omega$ conserved. Using coordinates $x_1, x_2$ for the base space, we can thus think of the foliations as being level curves of two functions $y_1 = y(x_1), y_2 = y(x_2)$ and can recalibrate these functions so that $dy_1 \wedge dy_2 = dx_1 \wedge dx_2$. The form on $M^4$ is then just $dx_1 \wedge dx_2 + dy_1 \wedge dy_2$.

In Samuelson’s case, once we know that $q_1, q_2, p^1, p^2$ is a symplectic manifold, we can find a Lagrangian submanifold and thus a profit function. Samuelson’s area condition is

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thus exactly the condition in q₁ p¹ space needed to make p¹ p² space a Lagrangian submanifold of q₁ q₂ p¹ p².

Thus the following are equivalent:

1) A firm maximizes profits with respect to the hiring of two inputs. i.e. given observed quantities and prices, q₁ q₂ p¹ p² there exists a profit function \( \Pi(p¹ p²) \) such that

\[
\frac{\partial \Pi}{\partial p^i} = q_i, \quad i=1,2.
\]

2) The constrained factor demands form a flat Lagrangian 2-web with respect to the canonical connection on q₁ p¹.

3) The S ratio condition holds.

**Legendrian submanifolds and the theory of the consumer.** In order to test for profit maximization it is not necessary to know the level of profits, so it is appropriate to drop the level of profits as an independent variable and to define equilibrium profit maximizing systems as Lagrangian submanifolds of even dimensional symplectic manifolds. For the consumer, matters are not as simple, because the level of the objective function, in this case income, enters the demand functions in a meaningful way.\(^5\)

For the theory of the consumer, therefore, the relevant result is as follows.

**Proposition 3:** Income minimizing equilibrium processes are flows on an n dimensional Legendrian sub-manifold of a \((2n+1)\) dimensional contact manifold with contact form \( \theta = dq_i dp^i \). Here M is the McKenzie income compensation function.

The proof of this mimics the proof of Proposition 1. On a Legendrian submanifold, the Arnold theorem guarantees that there exists a function \( M(p) \) with the properties that

\[
\frac{\partial M}{\partial p^i} = q_i \quad \text{and} \quad M = q_i dp^i
\]

One way to try to describe these flows would be to try to mimic the methods used in symplectic geometry to derive the behavior of a mechanical system. There, if we have a flow which preserves the symplectic form, it is straightforward to show there is a so-called Hamiltonian function with the property that mechanical systems evolve along the

\(^5\) As the old economics textbooks used to say, there are no income effects in the theory of the firm.
level curves of this Hamiltonian, that is that they satisfy Hamilton’s equation. Put differently, flows which lie on a Lagrangian submanifold are exactly those flows which preserve the symplectic form.

It would be very convenient if, by analogy, flows on a contact manifold which preserved the contact structure would be those which were parallel to a Legendrian submanifold. Then all individual systems of demand curves would be generated by contact structure preserving flows. Unfortunately, however, this is not the case. It turns out that indeed some flows which preserve the contact structure are the economic demand functions of a single consumer, but some are not, representing instead whereas the demands of a one parameter family of economic agents. This situation is exactly as in thermodynamics where it has been extensively studied by Mrugula and his coauthors, see Mrugula et al [1991], Mrugala [2005].

To see this, we use the standard mathematical theory of contact preserving flows. On a contact manifold $(\mathbb{R}^{2n+1}, \theta = dM - q dp)$, (here we drop the summation sign) a function $K$ which preserves the contact structure has an associated contact structure preserving vector field $V_K$ given as follows;

with the notation $dK=K_M dM+K_q dq+K_p dp$

$$V_K = (K-qK_q) \frac{\partial}{\partial M} + (K_p + qK_M) \frac{\partial}{\partial q} - K_q \frac{\partial}{\partial p}.$$

The function $K$ is called a “contact Hamiltonian”.

It defines a flow on $\mathbb{R}^{2n+1}$ given by

$$\dot{M} = K - qK_q \quad \dot{q} = K_p + qK_M \quad \dot{p} = -K_q$$

Now, in contrast to the symplectic situation, $V_K$ contains a term $K$, the value of the function. When we apply this vector field to $K$, i.e. calculate $dK (V_K)$ this term does not go away, and we obtain an expression $dK (V_K) = K()$. So the flow given by $V_K$ will only be parallel to a Legendrian manifold i.e. along the level curves of $K$ if $K=0$ see Mrugala (1991). Thus we have

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6 Weinkove [2006] provides a clear presentation of these ideas in video form
**Proposition 4:** A system of demand functions which minimize an income compensation function is a flow on a Legendrian submanifold $S$ which lies in the zero set of a function which preserves the contact structure. In this case, the flow is tangent to $S$.

Looking at the flows given above, and interpreting the flow as a demand function, we see that we have $q = K_p + qK_M$, the standard textbook income compensated demand function.

**Contact Structure Preserving Flows and Aggregation.** Now that we have identified an individual system of demand functions with a contact structure preserving vector field, it is natural to ask what it would mean for this system of demand functions to have some special form. For example, if we have an ensemble of $m$ individuals, we may seek conditions on the individual demands such that the average demand functions depend on less than $m$ income terms. In this way we are led to the special demand functions of Gorman (1981).

What is needed is a reduction of the dimension of the Legendrian submanifolds to which the vector fields are parallel. This process is known as contact reduction and in general is associated with a Lie group action on the manifold. Weinkove [op.cit.] discusses this process of reduction in the Hamiltonian case. The role of Lie groups in the Gorman case is discussed in Russell et al [1998].

This approach to aggregation places restrictions on the form of individual demand functions. Following Grandmont [1992] and Hildenbrand[1983,1994], dimension reduction can be engineered by instead placing restrictions on the distributions over individual types. For example, the distribution family may be taken to be invariant under group action as in Grandmont [op.cit.].

As we have just seen, however, one–parameter families of individual demand functions are contact preserving flows for which do not lie on the zero value level of the contact Hamiltonian. Thus the notion of a group action on a contact manifold unifies these two approaches to aggregation.

It may be noted that in both the Gorman and Hildenbrand /Grandmont approaches to aggregation it is assumed that the demand functions are given by maximizing behavior. There is yet a third approach to aggregation in which no maximizing assumptions are place on the source of demand functions, the only constraint being that they satisfy a
budget constraint, Marris [1957], Becker[1962]. In this case the ensemble of individuals is still a flow on a contact manifold but the flow does not preserve the contact structure. Group structures on the distribution of types can still be used, however to reduce dimension, see Kneip [1999]. Thus the mathematics of groups on a contact manifold provides a unifying framework for all three approaches to aggregation.

**Conclusion:** In this paper we have shown how techniques from symplectic and contact geometry underlie the duality approach to economic maximization. For production theory the natural geometry is symplectic geometry and we have shown that Samuelson’s area ratio test is just a test for the existence of a Lagrangian submanifold of an even dimensional manifold of prices and quantities. For consumer theory the natural geometry is contact geometry. Contact structure preserving transformations generate both demand functions for individual consumers and demand functions across a one-parameter group of consumers and we have provided a criterion which separates out the two cases. In either case dimension reduction (aggregation) is associated with group actions which preserve the contact structure.
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