Seigniorage, Gesell Taxes and Monetary Policy in the Middle Ages*

Preliminary - Please do not quote!

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Abstract

A common form of government revenue during the Middle Ages was Gesell taxes on money (in the form of coins). Gesell taxes were implemented through coins being legal tender only for a limited time period and, at the end of that period, they had to be exchanged for new ones for a fee - an institution known as periodic re-coinage. Empirical evidence based on several methods shows that periodic re-coinage: 1) could occur as often as twice a year within a currency area; and 2) was the dominating method used to generate seigniorage during 150-200 years in large parts of medieval Europe. We set up a cash-in-advance model to analyze the consequences of periodic re-coinage on prices, re-minting and people's choice to use new or old coins for transactions. We find that prices increase over time during an issue period and fall just after the re-coinage date. People prefer to re-mint their old coins into new ones: 1) the lower is the exchange fee; 2) the longer is the time period between two instances of re-coinage; and 3) the higher is the probability of being detected using old coins.

Keywords: Seigniorage, Gesell tax, Periodic re-coinage, short-lived coinage system, cash-in-advance model

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1 Introduction

In the Middle Ages, seigniorage from minting was an important source of government revenue.[1] According to Allen (2012, p. 210), seigniorage from minting accounted for 2-5 percent of the total annual royal revenues in England 1158-1509. However, in periods of coinage reforms or debasement, this share could increase substantially, for example to 7-10 percent and 46 percent during the English coinage reforms in 1205-06 and 1279-80, respectively.[2] One source of seigniorage was that, due to legal restrictions on the means of payment used in transactions, foreign coins and raw silver (bullion) had to be exchanged for current coins at the mints, generating revenue through exchange and coinage fees. Here, the minting authority had an exchange monopoly and could thereby generate revenues; see Kluge (2007, p. 62–63). However, there were two other important sources of revenues from minting: debasement and a system akin to Gesell (1906) taxes.

Debasement means that the content of precious metal in the coins is decreased – either through lower weight or fineness. For the mechanisms through which seigniorage is increased during debasement, see e.g., Sussman and Zeira (2003) and Rolnick, Velde, and Weber (1996).

From the vantage point of today's economists, a less well-known way to profit from minting was Gesell taxes. Gesell taxes were implemented through coins being legal tender only for a limited time period and, at the end of that period, they had to be exchanged for new ones for a fee - an institution known as periodic re-coinage, see Allen (2012, p.35).[3] In Gesell's original proposal, holders of money had to buy and attach stamps to bank notes in order for them to retain their full nominal value.[4]

The monetary systems in the High Middle Ages of Europe (1000–1300 A.D.) can be divided into two main systems; see Kluge (2007, p. 62ff). One system had long-lived coins where the period in which coins were legal was not fixed.[5] The other system had short-lived coins that were only valid for specific time intervals and often relied on Gesell taxes to generate revenues in the form of frequent and systematic re-coinage. There was a substantial variation in the level of Gesell taxes. The revenues do

[1] When medieval documents about minting revenues are rare, the critical importance of these revenues is underscored by the value of pawned minting rights (Nau (1977), p. 92) and the severe penalties for counterfeiting; see Nathorst-Böös (1973, p. 51ff). Both together indicate the value of minting.

[2] There are also other indications, albeit from the 16th century, that the revenues from seigniorage could be high. During The Great Debasement in England in the 1540s, the profits from minting rose to as much as 57 percent of total Government revenues, compared to less than 2 percent normally; see Velde, Weber, and Wright (1999). Moreover, Edvinsson (2011) estimates that the revenue from minting during a debasement period in 1591 was four times higher than the value of the Crown tithes in Sweden (excluding Finland) in the same year. Finally, during a period of substantial debasement, in France in 1419, the seigniorage revenues amounted to six times the ordinary royal revenues; see Sussman and Zeira (2003, p. 1776).

[3] Also known as *coin renewals* or lat. *renovatio monetae*.

[4] In a system with re-coinage, the monetary authority had to ensure that new coins could be distinguished from old coins by their physical appearance so that it would be easy to verify that only new coins were legal.

[5] Sometimes, these coins were valid for the duration of the reign of the coin issuer. In these cases, successors occasionally minted variants of the same coin type. These are called immobilized types and could be valid for very long periods – occasionally centuries – surviving through the reigns of several new rulers.

not only depend on the fee charged at the time of the re-coinage, but also on the duration of an issue. In Germany, four old coins were usually exchanged for three new ones, and the Gesell tax was thus 25 percent; see Kluge (2007, p. 105ff). In specific currency areas, re-coinage could occur up to twice a year and entail Gesell taxes of up to 33 percent (see Kluge (2007) and Grinder-Hansen (2000)), implying annualized rates up to above 50 percent. Note also that Gesell taxes have been suggested as a remedy to the problem with the zero lower bound on the nominal interest during the last recession; see e.g., Mankiw (2009), Goodfriend (2000), Buiter and Panigirzoglou (1999) and Menner (2011).

In a system with periodic re-coinage, the monetary authority does not only compete with foreign coin issuers, but also with older issues of its own. It is important for the issuer to ensure that his current coins are used in transactions, since the presence of old and foreign coins could substantially reduce the seigniorage from re-coinage. To limit the circulation of illegal coins, the monetary authorities penalized the usage of invalid coins. Furthermore, fees, rents and fines were to be paid with current coins; see Haupt (1974, p. 29), Grinder-Hansen (2000, p. 69) and Hess (2004, p. 16-19). The disciplines of archaeology and numismatics have long been familiar with periodic re-coinage which was used for almost 200 years in large parts of medieval Europe (Kluge (2007), Allen (2012), Bolton (2012)). Yet, remarkably, this monetary system is seldom if ever analyzed theoretically in the literature of economics or economic history. Although scientific methods in archaeology and numismatics identify the presence of re-coinage, empirical evidence in written sources is scarce on the consequences of re-coinage with respect to prices and people's usage of new and old coins. However, written documents tell about complaints against such a monetary tax (Grinder-Hansen (2000, p. 52)) and evidence from coin hoards indicates that old (illegal) coins did often (but not always) circulate along with new ones; see Allen (2012, p. 520-23) and Haupt (1974, p. 29).

The purpose of the present study is to make up for this absence. We formulate a cash-in-advance model, along the lines of Velde and Weber (2000), in order to capture the implications of Gesell taxation in the form of periodic re-coinage on prices and on people's choice to use new or old coins for transactions. In the model, there are households, firms and a lord. Households care about goods and jewelry consumption. When trading jewelry and consumption, households face a cash-in-advance constraint. Households can hold both new and old coins, where only new coins are legal tender. Jewelry can be melted and minted into coins and coins can be melted and made into jewelry. Thus, jewelry captures the commodity property of coins. An issue of coins is only legal for a finite number of time periods. Old coins have to be re-minted at the re-coinage date in order to be regarded as legal tender. The lord charges a fee when there is a re-coinage so that, for each old coin handed in, the household only receives a fraction in return. While illegal, old coins can be used for transactions. In order to deter the use of illegal coins, there are lord plaintiffs that check whether legal means of payment are used in transactions. In case old coins are discovered in a transaction by lord plaintiffs, they are confiscated by the plaintiff and re-minted into new coins. Thus, whether illegal coins circulate is endogenous in the model. Lord revenues depend on the re-coinage (and mintage) fee, old coin confiscations and the duration of each coin issue. The lord uses the revenues to finance consumption expenditures.

We continue to analyze equilibria in the model. Since re-coinage occurs at a given frequency, and not necessarily in each time period, a steady state need not exist. Instead of analyzing steady states, as in Velde and Weber (2000), we analyze a model where re-coinage occurs infrequently, i.e., over time, one coin issue is replaced by another at fixed (and equal) time intervals. In order to focus on steady state-like properties, we analyze cyclical equilibria, i.e., equilibria where the price level, money holdings, consumption etc., are the same at a given point in different coins issues.

Our key results are that, in equilibrium, prices increase over time during an issue period and fall just after the re-coinage date and, the higher the Gesell tax, the higher are the price increases (as long as coins are handed in for re-coinage). Furthermore, households re-mint their old coins into new ones: 1) the lower is the exchange fee; 2) the longer is the time period between re-coinage dates; and 3) the higher is the risk of being detected using old invalid coins.

The paper is organized as follows. In section 2, we describe some basic facts about medieval European coins and also the extension of short-lived coinage systems through time and space. Seigniorage and enforcement of short-lived coinage systems are outlined in section 3. In section 4, we set up a cash-in-advance model to analyze the consequences of re-coinage. The final section 5 delineates the conclusions.

2 Short-lived coinage systems through time and space

2.1 Basics about medieval money

Money in medieval Europe was overwhelmingly in the form of commodity money and fiat money did not exist in its pure form. During the main period of the Middle Ages (700–1300), silver was almost the only key raw material in European coins.[6,7] The reason for this was the relative abundance of silver mines leading to a high supply of silver; see Spufford (1988, p.109ff, 119ff).

As for money in general, commodity money might have value for other reasons than the value of the commodity that it is based on, e.g., liquidity reasons and that it is costly to mint coins from the commodity.[8] In the case of commodity money, if households hold coins for liquidity reasons, the value

[6] Precious metals (gold and silver) have specific characteristics that correspond to the functions of commodity money and were used as raw materials in medieval coins. These metals: 1) exist in limited quantities, are 2) well-known, 3) of stable value and 4) relatively soft and thereby easy to work up.

[7] Mainly with the exception of the Byzantine empire.

[8] If, due to e.g., production costs, it is costly to mint coins, then, if coins still are held in equilibrium e.g. for liquidity

of the coins would then be higher than their intrinsic value due to the metal content.[9] People might then be willing to pay a premium to have their silver transformed into standard coins; see Sussman (1993, p.50).

In the Middle Ages, the right to mint – as well as the rights to charge market customs and run mines – belonged to the droit de régale, i.e. the king/emperor possessed these rights. Besides the right to determine e.g. the design and monetary standard, the coinage right encompassed the right to use the profits from minting and to decide which coins are legal tender, i.e., which coins are legitimate and valid as a medium of exchange; see Kluge (2007, p. 52). The right to mint for a region could be delegated, sold or pawned to other local authorities (local lords, laymen, churchmen, citizens) for a limited or unlimited time period; see Kluge (2007, p. 53). In general, these local authorities had to observe the king's guidelines for valid coins and the monetary standard.

The size of the currency areas could vary substantially in the High Middle Ages. The whole of England was a single currency area (after 975), while Sweden and Denmark each had 2–3 areas. The currency areas in these countries were thus relatively large, each having several mints. In contrast, in France the minting right was delegated to many civil authorities and there were many small currency areas. Germany in the High Middle Ages was extremely politically decentralized with a weak emperor. One method that the German emperor used to strengthen feudal loyalties was to delegate land; another was to delegate the rights to mint and charge market customs. Unlike in France, ecclesiastical authorities in Germany frequently received the coinage rights. Therefore, the best examples of many small currency areas can be found in Germany and eastern Europe where a city (mint) with its surroundings could constitute a currency area.

2.2 Geographical extension of short-lived coinage systems

A commonly used system in the middle ages was Gesell taxation, in the form of re-coinage. A re-coinage system has as its main features that coins circulate for a limited time and, at the end of the period, coins have to be handed in to the monetary authority and be re-minted for a fee, i.e., a Gesell tax. Thus, coins can be said to have been "short-lived", in contrast to a "long-lived" monetary system where coins did not have a given fixed period of being legal means of payment. There is an extensive historical and numismatic literature on re-coinage. Three methods have been used to identify re-coinage and its frequency; written documents, the number of coin types per ruler and years and the distribution of coin types in hoards (for details, see Appendix A.1).

reasons, users have to be compensated both for forfeiting the commodity value and the production cost, thus implying that coins have a value over and above the metal content.

[9] Another reason is that coins might work better as a medium of exchange and standard of value than un-minted silver does. It is manifestly easier to count coins than to weigh silver and try somehow to check its fineness.

There is a reasonable consensus in drawing conclusions about the extension of long-lived and short-lived coinage systems through time and space. Long-lived coins were common in northern Italy, France and Christian Spain in the period 900–1300 (see Map 1). This system spread to England when the sterling was introduced during the second half of the 12th century. In France in the 11th and 12th centuries, long-lived coins dominated in most regions where the rights to mint were distributed to many civil authorities. It was not until the 13th century that the French king expanded his control over the coinage; see Kluge (2007, p. 136ff). In northern Italy, where towns took over the minting rights from the 12th century, long-lived coins likewise dominated.

Map 1 Here

A well-known example where short-lived coinage systems were used is England. Re-coinage was introduced in the English kingdom around 973 and occurred every sixth year until 1035. For around a century after 1035, English kings continued to renew their coinage at a higher frequency. These coins were valid throughout England, i.e. a large geographical area; see Spufford (1988, p. 92) and Bolton (2012, p. 87ff).

Short-lived coinage systems were the dominant monetary system in central, northern and eastern Europe in the period 1000–1300. The eastern parts of France and the western parts of Germany had re-coinage in the 11th and 12th centuries; see Hess (2004, p. 19–20). However, the best examples of short-lived and geographically constrained coins can be found in central and eastern Germany and eastern Europe where the currency areas were relatively small. Here, re-coinage started in the middle of the 12th century and lasted until around 1300 and was especially frequent in areas where uni-faced bracteates were minted;[10] usually annually but sometimes twice a year; see Kluge (2007, p. 63).

Sweden had re-coinage of bracteates in two of three currency areas (especially in Svealand and to some extent in western Götaland) for more than a century, from 1180 to 1290. This conclusion is supported by evidence of numerous coin types per period and the composition of coin hoards; see Svensson (2013, p. 223-24). The Kings of Denmark introduced re-coinage in all currency areas from the mid 12th century that continued for 200 years with some interruptions; see Grinder-Hansen (2000, p. 61ff). In Poland, irregular re-coinages started in the early 12th century. Like Germany, Poland had many currency areas and minting authorities, see Suchodolski (2012).

2.3 Other facts and the concept of re-coinage

In a short-lived coinage system, the minting authority does not only face competition with other coin issuers but also with old issues minted by itself. To create a monopoly position for its coins, legal tender

[10] Bracteates are thin uni-faced coins that were struck with only one die. A piece of soft material, such as leather or lead, was placed under the thin flan. Consequently, the design of the obverse can be seen as a mirror image on the reverse of the bracteates.

laws stated that foreign coins were ipso facto invalid and to be exchanged for local current coins along with the payment of an exchange fee, the amount determined by the coin issuer (exchange monopoly). Moreover, only one local coin type can be considered to be legal at a given point in time. For it to be possible to determinate which local coins that are legal, coin types representing various issues must have clearly visible markers differentiating them so that people can easily distinguish between valid and invalid types. The main design was usually changed but the monetary standard (weight, fineness, diameter, shape of the flan) remained the same between issues.[11] The frequency and exchange fee of the re-coinage could and did vary (see section 3.1 below). Re-coinage normally occurred on a specific date. Afterwards, new local coins were the only legal tender in the city.[12] Similar to Gesell's original proposal, where stamps had to be attached to a bank note for it to retain its full value, thus making it easy to verify whether the tax had been paid, under re-coinage, the main design of the coin was changed (e.g. the portrayed figure is in a different position, or there are different attributes in the hands of the figure), whereas the monetary standard (e.g. weight, fineness, diameter, shape of the flan, denomination) usually remained about the same.

When analyzing the empirical evidence for periodic re-coinages, coins were usually exchanged at recurrent dates at a substantial fee and coins were only valid for a limited (and ex ante known) time period. The withdrawals were systematic and recurrent. Thus, when analyzing re-coinage in section 4 below, we assume that both the exchange fee and the re-coinage dates are known in advance. One may also want to make a distinction between *periodic re-coinage* and *coinage reform*, a distinction not necessarily made explicit by historians and numismatists.[13] When a coinage reform is undertaken, coin validity is not constrained in time. A coinage reform also includes a re-mintage, but is announced infrequently and the validity period of coins is not (explicitly) known in advance. Moreover, the coin and the monetary standard are in general considerably changed.[14] Note that in case the issuer charges a fee at the time of the reform, the coinage reform shares some features of re-coinage, but because the monetary standard is changed, there might be additional effects **on,** e.g., the price level at the time

[11] Note that in order to obtain revenues from seigniorage, the coin issuer benefits from having an exchange monopoly in both long-lived and short-lived coinage systems. The coin issuer then has incentives to ensure that foreign coins are not allowed to circulate. Moreover, to avoid that illegal coins circulate, the minting authority must control both the local market and the coinage; see Kluge (2007).

[12] In 1231 the German king Henry VII (1222–35) published an edict in Worms stating that in towns in Saxony with their own mints, goods could only be exchanged for coins from the local mint; see Mehl (2011, p. 33). However, when this edict was published, the system with coins constrained through time and space had been in force for a century in large parts of Germany.

[13] In fact, historians often use the term re-coinage for both periodic re-coinage and coinage reform.

[14] England had two re-mintings in the thirteenth century when the coinage was long-lived, but these events had other purposes than to simply charge a gross seigniorage. The short-cross pennies minted in the twelfth and thirteenth centuries were often clipped. A re-minting occurred in 1247. A new penny ('long-cross') with the cross on the reverse extended to the edge of the coin to help safeguard the coins against clipping was introduced. Another coinage reform occurred in 1279. Before 1279, the double-lined cross on the long-cross pennies was used when cutting the coins into halves to get small change to the penny. New denominations were introduced in 1279 – all with single-lined crosses on the reverse. In addition to the new penny, groat, halfpence and farthing were also minted.

of the reform.

3 Seigniorage and enforcement of short-lived coinage systems

3.1 Seigniorage and prices in systems with re-coinage

The seigniorage under re-coinage does not only depend on the fee charged at the time of the re-coinage, but also on the duration of an issue. Given a fee of, say, 25 percent at each re-coinage, the shorter the duration, the higher the revenues, given that the money holdings are not affected. Any reduction in money holdings due to the shortening of issue time would push revenues the other way.

Table 1: Exchange fees and duration of re-coinage in different areas

Region	Currency area♦	Period	Gesell tax (Annualized)	Duration	Method/Source†
England	Large	973-1035	n.a.	6 years	1-3, Spufford (1988)
	Large	1035-1125	n.a.	2-3 years	2-3, Bolton (2012)
Germany, western✠	Small	Ca. 1000-ca. 1300	mostly 25% (4.6%-25%)‡	1-5 years	1-3, Hess (2004)
Germany, eastern, northern✠	Small	ca. 1130-ca. 1330, sometimes until 15th cent.	mostly 25% (25%-56%)‡	$\frac{1}{2}$ or 1 year	1-3, Kluge (2007)
Teutonic Order in Prussia	Medium	1237-1364	17% (1.6%)	10 years	1-3, Paszkiewicz (2008)
Austria	Small	Ca. 1200-ca. 1400	n.a.	1 year	2-3, Kluge (2007)
Denmark	Medium	1140s-1330s.	33% (33%)	1 year, with some interruptions	1-3, Grinder-Hansen (2000)
Sweden, Svealand	Large	1180-1290	n.a.	1-5 years	2-3, Svensson (2013)
Sweden, Götaland	Large	1180-1290	n.a.	3-7 years	
Poland	Small	Ca. 1100-ca. 1150	n.a.	3-7 years	1-3, Suchodolski (2012)
	Small	Ca. 1150-ca. 1200	n.a.	1 year	
	Small	Ca. 1200-ca. 1300	n.a.	$\frac{1}{3}$ or $\frac{1}{2}$ year	
Bohemia-Moravia	Large	Ca. 1150-1225	n.a.	1 year	Sejbal (1997) and Vorel (2000)
	Large	1225-ca. 1300	n.a.	$\frac{1}{2}$ year	

Notes: ♦ We do not use a formal definition of the area size. By a large area, we mean a country, or a substantial part of a country, like England or Svealand. A small area is usually a city and its hinterland. A medium-size area is somewhere in between, exemplified by the kingdom of Wessex. †As in Appendix A.1. ✠ Various mints and authorities. ‡Annualized rate based on a fee of 25 percent.

There was a substantial variation in the level of the seigniorage. In England 973-1035, re-coinage occurred every sixth year. For about a century after 1035, English kings renewed their coinage every second or third year; see Spufforf (1988, p. 92) and Bolton (2012, p. 99ff). The level of the fee is uncertain. According to Spufford (1988), four old coins were exchanged for three new ones, although this is based on a rather uncertain weight analysis. A possible alternative indication of the level of the fee can be found from the fee at coinage reforms after the system of periodic re-coinage had been abandoned when the fee was substantially lower; see Allen (2012, p. 170ff). In case the gross seigniorage was 25 percent every sixth year, the annualized rate was almost 4 percent.

In other areas in Europe, the duration was often significantly shorter. Austria had annual re-coinage until the end of the 14th century and Brandenburg until 1369 (Kluge (2007, p. 119)). Individual German mints mostly had bi-annual or annual renewals until the 14th or 15th centuries (e.g. Brunswick until 1412); see Kluge (2007, p. 105). In Denmark, re-coinage was frequent (mostly annual) from the mid 12th century and continued for 200 years with some interruptions; see Grinder-Hansen (2000, p. 61ff). Sweden had re-coinage starting around 1180 and continuing for about a century; see Svensson (2013, p. 225). In Poland, King Boleslaw (1102–38) started with irregular re-coinages – every third to seventh year, but later these became far more frequent. At the end of the 12th century, coin renewals were annual and in the 13th century, they occurred twice a year; see Suchodolski (2012). Bohemia also had re-coinage at least once **a** year in the 12th and 13th centuries; see Sejbal (1997, p. 83) and Vorel (2000, p. 26). On the other hand, the Teutonic Order in Eastern Prussia had re-coinages only every tenth year between 1237 and 1364; see Paszkiewicz (2008, p. 178).

The exchange fee in Germany was in general four old coins for three new ones, i.e. a Gesell tax of 25 percent, see e.g., Magdeburg (12 old for 9 new coins). In Denmark, the Gesell tax for people – three old coins for two new ones – was higher, 33 percent. The annualized tax in Germany and Denmark could be very high – up to more than 50 percent. The Teutonic Order in Prussia had a relatively generous exchange fee of seven old coins for six new ones; see Svensson (2013, p. 97-98), i.e., a tax rate of almost 17% or, in annualized terms, 1.6 percent.

Unfortunately, evidence is scant on prices for monetary systems with re-coinage. Finding price indices for the period at hand is hardly possible. However, there is some evidence from the Frankish empire that indicates that prices rose during an issue.[15] Specifically, several attempts at price regulations following a re-coinage/coinage reform in 793/4 seem to indicate problems with rising prices; see Suchodolski (1983).

3.2 Success, monitoring and enforcement of re-coinage

There was considerable variation in the success of re-coinage. The coin hoards discovered to date can tell us a great deal about the success of re-coinage. Hoards in Germany from this period (1100–1300) usually contain many different issues of the local coinage, as well as many issues of foreign coinage, i.e. locally invalid coins; see Haupt (1954), Hävernick (1955) and Gaettens (1963), indicating that the monetary authorities had problems in enforcing the circulation of their coins. By skipping some coin

[15] The Frankish empire seems to have had a system akin to re-coinage in the 8th and 9th century, although the weight of coins was often changed when coins were exchanged in that system.

renewals and saving their retired coins, people could accumulate silver or use the old coins illegally. On the other hand, hoard evidence from England indicates that the re-coinage systems were partly successful; see Dolley (1983). In table 2, almost all coins in hoards are of the last type during the period 973-1035 when coins were exchanged every sixth year, while in the period 1035-1125, only slightly more than half of the coins were of the last type, indicating that the system worked well up to 1035 but less so after. One reason for this might be that the seigniorage for the latter period was higher, due to the shorter time period between withdrawals (at an unchanged exchange fee; see Table 1).

Since hoards often contain illegal coins, the incentives to try to avoid the re-coinage fees occasionally seem to have been fairly high. To try to curb the circulation of illegal coins, the monetary authority used different methods to try to control the usage of coins. The usage of invalid coins was deemed illegal and penalized, although the possession of invalid coins was mostly legal.[16] If an inhabitant used foreign coins or old local coins for transactions and was detected, the penalty could be severe. Moreover, sheriffs and other administrators who accepted taxes or fees in invalid coins were penalized; see Haupt (1974, p. 29), Grinder-Hansen (2000, p. 69), and Hess (2004, p. 16). Controlling the usage of current coins was probably easier in cities than **in** the countryside.[17]

Table 2: English hoards 979-1125, the number of coins and shares†

Period	973-1035		1035-1125	
Duration	6 years		2-3 years	
	Number of coins	Share	Number of coins	Share
Last coin type	883	97.4%	11 910	53.1%
Second last coin type	17	1.9%	1 947	8.7%
Third last coin type	1	0.1%	827	3.7%
Earlier types	6	0.7%	7 737	34.5%

Notes: †Allen (2012, p. 520-23).

The minting authority could also indirectly control the coin circulation in an area. Documents show that fees, rents and fines were to be paid with current coins, apart from traditional situations

[16] From city laws in Germany and Denmark, one can read that neither the mint master nor the judge was allowed to enter homes and search for invalid coins.

[17] As noted in sections 2.1 and 2.2, the medieval currency areas could be large, as in England and Sweden, or small, as in Germany and Poland. However, irrespective of the size of the currency area, the systems with short-lived coins as legal tender could often only be strictly enforced in a limited area of the authority's domain, such as within the cities. If most trade took place within cities, this restriction might not be a strong constraint, though. Normally, the city-border demarcated the area that included the jurisdiction of the city in the Middle Ages. The use of foreign and retired local coins within the city-border was forbidden. This state of affairs is well documented in an 1188 letter from Emperor Friedrich I (1152–90) to the Bishop of Merseburg (Thuringia) regarding an extension of the city. The document plainly states that the market area boundary includes the whole city, and not just the physical marketplaces; see Hess (2004, p. 16). However, retired local coins as well as foreign coins were often allowed for transactions outside the city-border; see Hess (2004, p. 16). A document from Erfurt (1248/51) shows that only current local coins could be used for transactions in the town, while retired local ones as well as foreign coins were allowed for transactions outside the city-border; see Hess (2004, p. 16).

where payment in kind was possible; see Grinder-Hansen (2000, p. 69), and Hess (2004, p. 19).

4 The model

In this section, we outline the model, define equilibria and analyze equilibrium outcomes in terms of how prices evolve and under what conditions on re-mintage fees and issue length old and new coins are used together.

4.1 The economic environment

The economy consists of households, firms and a lord. Households care about goods consumption c_t and jewelry consumption d_t. When trading jewelry and consumption, households face a cash in advance constraint. Money holdings consist of new and old coins, made of silver.[18] Only new coins are legal tender, but the household can use both types in transactions. Thus, whether illegal (old) coins circulate is endogenous in the model. The new coins are withdrawn from circulation every T'th period. Specifically, in order to be regarded as legal tender after the withdrawal, coins have to be handed in to be re-minted. Any coin that is not handed in for re-mintage is not legal tender and is thus treated as an old coin after the withdrawal. The lord charges a fee for the withdrawal. Specifically, for each coin handed in for re-mintage, the household gets $k < 1$ new coins in return and the lord gets the remainder. Thus, $1 - k$ is the Gesell tax. As mentioned above, while illegal, old coins can be used for transactions but, due to punishments for using illegal coins, it is costly to do so. We model the legal punishment of using illegal coins as follows. There are lord plaintiffs that check whether legal means of payment are used in transactions. In case old coins are discovered in a transaction by lord plaintiffs, they are confiscated by the plaintiff, re-minted (costlessly) as new coins and used to fund lord expenditures. The plaintiff finds old coins with probability $1 - \chi$. Due to e.g., the confiscation of old coins by the lord plaintiff, old and new coins need not circulate at par. We let e_t denote the exchange rate between old and new coins. Households can also sell jewelry to the representative firm (mint) in return for new coins. Lord revenues, i.e., from minting, re-mintage and confiscations, are spent on lord consumption, denoted g_t.

As in Velde and Weber (2000), competitive firms can produce the consumption good using the endowment, jewelry by melting new and old coins and new coins by minting.[19] At the beginning of a period t, households own an endowment ξ_t, jewelry d_t, and the stock of new and old coins, m_t^n and

[18] For simplicity, we ignore foreign coins.

[19] A motivation for competitive mints is that, in e.g., 11th-12th century England at some points in time up to around 70 mints were active, see Allen (2012, p. 16 and p. 42f). Moreover, these mints were sometimes farmed out; see Allen (2012, p. 9).

m_t^o, respectively. The stock of silver in the economy is S_t and the change in stock $S_t - S_{t-1}$ is added to the household jewelry stock at the start of period t. The endowment of the household is sold to the firms in return for a claim on firm profits. Then, shopping begins with households using coin balances to buy consumption and jewelry at competitively determined prices p_t and q_t. Firms sell the endowment to households and the Lord and get coins in exchange. Moreover, n_t^n coins are minted and μ_t^n new coins and μ_t^o old coins are melted. In case coins are minted, households pay the same fee as when handing in coins at the re-coinage date.[20] Then, the profits are returned to households in the form of dividends. Finally, at the re-coinage date, households decide on the share s_t^n of coins that is to be handed in to the firm for re-minting into new coins.

4.1.1 The firm

The firm profits are

$$\Pi_t = p_t \left(c_t + g_t\right) + n_t^n - \mu_t^n + q_t h_t - e_t \mu_t^o, \tag{1}$$

where g_t is lord consumption in period t, μ_t^n and μ_t^o denote melting of new and old coins, n_t^n minting of new coins and h_t jewelry supply. Note that coin holdings m_t^n and m_t^o are determined before firm dividends are disbursed to households and are chosen in period $t-1$ for use in period t. Mintage must be non-negative and melting cannot exceed the stock of new and old coins $m_t^n + \Pi_{t-1}$ and m_t^o, respectively. Moreover, coins are defined by the number b of grams of silver per coin. Thus, the firm faces the following constraints, related to mintage and melting,

$$\begin{aligned} n_t^n &\geq 0 \\ m_t^n + \Pi_{t-1} &\geq \mu_t^n \geq 0 \\ m_t^o &\geq \mu_t^o \geq 0 \end{aligned} \tag{2}$$

and

$$h_t = b\left(\mu_t^n + \mu_t^o - n_t^n\right). \tag{3}$$

The firm maximizes its profits in (1) subject to constraints (2), (3) and

$$c_t + g_t \leq \xi_t. \tag{4}$$

From the firm's first-order conditions, if

$$\frac{k}{b} \geq q_t \tag{5}$$

[20] The results hold as long as the mintage fee is weakly larger than the re-coinage fee.

then $n_t^n \geq 0$ and if

$$\frac{k}{b} < q_t \tag{6}$$

then $n_t^n = 0$. Thus, as long as the price of jewelry is too high, i.e., $q_t > \frac{k}{b}$, it is not profitable for the firm to buy jewelry in order to mint new coins. On the other hand, if the jewelry price were lower than $\frac{k}{b}$, firms would make positive profits on mintage. Due to the constant returns technology, this would lead to an infinite demand for jewelry. Equilibrium then requires that $\frac{k}{b} \leq q_t$ with equality, whenever $n_t^n > 0$

Firm optimization leads to the following conditions for the melting of new coins. If we have

$$\frac{1}{b} > q_t \tag{7}$$

then $\mu_t^n = 0$. When

$$\frac{1}{b} < q_t \tag{8}$$

then $\mu_t^n = m_t^n + \Pi_{t-1}$. Finally, in equilibrium, as long as $\mu_t^n \in (0, m_t^n + \Pi_{t-1})$ we have

$$\frac{1}{b} = q_t. \tag{9}$$

Hence, if the jewelry price is too low, i.e., $\frac{1}{b} > q_t$, it is not profitable for the firm to melt coins and transform them into jewelry. If the price is higher than $\frac{1}{b}$ then the firm makes a positive profit on each new coin that it melts. Once more, due to the constant returns technology, the demand for new coins to be melted by the firm is infinite. Competition then forces the jewelry price to $\frac{1}{b}$.

Repeating the same for μ_t^o gives

$$\text{if } \frac{e_t}{b} > q_t \text{ then } \mu_t^o = 0 \tag{10}$$

$$\text{if } \frac{e_t}{b} < q_t \text{ then } \mu_t^o = \chi m_t^o. \tag{11}$$

Thus, in equilibrium, we have $\frac{e_t}{b} = q_t$ when μ_t^o is interior. The intuition is similar to expressions (7)-(9), with the modification that the cost of buying old coins is e_t instead of one.

4.1.2 The household

The household preferences are

$$\sum_{t=0}^{\infty} \beta^t \left[u\left(c_t\right) + v\left(d_{t+1}\right)\right]. \tag{12}$$

Both u and v are assumed to be strictly increasing and strictly concave. We impose the standard Inada conditions so that $\lim_{c_t \to 0} u'(c_t) \to \infty$ and $\lim_{d_t \to 0} v'(d_t) \to \infty$. Households own an endowment ξ_t of the consumption good and silver S_t. Following Velde and Weber (2000), the endowment is transferred to firms in return for a claim on profits. The household maximizes utility in (12), subject to the law of motion for jewelry

$$d_{t+1} = d_t + h_t + S_{t+1} - S_t \tag{13}$$

the CIA constraint

$$p_t c_t + q_t h_t \leq K_t \left(m_t^n + \Pi_{t-1}\right) + e_t \chi \left(m_t^o + L_t \left(m_t^n + \Pi_{t-1}\right)\right) \tag{14}$$

where

$$K_t = \begin{matrix} 1 & \text{if } t \neq T, 2T, 3T, \ldots \\ k s_{t-1}^n & \text{otherwise} \end{matrix} \tag{15}$$

and

$$L_t = \begin{matrix} 0 & \text{if } t \neq T, 2T, 3T, \ldots \\ 1 - s_{t-1}^n & \text{otherwise} \end{matrix} \tag{16}$$

where s_{t-1}^n is the share of new coins handed in by households for re-coinage at the time of withdrawal and the budget constraint

$$m_{t+1}^n + e_t m_{t+1}^o \leq K_t \left(m_t^n + \Pi_{t-1}\right) + e_t \chi \left(m_t^o + L_t \left(m_t^n + \Pi_{t-1}\right)\right) - p_t c_t - q_t h_t. \tag{17}$$

Also, for $t = T, 2T, \ldots$

$$s_t^n \in [0, 1] \tag{18}$$

and

$$\begin{aligned} c_t &\geq 0 \\ m_{t+1}^n &\geq 0 \\ m_{t+1}^o &\geq 0. \end{aligned} \tag{19}$$

and

$$h_t \geq -d_t - \left(S_{t+1} - S_t\right). \tag{20}$$

Note that the Inada condition with respect to jewelry implies that the last constraint never binds.

We now derive the household Euler equation. Using the first-order condition with respect to c_t

and h_t, the first-order condition with respect to d_{t+1} can be written as[21]

$$u'(c_t)\frac{q_t}{p_t} = \beta u'(c_{t+1})\frac{q_{t+1}}{p_{t+1}} + v'(d_{t+1}). \tag{21}$$

As usual, the Euler equation describes the consumption-savings trade-off in the model. However, since the model has commodity money, the expression is slightly different than in standard macro models; see e.g., Gali (2008) equation (7). To get the intuition behind expression (21), consider a consumer that chooses to save some more by reducing consumption today and holding some extra jewelry, in order to increase consumption tomorrow. The decrease in consumption today leads to a decrease in utility through $u'(c_t)$, and is transformed into jewelry at the relative price $\frac{q_t}{p_t}$. When holding some extra jewelry, this gives the consumer a direct payoff effect through $v'(d_{t+1})$ and an indirect effect through an increase in consumption tomorrow. The increase in c_{t+1} is discounted by β and the stored jewelry is sold at the relative price $\frac{q_{t+1}}{p_{t+1}}$. Note that the real interest rate in this model is given by

$$\frac{q_{t+1}/q_t}{p_{t+1}/p_t}, \tag{22}$$

i.e., gross jewelry inflation divided by gross goods inflation.

The first-order conditions are illustrated in Appendix A.2. Here, we describe those used in the analysis below and assume that $c_t > 0$ and $p_t > 0$ for all t, which holds in equilibrium.

Whether old or new coins are held depend on how exchange rates affect their relative return. We now state conditions on how exchange rates change, depending on whether old or new coins are held. Using the first-order conditions with respect to c_t and m^n_{t+1}, if $m^o_{t+1} > 0$ then, if $t \neq T, 2T$ etc.,

$$\frac{e_{t+1}\chi}{e_t} \geq 1 \tag{23}$$

and, if $t = T, 2T$ etc.,

$$\frac{e_{t+1}\chi}{e_t} \geq s^n_t k + e_{t+1}\chi(1 - s^n_t), \tag{24}$$

otherwise. Since the consumer holds old coins in period $t+1$, the exchange rates in periods t and $t+1$ have to give the consumer incentives not to only hold new coins. Then, it follows that the exchange rate has to increase by at least $\frac{1}{\chi}$ between adjacent periods, except in the withdrawal period. The appreciation of the exchange rates compensates the consumer for the loss due to confiscations by the lord plaintiff so that the consumer does not lose in value terms by holding an old coin, relative to new coins, for an additional period. The condition is slightly different for the withdrawal period, due to

[21]Note that since jewelry is a consumer durable good, the Euler equation here is similar to Euler equations in such models; see e.g., equation (5) in Barsky, House, and Kimball (2007).

the fact that the return on holding new coins changes for two reasons. First, holding a new coin for an additional period relaxes the cash in advance constraint by k instead of one. Second, the consumer can choose not to hand in new coins for re-mintage, rendering them old coins in the next period, valued at e_{t+1} and subject to confiscation by the plaintiff at rate $1-\chi$. Between these two options, the consumer optimally chooses the fraction s_t^n to hand in for re-mintage; see equations (27)-(29) below.

If $m_{t+1}^n > 0$, if $t \neq T, 2T$ etc.,

$$\frac{e_{t+1}\chi}{e_t} \leq 1. \tag{25}$$

Since the consumer now holds new coins in period $t+1$, the exchange rates in period t and $t+1$ have to give the consumer incentives to not only hold old coins. For this to be the case, the exchange rate increase cannot be too large and is bounded above by $\frac{1}{\chi}$. If $t = T, 2T$ etc.,

$$\frac{e_{t+1}\chi}{e_t} \leq \left(s_t^n k + e_{t+1}\chi\left(1 - s_t^n\right)\right). \tag{26}$$

Finally, the household also optimally chooses the share of coins to be handed in for re-coinage, s_t^n in periods $t \neq T, 2T$ etc. If $s_t^n \in (0,1)$ then

$$e_{t+1}\chi = k. \tag{27}$$

If $s_t^n = 0$ then

$$e_{t+1}\chi \geq k. \tag{28}$$

Finally, if $s_t^n = 1$ then

$$e_{t+1}\chi \leq k. \tag{29}$$

When choosing how to allocate the new coins in period T to new and old coins in the next period, the household takes into account **its** relative returns. When handing in a coin for re-mintage, the return is k, while when not handing it in and letting it become an old coin in the next period, it is valued to e_{t+1} and risks confiscation with probability $1-\chi$, rendering the return $e_{t+1}\chi$. Thus, if k is high enough, all new coins are re-minted (when $k > e_{t+1}\chi$) and if it is too low, no new coins are re-minted (when $k < e_{t+1}\chi$).

4.1.3 The lord

The lord gets revenue from coin withdrawals and confiscation of illegal coins. The lord costlessly re-mints all confiscated old coins into new ones. Letting $m_t^L \geq 0$ denote coins stored by the lord, the lord budget constraint is

$$p_t g_t + m_{t+1}^L = (1-k) N_t (m_t^n + \Pi_{t-1}) + (1-k) n_t^n + (1-\chi) m_t^o + m_t^L \tag{30}$$

where

$$N_t = \begin{cases} 0 & \text{if } t \neq T, 2T, 3T, \ldots \\ s_{t-1}^n & \text{otherwise} \end{cases} \tag{31}$$

Thus, the lord uses revenues from money withdrawals through m_t^n, from new mintage through n_t^n, confiscations through m_t^o and previously stored coins m_t^L to spend on consumption g_t and coins stored to the next period m_{t+1}^L. In equilibrium, government spending is determined by the revenues generated by the Gesell tax k and the plaintiff confiscation probability $1-\chi$. Given sequences $\{p_t\}$, $\{m_t^L\}$, $\{N_t\}$, $\{m_t^n\}$, $\{\Pi_t\}$, $\{n_t^n\}$, $\{m_t^o\}$, a feasible sequence of government spending $\{g_t\}$ satisfies (30) for all t. Below, when analyzing equilibria, we put additional restrictions on g_t.

4.1.4 Money transition and resource constraints

The household stocks of new and old coins evolve according to

$$m_{t+1}^n = K_t (m_t^n + \Pi_{t-1}) + k n_t^n - \mu_t^n \tag{32}$$

$$m_{t+1}^o = \chi m_t^o - \mu_t^o + Q_t (m_t^n + \Pi_{t-1}) \tag{33}$$

where

$$Q_t = \begin{cases} 0 & \text{if } t \neq T, 2T, 3T, \ldots \\ 1 - s_{t-1}^n & \text{otherwise.} \end{cases} \tag{34}$$

The holdings of new coins, m_{t+1}^n, in expression (32) depend on the previous stock net the Gesell tax $K_t m_t^n$, net dividends from firms $\Pi_{t-1} = p_{t-1} g$ (noting that the Lord only spends new coins; see section 4.1.3), mintage net of mintage fee $k n_t^n$ and melting μ_t^n. The holdings of old coins, m_{t+1}^o, in expression (33) depend on the previous stock net of plaintiff confiscations χm_t^o, melting μ_t^o and new coins not handed in for re-coinage $Q_t m_t^n$.

Finally, we have the goods resource constraint

$$c_t + g_t = \xi_t \tag{35}$$

and the silver resource constraint

$$b \left(m_t^n + m_t^L \right) + d_t = S_t. \tag{36}$$

4.2 Equilibria

Definition 1 *An equilibrium is a collection* $\{m_{t+1}^n\}$, $\{m_{t+1}^o\}$, $\{n_t^n\}$, $\{\mu_t^n\}$, $\{\mu_t^o\}$, $\{c_t\}$, $\{d_{t+1}\}$, $\{h_t\}$, $\{p_t\}$, $\{q_t\}$ *and* $\{e_t\}$ *such that i) the household maximizes (12) subject to (20), (13), (14), (17), (18) and (19); ii) the firm maximizes (1) subject to (2), (3) and (4); iii)* $c_t + g_t = \xi_t$ *and that (32), (33), (36) and (30) hold.*

For the rest of the analysis, we assume that the endowment is constant; $\xi_t = \xi$. Furthermore, $S_{t+1} = S_t = S$ and hence, the jewelry stock evolves according to

$$d_{t+1} = d_t + h_t. \tag{37}$$

For the lord, we assume e.g. that the Lord keeps a standing army of the same size every period that needs to be sustained by consumption goods. Specifically, we assume that revenues from withdrawals are spread equally across periods and hence $g_t = g$ in every period. Moreover, the budget is balanced over the cycle and there is no excess storage, i.e., $m_{sT+t}^L = m_t^L$ for all t and $s \in Z$. Thus, summing the lord constraint (30) over $t = 1$ to T

$$\sum_{t=1}^{T} p_t g_t = (1-k)(m_1^n + \Pi_T) + \sum_{t=1}^{T} (1-\chi) m_t^o. \tag{38}$$

Note that due to the fact that money withdrawals occur infrequently, i.e., every T'th period, a steady state cannot be expected to exist. Therefore, we instead restrict the attention to *cyclical equilibria*, as defined below. Let $C^T = \{T+1, T+2, \ldots, T+T\}$ denote an issue period, or cycle, with C_i^T denoting the i'th time period of the cycle C^T. An issue period starts just after a withdrawal and ends just before the next withdrawal.

Definition 2 *Given that money withdrawals occur every T'th period, an equilibrium is said to be* ***cyclical*** *if it satisfies* $m_{C_i^T}^n = m_{C_i^R}^n$, $m_{C_i^T}^o = m_{C_i^R}^o$, $n_{C_i^T}^n = n_{C_i^R}^n$, $\mu_{C_i^T}^n = \mu_{C_i^R}^n$, $\mu_{C_i^T}^o = \mu_{C_i^R}^o$, $d_{C_i^T} = d_{C_i^R}$, $h_{C_i^T} = h_{C_i^R}$, $p_{C_i^T} = p_{C_i^R}$, $q_{C_i^T} = q_{C_i^R}$ *and* $e_{C_i^T} = e_{C_i^R}$ *for all* T, R *and* i.

The definition of cyclicality requires that, at the same point in two different issue periods C^T and C^R , the variables attain the same value, i.e., e.g., $m_{C_i^T}^n = m_{C_i^R}^n$. Note that using that government spending is constant over time and since $c_t = \xi - g$, we do not need to put any restrictions on consumption.

4.3 An example

The **below** example illustrates how to find a cyclical equilibrium when there is a withdrawal of coins every second period. As we will see in section 4.4, many of the results carry over to the general case.

Example 1 *Withdrawals occur every second period and only new coins are held in equilibrium. Noting that if* $n_1^n > 0$ ***then, by cyclicality, we have*** $\mu_2^n = n_1^n > 0$***, and hence, using (5) and (9),*** $q_1 = \frac{k}{b}$ ***(from competition between firms) and*** $q_2 = \frac{1}{b}$ ***and thus****, using the CIA constraint (14) and the money transition equation (32) we have,* ***using cyclicality*** *(i.e.,* $m_3^n = m_1^n$*),* $\Pi_2 = p_2 g$ *and setting* $c = c_1 = c_2$*,*

$$\begin{aligned} p_1 c &= m_2^n \\ p_2 c &= m_1^n \end{aligned} \tag{39}$$

for $t = \{1, 2\}$*. A similar result can be established when* $n_2^n > 0$ *and when* $n_1^n = n_2^n = 0$*. Note also that* $s_2^n = 1$*, since no old coins are held.*

There are three candidate equilibria; i) $n_1^n > 0$*,* $n_2^n = 0$ *and* $\mu_1^n = 0$*,* $\mu_2^n = n_1^n$*; ii)* $n_2^n > 0$*,* $n_1^n = 0$ *and* $\mu_1^n = n_2^n$*,* $\mu_2^n = 0$*; iii)* $n_t^n = \mu_t^n = 0$ *for* $t = 1, 2$*.*

First, suppose that $n_1^n > 0$ *so that* $q_1 = \frac{k}{b}$ *and* $q_2 = \frac{1}{b}$*. Using the money transition equation (32), we have*

$$\begin{aligned} m_2^n &= k\left(m_1^n + \Pi_2\right) + k n_1^n \\ &= k m_1^n + k m_1^n \frac{g}{c} + k n_1^n \end{aligned} \tag{40}$$

so that

$$m_2^n > k\left(1 + \frac{g}{c}\right) m_1^n. \tag{41}$$

Then, using (39),

$$p_1 > k\left(1 + \frac{g}{c}\right) p_2 \tag{42}$$

so that the return on jewelry holdings is

$$\frac{q_2/q_1}{p_2/p_1} > k\left(1 + \frac{g}{c}\right)\frac{1}{k} = 1 + \frac{g}{c}. \tag{43}$$

Since $n_1^n > 0$*, using (3), (13) and (37), we have* $d_2 < d_1$ *so that* $v'(d_2) > v'(d_1)$ *and hence, using (21),*

$$\frac{q_1}{p_1} - \beta\frac{q_2}{p_2} > \frac{q_2}{p_2} - \beta\frac{q_1}{p_1}. \tag{44}$$

Then, we have $\frac{q_1}{p_1} > \frac{q_2}{p_2}$, a contradiction.

The reason why an equilibrium does not exist is that the high return in (43) implies that households have incentives to save in jewelry in period 1, contradicting $n_1^n > 0$. The equilibrium where $n_2^n > 0$ can also be ruled out.[22] *Thus, the only equilibrium has $n_t^n = \mu_t^n = 0$ for $t = 1, 2$.*

Since the equilibrium entails neither minting nor melting, using money transition (32) $m_1^n = m_2^n + \Pi_1$, we get that $m_1^n > m_2^n$, in turn implying that prices increase over the cycle (i.e., $p_2 > p_1$) following from a modified quantity theory argument using expression (39).[23] To get an idea **of** why we have zero minting and melting, consider the case that we did not analyze in detail in the above example and thus suppose that there is positive mintage in period 2 so that $n_2^n > 0$ and hence, by cyclicality, melting in period 1 is positive; $\mu_1^n > 0$. Then we have $q_1 = \frac{1}{b}$ and $q_2 = \frac{k}{b}$. This tends to lower the returns on savings in jewelry in period 1, as indicated by expression (22). Moreover, the direct utility payoff, $v'(d_2)$, from holding jewelry between periods 1 and 2 is low in relative terms, since $d_1 < d_2$ through the selling off of jewelry in period 2 (recall that $d_1 = d_2 - bn_2^n$ from (13)). Due to the low return of savings in jewelry between periods 1 and 2, as indicated by the fall in the jewelry price (see also (22)), households do not want to melt coins in period 1 and thus, this cannot be an equilibrium.

Example 1 continued. *We now describe prices in equilibria where $n_t^n = \mu_t^n = 0$* for $t = 1, 2$. *From* ***cyclicality****, money transition (32) and the CIA constraint (14) $m_1^n = \frac{\xi}{\xi - g} m_2^n$. Moreover, using money transition (32) and (39), we have*

$$m_2^n = k \frac{\xi}{\xi - g} m_1^n. \tag{45}$$

Since, using (39), $p_1 = k \frac{\xi}{\xi - g} p_2$ and $p_2 = \frac{\xi}{\xi - g} p_1$ we have $\frac{\xi - g}{\xi} = \sqrt{k}$ so that, using (35), $c = \sqrt{k}\xi$. Then, goods prices increase by $\frac{1}{\sqrt{k}}$ between periods 1 and 2;

$$p_2 = \frac{1}{\sqrt{k}} p_1. \tag{46}$$

Finally, the relative jewelry price can be determined by using $d_1 = d_2$ in the Euler equation, implying that the direct marginal utility payoff from jewelry holdings is constant over the cycle, i.e., $v'(d_1) = v'(d_2)$. Then, we have $\frac{q_1}{p_1} = \frac{q_2}{p_2}$ and thus, using (46), $q_2 = \frac{\xi}{\xi - g} q_1$.
Since $q_2 \leq \frac{1}{b}$, any combination of jewelry prices such that $q_1 \in [\frac{k}{b}, \frac{\sqrt{k}}{b}]$ is feasible. Each such jewelry price is associated with a unique level of money holdings via the Euler equation. These equilibria can be Pareto ranked with the equilibrium yielding the highest welfare being associated with the lowest

[22] If $n_2^n > 0$ **then, using cyclicality,** $\mu_1^n > 0$ **and hence** $q_1 = \frac{1}{b}$ and $q_2 = \frac{k}{b}$. Then, from the money transition equations, we have $m_1^n > m_2^n$ so that $p_2 > p_1$ from the CIA constraints. Thus, $\frac{q_1}{p_1} > \frac{q_2}{p_2}$. Since $d_2 > d_1$ so that $v'(d_2) < v'(d_1)$ we have, using (44) with the inequality reversed, $\frac{q_1}{p_1} < \frac{q_2}{p_2}$, again a contradiction.

[23] Instead of the usual (14).

jewelry price.[24] *Finally, consider exchange rate restrictions for the equilibrium. Since households hold only new coins and* $s_2^n = 1$, *from* ***cyclicality****, (25), (26) and (29), we have* $e_1\chi \leq e_2 k$, $e_2\chi \leq e_1$ and $e_1\chi \leq k$. *Combining gives the following requirement for households to hold only new coins in equilibrium;*

$$k \geq \chi^2. \tag{47}$$

Since there is neither mintage nor melting, household coin holdings increase by $\frac{1}{\sqrt{k}}$ between period 2 and 1 and decrease by $\sqrt{k}$ between period 1 and 2. The modified Cash in Advance constraint (39) then implies that prices decrease by $\sqrt{k}$ between period 2 and 1 and increase by $\frac{1}{\sqrt{k}}$ between period 1 and 2. Thus, during a cycle, prices increase and fall at the start of a new cycle. Since jewelry holdings are constant, jewelry relative prices $\frac{q_t}{p_t}$ are constant over the cycle so that the jewelry price changes one to one with goods prices.

Now consider the case where old coins are held.

Example 1 continued. *We now consider equilibria where both new and old coins are held. From* ***cyclicality*** *and the money transition equation (32), we have*

$$\begin{aligned} p_1 c &= m_2^n + e_1 \left(\chi \left(m_1^o + (1 - s_2^n)(m_1^n + \Pi_2)\right) - \mu_1^o\right) \qquad (48) \\ p_2 c &= m_1^n + e_2 \left(\chi m_2^o - \mu_2^o\right). \end{aligned}$$

As in the case when only new coins are held, we can show that mintage and melting are always zero so that $n_t^n = \mu_t^n = \mu_t^o = 0$ *for* $t = 1, 2$*; see Lemmata 2 - 3 below for details. Moreover, for cyclicality, we require that* $s_2^n < 1$, *since otherwise* $m_1^o = m_3^o \leq \chi m_2^o \leq \chi^2 m_1^o$, *a contradiction. Using* ***cyclicality****, (23) - (25) and (27) - (28), the conditions on exchange rates are* $\frac{e_1\chi}{e_2} = s_2^n k + (1 - s_2^n) e_1\chi$, $e_2\chi = e_1$ and $e_1\chi \geq k$. *If* $s_2^n > 0$ *then* $e_1\chi = k$ *and thus* $e_2 = 1$ and $k = \chi^2$. *If* $s_2^n = 0$ *so that* $e_1\chi \geq k$ *we have* $k \leq \chi^2$, but again $e_2 = 1$. *Focusing on the case when* $k < \chi^2$ *so that* $s_2^n = 0$ *gives, using* ***cyclicality****,* $m_1^o = \chi m_2^o$ *and* $m_2^o = \chi(m_1^o + m_1^n + \Pi_2)$ *so that* $m_1^o = \frac{\chi^2}{1-\chi^2}(m_1^n + \Pi_2)$. *Then, the Cash in Advance constraint in period 2 is, using (32),* $e_2 = 1$ *and* $e_1 = \chi$,

$$p_2 c = m_1^n + \frac{\chi^2}{1 - \chi^2}(m_1^n + p_2 g) \iff p_2 = \frac{1}{\xi(1 - \chi^2) - g} m_1^n. \tag{49}$$

Since, ***using cyclicality****,* $m_1^n = m_2^n + p_1 g = m_2^n + \frac{g}{\xi}\frac{1}{1-\chi^2} m_1^n$, *the Cash in Advance constraint in period 1 is,* using $p_2 g = \frac{g}{\xi(1-\chi^2)-g} m_1^n$,

$$p_1 \xi = \frac{\xi - g}{\xi(1 - \chi^2) - g} m_1^n \tag{50}$$

[24] See the proof of Theorem 1 in the Appendix.

so that

$$p_1 = \frac{\xi - g}{\xi} p_2. \tag{51}$$

Akin to the case where only new coins are held, c.f., expression (46), we now find an expression for the price change in terms of parameters of the model. For this purpose, first note that the government revenues over a cycle are $(1-\chi)\left(m_1^o + m_1^n + \Pi_2 + m_2^o\right)$. *Using* $m_1^o = \frac{\chi^2}{1-\chi^2}\left(m_1^n + \Pi_2\right)$ and $m_2^o = \frac{\chi}{1-\chi^2}\left(m_1^n + \Pi_2\right)$, *the revenues are* $\frac{\xi(1-\chi^2)}{\xi(1-\chi^2)-g} m_1^n$. *Using the Cash in Advance constraints (49) and (50), government spending is*

$$p_1 g + p_2 g = \frac{g}{\xi} \frac{\xi}{\xi\left(1-\chi^2\right) - g}\left(2 - \frac{g}{\xi}\right) m_1^n. \tag{52}$$

For revenues to equal spending, we require that $\chi = 1 - \frac{g}{\xi}$ *and thus*

$$p_2 = \frac{1}{\chi} p_1. \tag{53}$$

Once more, prices increase during the cycle and fall at the start of a new cycle. Then, from the Euler equation (21) we have, using $p_1 = \chi p_2$, *(49) and (50), and that jewelry holdings are constant over the cycle, i.e.,* $d_1 = d_2$, $q_2 = \frac{1}{\chi} q_2$. *Since* $q_2 \leq \frac{1}{b}$, *any combination of jewelry prices such that* $q_1 \in [\frac{k}{b}, \frac{\chi}{b}]$ *is feasible. Note that both new and old coins are held in equilibrium, since from (32), we have* $m_1^n = \Pi_2 + m_2^n > 0$.

4.4 The general case

This section shows the existence of and analyzes properties of equilibria in the general case. By using money transition (32) in the CIA constraint (14), we can derive the following Lemma, akin to expression (39) in example 1.

Lemma 1 *The CIA constraint (14) is, when* $t \neq T+1$

$$p_t c = m_{t+1}^n + e_t\left(\chi m_t^o - \mu_t^o\right) \tag{54}$$

and, when $t = T+1$

$$p_t c = m_{t+1}^n + e_t\left(\chi\left(m_t^o + \left(1 - s_T^n\right)\left(m_t^n + \Pi_{t-1}\right)\right) - \mu_t^o\right). \tag{55}$$

Proof: See the appendix. ■

We now show that there is neither minting nor melting in equilibrium. First, we show that there can only be minting in the first period of a cycle.

Lemma 2 *Mintage can be positive only in period* 1.

To see this, suppose that only new coins are held so that $m_t^o = \mu_t^o = 0$ for all t. It is convenient to rearrange the Euler equation (21) as

$$p_t = Q_t\left(q_t, q_{t-1}, d_t, p_{t-1}\right) p_{t-1} \tag{56}$$

where

$$Q_t\left(q_t, q_{t-1}, d_t, p_{t-1}\right) = \beta \frac{q_t u'\left(c\right)}{q_{t-1} u'\left(c\right) - v'\left(d_t\right) p_{t-1}}. \tag{57}$$

To get a clearer idea about expression (57), it might be advantageous to think about the intertemporal consumption choice in terms of calculus of variation. The denominator in (57) is the (negative of the) nominal change in payoff when decreasing consumption at $t-1$ by increasing the jewelry holdings and the numerator is the change in payoff from the resulting increase in tomorrow's consumption that follows if future jewelry holdings d_{t+s} and consumption decisions c_{t+1+s} are unchanged for $s > 0$.

Now, let us look at why the mintage must be zero, except at the initial period of the cycle. If mintage is positive in some period $t > 1$, i.e., $n_t^n > 0$, then the jewelry price q_t is equal to $\frac{k}{b}$ and is thus low in that period, c.f. equation (43) in example 1. Then $\mu_t^n = 0$. Moreover, using money transition (32) and Lemma 1, we have $m_{t+1}^n > \frac{\xi}{\xi - g} m_t^n$ and then, by Lemma 1, prices increase so that $Q_t > 1$. Since prices increase ($p_t > p_{t-1}$), jewelry holdings decrease ($d_{t+1} < d_t$) so that $v'\left(d_t\right) < v'\left(d_{t+1}\right)$ and jewelry prices weakly increase ($q_{t+1} \geq q_t$) in period $t+1$ households have even stronger incentives to postpone consumption and, using (57), we have $Q_{t+1} > Q_t$. Then, prices in the next period increase even more. Money transition (32) and Lemma 1 then imply that there is positive mintage also in the next period. Induction then establishes that mintage is positive in all periods, thus violating cyclicality. Since the Gesell tax is collected in the first period, this argument does not work starting at $t = 1$, since even if we have positive minting in the first period, then (32) and Lemma 1 only imply that $m_2^n > k \frac{\xi}{\xi - g} m_1^n$ so that m_2^n can be lower than m_1^n, in turn allowing p_2 to be lower than p_1.[25]

The next lemma shows that there is no melting of coins during a cycle. As a corollary, it then follows by cyclicality that there cannot be minting in the first period of a cycle.

Lemma 3 *There is no melting of either new or old coins.*

Proof: See the appendix. ■

[25] Note that this initial period argument does not affect the induction argument when starting at $n_t^n > 0$ for $t > 1$. The reason for this is twofold. First, since we only use the the Cash in Advance constraint from Lemma 1 at the start of the induction argument, we must have $m_{t+1}^n > \frac{\xi}{\xi - g} m_t^n$ for that period and thus $Q_t > 1$ (instead of $Q_t > k \frac{\xi}{\xi - g}$). Second, the rest of the induction argument is based on the Euler equation as described in (57), which does not depend on the Gesell tax k.

Note first, akin to models with durable consumption goods, that we can rewrite the relative jewelry price, by repeatedly using future Euler equations (21) **as** the discounted value of future jewelry holdings, measured in monetary terms[26]

$$\frac{q_t}{p_t} = \frac{1}{1-\beta^T} \sum_{s=0}^{T} \beta^s \frac{v'(d_{t+s+1})}{u'(\bar{c})}. \tag{59}$$

We here focus on the case where only new coins are held during the entire cycle (so that $s_T^n = 1$). Assume that $\mu_t^n > 0$ for some t. Then, by cyclicality, we must have $n_1^n > 0$ from Lemma 3. Since there can be minting only in the first period, using (13), we have that jewelry holdings are increasing over the cycle; $d_2 \leq d_3 \leq \ldots \leq d_{T+1}$ where the inequality is strict for some t. Then, using strict concavity of v in (59), we have $\frac{q_{T+1}}{p_{T+1}} > \frac{q_T}{p_T}$, since jewelry holdings are valued **higher in period** $T+1$ **than in period** T. In fact, as is shown in the proof of Lemma 3, for $t = 2, \ldots, T$

$$\frac{q_{t+1}}{p_{t+1}} > \frac{q_t}{p_t}. \tag{60}$$

Thus, the real return, as defined in (22), increases during a cycle. Since mintage is positive in the first period, we have $q_1 = \frac{k}{b}$. Then, using money transition (32) and rewriting $\frac{\xi}{\xi - g}$,

$$m_2^n > \frac{k}{1 - \frac{g}{\xi}} m_1^n \tag{61}$$

and thus, using Lemma 1 and that the relative goods price between T and $T+1$, $\frac{p_T}{p_{T+1}}$ is smaller than $\frac{1-\frac{g}{\xi}}{k}$. Then, using $q_{T+1} = \frac{k}{b}$ and that the real return is high, as can be seen by (60), q_T is bounded from above;

$$q_T < \left(1 - \frac{g}{\xi}\right) \frac{1}{b}. \tag{62}$$

Thus, since the relative price between T and $T+1$ changes by less than $\frac{1-\frac{g}{\xi}}{k}$ and $q_{T+1} = \frac{k}{b}$**,** it follows that q_T has to be smaller than $\frac{1}{b}$, in turn making it unprofitable to melt coins by (7); $\mu_T^n = 0$. From money transition $m_{T+1}^n = \frac{\xi}{\xi - g} m_T^n$, in turn implying that $p_T \left(1 - \frac{g}{\xi}\right) = p_{T-1}$ from Lemma 1, so that prices increase by $\frac{1}{1-\frac{g}{\xi}}$. Then, again using that real returns are increasing, i.e., we have $q_T \frac{p_{T-1}}{p_T} > q_{T-1}$ from (60), and repeating a similar argument we then have $q_{T-1} < \left(1 - \frac{g}{\xi}\right)^2 \frac{1}{b}$ and thus, $\mu_{T-1}^n = 0$. Induction then establishes that $q_t < \left(1 - \frac{g}{\xi}\right)^{T-t+1} \frac{1}{b}$, in turn implying that melting is zero for all t

[26] In general, we have

$$\frac{q_t}{p_t} = \sum_{s=0}^{\infty} \beta^s \frac{u'(c_{t+s})}{u'(c_t)} \frac{v'(d_{t+s+1})}{u'(c_{t+s})}. \tag{58}$$

However, using cyclicality and the fact that consumption is constant, we can write the relative price as functions of jewelry holdings during a cycle as in (59).

by (7). The argument when old coins are held is slightly more complicated and is dealt with in the appendix.

Combining Lemmata 2 - 3 implies that mintage must be zero in period 1 as well. The reason is that, since coins are never melted, for cyclicality to hold, there cannot be any mintage in any period. We then have the following theorem.

Theorem 1 *A cyclical equilibrium exists and entails $n_t^n = \mu_t^n = \mu_t^n = 0$ for all t. If $k > \chi^T$ ($k < \chi^T$), in any cyclical equilibrium, only new (both new and old) coins are held. If $k = \chi^T$ either only new or both new and old coins are held. In any equilibrium, prices increase during an issue, i.e., $p_t > p_{t-1}$ for $t = 2, \ldots, T$ and drop between periods T and $T+1$. If $k \geq \chi^T$ prices increase at the rate $k^{-\frac{1}{T}}$ during a cycle and if $k < \chi^T$ prices increase at the rate χ^{-1} and no coins are handed in for re-coinage.*

Proof: See the appendix. ■

Suppose that only new coins are held. The results for increasing prices follow from the fact that money transition (32) implies that household money holdings increase over the cycle, due to the fact that firm dividends from government consumption increase household money holdings, so that, using a quantity theory argument and Lemma 1, prices increase. A modification of this argument establishes a similar result when also old coins are held. As long as only new coins are held, price increases are higher the higher is the Gesell tax, since a higher Gesell tax leads to higher government spending and, in turn, a higher increase in household money holdings during a cycle. When $k < \chi^T$ so that old coins are also held, price increases only depend on the plaintiff confiscation rate χ. The reason is that since no coins are handed in for re-coinage, the only source of government revenues is the confiscation of illegal coins and thus, χ is the sole determinant of government spending and hence, of the increase in money holdings during a cycle.

The cutoff values for whether old coins are held depend on χ and k. The reason for these cutoffs is that, assuming that both types are held, using (23) and (25), the exchange rate must appreciate at rate $\frac{1}{\chi}$ so that $e_{t+1}\chi = e_t$ and, from (24), (26) together with (27) when s_T^n is interior, that $e_T = 1$. We then have

$$e_1 = \frac{1}{\chi} e_2 = \cdots = \frac{1}{\chi^{T-1}}. \tag{63}$$

Since not all new coins are handed in for re-coinage, households must weakly prefer not to hand in new coins and hence $e_1\chi \geq k$. Thus, $k \leq \chi^T$. When only new coins are held, appreciation is bounded above by $\frac{1}{\chi}$, implying that $k \geq \chi^T$.

4.5 Relationship to empirical evidence

The empirical evidence in section 3.2 indicates that only new coins circulated in England during a period when withdrawals occurred rather infrequently. In the 11th century, the intervals became shorter, thus tightening the condition that $k > \chi^T$, and, in case the fee was unchanged, also increasing the implied yearly fee. When that happened, old coins tend to be much more frequent in hoards, thus indicating that both old and new coins circulated together. In Germany, where re-coinage could occur as often as twice a year, hoards contain old coins even more frequently, indicating that old coins tend to circulate along with new coins.

Regarding prices, the evidence is scant. However, some evidence of price regulation from the Frankish empire in the late 7th century seems to indicate that prices rose during a cycle, in line with Theorem 1 (see also section 3.1).

5 Conclusions

A frequent method for generating revenues from seigniorage in the Middle Ages was to use Gesell taxes through periodic re-coinage where coins are legal tender for a limited period of time. In such a short-lived coinage system, old coins are declared invalid and exchanged for new ones at publicly announced dates and exchange fees, akin to Gesell taxes. Empirical evidence based on several methods shows that re-coinage could occur as often as twice a year within a currency area in the Middle Ages. In contrast, in a long-lived coinage system, coins did not have a given fixed period of being legal means of payment. Long-lived coins were common in western and southern Europe in the High Middle Ages, whereas short-lived coins dominated in central, northern and eastern Europe. Although the short-lived coinage system defined legal tender for almost 200 years in large parts of medieval Europe, it has seldom if ever been mentioned or analyzed in the economic literature.

The main purpose of this study has been to discuss the evidence for, and analyze the consequences of, short-lived coinage systems. In a short-lived coinage system, only one type may circulate in the currency area, and different types reflecting various issues need to be clearly distinguishable for everyday users of the coins. The coin issuing authority had several methods to monitor and enforce the re-coinage. First, they had exchangers and other administrators at the city markets. Second, payment of any fees, taxes, rents, tithes or fines had to be made in new coins. Although only new coins were allowed to be used for transactions, evidence from coin hoards indicates that agents used illegal coins.

A cash-in-advance model is formulated to capture the implications of this monetary institution. There are households, firms and a lord, where households care about goods and jewelry consumption and firms about profits. The lord uses seigniorage to finance consumption. When trading jewelry and consumption, households face a cash-in-advance constraint. Households can hold both new and old coins, so that the equilibrium choice of what coins to hold is endogenous.

Our key results are that, in equilibrium, prices increase over time during an issue period and fall just after the re-coinage date and, the higher the Gesell tax, the higher the price increases (as long as coins are handed in for re-coinage). Furthermore, households re-mint their old coins into new ones: 1) the lower is the exchange fee; 2) the longer is the time period between re-coinage dates; and 3) the higher is the risk of being detected using old invalid coins.

In future research, it would be interesting to analyze under which circumstances it is optimal for the minting authority to use Gesell taxes vis-à-vis debasement to generate seigniorage. Although debasement and re-coinage are not mutually exclusive, Kluge (2007, p. 64), has suggested that debasement primarily occurred in areas with long-lived rather than short-lived coins. A further characteristic for cities and regions where the short-lived coinage system was in force is that the local economy was relatively undeveloped; see Spufford (1988, p. 104). There are several explanations for why re-coinage would work particularly well in relatively undeveloped economies. Such economies had a small volume of coins in circulation, which facilitates re-minting. Furthermore, there tended to be few places where coins were used for transactions and few groups in society that use the coins, i.e. low monetization, thus facilitating monitoring.

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A Appendix

A.1 Methods to identify re-coinage

From archaeology and numismatics, there are three basic methods for identifying re-coinage. In Table 3, we have ranked the methods by confidence. The most confident way of identifying re-coinage is through written documents that may contain explicit information about dates of re-coinage and/or exchange fees. However, for most currency areas and mints, there are no written sources about recurrent re-coinage and other methods must be used.

By classifying different coin types as originating from a specific coin issuer and mint, it is relatively straightforward to establish whether re-coinage must have occurred. When and if there is only one type per reign, the coinage system is long-lived. However, in the event that there are as many types as years of a specific reign and mint, the evidence indicates annual renewals. If the number of types exceeds (falls short of) the number of years, the renewals are more (less) frequent.

Table 3: Methods for identifying short-lived and long-lived coinage systems

Method	Long-lived coins	Short-lived coins	Confidence of method
1. Written documents	-	-	Very strong
2. Coin types per reign, years and currency area	One	At least two	Strong
3. Distribution of coin types in hoards	One or a few types from each mint	Many types from each mint, but a few late dominate	Medium

A third method for identifying re-coinage involves carefully analyzing the concentration and distribution of types in coin hoards. Coin hoards from the Middle Ages may contain few or many issues from each mint represented in the hoard. If re-coinage has occurred, one would expect many types in the hoard from a specific mint, but a few types to strongly dominate the composition of the hoard. These types in such cases would be relatively young, while older types should have a more sparse representation. In those cases where there are several coin hoards from a specific coin issuer, one can expect types existing in many hoards to be older and types in a few hoards to be younger.

A.2 Household optimization

Using the first-order conditions with respect to c_t and m^n_{t+1}, if $m^o_{t+1} > 0$ then, if $t \neq T, 2T$ etc.,

$$\left(\frac{e_{t+1}\chi}{e_t} - 1\right) \frac{u'\left(c_{t+1}\right)}{p_t} \geq 0 \tag{A.1}$$

and

$$\left(\frac{e_{t+1}\chi}{e_t} - \left(s_t^n k + e_{t+1}\chi\left(1-s_t^n\right)\right)\right)\frac{u'\left(c_{t+1}\right)}{p_t} \geq 0, \tag{A.2}$$

otherwise.

If $m_{t+1}^n > 0$, if $t \neq T, 2T$ etc.

$$\beta\left(\frac{e_{t+1}\chi}{e_t} - 1\right)\left(\frac{u'\left(c_{t+1}\right)}{p_t}\right) \leq 0. \tag{A.3}$$

Since the consumer now holds new coins in period $t+1$, the exchange rates in period t and $t+1$ have to give the consumer incentives to **not** only hold old coins. For this to be the case, the exchange rate increase cannot be too large and is bounded above by $\frac{1}{\chi}$. If $t = T, 2T$ etc.,

$$\beta\left(\frac{e_{t+1}\chi}{e_t} - \left(s_t^n k + e_{t+1}\chi\left(1-s_t^n\right)\right)\right)\left(\frac{u'\left(c_{t+1}\right)}{p_t}\right) \leq 0. \tag{A.4}$$

Furthermore, using the first-order condition with respect to m_{t+1}^n gives, if $t \neq T, 2T$ etc.

$$\beta \max\left\{\frac{u'\left(c_{t+1}\right)}{p_{t+1}}, \frac{e_{t+1}\chi}{e_t}\frac{u'\left(c_{t+1}\right)}{p_{t+1}}\right\} \leq \frac{u'\left(c_t\right)}{p_t} \tag{A.5}$$

and

$$\beta \max\left\{\left(s_t^n k + e_{t+1}\chi\left(1-s_t^n\right)\right)\frac{u'\left(c_{t+1}\right)}{p_{t+1}}, \frac{e_{t+1}\chi}{e_t}\frac{u'\left(c_{t+1}\right)}{p_{t+1}}\right\} \leq \frac{u'\left(c_t\right)}{p_t}. \tag{A.6}$$

The conditions hold with equality only if the cash in advance constraint does not bind. One way of thinking about the above expression is the following. Consider a relaxation of the cash in advance constraint today and a corresponding tightening tomorrow. The relaxation today yields an increase in utility (readjusted for the relative price) of $\frac{u'(c_t)}{p_t}$ or $e_t\frac{u'(c_t)}{p_t}$, while the strengthening tomorrow leads to a decrease of either $\beta\frac{u'(c_{t+1})}{p_{t+1}}$ or $\beta e_{t+1}\chi\frac{u'(c_{t+1})}{p_{t+1}}$, depending on whether it is new or old coins that are used. To ensure that this deviation is not optimal, the above inequality must hold.

Finally, the household also optimally chooses the share of coins to be handed in for re-coinage, s_t^n. If $s_t^n \in (0,1)$ then

$$\left(-k + e_{t+1}\chi\right)\left(m_{t+1}^n + \Pi_t\right) = 0. \tag{A.7}$$

If $s_t^n = 0$ then

$$\left(-k + e_{t+1}\chi\right)\left(m_{t+1}^n + \Pi_t\right) \geq 0. \tag{A.8}$$

Finally, if $s_t^n = 1$ and hence

$$\left(-k + e_{t+1}\chi\right)\left(m_{t+1}^n + \Pi_t\right) \leq 0. \tag{A.9}$$

A.3 Proofs

Before proceeding with the proofs, we state and prove the following Lemma.

Lemma 4 *In a cyclical equilibrium,* $\frac{k}{b} \leq q_t \leq \frac{1}{b}$.

Proof

Suppose first that $q_t > \frac{1}{b}$. Then, using (8), $m^n_{t+1} = 0$. Moreover, we have $\frac{e_t}{b} \geq q_t$, since otherwise, using (11), $\mu^o_t = \chi m^o_t$ and thus $m^o_{t+1} = 0$ and hence $p_t = 0$. Selling a small amount of jewelry allows the consumer infinite consumption, contradicting the resource constraint. As long as a $m^o_s > 0$, we have, using (23) $e_s\chi \geq e_{s-1}$ so that, for $s > t$, $e_s > 1$. If $m^o_s > 0$ for all $s \in \{t+1, \ldots, T\}$ then $e_T > 1$. From (24), we have, using that holding of old coins implies $s^n_t < 1$ and hence $e_{T+1}\chi \geq k$ from (27) and (28) with equality if s^n_T is interior,

$$\frac{e_{t+1}\chi}{s^n_t k + e_{t+1}\chi\left(1 - s^n_t\right)} = 1 \geq e_t, \tag{A.10}$$

a contradiction. If $m^o_s = 0$ for some $s \in \{t+2, \ldots, T\}$ then $\frac{e_{s-1}}{b} \leq q_t$ and $e_r\chi \geq e_{r-1}$ for all $t+1 < r < s-1$ so that $e_{s-1} > 1$. Then $q_{s-1} > \frac{1}{b}$, implying that $m^n_s = 0$ and thus $p_{s-1} = 0$, a contradiction.

Suppose now that $q_t < \frac{k}{b}$. Then, $n^n_t = d_t$ so that $d_{t+1} = 0$. The Inada conditions then establish a contradiction.■

Proof of Lemma 1:

Case 1**.** First, suppose that $t \neq T+1$.

Suppose that $\mu^o_t = 0$. If $n^n_t > 0$ then $h_t = -bn^n_t$ and $q_t = \frac{k}{b}$ from (5) and thus, $\mu^n_t = 0$. Using that the Inada conditions imply that (14) holds with equality, the CIA constraint (14) is,

$$p_t c_t + \frac{k}{b}\left(-bn^n_t\right) = m^n_t + \Pi_{t-1} + e_t\chi m^o_t. \tag{A.11}$$

Using money transition (32), we get

$$m^n_{t+1} = m^n_t + \Pi_{t-1} + kn^n_t, \tag{A.12}$$

and thus

$$p_t c_t + \frac{k}{b}\left(-bn^n_t\right) = m^n_{t+1} - kn^n_t + e_t\chi m^o_t \tag{A.13}$$

establishing that

$$p_t c_t = m^n_{t+1} + e_t\chi m^o_t. \tag{A.14}$$

A similar argument holds if $n_t^n = \mu_t^n = 0$. Suppose that $\mu_t^n > 0$ so that $q_t = \frac{1}{b}$ from (8) - (9) and Lemma 4. Then, $h_t = b\mu_t^n$ and

$$p_t c_t + \frac{1}{b} b\mu_t^n = m_t^n + \Pi_{t-1} + e_t \chi m_t^o. \tag{A.15}$$

Using money transition (32) we get

$$p_t c_t = m_{t+1}^n + e_t \chi m_t^o. \tag{A.16}$$

Suppose that $\mu_t^o > 0$. Then, $\frac{e_t}{b} = q_t$ from (11) since competition among firms forces the jewelry price to $\frac{e_t}{b}$. If $n_t^n = \mu_t^n = 0$ then, using that the Inada conditions imply that (14) holds with equality, the CIA constraint is

$$p_t c_t + \frac{e_t}{b} b\mu_t^o = m_t^n + \Pi_{t-1} + e_t \chi m_t^o. \tag{A.17}$$

Using money transition (32) gives

$$p_t c_t = m_{t+1}^n + \Pi_{t-1} + e_t \left(\chi m_t^o - \mu_t^o \right). \tag{A.18}$$

If $\mu_t^n > 0$ then, from Lemma 4, $q_t = \frac{1}{b} = \frac{e_t}{b}$ so that the CIA constraint is

$$p_t c_t + \frac{e_t}{b} b\mu_t^o + \frac{1}{b} b\mu_t^n = m_t^n + \Pi_{t-1} + e_t \chi m_t^o. \tag{A.19}$$

Using money transition (32) gives

$$p_t c_t = m_{t+1}^n + e_t \left(\chi m_t^o - \mu_t^o \right). \tag{A.20}$$

A similar argument establishes the same expression whenever $n_t^n > 0$.

Case 2. Now, suppose that $t = T + 1$.

Suppose that $\mu_t^o = 0$. Then, using that the Inada conditions imply that (14) holds with equality,

$$p_t c_t + q_t h_t = k s_T^n \left(m_t^n + \Pi_{t-1} \right) + e_t \chi \left(m_t^o + \left(1 - s_T^n\right) \left(m_t^n + \Pi_{t-1} \right) \right). \tag{A.21}$$

If $n_t^n > 0$, we can proceed as in case 1 to establish

$$p_t c_t + \frac{k}{b} \left(-b n_t^n \right) = k s_T^n \left(m_t^n + \Pi_{t-1} \right) + e_t \chi \left(m_t^o + \left(1 - s_T^n\right) \left(m_t^n + \Pi_{t-1} \right) \right). \tag{A.22}$$

Again using money transition establishes (32) that

$$p_t c_t + \frac{k}{b}\left(-b n_t^n\right) = m_{t+1}^n - k n_t^n + e_t \chi \left(m_t^o + \left(1 - s_T^n\right)\left(m_t^n + \Pi_{t-1}\right)\right) \tag{A.23}$$

and thus

$$p_t \xi = m_{t+1}^n + e_t \left(\chi m_t^o + \left(1 - s_T^n\right)\left(m_t^n + \Pi_{t-1}\right)\right). \tag{A.24}$$

A similar argument holds if $\mu_t^n > 0$ and if $\mu_t^n = n_t^n = 0$.

Suppose that $\mu_t^o > 0$ so that $\frac{e_t}{b} = q_t$ from (11) since competition among firms forces the jewelry price to $\frac{e_t}{b}$. If $n_t^n = \mu_t^n = 0$ then the CIA constraint is

$$p_t c_t + \frac{e_t}{b} b \mu_t^o = k s_T^n \left(m_t^n + \Pi_{t-1}\right) + e_t \left(\chi \left(m_t^o + \left(1 - s_T^n\right)\left(m_t^n + \Pi_{t-1}\right)\right)\right). \tag{A.25}$$

Again, proceeding as in Case 1 establishes that

$$p_t \xi = m_{t+1}^n + e_t \left(\chi \left(m_t^o + \left(1 - s_T^n\right)\left(m_t^n + \Pi_{t-1}\right)\right) - \mu_t^o\right). \tag{A.26}$$

The cases when $n_t^n > 0$ and $\mu_t^n > 0$ follow analogously.■

Lemma 5 *Suppose that $m_{T+1}^o > 0$. Then, $e_T = 1$.*

Proof:

If $m_{T+1}^o > 0$ and $s_T^n = 1$ or $m_{T+1}^n = 0$, using money transition 33, $m_{T+1}^o = \chi^T m_1^o$, thus contradicting cyclicality. Thus, $m_{T+1}^n > 0$ and $s_T^n < 1$, since there must be some coins that are not handed in for re-coinage for old coins to be held in a cyclical equilibrium. Since $m_{T+1}^o > 0$ and $m_{T+1}^n > 0$ then, using (24), (26) and, if $s_T^n \in (0,1)$, (27) so that $e_{T+1}\chi = k$ implies

$$\frac{e_{T+1}\chi}{e_T} = \left(s_T^n k + e_{T+1}\chi\left(1 - s_T^n\right)\right), \tag{A.27}$$

and thus $e_T = 1$.■

Proof of Lemma 2.

We prove this by contradiction. Suppose that $n_s^n > 0$ for some $s \leq T$.

Step 1. Finding an intertemporal relationship between money holdings and showing $\mu_s^o = 0$ whenever $m_s^o > 0$.

Since $n_s^n > 0$ we have, from (5) and Lemma 4, that $q_s = \frac{k}{b}$. Using that in case $m_s^o > 0$ (requiring $s_T^n < 1$ for cyclicality to be satisfied; see the proof of Lemma 5) we have, from (27), (28) and (23) that $e_1 \chi \geq k$ and $e_r \chi \geq e_{r-1}$ so that $e_r > k$ for $r \leq s$ and hence $\frac{e_s}{b} > q_s$. Then, using (10), we have

$\mu_s^o = 0$. Moreover, $m_{s+1}^o = \chi m_s^o - \mu_s^o = \chi m_s^o$. Then, $e_s m_{s+1}^o = e_s \chi m_s^o \geq e_{s-1} m_s^o$. Using (32),

$$m_{s+1}^n + e_s \chi m_s^o = m_{s+1}^n + e_s m_{s+1}^o = m_s^n + e_s m_{s+1}^o + \Pi_{s-1} + n_s^n > m_s^n + e_{s-1} m_s^o + \Pi_{s-1} \tag{A.28}$$

and, using that we from Lemma 1 have

$$p_s c_s = m_{s+1}^n + e_s m_{s+1}^o \tag{A.29}$$

gives

$$m_{s+1}^n + e_s m_{s+1}^o > \frac{\xi}{\xi - g} \left(m_s^n + e_{s-1} m_s^o \right), \tag{A.30}$$

so that, using Lemma 1, we have $p_s > \frac{\xi}{\xi - g} p_{s-1}$ and hence $Q_s > \frac{\xi}{\xi - g}$.

Since $q_{s-1} \geq \frac{k}{b}$ and $n_s^n > 0$ implies $d_{s+1} < d_s$ we have, using concavity of v,

$$b q_{s-1} u'(\bar{c}) - b v'(d_s) p_{s-1} > b q_s u'(\bar{c}) - b v'(d_{s+1}) p_s. \tag{A.31}$$

Finally, since $q_{s+1} \geq \frac{k}{b}$ we have, using (57), that

$$Q_{s+1} > Q_s > \frac{\xi}{\xi - g}, \tag{A.32}$$

implying that $p_{s+1} > \frac{\xi}{\xi - g} p_s$ and, using $p_t c = m_{t+1}^n + e_t m_{t+1}^o$,

$$m_{s+2}^n + e_{s+1} m_{s+2}^o > \frac{\xi}{\xi - g} \left(m_{s+1}^n + e_s m_{s+1}^o \right) \tag{A.33}$$

and hence, following similar steps as above, using Lemma 1 and that $m_{s+1}^o = \chi m_s^o - \mu_s^o$, we have

$$m_{s+2}^n + e_{s+1} m_{s+2}^o > m_{s+1}^n + e_s m_{s+1}^o + \Pi_s. \tag{A.34}$$

Step 2. Showing that $n_{s+1}^n > 0$.

Case 1. Suppose that $m_{s+2}^o = 0$.

If $m_{s+2}^o = 0$ then

$$m_{s+2}^n > m_{s+1}^n + e_s m_{s+1}^o + \Pi_s > m_{s+1}^n + \Pi_s. \tag{A.35}$$

From money transition (32), we have

$$m_{s+2}^n - \left(m_{s+1}^n + \Pi_s \right) = k n_{s+1}^n - \mu_{s+1}^n > 0, \tag{A.36}$$

and hence, since $\mu_{s+1}^n \geq 0$, it follows that $n_{s+1}^n > 0$.

Case 2. Suppose that $m^o_{s+2} > 0$.

If $m^o_{s+2} > 0$ we have $m^o_{s+2} \leq \chi m^o_{s+1}$.

First, suppose that $m^n_{s+2} = 0$ then, using that $n^n_s > 0$ implies that $m^n_{s+1} > 0$, $\mu^n_{s+1} > 0$ so that, using Lemma 4, $q_{s+1} = \frac{1}{b}$. If $m^o_{s+i} > 0$ for all $T - s + 1 > i > 2$ then, from Lemma 5, $e_T = 1$ and by (23), $e_{s+i}\chi \geq e_{s+i-1}$ for all i such that $s + i \leq T$ and thus $e_{s+1} < 1$. Then, from (11), we have $q_{s+1} = \frac{1}{b} > \frac{e_{s+1}}{b}$ so that $\mu^o_{s+1} = \chi m^o_{s+1}$ implying that $m^o_{s+2} = 0$, a contradiction. If $m^o_{s+i} = 0$ (and $m^o_{s+i-1} > 0$) for some $T - s + 1 > i > 2$, then, using (11), $\frac{e_{s+i-1}}{b} \leq q_{s+i-1}$ and $\mu^o_{s+i-1} = \chi m^o_{s+i-1}$ and also since $m^o_{s+i-1} > 0$, $m^o_{s+k} > 0$ and, using (10), $\frac{e_{t+k}}{b} \geq q_{s+k}$ for all $i - 1 > k > 2$. Since $q_{s+1} = \frac{1}{b} \leq \frac{e_{s+1}}{b}$ we get $e_{s+1} \geq 1$. Moreover, since $m^o_{s+k} > 0$ for all $i - 1 > k > 2$ then, by (23), $e_{s+k}\chi \geq e_{s+k-1}$ for all i such that $s + k \leq s$, implying that $e_{s+i-1} > 1$ and thus $q_{s+i-1} \geq \frac{e_{s+i-1}}{b} > \frac{1}{b}$ in turn implying that $\mu^n_{s+i-1} = m^n_{s+i-1} + p_{s+i-2}g$ so that $m^n_{s+i} = 0$. Then, using Lemma 1, we have $p_{s+i} = 0$, a contradiction.

Second, suppose that $m^n_{s+2} > 0$. Thus, by (23) and (25), we have $e_{s+2}\chi = e_{s+1}$ and, using (33),

$$e_{s+1}m^o_{s+2} = e_{s+1}\left(\chi m^o_{s+1} - \mu^o_{s+1}\right) \leq e_s m^o_{s+1} \tag{A.37}$$

and thus, using (A.34),

$$m^n_{s+2} + e_s m^o_{s+1} > m^n_{s+2} + e_{s+1}m^o_{s+2} > m^n_{s+1} + e_s m^o_{s+1} + \Pi_s \tag{A.38}$$

in turn, using money transition (32), implying that

$$m^n_{s+2} - \left(m^n_{s+1} + \Pi_s\right) = kn^n_{s+1} - \mu^n_{s+1} > 0 \tag{A.39}$$

and hence, since $\mu^n_{s+1} \geq 0$, it follows that $n^n_{s+1} > 0$.

Step 3. Suppose that $s = T + 1$.

If $n^n_r > 0$ for $r \leq T$ then, by using induction on (A.32) we have, when $s - 1 = T$, that $Q_T > 1$. Since from step 1 we have $\mu^o_{s-1} = 0$ if $m^o_{s-1} > 0$ and $n^n_{s-1} > 0$ (and thus $q_{s-1} = \frac{k}{b}$ and $\frac{e_{s-1}}{b} > q_{s-1}$), it follows that $d_s < d_{s-1}$. Then, since $q_{s-1-1} \geq \frac{k}{b} = q_{s-1}$ and by using concavity of v, we have

$$bq_{s-1-1}u'(c) - bv'(d_{s-1})\,p_{s-1-1} > bq_{s-1}u'(c) - bv'(d_s)\,p_s. \tag{A.40}$$

Finally, since $q_s \geq \frac{k}{b}$ from Lemma 4, we have, using (57) that

$$Q_{T+1} > Q_T > \frac{\xi}{\xi - g}, \tag{A.41}$$

so that $p_{T+1} > p_T$. Then, from Lemma 1,

$$m^n_{T+2} + e_{T+1} m^o_{T+2} > \frac{\xi}{\xi - g} \left(m^n_{T+1} + e_T m^o_{T+1} \right). \tag{A.42}$$

If $m^o_{T+2} = 0$ then, from (A.42) and Lemma 1,

$$m^n_{T+2} > \left(\frac{\xi - g}{\xi - g} + \frac{g}{\xi - g} \right) \left(m^n_{T+1} + e_T m^o_{T+1} \right) > m^n_{T+1} + e_T m^o_{T+1} + \Pi_T \geq m^n_{T+1} + \Pi_T > k s^n_T m^n_{T+1} + \Pi_T \tag{A.43}$$

From money transition (32), we have

$$m^n_{T+2} - \left(k s^n_T m^n_{T+1} + \Pi_T \right) = k n^n_{T+1} - \mu^n_{T+1} > 0 \tag{A.44}$$

and hence, since $\mu^n_{T+1} \geq 0$, it follows that $n^n_{T+1} > 0$.

If $m^o_{T+2} > 0$ then, using (33),

$$e_{T+1} m^o_{T+2} = e_{T+1} \left(\chi \left(m^o_{T+1} + (1 - s^n_T) \left(m^n_{T+1} + \Pi_T \right) \right) - \mu^o_{T+1} \right) \leq e_{T+1} \chi \left(m^o_{T+1} + (1 - s^n_T) \left(m^n_{T+1} + \Pi_T \right) \right) \tag{A.45}$$

and thus, using (32), we can write (A.42) as

$$\begin{aligned} m^n_{T+2} &> m^n_{T+1} + e_T m^o_{T+1} - e_{T+1} m^o_{T+2} + \Pi_T \\ &= m^n_{T+1} + e_T m^o_{T+1} - e_{T+1} \left(\chi \left(m^o_{T+1} + (1 - s^n_T) \left(m^n_{T+1} + \Pi_T \right) \right) - \mu^o_{T+1} \right) + \Pi_T \\ &\geq m^n_{T+1} + e_T m^o_{T+1} - e_{T+1} \left(\chi \left(m^o_{T+1} + (1 - s^n_T) \left(m^n_{T+1} + \Pi_T \right) \right) \right) + \Pi_T. \end{aligned} \tag{A.46}$$

Suppose that $s^n_T > 0$ so that, using (27), we have $e_{T+1}\chi = k$. Then, using $k < 1$ and, whenever $m^o_{T+1} > 0$, from Lemma 5, $e_T = 1$,

$$\begin{aligned} m^n_{T+2} &> \left(1 - e_{T+1} \chi \left(1 - s^n_T \right) \right) m^n_{T+1} + \left(e_T - e_{T+1} \chi \right) m^o_{T+1} + \Pi_T - e_{T+1} \chi \left(1 - s^n_T \right) \Pi_T \\ &> \left(1 - e_{T+1} \chi + e_{T+1} \chi s^n_T \right) \left(m^n_{T+1} + \Pi_T \right) \\ &> k s^n_T \left(m^n_{T+1} + \Pi_T \right). \end{aligned} \tag{A.47}$$

Suppose that $s^n_T = 0$. First, consider the case when $m^o_{T+1} > 0$. If $m^o_{T+1} > 0$ then $m^o_t > 0$ for all t. Moreover, $m^n_{T+1} > 0$ for $m^o_t > 0$ in a cyclical equilibrium, since otherwise no new coins can be transformed into old ones, implying that the old coin holding decreases to zero, thus contradicting cyclicality. Thus, using (23), we have $e_{t+1}\chi \geq e_t$, and, from Lemma 5, $e_T = 1$. Then, using that $m^n_{T+1} > 0$ which, in turn, implies that $e_1 \leq \chi^T e_T < 1$ from (23). Now suppose that $m^o_{T+1} = 0$ and

$e_{T+1}\chi > 1$. If $m^o_{T+1} = 0$ then, since $s^n_T = 0$, using (33), $\mu^o_t = \chi m^o_t > 0$ for some t. Then, using (11), $q_t \geq \frac{e_t}{b}$. Since, for any $s < t$, we have $m^o_s > 0$, we have, using (23), that $e_{s+1}\chi \geq e_s$ and thus, using (23), $e_2 \geq \frac{1}{\chi}e_1 = \frac{1}{\chi}e_{T+1} > 1$ so that $e_t > 1$. Then $\mu^n_t = m^n_t + \Pi_{t-1}$ so that $m^n_{t+1} = 0$, in turn implying that $p_t = 0$ from Lemma 1, a contradiction. Thus, we either have $m^o_{T+1} > 0$ or $m^o_{T+1} = 0$ and $e_{T+1}\chi \leq 1$.

Since $s^n_T = 0$, using (28), we have $e_{T+1}\chi \geq k$. Then, since $e_{T+1}\chi \leq 1$, **a** similar argument as when $s^n_T > 0$ in (A.47) above establishes that

$$m^n_{T+2} > ks^n_T \left(m^n_{T+1} + \Pi_T\right). \tag{A.48}$$

From the above cases, using (32),

$$m^n_{T+2} - \left(ks^n_T m^n_{T+1} + \Pi_T\right) = kn^n_{T+1} - \mu^n_{T+1} > 0 \tag{A.49}$$

and hence, since $\mu^n_{T+1} \geq 0$, it follows that $n^n_{T+1} > 0$.

Step 4. Induction.

By induction we have $n^n_t > 0$ for all $t > 1$, contradicting cyclicality.■

Proof of Lemma 3.

Preliminaries

Note first that we cannot have $\mu^n_1 > 0$, implying that $q_1 = \frac{1}{b}$, in turn implying that $n^n_1 = 0$, since this violates cyclicality.

From money transition (32), we have, except when $t = 1$,

$$m^n_{t+1} = m^n_t + p_{t-1}g - \mu^n_t \iff m^n_{t+1} = m^n_t + \left(m^n_t + e_{t-1}\left(\chi m^o_{t-1} - \mu^o_{t-1}\right)\right)\frac{g}{c} - \mu^n_t \tag{A.50}$$

and hence

$$\left(1 + \frac{g}{c}\right) m^n_t = m^n_{t+1} - e_{t-1}\left(\chi m^o_{t-1} - \mu^o_{t-1}\right)\frac{g}{c} - \mu^n_t. \tag{A.51}$$

Note that, if $m_1^n = 0$ then, if $m_1^o = 0$ we have, using Lemma 1, $p_T = 0$, a contradiction. If $m_1^o > 0$ then, using money transition (33), $m_{T+1}^o = \chi^T m_1^o$, contradicting cyclicality. Using Lemma 1 gives

$$\begin{aligned}
\frac{p_t}{p_{t-1}} &= \frac{m_{t+1}^n + e_t\left(\chi m_t^o - \mu_t^o\right)}{m_t^n + e_{t-1}\left(\chi m_{t-1}^o - \mu_{t-1}^o\right)} = \frac{\left(1+\frac{g}{c}\right) m_t^n + e_{t-1}\left(\chi m_{t-1}^o - \mu_{t-1}^o\right)\frac{g}{c} - \mu_t^n + e_t\left(\chi m_t^o - \mu_t^o\right)}{m_t^n + e_{t-1}\left(\chi m_{t-1}^o - \mu_{t-1}^o\right)} \\
&= \frac{\left(1+\frac{g}{c}\right) m_t^n + \left(1+\frac{g}{c}\right) e_{t-1}\left(\chi m_{t-1}^o - \mu_{t-1}^o\right) - \left(1+\frac{g}{c}\right) e_{t-1}\left(\chi m_{t-1}^o - \mu_{t-1}^o\right)}{m_t^n + e_{t-1}\left(\chi m_{t-1}^o - \mu_{t-1}^o\right)} \\
&\quad + \frac{e_{t-1}\left(\chi m_{t-1}^o - \mu_{t-1}^o\right)\frac{g}{c} - \mu_t^n + e_t\left(\chi m_t^o - \mu_t^o\right)}{m_t^n + e_{t-1}\left(\chi m_{t-1}^o - \mu_{t-1}^o\right)} \\
&= \left(1+\frac{g}{c}\right) + \frac{e_t\left(\chi m_t^o - \mu_t^o\right) - e_{t-1}\left(\chi m_{t-1}^o - \mu_{t-1}^o\right) - \mu_t^n}{m_t^n + e_{t-1}\left(\chi m_{t-1}^o - \mu_{t-1}^o\right)}.
\end{aligned}$$

Then, if $m_t^o > 0$ and $m_t^n > 0$ implies that $e_t\chi = e_{t-1}$, and that $m_t^o = \chi m_{t-1}^o - \mu_{t-1}^o$,

$$\frac{p_t}{p_{t-1}} = \frac{\xi}{\xi - g} - \frac{e_t\mu_t^o + \mu_t^n}{m_t^n + e_{t-1}\left(\chi m_{t-1}^o - \mu_{t-1}^o\right)}. \tag{A.52}$$

If $m_t^o = 0$ and $m_t^n > 0$ then, using $\chi m_{t-1}^o - \mu_{t-1}^o = 0$ establishes the same expression. If $m_t^o > 0$ and $m_t^n = 0$ then, since $m_t^n = 0$ requires $\mu_{t-1}^n > 0$, we have $q_{t-1} = \frac{1}{b}$ and $\frac{e_{t-1}}{b} \geq q_{t-1}$. Using $e_t\chi \geq e_{t-1}$ from (23), we have $\frac{e_s}{b} > q_s$ for $s \geq t$ and thus $\mu_s^o = 0$ and $m_{t+1}^o > 0$ and thus $m_{T+1}^o > 0$. Then, since $e_s \geq \chi^{-(s-t+1)} e_{t-1} > 1$ from Lemma 5, $e_T = 1$, a contradiction. Thus, $\frac{p_t}{p_{t-1}} \leq \frac{\xi}{\xi - g}$ where the inequality is strict whenever $e_t\mu_t^o + \mu_t^n > 0$. Hence,

$$\frac{p_{t-1}}{p_t} \geq 1 - \frac{g}{\xi} \tag{A.53}$$

where the inequality is strict whenever $e_t\mu_t^o + \mu_t^n > 0$.

Step 1. Jewelry prices.

The Euler equations (21) for periods $1, \ldots, T$ in the general case are

$$\begin{aligned}
\frac{q_1}{p_1} &= \beta\frac{q_2}{p_2} + \frac{bv'(d_2)}{\xi u'(\xi - g)} \\
\frac{q_2}{p_2} &= \beta\frac{q_3}{p_3} + \frac{bv'(d_3)}{\xi u'(\xi - g)} \\
&\vdots \\
\frac{q_{T-1}}{p_{T-1}} &= \beta\frac{q_T}{p_T} + \frac{bv'(d_T)}{\xi u'(\xi - g)} \\
\frac{q_T}{p_T} &= \beta\frac{q_1}{p_1} + \frac{bv'(d_1)}{\xi u'(\xi - g)}.
\end{aligned} \tag{A.54}$$

Since, using that we from Lemma 2 have $n_t^n = 0$ for $t = 2, \ldots, T$, we have $d_2 \leq d_3 \leq \ldots \leq d_T \leq d_1$ and thus, using the properties of v,

$$\frac{q_1}{p_1} - \beta \frac{q_2}{p_2} \geq \frac{q_2}{p_2} - \beta \frac{q_3}{p_3} \geq \ldots \geq \frac{q_T}{p_T} - \beta \frac{q_1}{p_1}. \tag{A.55}$$

In matrix form, (A.54) is

$$\begin{pmatrix} 1 & -\beta & 0 & & & \\ 0 & 1 & -\beta & 0 & & \\ & & & & & \\ & & 0 & 1 & -\beta \\ -\beta & 0 & & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{q_1}{p_1} \\ \\ \\ \\ \frac{q_T}{p_T} \end{pmatrix} = \begin{pmatrix} y_1 \\ \\ \\ \\ y_T \end{pmatrix}, \tag{A.56}$$

where $y_i = \frac{bv'(d_{i+1})}{\xi u'(\xi - g)}$ for $i = 1, \ldots, T-1$ and $y_T = \frac{bv'(d_1)}{\xi u'(\xi - g)}$. Note that, using $d_2 \leq d_3 \leq \ldots \leq d_T \leq d_1$ and the properties of v, we have

$$y_1 \geq y_2 \geq \ldots \geq y_T. \tag{A.57}$$

The inverse of the matrix in (A.56) is

$$\frac{1}{1-\beta^T} \begin{pmatrix} 1 & \beta & \beta^2 & & \beta^{T-1} \\ \beta^{T-1} & 1 & \beta & & \beta^{T-2} \\ & & & & \\ \beta^2 & & \beta^{T-1} & 1 & \beta \\ \beta & \beta^2 & & \beta^{T-1} & 1 \end{pmatrix}. \tag{A.58}$$

Then, using (A.57),

$$\begin{aligned} \frac{q_1}{p_1} - \frac{q_T}{p_T} &= \frac{(1-\beta) y_1 + \beta (1-\beta) y_2 + \ldots + (1-\beta) \beta^{T-2} y_{T-1} + \left(\beta^{T-1} - 1\right) y_T}{1 - \beta^T} \qquad \text{(A.59)} \\ &\geq \frac{(1-\beta) + \beta (1-\beta) + \ldots + (1-\beta) \beta^{T-2} + \left(\beta^{T-1} - 1\right)}{1 - \beta^T} y_T = 0, \end{aligned}$$

with strict inequality if $y_t > y_T$ for some t. Note that, if $y_s = y_t$ for all s, t, then, from (A.56), $\frac{q_s}{p_s} = \frac{q_t}{p_t}$ for all s, t.

Moreover, from the Euler equations (A.55) for periods $T-1$ and T,

$$\frac{q_T}{p_T} - \beta \frac{q_1}{p_1} \leq \frac{q_{T-1}}{p_{T-1}} - \beta \frac{q_T}{p_T} \iff \frac{q_{T-1}}{p_{T-1}} \geq \frac{q_T}{p_T} - \beta \left(\frac{q_1}{p_1} - \frac{q_T}{p_T} \right). \tag{A.60}$$

For remaining time periods, using (A.57),

$$\frac{q_{T-j}}{p_{T-j}} \geq \frac{q_{T-j+1}}{p_{T-j+1}} - \beta\left(\frac{q_{T-j+2}}{p_{T-j+2}} - \frac{q_{T-j+1}}{p_{T-j+1}}\right) \tag{A.61}$$

again with equality if $y_s = y_t$ for all s, t.

Furthermore, if $\mu_t^o = \mu_t^n = 0$ for $T > t \geq s \geq 2$ then $d_s = \ldots = d_1$ so that

$$\frac{q_s}{p_s} - \beta\frac{q_{s+1}}{p_{s+1}} = \ldots = \frac{q_T}{p_T} - \beta\frac{q_1}{p_1}. \tag{A.62}$$

Then, using (A.59), expression (A.61) holds with equality so that, for $j \geq 0$,

$$\frac{q_t}{p_t} = \frac{q_{t+1}}{p_{t+1}} - \beta\left(\frac{q_{t+2}}{p_{t+2}} - \frac{q_{t+1}}{p_{t+1}}\right). \tag{A.63}$$

By (A.59) and using induction, we have, if $y_T < y_1$**,** and $\mu_r^n = \mu_r^o = 0$ for $r \geq t$,

$$\frac{q_t}{p_t} < \frac{q_{t+1}}{p_{t+1}}. \tag{A.64}$$

The above expression holds with equality if $y_T = y_1$.

Step 2. Showing $\mu_t^n = 0$ for all t.

If $n_1^n = 0$ cyclicality requires that $\mu_t^n = 0$ and $\mu_t^o = 0$ for all t.

Suppose now that $n_1^n > 0$. From (5) and Lemma 4, we have $q_1 = \frac{k}{b}$ and, from (7), $\mu_1^n = 0$.

Case 1. $m_t^o = 0$ for all t. Then, $d_T < d_1$ and $y_1 > y_T$ and thus $\frac{q_1}{p_1} > \frac{q_T}{p_T}$. Then, if only new coins are held, using the Cash in Advance constraints (14) and money transition (32) and that $m_t^o = 0$ for all t implies that $s_T^n = 1$,

$$\begin{aligned} p_T c &= m_1^n \\ p_1 c &= m_2^n = k\left(m_1^n + p_T g\right) + k n_1^n. \end{aligned} \tag{A.65}$$

Then

$$m_2^n = k\left(\frac{\xi}{\xi - g}m_1^n + n_1^n\right) > k\frac{\xi}{\xi - g}m_1^n. \tag{A.66}$$

and thus, using $q_1 = \frac{k}{b}$ and (A.59),

$$q_1\frac{m_1^n}{k\frac{\xi}{\xi-g}m_1^n} > q_1\frac{p_T}{p_1} > q_T \iff q_T < \left(1 - \frac{g}{\xi}\right)\frac{1}{b}. \tag{A.67}$$

Then, using expression (7), $\mu_T^n = 0$ and $d_T = d_1$ so that, using (14), (32) and, from (A.53), $p_{T-1} = \left(1 - \frac{g}{\xi}\right) p_T$,

$$m_1^n \left(1 - \frac{g}{\xi}\right) = m_T^n. \tag{A.68}$$

Then, using (A.64) and (A.53),

$$q_{T-1} < q_T \frac{p_{T-1}}{p_T} = \left(1 - \frac{g}{\xi}\right) q_T < \left(1 - \frac{g}{\xi}\right)^2 \frac{1}{b}. \tag{A.69}$$

Hence, from (7) - (9), $\mu_{T-1}^n = 0$. Using (A.64) and Lemma 1 repeatedly, we have

$$q_{T-j} < \left(1 - \frac{g}{\xi}\right)^{j+1} \frac{1}{b} \tag{A.70}$$

and $\mu_{T-j}^n = 0$. Thus, there is no melting for $t \geq 2$, contradicting cyclicality.

Case 2. $m_t^o > 0$ for some t. Since $m_t^o > 0$ for some t, we have $s_T^n < 1$ and, if $m_1^o > 0$ by Lemma 5, $e_T = 1$. If $m_1^o > 0$ and $m_1^n = 0$ then, using money transition 33, $m_{T+1}^o = \chi^T m_1^o$, contradicting cyclicality. Thus, $s_T^n \geq 0$, $m_t^o > 0$ and $m_{T+1}^n = m_1^n > 0$ gives, using (27) and (28), $e_1 \chi \geq k$. Then $\frac{e_1}{b} > \frac{k}{b} = q_1$ so that, from (10), $\mu_1^o = 0$. If $m_1^o = 0$ then $e_T \geq 1$ by (26).

We have, using (32) and Lemma 1,

$$\begin{aligned} p_T c &= m_1^n + e_T \left(\chi m_T^o - \mu_T^o\right) = m_1^n + m_1^o \\ p_1 c &= m_2^n + e_1 \chi \left(m_1^o + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right). \end{aligned} \tag{A.71}$$

Since $s_T^n < 1$ and using (27) - (28), we have $e_1 \chi \geq k$ and thus, using (32),

$$\begin{aligned} p_1 c &\geq k s_T^n \left(m_1^n + \Pi_T\right) + k\left(m_1^o + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right) + k n_1^n \\ &= k\left(m_1^n + \Pi_T\right) + k m_1^o + k n_1^n > k m_1^n + k m_1^o \end{aligned} \tag{A.72}$$

so that, using (A.64) and that, using Lemma 4, $n_1^n > 0$ implies that $q_1 = \frac{k}{b}$, we have

$$q_1 \frac{m_1^n + m_1^o}{k\left(m_1^o + m_1^n\right)} > q_1 \frac{p_T}{p_1} > q_T \iff q_T < \frac{1}{b}, \tag{A.73}$$

implying $\mu_T^n = 0$. Moreover, since $q_T < \frac{1}{b} \leq \frac{e_T}{b}$ we have $\mu_T^o = 0$ and thus $d_T = d_1$.

For $T = 2$ the conclusion follows immediately, since $\mu_T^n = \mu_T^o = 0$ requires $n_1^n = 0$ for cyclicality to be satisfied.

Suppose that $T \geq 3$.

First, suppose that $\mu_t^o = 0$ for all t. Then, using (A.53) and (A.64), we have

$$q_t < \left(1 - \frac{g}{\xi}\right)^{T-t} \frac{1}{b} \tag{A.74}$$

so that $q_t < \frac{1}{b}$, inductively establishing $\mu_t^n = 0$ for all t.

Second, suppose that $\mu_{\hat{t}-1}^o > 0$ for some $\hat{t}$ and $\mu_s^o = 0$ for $s \geq \hat{t}$.

If $\hat{t} - 1 = T$ then, proceeding as above using (A.59) establishes $q_T < \left(1 - \frac{g}{\xi}\right)\frac{1}{b}$ so that $\mu_T^n = 0$. Moreover, if $m_1^o = 0$ then, from the argument at the start of Case 2, $e_T \geq 1$, and, if $m_1^o > 0$ then, from Lemma 5, $e_T = 1$. We then have $\frac{e_T}{b} > q_T$ so that, from (10), $\mu_T^o = 0$, a contradiction.

Now suppose that $\hat{t} - 1 < T$. We first prove that $\mu_t^n = 0$ for $t \geq \hat{t}$.

From $\mu_{\hat{t}}^o > 0$ we have, using (11), that

$$\frac{e_{\hat{t}}}{b} \leq q_{\hat{t}}. \tag{A.75}$$

Moreover, since $\mu_s^o = 0$ for $s > \hat{t}$, we have, using (A.53) and (A.64), **for** $t \geq \hat{t}$,

$$q_t < \left(1 - \frac{g}{\xi}\right)^{T-t} \frac{1}{b} \tag{A.76}$$

so that $q_t < \frac{1}{b}$, inductively establishing $\mu_t^n = 0$ for $t \geq \hat{t}$.

We now show that $\mu_t^n = 0$ for all t. Suppose first that $\mu_{T-1}^o = 0$.

For all $t > \hat{t}$ we have $\mu_t^o = 0$ so that $d_{t+1} = d_{t+2} = \ldots = d_{T+1}$. Then, using (A.63), we have

$$\frac{q_t}{p_t} - \frac{q_{t+1}}{p_{t+1}} = \beta\left(\frac{q_{t+1}}{p_{t+1}} - \frac{q_{t+2}}{p_{t+2}}\right) = \beta^s\left(\frac{q_{t+s}}{p_{t+s}} - \frac{q_{t+s+1}}{p_{t+s+1}}\right) = \ldots = \beta^{T-t}\left(\frac{q_T}{p_T} - \frac{q_{T+1}}{p_{T+1}}\right). \tag{A.77}$$

Moreover,

$$\frac{q_{t+1}}{p_{t+1}} - \frac{q_{t+2}}{p_{t+2}} = \beta^{T-t+1}\left(\frac{q_T}{p_T} - \frac{q_{T+1}}{p_{T+1}}\right). \tag{A.78}$$

Then

$$\frac{1}{p_t}\left(q_t - q_{t+1}\frac{p_t}{p_{t+1}}\right) = \frac{1}{p_{t+1}}\beta\left(q_{t+1} - q_{t+2}\frac{p_{t+1}}{p_{t+2}}\right) \tag{A.79}$$

or, using (A.53)

$$q_t - \left(1 - \frac{g}{\xi}\right) q_{t+1} = \left(1 - \frac{g}{\xi}\right)\beta\left(q_{t+1} - q_{t+2}\left(1 - \frac{g}{\xi}\right)\right). \tag{A.80}$$

Then, computing the solution gives, using Sydsæter (1990, p. 294),

$$m_{1,2} = \begin{cases} \left(1 - \frac{g}{\xi}\right)^{-1} \\ \frac{1}{\beta}\left(1 - \frac{g}{\xi}\right)^{-1} \end{cases} \tag{A.81}$$

where

$$q_t = C_1 m_1^{t-\hat{t}} + C_2 m_2^{t-\hat{t}}. \tag{A.82}$$

We have two boundary conditions

$$\begin{aligned} q_{\hat{t}} &= C_1 + C_2 \\ q_T &= C_1 m_1^{T-\hat{t}} + C_2 m_2^{T-\hat{t}} \end{aligned} \tag{A.83}$$

so that, using $C_1 = q_{\hat{t}} - C_2$,

$$q_T = (q_{\hat{t}} - C_2)\left(1 - \frac{g}{\xi}\right)^{-(T-\hat{t})} + C_2\left(\beta\left(1 - \frac{g}{\xi}\right)\right)^{-(T-\hat{t})} \tag{A.84}$$

or

$$C_2 = \frac{q_T\left(1 - \frac{g}{\xi}\right)^{(T-\hat{t})} - q_{\hat{t}}}{\left((\beta)^{-(T-\hat{t})} - 1\right)}. \tag{A.85}$$

Then

$$C_1 = \frac{q_{\hat{t}}\,(\beta)^{-(T-\hat{t})} - q_T\left(1 - \frac{g}{\xi}\right)^{(T-\hat{t})}}{(\beta)^{-(T-\hat{t})} - 1}. \tag{A.86}$$

We also have, using that $d_{\hat{t}} < d_{\hat{t}+1}$, (A.61) and that $\frac{p_{\hat{t}-1}}{p_{\hat{t}}} > 1 - \frac{g}{\xi}$ from (A.53),

$$q_{\hat{t}-1} > \left(1 - \frac{g}{\xi}\right)\left((1+\beta)\,q_{\hat{t}} - q_{\hat{t}+1}\beta\left(1 - \frac{g}{\xi}\right)\right). \tag{A.87}$$

Note that, for $s \geq \hat{t}$ and using (A.82),

$$\frac{q_{s+1}}{q_s} - \frac{q_s}{q_{s-1}} = C_1 C_2 \left(\frac{1}{\beta}\right)^{s+1-\hat{t}} \frac{(1-\beta)^2}{\left(C_1 + C_2\left(\frac{1}{\beta}\right)^{s-\hat{t}}\right)\left(C_1 + C_2\left(\frac{1}{\beta}\right)^{s-1-\hat{t}}\right)}\left(1 - \frac{g}{\xi}\right)^{-1}. \tag{A.88}$$

Thus $\frac{q_{s+1}}{q_s} > \frac{q_s}{q_{s-1}}$. Moreover,

$$\frac{q_T}{q_t} = \frac{q_T}{q_{T-1}} \cdot \ldots \cdot \frac{q_{t+1}}{q_t}. \tag{A.89}$$

Since $\frac{q_{s+1}}{q_s} > \frac{q_s}{q_{s-1}}$, we have

$$\frac{q_T}{q_t} > \left(\frac{q_{t+1}}{q_t}\right)^{T-t} \tag{A.90}$$

Moreover, from (A.73)

$$\frac{1}{b} > q_T > \left(\frac{q_{t+1}}{q_t}\right)^{T-t} q_t \tag{A.91}$$

and $q_{\hat{t}} \geq \frac{e_{\hat{t}}}{b} \geq \frac{\chi^{T-\hat{t}}}{b}$, where the last equality follows from $m_t^n > 0$ for $t > \hat{t}$, $e_T \geq 1$ and $\frac{e_{t+1}\chi}{e_t} \leq 1$ (25). Then

$$\left(\frac{q_{\hat{t}+1}}{q_{\hat{t}}}\right)^{T-\hat{t}} q_{\hat{t}} \geq \left(\frac{q_{\hat{t}+1}}{q_{\hat{t}}}\right)^{T-\hat{t}} \frac{\chi^{T-\hat{t}}}{b} \tag{A.92}$$

so that, using (A.91),

$$\left(\frac{q_{\hat{t}+1}}{q_{\hat{t}}}\right)^{T-\hat{t}} \chi^{T-\hat{t}} < 1, \tag{A.93}$$

implying that $\chi < \left(\frac{q_{\hat{t}+1}}{q_{\hat{t}}}\right)^{-1}$. Let the left-hand side of (A.87) be denoted $q_{\hat{t}-1}^*$. We have, using (A.82),

$$q_{\hat{t}-1}^* = C_1 m_1^{-1} + C_2 m_2^{-1} \tag{A.94}$$

so that, proceeding as in (A.88) above

$$\frac{q_{\hat{t}}}{q_{\hat{t}-1}^*} < \frac{q_{\hat{t}+1}}{q_{\hat{t}}}. \tag{A.95}$$

Then, using $\chi < \left(\frac{q_{\hat{t}+1}}{q_{\hat{t}}}\right)^{-1}$,

$$q_{\hat{t}-1}^* \frac{q_{\hat{t}+1}}{q_{\hat{t}}} > q_{\hat{t}} > \chi^{T-\hat{t}} \iff q_{\hat{t}-1}^* > \chi^{T-\hat{t}} \left(\frac{q_{\hat{t}+1}}{q_{\hat{t}}}\right)^{-1} > \chi^{T-\hat{t}+1}, \tag{A.96}$$

and thus $q_{\hat{t}-1} > \chi^{T-\hat{t}+1}$ implying that $\mu_{\hat{t}-1}^o = \chi m_{\hat{t}-1}^o$, a contradiction.

Hence, $\mu_{\hat{t}}^o = 0$. Moreover, since $q_{\hat{t}} < \left(1 - \frac{g}{\xi}\right)^{T-\hat{t}} \frac{1}{b}$, we have $\mu_{\hat{t}}^n = 0$. Induction then establishes $\mu_s^n = \mu_s^n = 0$ for all s.

Consider now the case when $\mu_{T-1}^o > 0$. Following similar arguments as in (A.76), we have $\mu_{T-1}^n = 0$. Moreover, using that we require that $m_1^n > 0$ for old coins to be held so that, from (25), we have $e_T\chi \leq e_{T-1}$ and thus, using that by Lemma 5 and (26), $e_T \geq 1$, we have $\frac{\chi}{b} \leq \frac{e_{T-1}}{b} \leq q_{T-1} < \left(1 - \frac{g}{\xi}\right)\frac{1}{b}$ where the last inequality follows from (A.76), using $\mu_T^o = 0$. Thus, $\chi < 1 - \frac{g}{\xi}$. Moreover, since $\frac{e_T}{b} \geq \frac{1}{b} \geq q_T$ and $q_{T-1} \geq \frac{\chi}{b}$ we have $\frac{q_{T-1}}{q_T} \geq \chi$. Consider period $T-2$. Then, using that the inequality in (A.61) is strict and that, from (A.53), $\frac{p_{T-2}}{p_{T-1}} > 1 - \frac{g}{\xi}$ and $\frac{p_{T-1}}{p_T} = 1 - \frac{g}{\xi}$, we have

$$\begin{aligned} q_{T-2} &> \frac{p_{T-2}}{p_{T-1}}\left((1+\beta) q_{T-1} - \beta q_T \frac{p_{T-1}}{p_T}\right) \\ &> \left(1 - \frac{g}{\xi}\right)\left((1+\beta) q_{T-1} - \beta q_T \left(1 - \frac{g}{\xi}\right)\right). \end{aligned} \tag{A.97}$$

Define q_{T-2}^* as the value of q_{T-2} corresponding to $\mu_{T-1}^o = 0$ so that $\frac{p_{T-2}}{p_{T-1}} = 1 - \frac{g}{\xi}$ given the jewelry prices q_{T-1} and q_T. Thus,

$$q_{T-2}^* = \left(1 - \frac{g}{\xi}\right)\left((1+\beta) q_{T-1} - \beta q_T \left(1 - \frac{g}{\xi}\right)\right). \tag{A.98}$$

This expression defines a difference equation. Proceeding as in (A.88) and (A.95) above, using $\frac{q_{T-1}}{q_T} > \chi$, establishes that $q^*_{T-2} > q_{T-1}\frac{q_{T-1}}{q_T} > \chi q_{T-1}$. Then, using that we from (23) have $e_{T-2} = \chi e_{T-1}$, we get

$$q_{T-2} > \chi q_{T-1} \geq \chi \frac{e_{T-1}}{b} = \frac{e_{T-2}}{b} \tag{A.99}$$

so that $\mu^o_{T-2} = \chi m^o_{T-2}$, a contradiction.

Once more, induction then establishes $\mu^n_s = \mu^n_s = 0$ for all s. ■

Proof of Theorem 1.

From Lemma 3, $n^n_t = 0$, $\mu^n_t = 0$ and $\mu^o_t = 0$ for all t. Thus, $d_t = \bar{d}$ for all t and thus, from (A.59) and that (A.64) holds with equality when $y_1 = y_T$, we have $\frac{q_s}{p_s} = \frac{q_t}{p_t}$ for all s, t.

Case 1. $m^o_{t+1} = 0$ for all t.

Step 1. Since $s^n_T = 1$ we have, from (25), (26) and (29), that $e_1\chi \leq e_T k$, $e_{t+1}\chi \leq e_t$ and $e_1\chi \leq k$ and hence

$$e_1\chi \geq e_2\chi^2 \geq \ldots \geq e_T\chi^T \geq \frac{e_1\chi}{k}\chi^T \iff k \geq \chi^T. \tag{A.100}$$

Step 2. Prices.

We have, using Lemma 1, (32) and that (A.53) holds with equality,

$$m^n_{t+1} = m^n_1 \left(1 - \frac{g}{\xi}\right)^{T-t+1} \tag{A.101}$$

and, using (30)

$$(1-k)\left(m^n_1 + \Pi_T\right) = \sum_{t=1}^{T} p_t g = \sum_{t=1}^{T} m^n_{t+1}\frac{g}{c} = \frac{g}{c}\sum_{t=2}^{T+1}\left(1 - \frac{g}{\xi}\right)^{T-t+1} m^n_1 \tag{A.102}$$

so that, using $\Pi_T = p_T g = m^n_1 \frac{g}{c}$,

$$(1-k)\frac{\xi}{\xi - g} = \frac{g}{\xi - g}\sum_{t=2}^{T+1}\left(1 - \frac{g}{\xi}\right)^{T-t+1} = \frac{\xi}{\xi - g}\left(1 - \left(1 - \frac{g}{\xi}\right)^T\right) \tag{A.103}$$

and hence

$$k = \left(1 - \frac{g}{\xi}\right)^T \iff \left(1 - \frac{g}{\xi}\right) = k^{\frac{1}{T}}. \tag{A.104}$$

Then, using (A.101)

$$\left(1 - \frac{g}{\xi}\right) m^n_{t+1} = m^n_t \tag{A.105}$$

so that, for $t = 2, \ldots, T$,

$$k^{\frac{1}{T}} p_t = p_{t-1} \tag{A.106}$$

and thus, using $\frac{q_t}{p_t} = \frac{q_{t-1}}{p_{t-1}}$,

$$k^{\frac{1}{T}} q_t = q_{t-1}, \tag{A.107}$$

and hence $q_1 = k^{\frac{T-1}{T}} q_T$. Since $q_T \leq \frac{1}{b}$ using Lemma 4, any $q_1 \in [\frac{k}{b}, \frac{k^{\frac{T-1}{T}}}{b}]$ is possible, implying that $q_T \in [\frac{k^{\frac{1}{T}}}{b}, \frac{1}{b}]$.

Step 3. Finding m_1^n.

Using that $c = k^{\frac{1}{T}}\xi$ from (A.104), $\frac{q_1}{p_1} = \frac{q_T}{p_T}$ from (A.106) and (A.107), the Cash in Advance constraint $p_T c = m_1^n$ and the silver market clearing condition $d_1 = \ldots = d_2 = S - b\left(m_1^n + \Pi_T\right) = S - b\frac{\xi}{\xi - g} m_1^n$, we can write the Euler equation (21) as

$$q_T \xi = \frac{1}{1-\beta} \frac{1}{u'\left(k^{\frac{1}{T}}\xi\right)} k^{-\frac{1}{T}} m_1^n v'\left(S - bk^{-\frac{1}{T}} m_1^n\right). \tag{A.108}$$

The right-hand side is continuous and increasing in m_1^n due to the concavity of v. Moreover, $\lim_{m_1^n \to 0} k^{-\frac{1}{T}} m_1^n v'\left(S - bk^{-\frac{1}{T}} m_1^n\right) = 0$ and $\lim_{m_1^n \to k^{\frac{1}{T}} \frac{S}{b}} k^{-\frac{1}{T}} m_1^n v'\left(S - bk^{-\frac{1}{T}} m_1^n\right) = \infty$. Then, for each $q_T \in [\frac{k^{\frac{1}{T}}}{b}, \frac{1}{b}]$, there is a unique m_1^n that satisfies the Euler equation. Furthermore, by differentiating the Euler equation, we have $\frac{dq_T}{dm_1^n} > 0$.[27]

Case 2. $m_{t+1}^o > 0$ for all t.

Step 1. Exchange rates.

Using that $\mu_t^o = 0$ from Lemma 3 and that $e_t \chi = e_{t-1}$ from (23) and (25) and $e_T = 1$ from Lemma 5, we have

$$e_t = \chi^{T-t}. \tag{A.109}$$

Moreover, if $s_T^n \in (0,1)$ then $e_{T+1}\chi = k$. Combining this and (A.109) establishes that $\chi^T = k$ whenever $s_T^n \in (0,1)$.

If $s_T^n = 0$ then $e_{T+1}\chi \geq k$ so that

$$\chi^T \geq k. \tag{A.110}$$

Step 2. Showing $\chi = 1 - \frac{g}{\xi}$.

Note that, using $\mu_t^o = 0$ for all t, we have $m_t^o = \chi m_{t-1}^o$. Then, using that we from (33) have $m_2^o = \chi\left(m_1^o + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right)$ and $m_t^o = \chi m_{t-1}^o$ we have

$$m_{t+1}^o = \chi^t \left(m_1^o + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right), \tag{A.111}$$

[27] Noting that consumption is independent of q_T and that jewelry holdings are decreasing in money holdings, the equilibrium yielding the highest welfare then has $q_T = \frac{k^{\frac{1}{T}}}{b}$.

by repeatedly using $m_t^o = \chi m_{t-1}^o$, and thus, setting $t = T$ above,

$$m_1^o = \frac{\chi^T}{1-\chi^T}\left(1-s_T^n\right)\left(m_1^n + \Pi_T\right) \tag{A.112}$$

and hence, using (A.111),

$$m_{t+1}^o = \frac{\chi^t}{1-\chi^T}\left(1-s_T^n\right)\left(m_1^n + \Pi_T\right). \tag{A.113}$$

Government revenues during a cycle are, in terms of new coins, noting that old confiscated coins are re-minted costlessly by the lord as new coins and using (A.113),

$$\begin{aligned} & (1-k)\, s_T^n \left(m_1^n + \Pi_T\right) + (1-\chi)\left(m_1^o + \left(1-s_T^n\right)\left(m_1^n + \Pi_T\right)\right) + \sum_{t=2}^{T}(1-\chi)\, m_t^o \\ = \;& (1-k)\, s_T^n \left(m_1^n + \Pi_T\right) + \sum_{t=1}^{T}(1-\chi)\frac{\chi^{t-1}}{1-\chi^T}\left(1-s_T^n\right)\left(m_1^n + \Pi_T\right) \\ = \;& (1-k)\, s_T^n \left(m_1^n + \Pi_T\right) + \left(1-s_T^n\right)\left(m_1^n + \Pi_T\right). \end{aligned} \tag{A.114}$$

To find government expenditures, using Lemma 1, we can write, using $e_{t-1} = \chi e_t$ and $m_t^o = \chi m_{t-1}^o$ from (23), (25) and (33),

$$p_t(\xi - g) = m_{t+1}^n + e_t\chi m_t^o = m_{t+1}^n + e_t\chi^t\left(m_1^o + \left(1-s_T^n\right)\left(m_1^n + \Pi_T\right)\right) = m_{t+1}^n + e_1\chi\left(m_1^o + \left(1-s_T^n\right)\left(m_1^n + \Pi_T\right)\right)$$

or

$$p_t\xi = m_{t+1}^n + \Pi_t + e_1\chi\left(m_1^o + \left(1-s_T^n\right)\left(m_1^n + \Pi_T\right)\right) \tag{A.115}$$

and, when $t = T+1$,

$$p_1\xi = m_2^n + \Pi_1 + e_1\left(\chi\left(m_1^o + \left(1-s_T^n\right)\left(m_1^n + \Pi_T\right)\right)\right). \tag{A.116}$$

The expenditures are, using (A.115) and (A.116)

$$\sum_{t=1}^{T} p_t g = \frac{g}{\xi}\left(\sum_{t=1}^{T}\left(m_t^n + \Pi_{t-1}\right) + Te_1\left(\chi\left(m_1^o + \left(1-s_T^n\right)m_1^n\right)\right)\right). \tag{A.117}$$

Using money transition,

$$m_{t+1}^n = m_t^n + p_{t-1}g = m_t^n + \frac{g}{\xi}\left(m_t^n + e_1\chi\left(m_1^o + \left(1-s_T^n\right)\left(m_1^n + \Pi_T\right)\right)\right). \tag{A.118}$$

Rearranging gives or

$$
\begin{aligned}
m_t^n &= \left(1-\frac{g}{\xi}\right) m_{t+1}^n - \frac{g}{\xi} e_1 \chi \left(m_1^o + \left(1-s_T^n\right)\left(m_1^n + \Pi_T\right)\right) \\
&= \left(1-\frac{g}{\xi}\right)^s m_{t+s}^n - \frac{g}{\xi} e_1 \chi \left(m_1^o + \left(1-s_T^n\right)\left(m_1^n + \Pi_T\right)\right) \sum_{v=0}^{s-1} \left(1-\frac{g}{\xi}\right)^v \qquad \text{(A.119)} \\
&= \left(1-\frac{g}{\xi}\right)^{T-t+1} m_1^n - e_1 \chi \left(m_1^o + \left(1-s_T^n\right)\left(m_1^n + \Pi_T\right)\right) \left(1 - \left(1-\frac{g}{\xi}\right)^{T-t+1}\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
\sum_{t=1}^{T} m_t^n &= \sum_{t=2}^{T+1} \left(1-\frac{g}{\xi}\right)^{T-t+1} m_1^n - e_1 \chi \left(m_1^o + \left(1-s_T^n\right)\left(m_1^n + \Pi_T\right)\right) \sum_{t=2}^{T} \left(1 - \left(1-\frac{g}{\xi}\right)^{T-t+1}\right) \\
&= \frac{1-\left(1-\frac{g}{\xi}\right)^T}{\frac{g}{\xi}} m_1^n - (T-1)\, e_1 \chi \left(m_1^o + \left(1-s_T^n\right)\left(m_1^n + \Pi_T\right)\right) \qquad \text{(A.120)} \\
&\quad + \frac{\left(1-\frac{g}{\xi}\right) - \left(1-\frac{g}{\xi}\right)^T}{\frac{g}{\xi}} e_1 \chi \left(m_1^o + \left(1-s_T^n\right)\left(m_1^n + \Pi_T\right)\right).
\end{aligned}
$$

Since expenditures equal revenues, using expressions (A.114), (A.117) and the above expression, we get

$$
(1-k)\, s_T^n \left(m_1^n + \Pi_T\right) + \left(1-s_T^n\right)\left(m_1^n + \Pi_T\right) = \left(1 - \left(1-\frac{g}{\xi}\right)^T\right) \left(m_1^n + e_1 \chi \left(m_1^o + \left(1-s_T^n\right)\left(m_1^n + \Pi_T\right)\right)\right). \qquad \text{(A.121)}
$$

Using that $e_1 m_2^o = e_2 \chi m_2^o = e_2 m_3^o = \ldots = e_T m_{T+1}^o$ and that, from Lemma 5, $e_T = 1$, we have

$$
(1-k)\, s_T^n \left(m_1^n + \Pi_T\right) + \left(1-s_T^n\right)\left(m_1^n + \Pi_T\right) = \left(1 - \left(1-\frac{g}{\xi}\right)^T\right) \left(m_1^n + m_{T+1}^o\right). \qquad \text{(A.122)}
$$

From (A.113), we have

$$
(1-k)\, s_T^n \left(m_1^n + \Pi_T\right) + \left(1-s_T^n\right)\left(m_1^n + \Pi_T\right) = \left(1 - \left(1-\frac{g}{\xi}\right)^T\right) \left(\left(m_1^n + \Pi_T\right) + \frac{\chi^T}{1-\chi^T}\left(1-s_T^n\right)\left(m_1^n + \Pi_T\right)\right). \qquad \text{(A.123)}
$$

Suppose that $s_n^T = 0$. Then

$$
1 = \left(1 - \left(1-\frac{g}{\xi}\right)^T\right) \left(\frac{1}{1-\chi^T}\right) \iff 1-\chi^T = 1 - \left(1-\frac{g}{\xi}\right)^T \qquad \text{(A.124)}
$$

so that

$$
\chi = 1 - \frac{g}{\xi}. \qquad \text{(A.125)}
$$

Suppose that $s_n^T > 0$. Then, using (27), $e_{T+1}\chi = k$ and, using (23) and (25), $e_{t-1} = \chi e_t$. Moreover, from Lemma 5 and (A.112), $e_T = 1$ so that $k = \chi^T$. Then

$$\begin{aligned} & \left(1-\chi^T\right) s_T^n \left(m_1^n + \Pi_T\right) + \left(1 - s_T^n\right) m_1^n \\ = & \left(1 - \left(1 - \frac{g}{\xi}\right)^T\right) \left(\left(1 - s_T^n + s_T^n\right)\left(m_1^n + \Pi_T\right) + \frac{\chi^T}{1-\chi^T}\left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right) \quad \text{(A.126)} \\ = & \left(1 - \left(1 - \frac{g}{\xi}\right)^T\right) \left(s_T^n \left(m_1^n + \Pi_T\right) + \frac{1}{1-\chi^T}\left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right). \end{aligned}$$

Once more, we get

$$\chi = 1 - \frac{g}{\xi}. \quad \text{(A.127)}$$

Step 3. Prices.

From money transition (32), for $t = 2, \ldots, T$,

$$\chi m_{t+1}^n = m_t^n \quad \text{(A.128)}$$

so that

$$\chi p_t = p_{t-1} \quad \text{(A.129)}$$

and thus

$$\chi q_t = q_{t-1}, \quad \text{(A.130)}$$

and hence, $q_1 = \chi^T q_T$. Since, using Lemma 4, $q_T \leq \frac{1}{b}$ any $q_1 \in [\frac{k}{b}, \frac{\chi^T}{b}]$ is possible.

Step 3. Finding m_1^n.

Fix s_T^n. Using Lemma 1, expression (A.113) and that expression (A.112) and Lemma 5 imply $e_T = 1$, we can write, using $\Pi_T = p_T g$,

$$p_T\left(\xi - g\right) = m_1^n + e_T \chi m_T^o = m_1^n + \chi m_T^o = m_1^n + \frac{\chi^T}{1-\chi^T}\left(1 - s_T^n\right)\left(m_1^n + p_T g\right) \quad \text{(A.131)}$$

or, defining

$$Z\left(\chi, s_T^n, \xi\right) = \frac{1 + \frac{\chi^T}{1-\chi^T}\left(1 - s_T^n\right)}{\xi\left(1 - \left(1-\chi\right)\left(1 + \frac{\chi^T}{1-\chi^T}\left(1 - s_T^n\right)\right)\right)} \quad \text{(A.132)}$$

we get

$$p_T = Z\left(\chi, s_T^n, \xi\right) m_1^n. \quad \text{(A.133)}$$

Moreover, using (A.112), we have

$$m_1^n + \Pi_T + m_1^o = \left(1 + \frac{\chi^T}{1-\chi^T}\left(1 - s_T^n\right)\right)\left(1 + Z\left(\chi, s_T^n, \xi\right)\frac{1-\chi}{\xi}\right) m_1^n. \tag{A.134}$$

Using that $c = \chi\xi$ from (35) and (A.127), $\frac{q_1}{p_1} = \frac{q_T}{p_T}$ from (A.129) and (A.130), the Cash in Advance constraint (A.133) and the silver market clearing condition $d_1 = d_2 = S - b\left(m_1^n + m_1^o\right)$ we can, using (A.112) and (A.134), write the Euler equation (21) as,

$$q_T\xi = \frac{1}{1-\beta}\frac{1}{u'\left(\chi\xi\right)}\left(1 + \frac{\chi^T}{1-\chi^T}\left(1 - s_T^n\right)\right)\left(1 + Z\left(\chi, s_T^n, \xi\right)\frac{1-\chi}{\xi}\right) m_1^n \tag{A.135}$$

$$\cdot v'\left(S - b\left(1 + \frac{\chi^T}{1-\chi^T}\left(1 - s_T^n\right)\right)\left(1 + Z\left(\chi, s_T^n, \xi\right)\frac{1-\chi}{\xi}\right) m_1^n\right). \tag{A.136}$$

The right-hand side is continuous and increasing in m_1^n due to concavity of v. Moreover,

$$\lim_{m_1^n \to 0} m_1^n v'\left(S - b\left(1 + \frac{\chi^T}{1-\chi^T}\left(1 - s_T^n\right)\right)\left(1 + Z\left(\chi, s_T^n, \xi\right)\frac{1-\chi}{\xi}\right) m_1^n\right) = 0 \tag{A.137}$$

and

$$\lim_{m_1^n \to \frac{S}{b\left(1+\frac{\chi^T}{1-\chi^T}\left(1-s_T^n\right)\right)\left(1+Z\left(\chi,s_T^n,\xi\right)\frac{1-\chi}{\xi}\right)}} m_1^n v'\left(S - b\left(1 + \frac{\chi^T}{1-\chi^T}\left(1 - s_T^n\right)\right)\left(1 + Z\left(\chi, s_T^n, \xi\right)\frac{1-\chi}{\xi}\right) m_1^n\right) = \infty. \tag{A.138}$$

Then, for each $q_T \in [\frac{k\chi^{-T}}{b}, \frac{1}{b}]$ there is a unique m_1^n that satisfies the Euler equation. Furthermore, by differentiating the Euler equation, we have $\frac{dq_T}{dm_1^n} > 0$. [28]

Suppose that $\chi^T > k$ so that $s_T^n = 0$. Then, using (A.135), m_1^n solves

$$q_T\xi = \frac{1}{1-\beta}\frac{1}{u'\left(\chi\xi\right)}\frac{1}{1-\chi^T}\left(1 + Z\left(\chi, 0, \xi\right)\frac{1-\chi}{\xi}\right) m_1^n \tag{A.139}$$

$$\cdot v'\left(S - b\frac{1}{1-\chi^T}\left(1 + Z\left(\chi, 0, \xi\right)\frac{1-\chi}{\xi}\right) m_1^n\right). \tag{A.140}$$

Suppose that $\chi^T = k$. Then, for any $s_T^n \in [0, 1]$, the equilibrium value of m_1^n solves (A.135). [29] ■

[28] Noting that consumption is independent of q_T and that jewelry holdings are decreasing in money holdings, the equilibrium yielding the highest welfare then has $q_T = \frac{k\chi^{-T}}{b}$.

[29] Pareto optimality now also requires $s_T^n = 1$.