Time Allocation in Friendship Networks*

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Abstract

We analyze stable and efficient investments into links (e.g. friendships) and the resulting network structure when agents are homogenous and endowed with a limited resource (e.g. time) which they can invest into themselves and distribute over all possible links. An agent's utility from a link follows a Cobb-Douglas technology which takes both linked agents' investments as inputs, and utility from self-investment follows a strictly concave function. Nash stable components are either symmetric, "reciprocal", or asymmetric, "non-reciprocal". In a reciprocal equilibrium every agent chooses the same self-investment and two agents' investments into one link match each other. In a non-reciprocal equilibrium the component is bipartite, with a larger set of "high intensity" agents who choose a higher self-investment than a smaller set of "low intensity" agents. Despite her lower overall social time, a "high intensity" agent always invests more into a link than her counterpart, a "low intensity" agent. We find that Nash stable regular networks are reciprocal and Nash stable trees are non-reciprocal. There is no strongly pairwise stable component consisting of more than two agents. The efficient network is reciprocal and features a lower self-investment than a Nash stable reciprocal equilibrium.

Keywords: network, social networks, link-specific investment, budget constraint, interdependence, time, friendship

JEL Classification Codes: D85, L14, Z13, C72

1 Introduction

In this paper, we analyze within the framework of network theory an agent's investment of a limited budget into productive relationships with other agents and into her own personal productive undertakings. The limited budget could be the time with which everyone is endowed and which can be spent on friendships with other persons or which can be kept to oneself for private purposes like work, sports or sleep. Another interpretation is research and development investment out of a limited budget by a firm into joint research projects with other firms and into in-house research. Throughout the following, we set our analysis in the context of friendship networks.

We investigate how much of their limited time utility-maximizing homogenous agents devote to their social network, how much they optimally keep for themselves, and how they distribute their "social time" across their social network. Friendships are productive and we assume that the value of each friendship is produced according to a Cobb-Douglas technology that takes both agents' time investments as inputs. This captures social interdependence: a person's optimal investment into a friendship is dependent on the friend's investment. Time which agents keep for themselves produces a utility for the agent which is strictly concave in "self-investment". We are particularly interested in which time allocations and network architectures are Nash stable, pairwise stable and efficient.

In real-life networks we observe that people, firms or other agents occupy different positions in a network: For example, one child can be an outsider at school, whereas another child is overly popular, takes a central role and receives a lot of attention. The question arises if these different positions result from an ex-ante inherent heterogeneity across agents in their preferences or other characteristics, or if asymmetries in positions could just be an equilibrium result with homogenous agents. In the former case, agents might have (been) intentionally selected into these positions. In the latter case, homogenous agents could end up in asymmetric positions for different reasons, and stay with these because the network is in equilibrium. Then, homogenous agents could have different utility levels in equilibrium. It is unclear if an asymmetric equilibrium with homogenous agents is desirable or efficient. A more symmetric equilibrium could be Pareto-dominant. If so, a social planner could help the network move from an asymmetric equilibrium to a more symmetric Pareto-dominant one. With the following analysis, we give some answers to these research questions. We provide a preview of our results in the next paragraphs.

As to Nash stable networks in which no agent has a strict incentive to unilaterally deviate, we find that a network is Nash stable iff all its components are Nash stable. There are exactly two types of Nash stable components. One is a symmetric, reciprocal Nash stable component in which every agent chooses the same self-investment and social time, and agents match each other's investment into each other. The other is an asymmetric, non-reciprocal Nash stable component which is bipartite with one smaller set of "low intensity" agents (l-agents) and one larger set of "high intensity" agents (h-agents). An l-agent chooses a lower self-investment than an h-agent, always invests less into a friendship than her friend, an h-agent, and has more friends on average than an h-agent. Hence, we obtain symmetric as well as asymmetric equilibria despite ex-ante homogenous agents. For a given component architecture (in terms of link existence) there either exists a reciprocal, a non-reciprocal or no Nash stable time allocation. We derive necessary and sufficient conditions on the component architecture for a Nash stable reciprocal time allocation to exist for this component architecture, and necessary conditions for the non-reciprocal case. From these follows for example that any Nash stable component which contains an agent who has only one friend is non-reciprocal and any Nash stable component that is regular and/or contains an odd length cycle is reciprocal. If there exists a Nash stable time allocation for a given component architecture, the Nash stable time allocation is often not unique. We can find infinitely many other Nash stable time allocations for a Nash stable component which contains an even length cycle because time investments are continuous and can be shifted on the even length cycle.

Comparing the utility levels of agents in Nash stable components reveals that an l-agent obtains the highest utility, an agent in a reciprocal Nash stable component the second highest and an h-agent the lowest utility. Thus, in the asymmetric equilibrium a smaller set of agents receives a high utility but a larger set of agents a low utility. In the symmetric equilibrium all agents receive an intermediate utility.

As to strongly pairwise stable networks in which no pair of agents has a strict incentive to deviate, we find that applying this different equilibrium concept reduces the number of stable networks dramatically. There exists no strongly pairwise stable network which contains a component of more than two agents.

Efficient networks in which the sum of utilities of all agents is maximized consist of reciprocal components only. Every reciprocal component in an efficient network features a lower self-investment than a Nash stable reciprocal component. An efficient time allocation for a given network architecture exists iff the network architecture is only made up of component architectures for which a Nash stable reciprocal time allocation exists. Hence, in our model only a symmetric structure is efficient with ex-ante homogenous agents.

This paper contributes to the research on network formation with endogenous link strength. This research can be divided into two strands: articles which treat link investment as specific, this means a quality, effort or investment level is specified for each link specifically; and articles which treat link investment as non-specific, this means an overall quality, effort or investment level is specified which then affects each link equally. This paper belongs to the first strand. Related articles of the first strand are Salonen (2014), Bloch and Dutta (2009), Brueckner (2006), Rogers (2006), and Goyal, Moraga-González and Konovalov (2008). In the following we point out the differences to our paper. Salonen (2014) studies the existence of reciprocal Nash stable time allocations for complete components and for a circle network only with a general Cobb-Douglas function and a general concave function for utility from self-investment. For the setup of the present paper, he proves existence. Bloch and Dutta (2009) consider a different form of agent interdependence: for the main part of the analysis, the link value is an additively separable and convex function of both agents' investments. In an extension, agents' investments act as perfect complements. Moreover, agents obtain utility from direct and indirect links. Self-investment is not allowed for. Brueckner (2006) also allows for utility from direct and indirect friendships but neither accounts for self-investment. Effort requires convex costs but is not resource-constrained and link formation is probabilistic depending positively on agents' effort levels. The author concentrates on symmetric solutions in which every agent chooses the same effort level towards every other agent. Rogers (2006) does not consider self-investment and an agent's link investment benefits only one of the two linked agents. Goyal et al. (2008) look at R&D networks and analyze a firm's choice of investment in in-house research and in each research project with another firm. Research reduces costs and firms compete in the market. Firms choose investment levels that maximize their profits. Again the authors focus on symmetric investment levels, when every firm invests the same into each project with another firm and every firm invests the same into in-house research.

Articles which treat investment as non-specific are for example Golub and Livne (2010), Cabrales, Calvó-Armengol and Zenou (2011), Galeotti and Merlino (2014), and Durieu, Haller and Solal (2011).

Our contribution to the existing literature is an extensive characterization of symmetric as well as asymmetric stable and efficient networks when agents have a limited budget which they can use for link-specific investments and private purposes, and are socially interdependent through the link-specific investments in the described way.

The structure of the paper is as follows. In section 2, we introduce the model. In section 3, we characterize Nash stable time allocations and network architectures. Strongly pairwise stable time allocations and network architectures are discussed in section 4. In section 5, we compare agents' utilities in the different Nash stable networks, and characterize efficient time allocations and network architectures. Section 6 concludes.

2 Model

The social network consists of a set of agents $N = \{1, ..., n\}$. Each agent i possesses the amount of time $T > 0$ which she can allocate to every other agent and to herself. We denote her time investment into the relationship with player $j \neq i$ as t_{ij} and her self-investment as t_{ii}. Time investments are naturally constrained to $t_{ij} \geq 0$ for all j and $\sum_j t_{ij} \leq T$. An agent's time investment strategy is the vector $t_i = (t_{i1}, ..., t_{in})$. The network time allocation and architecture are fully characterized by the matrix of all time investment strategies $\mathscr{T} = [t_i]_{i=1,\ldots,n}$. We denote a link (friendship) between agent i and agent j as link ij. The existence of a link is undirected, hence, ij and ji denote the same link. We define a link (friendship) between two agents i and j to exist iff both $t_{ij} > 0$ and $t_{ji} > 0$.

We assume that the value $v(t_{ij}, t_{ji})$ of friendship ij for agent i follows a Cobb-Douglas function which takes both friends' investments as inputs:

$$v(t_{ij}, t_{ji}) = a\, t_{ij}^{\beta}\, t_{ji}^{1-\beta}$$

with $a > 0$ and $\beta \in (0, 1)$. This Cobb-Douglas technology captures certain properties of value creation in friendship well. The friendship gets more valuable if time investments (or equivalently, efforts in friendship) are increased. However, the positive effect of a unilateral increase for a given positive investment level of the friend is decaying. Moreover, one friend's investment is dependent on the other's investment: The marginal utility of an agent's time investment is increasing in the level of investment of her friend. These properties are reflected by positive, but diminishing partial derivatives and positive cross-derivatives for positive investment levels. Note that the marginal utility from friendship is zero for a unilateral increase in time investment by agent i (j), if j (i) does not invest. Furthermore, $v(t_{ij}, t_{ji}) = 0$, if $t_{ij} = 0$ and/or $t_{ji} = 0$ and the link ij does not exist. Thus, a friendship cannot be established and friendship benefits cannot be enjoyed without some consent of the counterpart. a can be seen as the intrinsic value of friendship. β is the value elasticity of agent i's time investment and measures the complementarity of agent i's and j's time investment in the value creation for agent i. Assuming that both exponents sum up to unity, as we do here, implies constant returns to scale to both agents' investments. Assuming decreasing returns to scale would bias our efficiency results towards agents diversifying their time investment over friendships, increasing returns to scale would bias our efficiency results towards agents concentrating their time investment, whereas constant returns to scale do not push our efficiency results into one direction a priori. Hence, this seems to be the least strict assumption, if one is to make an assumption on returns to scale.

Agent i's utility from self-investment is assumed to be an increasing, strictly concave and differentiable function $f(t_{ii})$ with $f'(t_{ii}) \to \infty$ if $t_{ii} \to 0$.

With these assumptions, both agent i's utility from a friendship ij and her utility from self-investment are strictly concave in t_{ij} and t_{ii}, respectively. Hence, an agent's time always features decreasing marginal returns. The joint investment of two agents has a higher productivity: decreasing marginal returns of the two individual investments add up to constant marginal returns of joint investment for the value of friendship as follows from the exponents in $v(t_{ij}, t_{ji})$.

The total utility of agent i from network $\mathscr{T}$ is defined to be the sum of her utility from her social network – the sum of all friendship values – and her utility from self-investment:

$$u_i(\mathscr{T}) = \sum_{j \neq i} a t_{ij}^{\beta} t_{ji}^{1-\beta} + f(t_{ii}).$$

3 Nash stable networks

3.1 Properties of Nash stable time allocations

To examine equilibrium time allocations and network architectures in this model, we use the concept of Nash stability based on Bloch and Dutta (2009).

Definition 1. *A network $\mathscr{T}$ is Nash stable if there exists no agent i who is strictly better off by unilaterally deviating from investment strategy t_i to another feasible investment strategy t_i'.*

A Nash stable network $\mathscr{T}$ is the Nash equilibrium of the game in which each agent simultaneously chooses t_i. Agents' best responses are such that in Nash equilibrium every agent i chooses $t_{ij} = 0$ if $t_{ji} = 0$, $t_{ij} > 0$ if $t_{ji} > 0$, and $t_{ii} > 0$. Moreover, in Nash equilibrium she invests her whole endowment T since self-investment is always utility enhancing.

Knowing that $t_{ij} = 0$ for all $t_{ji} = 0$ in Nash equilibrium, the agent's time allocation problem reduces to choosing her utility-maximizing level of $t_{ij} > 0$ for each friendship ij for which $t_{ji} > 0$, and her utility-maximizing level of $t_{ii} > 0$. For every agent i, all $t_{ij} > 0$ and $t_{ii} > 0$ satisfy the following First Order Conditions in Nash equilibrium:

$$a\beta \left(\frac{t_{ji}}{t_{ij}}\right)^{1-\beta} = f'(t_{ii}) \quad \forall \quad j \neq i \ \text{ with } \ t_{ji} > 0 \tag{1}$$

$$\sum_j t_{ij} = T \tag{2}$$

Thus, in Nash equilibrium every agent i chooses an investment strategy such that her marginal utility of investment in each of her existing friendships is equal to her marginal utility from self-investment (1) and the budget constraint is binding (2). We obtain the following Lemma:

Lemma 3.1. *The network $\mathscr{T}$ is Nash stable if and only if each agent i chooses t_i such that*

(a) (2) holds,

(b) $t_{ij} = 0$ for all $j \neq i$ for which $t_{ji} = 0$,

(c) $t_{ii} > 0$, and $t_{ij} > 0$ for all $j \neq i$ for which $t_{ji} > 0$ such that (1) is satisfied.

The total differential of (1) shows that in a Nash equilibrium with a higher level of self-investment of agent i also her investment into each of her existing friendships is relatively higher:

$$\frac{dt_{ij}}{dt_{ii}} = \frac{f''(t_{ii})}{a\beta(\beta-1)t_{ji}^{1-\beta}t_{ij}^{\beta-2}} > 0.$$

Since the budget constraint is binding in Nash equilibrium and the time budget is fixed, higher investment levels could be possible by reducing the number of existing friendships such that some time budget is available for increased investment levels.

From Lemma 3.1 follows that the empty network with $t_{ij} = 0$ for all $j \neq i$ and $t_{ii} = T$ for all agents i is Nash stable.
In the following, we derive further characteristics of Nash stable networks different from the empty network. First, we introduce necessary definitions. A *path* in a network $\mathscr{T}$ between agents i and j is a sequence of links $i_1i_2, i_2i_3, ..., i_{K-1}i_K$ such that $i_1 = i$ and $i_K = j$, each link i_ki_{k+1} with $k \in \{1, ..., K-1\}$ exists in $\mathscr{T}$, and no agent appears twice in the sequence $i_1, ..., i_K$. In $\mathscr{T}$ two agents i and j are *connected* if there exists a path between i and j in $\mathscr{T}$. A *component* of $\mathscr{T}$ is a non-empty subnetwork (at least one agent) of $\mathscr{T}$ such that all agents in the subnetwork are connected and no agent in the subnetwork is linked to an agent who is not in the subnetwork. We say *a component C is Nash stable* if every agent $i \in C$ chooses t_{ii} and all t_{ij} with $j \neq i$ and $j \in C$ such that (a), (b), and (c) of Lemma 3.1.

Observe first that because of Lemma 3.2 we only need to determine properties of Nash stable components to describe all Nash stable networks.

Lemma 3.2. *Network $\mathscr{T}$ is Nash stable if and only if it consists of Nash stable components.*

If network $\mathscr{T}$ is Nash stable, then according to Lemma 3.1 every agent i in component C chooses $t_{ij} = 0$ for all $j \notin C$ because $t_{ji} = 0$ for all $j \notin C$, and t_{ii} and all t_{ij} with $j \neq i$ and $j \in C$ such that (a), (b), and (c) of Lemma 3.1. Hence, also component C is Nash stable. This is true for all components C.

If a component C is Nash stable, then every agent $i \in C$ chooses t_{ii} and all t_{ij} with $j \neq i$ and $j \in C$ such that (a), (b), and (c) of Lemma 3.1. Moreover, the best response of agent i to every agent $j \notin C$ is $t_{ij} = 0$ because $t_{ji} = 0$. This is in line with the non-existence of links between components in network $\mathscr{T}$. Hence, if all components of network $\mathscr{T}$ are Nash stable, then network $\mathscr{T}$ is Nash stable. $\square$

The number of agents in component C will be denoted by n^C. A component with $n^C = 1$, which is just one agent without any links, is Nash stable iff $i \in C$ chooses $t_{ii} = T$. From here on, we analyze components with $n^C \geq 2$ unless stated otherwise.

Since in a Nash stable component every agent $i \in C$ chooses t_i such that the marginal utilities across all of her friendships are equal, the ratios of time investment across all of her friendships must be equal. This follows from the First Order Conditions (1). In other words, in a Nash stable component for every agent i, $\frac{t_{ji}}{t_{ij}} = q_i$ for all j for which $t_{ji} > 0$ and $t_{ij} > 0$. Every friend j of agent i then just faces the reverse time ratio $\frac{t_{ij}}{t_{ji}} = q_j = \frac{1}{q_i}$ in all of her friendships. Every friend k of agent j faces again the reverse time ratio of agent j, $q_k = \frac{1}{q_j} = q_i$, in all of her friendships and so on. Since in one component all agents are connected we arrive at the following Proposition.

Lemma 3.3. *For every Nash stable component C, there exists exactly one real $q \geq 1$ such that*

(a) for every agent $i \in C$, $\frac{t_{ji}}{t_{ij}} = 1$ for all $j \neq i$ for which $t_{ji} > 0$ and $t_{ij} > 0$, if $q = 1$.

(b) for every agent $i \in C$, either $\frac{t_{ji}}{t_{ij}} = q$ or $\frac{t_{ji}}{t_{ij}} = \frac{1}{q}$ for all $j \neq i$ for which $t_{ji} > 0$ and $t_{ij} > 0$, if $q > 1$. Every friend j of agent i faces the reverse time ratio $\frac{t_{ij}}{t_{ji}}$ in all of her friendships.

Note that the fact that the ratios of time investments in all friendship of agent i are equal in Nash equilibrium still allows the agent to have friendships of different strengths. An agent can have good friendships, where both linked agents invest relatively much, and bad friendships, where both linked agents invest relatively little, as long as the ratios of time investment are equal across friendships.

If $q = 1$, friends i and j invest the same amount of time into each other. We call this situation "reciprocal". If $q > 1$ and w.l.o.g $\frac{t_{ji}}{t_{ij}} = \frac{1}{q}$, then i invests more into each friend j than each j into i, and each j invests less into each of her friends (among which is also i) than her friends into j. We call this situation "non-reciprocal". Thus, there are exactly two types of Nash stable components. Depending on the value of q, a Nash stable component is either reciprocal or non-reciprocal.

Proposition 3.4. *A Nash stable component C is either*

> ***reciprocal** ($q = 1$): Every agent $i \in C$ is a reciprocal (r-) agent. Every r-agent chooses self-investment t_{rr} such that $a\beta = f'(t_{rr})$ and social time $T_r^S = T - t_{rr}$. Friends invest the same amount of time into each other.*

or

> ***non-reciprocal** ($q > 1$): Every agent $i \in C$ is either a high intensity (h-) agent or a low intensity (l-) agent, and friends are of different type. For all friendships, $\frac{t_{hl}}{t_{lh}} = q > 1$, hence, every h invests more into the friendship than l does.*
>
> *Every l-agent chooses self-investment t_{ll} such that $a\beta q^{1-\beta} = f'(t_{ll})$ and social time $T_l^S = T - t_{ll}$. Every h-agent chooses self-investment t_{hh} such that $a\beta\left(\frac{1}{q}\right)^{1-\beta} = f'(t_{hh})$ and social time $T_h^S = T - t_{hh}$.*

Proposition 3.4 follows from Lemma 3.3 and from substituting q into the FOCs.

Observe that $a\beta\left(\frac{1}{q}\right)^{1-\beta} = f'(t_{hh}) < a\beta = f'(t_{rr}) < a\beta q^{1-\beta} = f'(t_{ll})$. Therefore, by the curvature of f, we know that $t_{hh} > t_{rr} > t_{ll}$ and hence $T_h^S < T_r^S < T_l^S$. In equilibrium, an h-agent faces the lowest marginal utility and hence the highest self-investment, an r-agent an intermediate marginal utility and hence an intermediate self-investment, and an l-agent the highest marginal utility and hence the lowest self-investment.

In a Nash stable reciprocal component agents are to some degree symmetric: every agent invests the same amount into a friendship as her friend, every agent chooses the same level of self-investment and hence the same level of social time. Yet, friendships can be of different strength. Links can feature a lower or higher investment. Moreover, agents can have a different number of friends.

In a Nash stable non-reciprocal component agents are more asymmetric. There are two types of agents with friends being of different type. An l-agent chooses lower self-investment and hence higher social time than h-agent, and an l-agent always invests less into a friendship than the h-agent does.

Next, we show examples of Nash stable time allocations for all components of $n^C = 4$ for which a Nash stable time allocation exists. We assume that the equilibrium levels of social time are $T_r^S = 11$ if $q = 1$, and $T_l^S = 13.5$ and $T_h^S = 9$ if $q = 2$. Next to each node (agent) we indicate her type. Time investments of each node into her friendships are written close to the node in red at the respective friendship link.

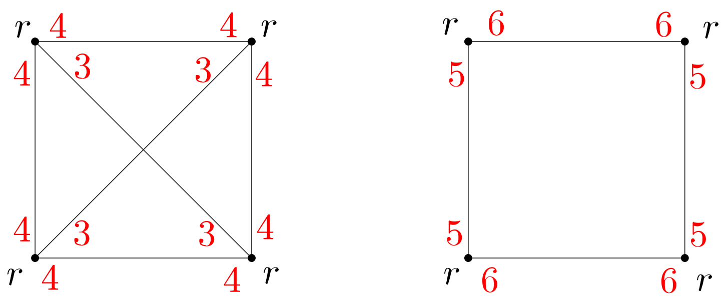

Figure 1: Nash stable reciprocal components with $n^C = 4$.

Figure 1 displays the complete component (the component in which all possible links exist) and the regular component (in a regular component all agents in the component have the same number of friends) which both have a Nash stable reciprocal time allocation. We see that every agent in the component is an r-agent who spends 11 hours on her friends. Some friendships are of different strength. The depicted time allocations are not the unique Nash stable reciprocal solutions for these components.

Figure 2 displays the star component (in a star component the "center" agent has $n^C - 1$ friends and every other agent is a "leaf" who is only friends with the center agent). Every leaf is an h-agent who invests twice as much in her friend than her friend, the center, who is an l-agent. The center spends overall 13.5 hours, and every leaf spends 9 hours on her friend(s). For this component the depicted time allocation is the unique Nash stable non-reciprocal solution.

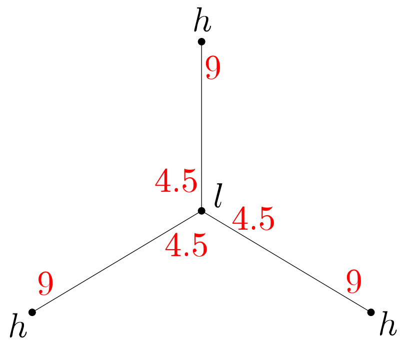

Figure 2: Nash stable non-reciprocal component with $n^C = 4$.

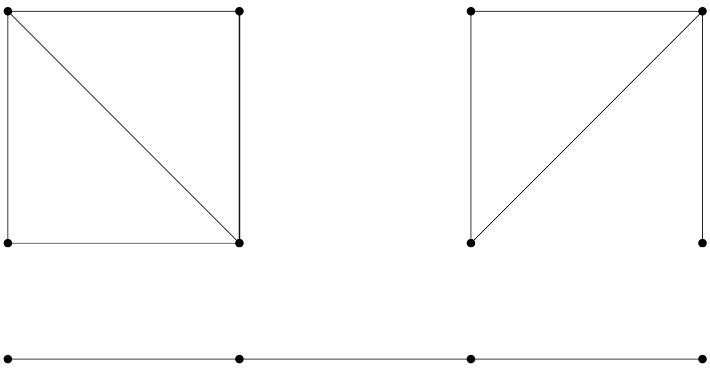

Figure 3: Component architecures with $n^C = 4$ for which no Nash stable time allocation exists.

Figure 3 displays all remaining component architectures with $n^C = 4$. There does not exist any Nash stable time allocation for (a), (b) and (c) of figure 3. The reasons become obvious in the next subsections.

In the following, we analyze for which component architectures a Nash stable reciprocal time allocation exists, for which a Nash stable non-reciprocal time allocation exists, and for which no Nash stable time allocation exists.

3.2 Existence of Nash stable time allocations for given component architectures

Let $C = (N^C, E^C)$ be a component architecture with a set of nodes N^C and a set of links E^C that exist among N^C. There exists a Nash stable time allocation for C iff there exists a Nash stable time allocation with $t_{ij} = 0$ for all $ij \notin E^C$, and $t_{ij} > 0$ for all $ij \in E^C$. Theorem 3.5[1] provides necessary and sufficient conditions on the component architecture for a Nash stable reciprocal time allocation to exist. For Theorem 3.5 we need to introduce some further notation. Let $C - U$ be the subgraph that results after deleting a set of nodes $U \subseteq N^C$ and all links ij with $i \in U$ from C. Denote by W the set of nodes in $C - U$ which do not have any link (set of isolated nodes in $C - U$).

Theorem 3.5. *There exists a Nash stable reciprocal time allocation for $C = (N^C, E^C)$ with $t_{ij} > 0$ if $ij \in E^C$, and $t_{ij} = 0$ if $ij \notin E^C$ if and only if for every $U \subseteq N^C$ either*

1. $|U| > |W|$,

or

2. $|U| = |W|$ *and for every link* $ij \in E^C$ *with* $i \in U$ *it is true that* $j \in W$.

The proof why $\Rightarrow$ is intuitive. First observe that every $i \in W$ is only linked to $j \in U$ in component architecture C because otherwise $i \in W$ would not be an isolate in $C - U$. Then, in a Nash stable reciprocal time

[1] Theorem 3.5 came to life with the help of Henning Bruhn-Fujimoto.

allocation for C the nodes in W choose an overall social time of $|W|\,T_r^S$ which they only spend on nodes in U and they require the same overall amount of social time from U because of reciprocity. Hence, if a Nash stable reciprocal solution exists for component architecture C, there is no $U \subseteq N^C$ for which $|U| < |W|$, or for which $|U| = |W|$ and not for every link $ij \in E^C$ with $i \in U$ it is true that $j \in W$. In the first case, the overall social time nodes in U could give to W, $|U|\,T_r^S$, would be strictly less than what the nodes in W require, and no Nash stable reciprocal solution would exist. Similarly, in the second case, the overall social time nodes in U choose, $|U|\,T_r^S$, would be equal to the amount the nodes in W required. Nodes in U are, however, not only linked to nodes in W and spend some social time on nodes which are not in W. Then again nodes in W would not receive the amount of social time they required but strictly less and no Nash stable reciprocal solution would exist. Thus, if a Nash stable reciprocal time allocation exists for C, then for every $U \subseteq N^C$ it is true that either $|U| > |W|$, or $|U| = |W|$ and for every link $ij \in E^C$ with $i \in U$ it is true that $j \in W$. $\square$

For the proof why $\Leftarrow$ we draw on Theorem 35.1 by Schrijver (2004, p. 584) which states necessary and sufficient conditions for a perfect b-matching with lower bounds on link values to exist for a graph. A Nash stable reciprocal time allocation for component C corresponds to a perfect b-matching for C with a value of $t_{ij} = t_{ji} > 0$ assigned to each $ij \in E^C$ by the matching and $\sum_{j \neq i} t_{ij} = T_r^S$ for all $i \in N^C$. Hence, if this perfect b-matching exists, then a Nash stable reciprocal solution exists. In the appendix, I show that every component C for which either 1. or 2. of Theorem 3.5 satisfies the necessary and sufficient conditions stated in Theorem 35.1 by Schrijver (2004, p. 584) for this perfect b-matching to exist. This proves sufficiency of the conditions in Theorem 3.5.

With the help of Theorem 3.5 we analyze the existence of Nash stable reciprocal time allocations for certain component architectures.

First, we look at component architectures which contain at least one leaf. A leaf is an agent who has only one link. Assume $n^C \geq 3$, then, figure 4 depicts the leaf, agent 1, and her only friend, agent 2, in any such component architecture. The other links of agent 2 are only hinted at. If $U = \{agent\ 2\}$,

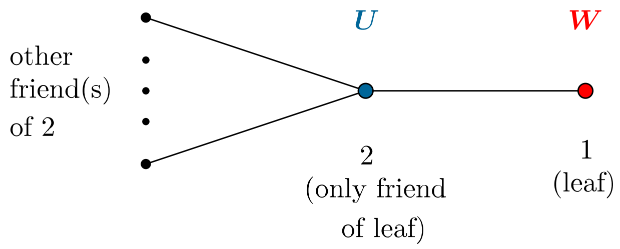

Figure 4: Component architecture which contains a leaf.

then $W = \{agent\ 1\}$ such that $|U| = |W|$ and not for every link $ij \in E^C$ with $i \in U$ it is true that $j \in W$. Thus, neither 1. nor 2. of Theorem 3.5 is true. In a component architecture with $n^C = 2$ both agents $i \in N^C$ are leaves by construction. In this case a reciprocal solution exists because for all $U \subseteq N^C$ either 1. or 2. of Theorem 3.5.

We conclude Lemma 3.6.

Lemma 3.6. *A Nash stable component with $n^C \geq 3$ which contains a leaf must be non-reciprocal. A Nash stable component with $n^C = 2$ can be reciprocal.*

Second, we look at bipartite component architectures. A component architecture C is bipartite if N^C can be partitioned into two sets A and B such that there only exist links between nodes across the sets and not within one set. Take any bipartite component architecture in which the two sets A and B are of unequal size. If U is equal to the smaller one of the two sets, then all nodes in the larger set are isolates in $C - U$. Thus, $|U| < |W|$ and no reciprocal solution exists. Figure 5 shows an example.

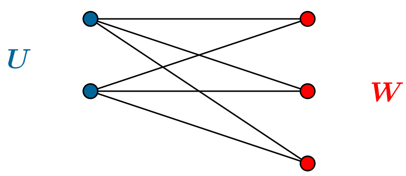

Figure 5: Bipartite component architecture with two unequally sized sets.

Lemma 3.7. *A Nash stable component which is bipartite with two unequally sized sets must be non-reciprocal.*

Third, we have a look at component architectures in which every agent is linked to more than half of the agents in the component.

Proposition 3.8. *For every component architecture in which each $i \in N^C$ is friends with more than half of the component population a Nash stable reciprocal time distribution exists.*

If the number of friends of each $i \in N^C$ is greater than $\frac{n^C}{2}$, then more than $\frac{n^C}{2}$ nodes have to be deleted in order to have at least one isolate in $C - U$. So for each $U \subseteq N^C$, $|U| > |W|$.

Fourth, we derive a result on *d-regular* component architectures in a similar way. A d-regular component architecture is a component in which every agent has d links.

Proposition 3.9. *For every d-regular component architecture a Nash stable reciprocal time allocation exists.*

The following is true for every $U \subseteq N^C$. In C every agent in $U \subseteq N^C$ has d links to other agents and every agent in W has d links only to agents in U. Hence, agents in W share a total of $d\,|W|$ links with agents in U. If U and W are such that for all links ij with $i \in U$ $j \in W$, then $d\,|U| = d\,|W|$ and hence $|U| = |W|$. If U and W are such that not for all links ij with $i \in U$ $j \in W$, then $d\,|U| > d\,|W|$ and hence $|U| > |W|$. Thus, for every $U \subseteq N^C$ either 1. or 2. of Theorem 3.5.

Next, we provide necessary conditions on the component architecture for the existence of a Nash stable non-reciprocal time allocation.

Proposition 3.10. *Every Nash stable non-reciprocal component is bipartite with two unequally sized sets of agents. The larger set contains all h-agents and the smaller set contains all l-agents.*

From Proposition 3.4 we know that in a Nash stable non-reciprocal component linked agents are of different type because of the alternating time ratio between friends. This means that every l-agent is only linked to h-agents and vice versa. Hence, the Nash stable non-reciprocal component is bipartite because the set of all agents in the component can be partitioned into the set of h-agents, H^C, and the set of l-agents, L^C. Then, there exist only links across the sets and not within the sets.

To see why the set of h-agents must be larger than the set of l-agents, we look at the properties of their time investment strategies (cf. Prop. 3.4). There only exist friendships between h-agents and l-agents, and for every friendship it is true that the l-agent invests less into the friendship than the h-agent: $t_{hl} > t_{lh}$ for every friendship (link). Thus, $\sum_h \sum_l t_{hl} > \sum_l \sum_h t_{lh}$ $\Leftrightarrow \sum_h T_h^S > \sum_l T_l^S$. However, $T_h^S < T_l^S$ because $t_{hh} > t_{ll}$. Then, it must be true that $\left|H^C\right| > \left|L^C\right|$.

As the number of links emanating from all nodes in each set is the same due to bipartiteness we know that an l-agent has on average more friends than an h-agent.

With Proposition 3.10 we can make a statement about component architectures which contain an odd length cycle. A *cycle* is a sequence of links $i_1i_2, i_2i_3, ..., i_{K-1}i_K$ where $i_1 = i_K$ and $i_k \neq i_{k'}$ for $k < k'$ unless $k = 1$ and $k' = K$. Hence, a cycle is a path except for having the same start and end node. *Length* refers to the number of links involved.

Corollary 3.11. *Every Nash stable component with an odd length cycle must be reciprocal.*

The reason is that a component architecture which contains an odd length cycle is not bipartite: Take any odd length cycle $i_1i_2, i_2i_3, ..., i_{K-1}i_K$. Because of odd length $K-1$ is an odd numbered agent. Even numbered agents in the cycle are only linked to odd numbered agents, all odd numbered agents except for i_1 and i_{K-1} are only linked to even numbered agents, and i_1 and i_{K-1} are each linked to one even numbered and one odd numbered agent. Thus, we cannot partition the nodes of the cycle in two sets such that there exist only links across and not within sets. From Proposition 3.10 we know that a component architecture which is not bipartite does not have a Nash stable non-reciprocal time allocation.

From the previous results we derive stronger conditions on leaves in Nash stable components.

Proposition 3.12. *In a Nash stable component of* $n^C \geq 3$ *every leaf must be an* h*-agent. In a Nash stable component of* $n^C = 2$ *both leaves must be an* r*-agent.*

First, we prove the first statement of Proposition 3.12. Let C be a Nash stable component of $n^C \geq 3$ which contains at least one leaf. From Lemma 3.6 we know that C must be non-reciprocal, such that every agent in C is either an h-agent or an l-agent. Assume by contradiction that a leaf in C is an l-agent and let this leaf be w.l.o.g. agent 1 in Figure 4. If 1 is an l-agent, then 2 is an h-agent. For their investment levels it must then be true that $t_{12} < t_{21}$ and $t_{11} < t_{22}$. Moreover, $t_{12} + t_{11} = T$ because in Nash equilibrium every agent spends her whole budget T. But then $T < t_{21} + t_{22}$. The budget constraint of 2 is violated and hence we have a contradiction. Agent 1 and thus every leaf must be an h-agent.

Second, we prove the second statement of Proposition 3.12. Let C be a Nash stable component of $n^C = 2$. From Lemma 3.6 we know that there exists a Nash stable reciprocal time allocation for C. Moreover, C is not bipartite with two unequally sized sets such that there does not exist a Nash stable non-reciprocal solution. Hence, C must be reciprocal and every agent in C is an r-agent.

Combining Lemma 3.7 and Proposition 3.10 reveals that the existence of a reciprocal and a non-reciprocal Nash stable solution for a given component architecture C are in fact mutually exclusive. If C is bipartite with two unequally sized sets of agents, then there does not exist a reciprocal solution. Bipartiteness with two unequally sized sets is, however, a necessary condition for a non-reciprocal solution to exist.

Proposition 3.13. *For every component architecture $C = (N^C, E^C)$ exactly one of the following is true:*

a) There exists a Nash stable reciprocal time allocation for C.

b) There exists a Nash stable non-reciprocal time allocation for C.

c) There does not exist a Nash stable time allocation for C.

Having analyzed conditions for existence and properties of reciprocal and non-reciprocal Nash stable solutions, we draw conclusions for which component architectures no Nash stable solution exists.

Corollary 3.14. *No component which contains both an odd length cycle and a leaf is Nash stable.*

On the one hand, if a component contains an odd length cycle, it must be reciprocal to be Nash stable (c.f. Cor. 3.11). On the other hand, if a component of $n^C \geq 3$ contains a leaf it must be non-reciprocal to be Nash stable because every leaf must be an h-agent in a Nash stable component of $n^C \geq 3$ (c.f. Prop. 3.12). Since a Nash stable component cannot be reciprocal and non-reciprocal at the same time, there does not exist a Nash stable time allocation for a component which contains both a leaf and an odd length cycle.

Corollary 3.15. *No component of $n^C \geq 3$ with an odd length path between two leaves is Nash stable.*

In a component of $n^C \geq 3$ every leaf must be an h-agent for the component to be Nash stable. However, if there exists an odd length path between two leaves, these two leaves cannot be both h-agents because the agent-type alternates between linked agents (cf. Prop. 3.4). Hence, there does not exist a Nash stable time allocation for a component of $n^C \geq 3$ with an odd length path between two leaves.

Proposition 3.12 and Corollary 3.15 have implications for *trees* which are a prominent class of networks. A component architecture which does not have a cycle is called a tree. Every tree has at least two leaves. A Nash stable tree of $n^C \geq 3$ is non-reciprocal, does not have an odd length path between two leaves and all its leaves are h-agents.

3.3 Non-uniqueness of Nash stable time allocations and component architectures

The constant returns to scale assumption inside the Cobb-Douglas function leads to non-uniqueness of Nash stable time allocations for certain component architectures and non-uniqueness of component architectures for which a Nash stable time allocation exists. By appropriately shifting time investments in one Nash stable component C, we find another Nash stable component C'.

Let C be a Nash stable component with time allocation $\mathscr{T}^C$ and time ratio q^C. Take an even length sequence S of agents $i_1i_2, i_2i_3, ..., i_{K-1}i_K$ of N^C where $i_1 = i_K$, $K \geq 5$ and $i_k \neq i_{k'}$ for $k < k'$ unless $k = 1$ and $k' = K$. Note that S is an even length "cycle of agents" which might also include two agents as a pair who are not linked in C. If $q^C > 1$, then agent i_k with k odd must be an h-agent and agent i_k with k even must be an l-agent.

Proposition 3.16. *Let C be a Nash stable component with time allocation $\mathscr{T}^C$ and time ratio q^C. Then, C' with $\mathscr{T}^{C'} \neq \mathscr{T}^C$ is another Nash stable component if*

1) $t'_{i_ki_{k+1}} = t_{i_ki_{k+1}} + x \geq 0$ and $t'_{i_ki_{k-1}} = t_{i_ki_{k-1}} - x \geq 0$ where $x = -a$ for k odd, $x = b$ for k even and $\frac{a}{b} = q^C$,

2) $t'_{ij} = t_{ij}$ for each pair $ij \notin S$.

If C is Nash stable and if we adjust the time allocation as defined in Proposition 3.16, then we reach component C' with $\mathscr{T}^{C'} \neq \mathscr{T}^C$ in which again the FOCs for a Nash stable time allocation (cf. Lemma 3.1) for every agent $i \in N^C$ are satisfied.

We give two examples in figures 6 and 7. Agents are numbered in black. Time investment t_{ij} by agent i into agent j is given in red, close to agent i

at the respective link. Each figure title indicates q^C characteristic for Nash stable component C and the levels of social time which we assume to prevail in equilibrium given q^C. In each figure, we depict C with $\mathscr{T}^C$ on the left and C' with $\mathscr{T}^{C'}$ on the right.

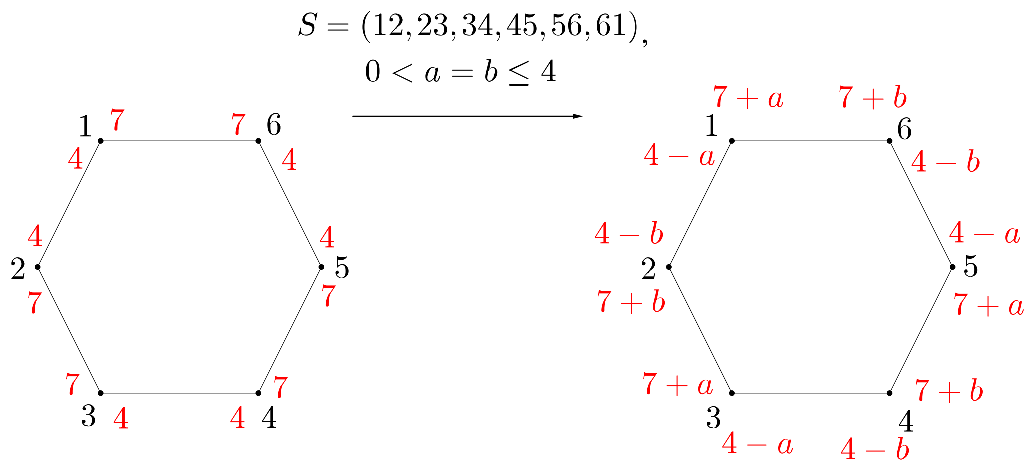

Figure 6: $q^C = 1$, $T_r^S = 11$.

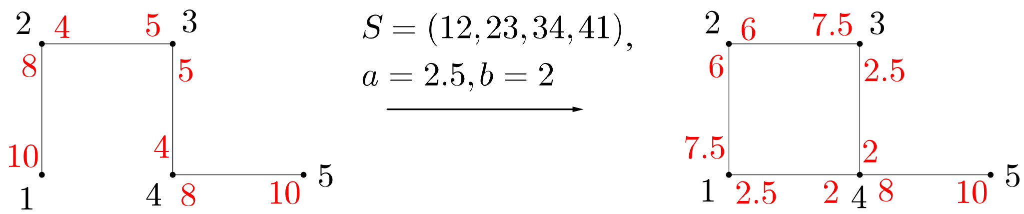

Figure 7: $q^C = \frac{5}{4}$, $T_h^S = 10$, $T_l^S = 12$.

4 Strongly pairwise stable networks

In the previous section we have seen that there exist infinitely many Nash stable time allocations and a large finite number of component architectures for which a Nash stable time allocation can be found. Applying the concept of strong pairwise stability based on Bloch and Dutta (2009) reduces the number of stable networks significantly.

Definition 2. *A network $\mathscr{T}$ is strongly pairwise stable if it is Nash stable and if there are no two individuals (i, j) which would be both strictly better off by a joint deviation from (t_i, t_j) to (t'_i, t'_j).*

Lemma 4.1. *Every r-agent r has a strict incentive to deviate jointly with any other agent j if both increase their investments into their friendship by ϵ and r unilaterally reduces her investment into another friend k by $\epsilon \in (0, t_{rk})$.*

A formal proof is provided in the appendix. The joint raise of investment in a reciprocal relationship increases an agent's utility more than the unilateral reduction of investment in a reciprocal relationship decreases it. After the deviation the r-agent receives a higher overall time investment from other agents than before.

Lemma 4.2. *Every h-agent h has a strict incentive to deviate jointly with any other h- or r-agent j if they establish a reciprocal link between them, each with a time investment ϵ, and h unilaterally reduces her investment into an l-agent l by $\epsilon \in (0, t_{hl})$.*

A formal proof is provided in the appendix. The joint raise of investment in a reciprocal relationship increases an agent's utility more than the unilateral reduction of investment in non-reciprocal relationship in which the own high investment level is only responded by a lower investment level.

From Lemma 4.1 and Lemma 4.2 we conclude the following result on strong pairwise stability of a network.

Proposition 4.3. *Every Nash stable network which contains at least two agents of whom each is either an r- or an h-agent is not strongly pairwise stable.*

Proposition 4.3 implies that every network which contains at least one component of $n^C \geq 3$ is not strongly pairwise stable. A Nash stable component of $n^C \geq 3$ is either reciprocal or non-reciprocal. In the former case, the network contains at least three r-agents, and, in the latter case, at least two h-agents. Moreover, every network which contains at least two components of size $n^C = 2$ is not strongly pairwise stable because each Nash stable component of $n^C = 2$ is reciprocal. Then, the network contains at least four r-agents.

5 Agent utility and welfare

In this section, we first compare the utility between the different types of agents in Nash stable components, and second characterize the efficient network.

In a Nash stable non-reciprocal component the utility of an l-agent l is

$$u_l = \sum_{j \neq l} a t_{lj}^{\beta} t_{jl}^{1-\beta} + f(t_{ll})$$

and the utility of an h-agent h is

$$u_h = \sum_{j \neq h} a t_{hj}^{\beta} t_{jh}^{1-\beta} + f(t_{hh}).$$

In Nash stable reciprocal component the utility of an r-agent r is

$$u_r = \sum_{j \neq r} a t_{rj}^{\beta} t_{jr}^{1-\beta} + f(t_{rr}).$$

Proposition 5.1. *In every Nash stable network, $u_l > u_r > u_h$.*

Among agent l, agent h and agent r, agent l spends the highest amount of time on socializing and she receives even more time from her friends since $T_l^S > T_r^S > T_h^S$ and $t_{jl} > t_{lj}$ for every friendship of l with another agent j. Agent l has the highest utility in a Nash stable network. Agent r spends the second highest amount of time on socializing and receives the same amount of time from her friends since $t_{rj} = t_{jr}$ for every friendship of r with another agent j. Agent r obtains the second highest utility. Agent h invests the smallest amount of time into socializing and receives even less time from her friends since $t_{hj} > t_{jh}$ for every friendship of h with another agent j. Agent h obtains the lowest level of utility. A formal proof for Proposition 5.1 can be found in the appendix.

Whether a reciprocal or a non-reciprocal Nash stable time allocation for a component of given size n^C dominates in terms of welfare is not obvious. The welfare from a non-reciprocal Nash stable time allocation is

$$W_{q>1} = \left|L^C\right| u_l + \left|H^C\right| u_h,$$

and from a reciprocal Nash stable time allocation

$$W_{q=1} = \left|N^C\right| u_r = \left(\left|L^C\right| + \left|H^C\right|\right) u_r.$$

Since $u_l > u_r > u_h$ and $\left|L^C\right| < \left|H^C\right|$, we cannot determine if $W_{q>1} > W_{q=1}$ or $W_{q>1} < W_{q=1}$ without further information. Yet, we can show that none of the Nash stable time allocations is welfare maximizing.

We call a network which is welfare maximizing an "efficient" network. With this efficiency notion, we follow Jackson and Wolinsky (1996) who term a network as "strongly efficient" if and only if it maximizes the sum of individual utilities. The sum of individual utilities in network $\mathscr{T}$ is

$$W(\mathscr{T}) = \sum_{i \in N} \left(\sum_{j \neq i} a t_{ij}^{\beta} t_{ji}^{1-\beta} + f(t_{ii}) \right).$$

In the following, we think of a social planner who implements the efficient time allocation. The efficient network is characterized by the solution to

$$\max_{t_{11}, t_{12}, \ldots, t_{nn}} W(\mathscr{T})$$

$$s.t. \quad \sum_{j} t_{ij} = T \quad \text{for all } i.$$

The time constraint has to be satisfied for each agent individually since time endowment is agent-specific and we do not grant the social planner the power to transfer time. Equally to individual best responses in Nash equilibrium, the social planner chooses $t_{ij} = 0$ if $t_{ji} = 0$, $t_{ij} > 0$ if $t_{ji} > 0$, and $t_{ii} > 0$.

If $t_{ij} = 0$ and $t_{ji} = 0$ the link ij does not exist in $\mathscr{T}$; if $t_{ij} > 0$ and $t_{ji} > 0$ the link ij exists. Hence, the social planner first has to decide which links to establish and then implement the efficient level of time investment on existing links. The FOCs for the efficient levels of time investment on existing links and the efficient level of self-investment are:

$$a\beta \left(\frac{t_{ji}}{t_{ij}}\right)^{1-\beta} + a(1-\beta)\left(\frac{t_{ji}}{t_{ij}}\right)^{\beta} = f'(t_{ii}) \qquad \text{for all } i \text{ and } j \neq i \text{ with } t_{ij}, t_{ji} > 0 \quad (3)$$

$$\sum_j t_{ij} = T \qquad \text{for all } i \quad (4)$$

As in the Nash equilibrium case, for every $\mathscr{T}$ which solves the system of linear equations the time investment strategies $\mathscr{T}^C := [t_i]_{i\in C}$ of agents within one component C of $n^C \geq 2$ can be again of two types. To distinguish between these two types of $\mathscr{T}^C$, we denote them as $\mathscr{T}_1^C$ and $\mathscr{T}_2^C$. $\mathscr{T}_1^C$ and $\mathscr{T}_2^C$ are candidates for $\mathscr{T}^C$ in an efficient network.

First, if $\mathscr{T}^C = \mathscr{T}_1^C$, then C is again reciprocal. More specifically, $\mathscr{T}_1^C$ is such that $t_{ij} = t_{ji} > 0$ for all links ij which exist in C, $t_{ij} = 0$ for all links ij which do not exist in C, t_{ii} satisfies $a = f'(t_{ii})$, and $\sum_j t_{ij} = T$ for all $i \in C$. Compared to the Nash stable investment levels of an r-agent r, $t_{ii} < t_{rr}$ because $a = f'(t_{ii}) > a\beta = f'(t_{rr})$ and f is strictly concave. An r-agent does not take into account the positive externality of her social time on her friends when choosing the level of self-investment. The social planner does and chooses a higher amount of social time than the individual agent does. $\mathscr{T}_1^C$ exists for all component architectures for which a Nash stable reciprocal time allocation exists because it exhibits the same properties as a Nash stable reciprocal time allocation except for another level of self-investment. The existence of a Nash stable reciprocal time allocation for a component architecture did not depend on the level of self-investment.

Second, if $\mathscr{T}^C = \mathscr{T}_2^C$, then C is again non-reciprocal. As a Nash stable non-reciprocal time allocation, $\mathscr{T}_2^C$ implies a bipartite component with one set of l-agents L^C and one set of h-agents H^C where $\left|L^C\right| < \left|H^C\right|$. Moreover, $\frac{t_{hl}}{t_{lh}} = q > 1$ for every friendship in C. t_{hh} is such that $f'(t_{hh}) = a\beta \left(\frac{1}{q}\right)^{1-\beta} + a(1-\beta)\left(\frac{1}{q}\right)^{\beta}$, and t_{ll} is such that $f'(t_{ll}) = a\beta q^{1-\beta} + a(1-\beta)q^{\beta}$. Hence, $t_{hh} > t_{ll}$. Since $\mathscr{T}_2^C$ implies a bipartition of the component, the non-existence of links among l-agents and among h-agents must be efficient for $\mathscr{T}^C = \mathscr{T}_2^C$ in an efficient network. Yet, we can show that welfare strictly improves if the high self-investment of two h-agents is reduced and a reciprocal link between these two agents is established. In $\mathscr{T}_2^C$, every h-agent has a relatively low marginal utility from friendship and self-investment. The linear increase in utility due to establishing a reciprocal relationship by a certain ϵ outweighs the loss in utility from reducing the high level of self-investment by the same ϵ. Therefore, $\mathscr{T}^C \neq \mathscr{T}_2^C$ in an efficient network. A rigorous proof is given in relation to Proposition 5.2 in the appendix. Hence, every component of $n^C \geq 2$ in an efficient network must be reciprocal with $\mathscr{T}^C = \mathscr{T}_1^C$.

Furthermore, there do not exist isolated agents, components of $n^C = 1$, in an efficient network. If an agent does not have any links, she should spend her whole time on self-investment to maximize her utility. Then, this isolated agent has an even lower marginal utility from self-investment than an h-agent had in the non-reciprocal component. Hence, the social planner can make two isolated agents strictly better off by reducing their high self-investment and establishing a reciprocal link between them. Another way to make an isolated agent k strictly better and no other agent worse off is to append k to two linked agents m and n in a reciprocal component: Reduce the time investment between m and n by a certain ϵ and the self-investment of k by 2ϵ. Establish one reciprocal link between k and m and one between k and n with investment ϵ on each new link. Now, the utility of k has increased and the utility of m and n has not changed. A formal proof can be constructed similarly to the proof that two h-agents can be made strictly better off.

Hence, an efficient network $\mathscr{T}$ only consists of reciprocal components of $n^C \geq 2$ with $\mathscr{T}^C = \mathscr{T}_1^C$. Observe that every $\mathscr{T}$ which only consists of such components yields the same sum of utilities independent of the underlying network architecture. If friendships are reciprocal, utility from each friendship is linear in the level of time investment in this friendship due to the constant returns to scale. This means that the sum of utilities is the same for different network architectures and different friendship intensities. Thus, every $\mathscr{T}$ which only consists of reciprocal components of $n^C \geq 2$ with $\mathscr{T}^C = \mathscr{T}_1^C$ and no other $\mathscr{T}$ is an efficient network.

Proposition 5.2.

1. *$\mathscr{T}$ is efficient if and only if $t_{ij} = t_{ji}$ for all $j \neq i$, t_{ii} such that $f'(t_{ii}) = a$, and $\sum_j t_{ij} = T$ for all i.*

2. *Let $G = (N, E)$ be a network architecture with a set of nodes N and a set of links E. There exists an efficient $\mathscr{T}$ for G (this means with $t_{ij} > 0$ for all $ij \in E$ and $t_{ij} = 0$ for all $ij \notin E$) if and only if G is composed of component architectures for which a Nash stable reciprocal time allocation exists.*

6 Conclusion

We investigated which networks are Nash stable, strongly pairwise stable and efficient when every agent can invest a limited budget into her private productive undertakings and specifically into productive links with all other agents in the network. Agents are interdependent as an agent's utility maximizing investment levels depend on the specific investments she receives from other agents. All agents are ex-ante homogenous in the sense that every agent has the same preferences and characteristics, and the same limited budget.

We found two different types of Nash stable components: first, the rather symmetric reciprocal Nash stable component in which agents match each others' investment, choose the same level of self-investment and receive the same intermediate utility; second, the rather asymmetric non-reciprocal Nash stable component with a smaller group of *low intensity* agents compared to a larger group of *high intensity* agents. *Low intensity* agents choose a low self-investment and invest an overall large amount into the network, but always less into each link than their link partner. *Low intensity* agents maintain more links on average than *high intensity* agents. *High intensity* agents contrarily choose a high self-investment and relatively little overall investment into the network. Yet, into every link they maintain they invest more than their partner. We could describe the different equilibrium behaviors of *low* and *high intensity* agents with several adjectives: *low intensity* agents are more diversified, more popular, more actively networking, free-riding on their link partner's higher effort; *high intensity* agents rely more on themselves, are more concentrated and thorough, and provide high efforts in relationships. *Low intensity* agents secure themselves a high utility with this behavior and *high intensity* agents only obtain a low utility. Thus, in equilibrium it is possible that agents display different behavior, end up with different network positions, and consequently obtain different utility although they are ex-ante homogenous. We show that an asymmetric situation with homogenous agents is not efficient from the perspective of the social planner. Welfare-maximizing networks are reciprocal with a lower level of self-investment compared to a Nash stable reciprocal component.

When two agents are able to coordinate their investment strategies, no network which contains at least one component of $n^C \geq 3$ and/or at least two components of $n^C = 2$ is stable. In both cases, there exist two agents who have a strict incentive to establish a reciprocal link between each other while each of them reduces her investment into some other friend. These two agents free-ride on another friend's time investment, and are in a sense disloyal to this friend in order to receive a larger overall amount of investment.

One avenue for future research is to pursue the analysis with heterogeneous agents. The models lends itself to introducing heterogeneity in the intrinsic value of friendship, the as, in the marginal utility from self-investment, the f's, and in the productivity of an agent's own investment relative to the other agent's investment, the βs. This would allow to derive results on assortativity. Moreover, it should be investigated which results can be added if a more general form of value production in friendship like $at_{ij}^{\alpha}t_{ji}^{\beta}$ with $0 < \alpha < 1$ and $0 < \beta < 1$ is assumed. This way agents would still face decreasing marginal returns to individual investment. Yet, we would account for decreasing, constant and increasing returns to scale of joint investment. A further way to go is to analyze existing data sets or run experiments and test if results of the model can be confirmed.

Appendix

Sufficiency proof of Theorem 3.5.

Let $C = (N^C, E^C)$ be a component with set of nodes N^C and set of links E^C. Let $E^C[X, Y]$ be the set of links xy in E^C with $x \in X \subseteq N^C$, $y \in Y \subseteq N^C$ and $X \cap Y = \emptyset$. Let $E^C[Y]$ be the set of links ij in E^C with $i, j \in Y \subseteq N^C$. Denote by $\delta(i)$ the set of links incident with node $i \in N^C$. $C[Y] = (Y, E^C[Y])$ with $Y \subseteq N^C$ is the subgraph induced by Y. For every vector $w \in \mathbb{R}^Y$ with vector components w_y, let $w(U) := \sum_{y \in U} w_y$ for any $U \subseteq Y$. $\mathbb{Z}$ is the set of integers.

Theorem 35.1 by Schrijver (2004, p. 584) can be reduced to the following because we are only interested in a special case.

Theorem 35.1, Schrijver (2004, p. 584). *Let $C = (N^C, E^C)$ be a component and let $b \in \mathbb{Z}^{N^C}$ and $c \in \mathbb{Z}^{E^C}$ with every $c_{ij} > 1$. Then there exists an $x \in \mathbb{Z}^{E^C}$ such that (i) $1 \leq x_{ij} \leq c_{ij}$ for all $ij \in E^C$ and (ii) $x(\delta(i)) = b_i$ for all $i \in N^C$ if and only if for each partition $\{T, U, W\}$ of N^C, the number of components K of $C[T]$ with*

(35.2) $b\,|K| + c\,\left|E^C[K, W]\right| + \left|E^C[K, U]\right|$

odd is at most

(35.3) $b\,|U| - 2\,|E[U]| - |E[T, U]| - b\,|W| + 2c\,|E[W]| + c\,|E[T, W]|\,.$

Observe that a Nash stable reciprocal time allocation exists for $C = (N^C, E^C)$ iff there exists a solution $\mathscr{T}^C = [t_i]_{i \in N^C}$ with $t_{ii} = T - T_r^S$ to the following system of linear equations with constraints:

$$\begin{aligned}
\sum_{j\neq i} t_{ij} &= T_r^S \quad \textit{for all } i \in N^C \\
t_{ij} &= t_{ji} \quad \textit{for all } i,j \in N^C \textit{ with } j \neq i \\
\textit{such that } t_{ij} &> 0 \quad \textit{for all } ij \in E^C \\
\textit{and } t_{ij} &= 0 \quad \textit{for all } ij \notin E^C
\end{aligned}$$

Dividing the whole system by T_r^S, and defining $\pi_{ij} := \frac{t_{ij}}{T_r^S}$, we obtain a system with only rational coefficients and constants to solve for all π_{ij} with $i \neq j$.

$$\begin{aligned}
\sum_{j\neq i} \pi_{ij} &= 1 \qquad \textit{for all } i \in N^C \\
\pi_{ij} &= \pi_{ji} \quad \textit{for all } i,j \in N^C \textit{ with } j \neq i \\
\textit{such that } \pi_{ij} &> 0 \qquad \textit{for all } ij \in E^C \\
\textit{and } \pi_{ij} &= 0 \qquad \textit{for all } ij \notin E^C
\end{aligned}$$

It is known that if a system of linear equations with only rational coefficients and constants has a solution, then it has a rational solution. By multiplying our system of equations with an appropriate and large enough scaling factor $s \in \mathbb{Z}^+$, and defining $\Pi_{ij} := s\pi_{ij}$, we obtain a system to be solved for all Π_{ij} with $i \neq j$ which has an integer solution if it has a solution.

$$\begin{aligned}
\sum_{j\neq i} \Pi_{ij} &= s \qquad \textit{for all } i \in N^C \\
\Pi_{ij} &= \Pi_{ji} \quad \textit{for all } i,j \in N^C \textit{ with } j \neq i \\
\textit{such that } \Pi_{ij} &> 0 \qquad \textit{for all } ij \in E^C \\
\textit{and } \Pi_{ij} &= 0 \qquad \textit{for all } ij \notin E^C
\end{aligned}$$

Now we return to the Theorem by Schrijver. Let every $c_{ij} = c$ with c very large and let every $b_i = s$ with s be the appropriate large enough scaling factor. Then $x_{ij} = \Pi_{ij} = \Pi_{ji}$, and a Nash stable reciprocal time allocation for C exists iff x for C exists.

Next we show that if it is true for C that for all $A \subseteq N^C$ and B being the set of isolates in $C - A$ either 1. $|A| > |B|$, or 2. $|A| = |B|$ and for every link $ij \in E^C$ with $i \in A$ it is true that $j \in B$, then x exists and hence a Nash stable reciprocal time allocation exists. This will be proven by contradiction.

Assume that if it is true for C that for any $A \subseteq N^C$ and B being the set of isolates in $C - A$ either 1. $|A| > |B|$, or 2. $|A| = |B|$ and for every link $ij \in E^C$ with $i \in A$ it is true that $j \in B$, then x does not exist. Then by the Theorem of Schrijver there must be a partition $\{T, U, W\}$ of N^C such that the number of components K of $C[T]$ with (35.2) odd is greater than (35.3).

For any partition with $E^C[W] \neq \emptyset$ and/or $E^C[T, W] \neq \emptyset$ the number of components K with (35.2) odd is always smaller than (35.3) because c is very large and the number of components K is finite. Then there must be a partition with $E^C[W] = E^C[T, W] = \emptyset$ with a number of K with (35.2) odd greater than (35.3).

For every partition $\{T, U, W\}$ with $E^C[W] = E^C[T, W] = \emptyset$ it must be true that every $i \in W$ has only links to nodes U and that every $i \in W$ has at least one link to nodes in U because C is a component, or, in other words, all nodes in C are connected. Then, W is a subset of the set of isolates in $C - U$. Hence, $W \subseteq B$ for $U = A$. We know that in C for all $A \subseteq N^C$ either 1. $|A| > |B|$, or 2. $|A| = |B|$ and for every link $ij \in E^C$ with $i \in A$ it is true that $j \in B$. This implies that also for all possible sets of U either 1. $|U| > |W|$, or 2. $|U| = |W|$ and for every link $ij \in E^C$ with $i \in U$ it is true that $j \in W$. Thus, there does not exist a partition $\{T, U, W\}$ of N^C for which $E^C[W] = E^C[T, W] = \emptyset$ and $|U| < |W|$.

Then there must be a partition $\{T, U, W\}$ of N^C for which $E^C[W] = E^C[T, W] = \emptyset$ and $|U| \geq |W|$ such that the number of K with (35.2) odd is greater than (35.3).

However, for any partition with $E^C[W] = E^C[T, W] = \emptyset$ and $|U| > |W|$ the number of components K with (35.2) odd is always smaller than (35.3) because $b_i = s$ is chosen large enough.

For every partition with $E^C[W] = E^C[T, W] = \emptyset$ and $|U| = |W|$ we know that for every link $ij \in E^C$ with $i \in U$ it is true that $j \in W$ because if $|A| = |B|$ for every link $ij \in E^C$ with $i \in A$ it is true that $j \in B$, and $U = A$ and in this case $W = B$. Then, $E^C[T, U] = E^C[T, W] = \emptyset$. This implies that $T = \emptyset$. If T was not be empty, nodes in T would not be connected to neither U nor W, and C would not be a component (where all nodes are connected) which is a contradiction. Then the number of K is zero. (35.3) is also zero ($E^C[U] = \emptyset$ because for every link $ij \in E^C$ with $i \in U$ it is true that $j \in W$). Hence, the number of K is not greater than (35.3).

Thus, there does not exist any partition $\{T, U, W\}$ of N^C such that the number of K with (35.2) odd is greater than (35.3). This is a contradiction and x must exist for C. This completes the sufficiency proof. □

Proof of Lemma 4.1. We show that in a Nash stable network every r-agent r has a strict incentive to jointly deviate with any other agent j if r and j increase each of their investments into their friendship by $\epsilon \in (0, t_{rk})$ and r takes the time for the joint deviation from a friendship with another agent k. By this deviation r's utility from her friendship with j and k changes while any other utility is not affected. r has a strict incentive to jointly deviate with j if

$$at_{rj}^{\beta}t_{jr}^{1-\beta} + at_{rk}^{\beta}t_{kr}^{1-\beta} < a\left(t_{rj}+\epsilon\right)^{\beta}\left(t_{jr}+\epsilon\right)^{1-\beta} + a\left(t_{rk}-\epsilon\right)^{\beta}t_{kr}^{1-\beta}. \qquad (5)$$

We know that $t_{rj} = t_{jr}$ because either r had a reciprocal link or no link with j before the deviation. Also, $t_{rk} = t_{kr}$ because r and k had a reciprocal link before the deviation. Hence, we can rewrite (5) as

$$t_{rj} + t_{rk} < \left(t_{rj}+\epsilon\right) + \left(t_{rk}-\epsilon\right)^{\beta}t_{rk}^{1-\beta}$$

$$\Leftrightarrow t_{rk} - (t_{rk}-\epsilon)^{\beta}t_{rk}^{1-\beta} < \epsilon.$$

Since $t_{rk} - (t_{rk}-\epsilon)^{\beta}t_{rk}^{1-\beta} < t_{rk} - \left(t_{rk}-\epsilon\right)^{\beta}\left(t_{rk}-\epsilon\right)^{1-\beta} = \epsilon$, r has a strict incentive to deviate jointly. □

Proof of Lemma 4.2. We show that in a Nash stable network every h-agent h has a strict incentive to jointly deviate with any other r- or h-agent j if h and j jointly establish a reciprocal link between each other by increasing each of their investments into their friendship by ϵ and if h takes the time for the joint deviation from her friendship with an l-agent l with $\epsilon \in (0, t_{hl})$.

By this deviation h's utility from her friendship with j and l changes while any other utility is not affected. h has a strict incentive to jointly deviate with j if

$$\begin{aligned} t_{hl}^{\beta} t_{lh}^{1-\beta} &< \epsilon^{\beta} \epsilon^{1-\beta} + (t_{hl} - \epsilon)^{\beta} t_{lh}^{1-\beta} \\ \Leftrightarrow \left(t_{hl}^{\beta} - (t_{hl} - \epsilon)^{\beta} \right) t_{lh}^{1-\beta} &< \epsilon. \end{aligned}$$

As $t_{hl} > t_{lh}$, it is sufficient to show that

$$\left(t_{hl}^{\beta} - (t_{hl} - \epsilon)^{\beta} \right) t_{hl}^{1-\beta} \leq \epsilon.$$

We have already shown that is true in the proof of Lemma 4.1. □

Proof of Proposition 5.1. We show first that in Nash equilibrium $u_l > u_r$, second that $u_r > u_h$ and third conclude that $u_l > u_h$. We rewrite u_l first:

$$\begin{aligned} u_l &= \sum_{j \neq l} a t_{lj}^{\beta} t_{jl}^{1-\beta} + f(t_{ll}) \\ &= \sum_{j \neq l} a t_{lj} \frac{t_{lj}^{\beta}}{t_{lj}} t_{jl}^{1-\beta} + f(t_{ll}) \\ &= a \sum_{j \neq l} t_{lj} \left(\frac{t_{jl}}{t_{lj}} \right)^{1-\beta} + f(t_{ll}) \\ &= a T_l^S q^{1-\beta} + f(t_{ll}) \end{aligned}$$

The last equality follows from the fact that in a Nash stable non-reciprocal component, $\frac{t_{jl}}{t_{lj}} = q > 1$ for all existing links lj. Similarly, we can rewrite u_r as $u_r = a T_r^S + f(t_{rr})$ by making use of the fact that $\frac{t_{jr}}{t_{rj}} = 1$ in a Nash stable reciprocal component for all existing links rj. Now observe that

$$a T_l^S q^{1-\beta} + f(t_{ll}) > a T_r^S + f(t_{rr})$$

$$\Leftrightarrow a\left(T_l^S q^{1-\beta} - T_r^S\right) > f(t_{rr}) - f(t_{ll}) = \int_{t_{ll}}^{t_{rr}} f'(t)dt.$$

Since f is strictly concave and $t_{rr} > t_{ll}$, it is sufficient to show that

$$a\left(T_l^S q^{1-\beta} - T_r^S\right) > f'(t_{ll})(t_{rr} - t_{ll}).$$

From the FOCs we know that $f'(t_{ll}) = a\beta q^{1-\beta}$ and hence

$$a\left(T_l^S q^{1-\beta} - T_r^S\right) > a\beta q^{1-\beta}(T_l^S - T_r^S)$$

which is true. Thus, $u_l > u_r$.

With the same argument we used above adjusted to the specific case, we can rewrite u_h as $u_h = aT_h^S\left(\frac{1}{q}\right)^{1-\beta} + f(t_{hh})$ and then show that $u_r > u_h$. □

Proof of Proposition 5.2. We show that a bipartition of a non-reciprocal component implied by $\mathscr{T}^C = \mathscr{T}_2^C$ is not efficient because two h-agents can both be made strictly better off by reducing their self-investment by $\epsilon \in (0, t_{hh} - c]$ with $f'(c) = a$ and establishing a reciprocal link with a mutual investment of ϵ between them. In every non-reciprocal component with $\mathscr{T}^C = \mathscr{T}_2^C$ there are at least two h-agents because $\left|H^C\right| > \left|L^C\right| \geq 1$. The change in utility due to this operation for an h-agent h is

$$\Delta u_h = a\epsilon + f(t_{hh} - \epsilon) - f(t_{hh}) = a\epsilon - \int_{t_{hh}-\epsilon}^{t_{hh}} f'(t)dt.$$

Because of the strict concavity of f

$$a\epsilon - \int_{t_{hh}-\epsilon}^{t_{hh}} f'(t)dt > a\epsilon - f'(t_{hh} - \epsilon)\epsilon$$

if $\epsilon > 0$.

Moreover, $f'(t_{hh}) = a\beta\left(\frac{1}{q}\right)^{1-\beta} + a(1-\beta)\left(\frac{1}{q}\right)^{\beta} < a$ as $\frac{1}{q} < 1$. Hence, for every $\epsilon \in (0, t_{hh} - c]$ with $f'(c) = a$

$$\Delta u_h > a\epsilon - f'(t_{hh} - \epsilon)\epsilon \geq a\epsilon - a\epsilon = 0.$$

No other agent besides the two h-agents is affected in her level of utility.

Thus, there does not exist a non-reciprocal component with $\mathscr{T}^C = \mathscr{T}_2^C$ in an efficient network.

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