Heterogeneous Countries in a Financial Union*

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First version: May 2014
This version: February 2015

Abstract

A financial union is a group of countries, each with its own nontradable goods sector, which can freely exchange tradable goods and debt contracts. In this paper, we establish the effects of shocks in a stylized financial union with heterogeneous regions—a lender North and a borrower South—and constraints on borrowing. We derive positive and normative results. First, when the degree of heterogeneity is high before the shock, the South is disproportionately hurt by the shock, no matter whether the shock strikes in the North or the South. Second, for a given value of the shock, when borrowing constraints bind in the South, the welfare of the North decreases while the welfare of the South may increase. Third, we characterize which policy interventions are able to generate Pareto improvements. Unconditional debt relief for the South fails to do so. Subsidized governmental loans succeed when Southern governments can commit to repay additional debt. Finally, whether or not Southern governments can commit to repay anything, a Pareto improvement is possible using a combination of conditional debt relief and a tax/subsidy package in the South.

JEL classification: F34, F36, F45

Keywords: Financial union, capital flows, heterogeneous countries, sudden stops, currency union, fiscal union, banking union

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We thank Olivier Blanchard, Marcos Chamon, Atish Ghosh, Francesco Giavazzi, Bernardo Huberman, Subir Lall, Alberto Martin and participants at presentations at the IMF and HP Labs for useful comments on various versions of this work. All errors remaining are our own. The views expressed in this document are those of the authors and do not necessarily represent those of the IMF, its Executive Board or Management, or those of HP and HP Labs.

1. Introduction

How do shocks affect the stability of economically and financially integrated groups of countries which are arranged in a core-periphery structure? And what policies are needed to deal with shocks that strike such country groupings?

These questions have come to the fore as European integration has deepened into a European Union and then a Eurozone. While early integration took the form of trade liberalization coupled with limited financial flows (the environment which stimulated Mundell 1961 and other theorists of optimal currency areas), the subsequent removal of barriers to private and public sector debt transactions brought forth a wave of large international flows which reshaped the net asset positions of member countries (documented in Giavazzi and Spaventa 2010 and Chen, Milesi-Ferretti and Tressel 2012). The Eurozone began to feature a Northern "core" supplying and/or intermediating funds to a Southern "periphery," echoing similar structures found within other groupings of heterogeneous countries and regions—such as East-West capital flows in the United States during the nineteenth century, flows between the United Kingdom and its colonies during the same period, and flows between the United States and the rest of the world today. The pattern of large debt flows from the core to the periphery, followed by a financial crisis with retrenchment in the core and deleveraging in the periphery, seems to prevail in the historical data no matter whether the crisis is triggered by a malfeasance in banks in the core, bursting bubbles in the periphery, or both. The periphery suffers reversals in the current account and collapses in investment, while its effective borrowing rates spike well above lending rates in the core. In fact, the severity of the crisis-era reversal of private capital inflows into Eurozone periphery economies, documented by Merler and Pisani-Ferry (2012), brings to mind findings from the literature on emerging market sudden stops (initiated by Dornbusch, Goldfajn and Valdés 1995 and Calvo 1998).

Our objective in this paper is to offer answers to the questions posed above, by designing and solving a model of a "financial union"—a collection of countries which enjoy in every period a single union-wide market in tradable goods and noncontingent debt contracts. We show how the degree of heterogeneity within the financial union is a crucial factor in determining how local shocks affect the consumption, investment, and welfare of each country. We prove that coalitions are better able than individual governments to agree on the implementation of Pareto-improving policy interventions after the shock. Finally, we design two instances of such policies, *i.e.*, subsidized government loans and debt relief programs, and evaluate the possibility for the coalitions to institutionalize them before the shock.

We consider a financial union with countries grouped into two heterogeneous regions—North and South—which begin with different levels of initial endowments. Each country consists of households, firms and an associated nontradable good, and all prices are flexible. Our model includes an initial period where consumption occurs, which can deplete assets in some countries or saddle them with noncontingent debt, and then there are two further cycles of debt transactions. The Northern countries endogenously emerge as lenders and the Southern countries as borrowers. In the first cycle, firms invest to produce nontradable goods, and a regional "financial shock" may strike and make some goods unavailable for investment purposes in one of the regions. In the second cycle, households finance the consumption of tradable and nontradable goods, subject to a borrowing constraint related to the future value of tradable and nontradable endowments.

The structure of our model combines elements that are characteristic of the optimal currency area literature together with features that are more common in studies about sudden stops in individual economies. As in the optimal currency area literature, we are interested in the effects of regional shocks in a setting where nominal exchange rates cannot change. However, prices are flexible in our model, so our single currency assumption matters to us only because it simplifies financial arbitrage, not because it distorts market clearing. From the sudden stops literature, we borrow the modeling of borrowing constraints. However, we analyze groups of interacting economies instead of individual countries, and we solve for the endogenous general equilibrium interest rates. Finally, the most distinctive feature of our model is the heterogeneity of countries in advance of shocks, rather than heterogeneity induced by asymmetric shocks.

Our first result is that the distributional effect of a shock depends greatly on the degree of heterogeneity that prevailed prior to the shock. When heterogeneity is low, i.e., regions have similar net asset positions before the shock strikes, the region which is hit by the shock suffers most strongly. This outcome echoes the result derived in the optimal currency area literature. On the other hand, when heterogeneity is high, the outcome changes starkly, and the Southern countries are always disproportionately hurt in terms of consumption, investment and welfare—whether the shock strikes the North or the South. The reason is that the shock has both a direct effect on the region where it strikes by making goods unavailable for investment, and an indirect effect by increasing the interest rate in the first cycle. When heterogeneity is high, the latter effect hurts the South hardest, because the Southern countries either have inherited debt or need to borrow in future periods. By contrast, the lender Northern countries benefit from the increase in the interest rate. The inclusion of investment to produce nontradable goods in our model means that the Southern countries may be disproportionately hurt even when they finish the first period not with debt, but with a small level of assets.

The second result is that when shocks strike anywhere in a heterogeneous union, borrowing constraints tend to bind for households in the South, and when they do so, it is the Northern countries' welfare which decreases. The reason is that the two cycles of borrowing and lending interact. A shock anywhere in the union which increases the interest rate in the first cycle also causes the Southern countries' debt burden to spiral upward by the second cycle. With more to roll over, the Southern countries are more likely to face binding borrowing constraints. It does not help that the market value of the Southern countries' future nontradable goods also falls. When constraints bind, there emerges a wedge between the lending rate and the implied rate of return calculated from the consumption profile over time in the South. The North's welfare decreases because as a lender region, it suffers on the infra-marginal dimension from the decrease in the union-wide interest rate in the period when the borrowing constraint binds in the Southern countries. The inclusion of investment to produce nontradable goods can generate a reverse feedback between cycles of borrowing and lending. Because such investment is tied to the future demand for nontradable goods in the same country, a binding constraint on households in the South (i.e., credit supply constraints) in the second cycle causes a collapse in Southern investment (i.e., credit demand) in the first cycle. Therefore, the interest rate may decrease in periods even before the constraint actually binds. This may hurt the Northern countries further. If the constraint is moderately binding, the Southern countries benefit in net terms from the decrease in interest rates; if the borrowing constraint is very stringent, the Southern countries' welfare decreases.

The third set of results relates to policy interventions: we outline a series of interventions and establish whether Pareto improvements are possible with each of them. We focus on financial unions with a relatively high degree of heterogeneity: the South is the region that suffers the most, no matter where the shock hits. We show that individual countries cannot easily implement Pareto-improving policies. Indeed, for each intervention, there is a collective action problem. The governments of the North have an incentive to free-ride on the actions of others, which means that the intervention will be under-provided. At the same time, governments of the South ignore the impact of their actions on the interest rates, and are individually willing to undertake policies that in the aggregate may make all the South worse off. We show that Pareto-improving policies can be designed and implemented if countries join two large regional coalitions (i.e., North and South), which then negotiate with each other. We examine three specific policies: unconditional gifts, subsidized governmental loans, and debt relief conditional on a fiscal policy package.

An unconditional gift from the North to the South, which can be interpreted as unconditional debt relief, cannot achieve a Pareto improvement. If the gift is made in the period when the borrowing constraint is binding in the South, it does succeed in relaxing the South's borrowing constraint and increasing the union-wide interest rates in all periods. The South benefits from this policy but the North is hurt, because the increase in interest rates is not enough to compensate the North for the cost associated with the gift. This negative result comes directly from the resource constraint and points us in the right direction for designing Pareto-improving policies.

A subsidized governmental loan from the North to the South (i.e., a loan between governments with partial repayment), within a context where each Southern government possesses some capacity to repay loans on its own, can achieve a Pareto improvement. Southern governments borrow from Northern governments and implement transfers within their own countries which in equilibrium help alleviate the borrowing constraints of households in the South. The consumption of Southern households increases in the period when the borrowing constraint is binding, but it decreases in the following period as the loan repayment is made. This loan repayment is essential to make the North better off, which is why an unconditional gift equal to the subsidy amount does not suffice to generate a Pareto improvement. At first glance, the subsidy on the loan may appear to run afoul of the transfer problem discussed by Keynes (1929) and Ohlin (1929), which posits that the donor country suffers both directly from the cost of the transfer it provides, and indirectly through a terms of trade deterioration. Therefore, a subsidized loan threatens to reduce the welfare of the North and make a Pareto improvement impossible. However, the original transfer problem does not consider inter-country transfers in a context where the private sector of the donor country has lent, or expects to lend, to the recipient country, and where borrowing constraints bind. The North benefits from providing a subsidy because by inducing the South to accept the governmental loans, it achieves increases in market interest rates, so that households in the North receive a higher return on their "exports" of international loans. Counter-intuitively, it may be that the larger is the nontradable goods sector, the larger the impact of the loan on interest rates in general equilibrium, so the larger the subsidy the North is willing to provide.

Debt relief for the South which is made conditional on a future tax on tradable consumption and a future subsidy on nontradable consumption can achieve a Pareto improvement. Differently from the subsidized governmental loan, this policy is based on the existence of the nontradable goods sector in the final period of our model. When governments in the South cannot commit to repay additional loans and thereby mitigate the constraints on private borrowing, they must try instead to directly increase the value of the borrowing limit of the Southern private sector. The value of the tradable endowment is fixed, so the only remaining option is to increase the price of future nontradable goods. To ensure that the North is sufficiently compensated for the debt relief, the higher future nontradable goods price must be achieved without increasing the future tradable consumption of Southern households. Therefore, debt relief for the South must be conditioned on the creation of a future price wedge between tradable and nontradable consumption. Our model directs future subsidies to be provided for nontradable goods which are collateralizable—such as housing and fixed capital—rather than for all nontradable goods, because the primary objective of the policy intervention is to increase the borrowing capacity of the South.

For both the proposed Pareto-improving policies, the set of interventions which regional coalitions negotiate to improve their welfares after the shock is realized (ex post Pareto improvements) may not overlap entirely with the set of interventions which would improve the welfares of both regional coalitions from a pre-shock perspective (ex ante Pareto improvements). Therefore, both regional coalitions may wish to meet before the shock to negotiate institutions which help shape the bargaining process after the shock. We offer a formal analysis to identify some Pareto-improving policies to be implemented at the ex ante stage.

We conclude by commenting on our results in terms of their relevance to the ongoing debate around the governance of financial unions such as the Eurozone. The first recommendation that emerges from our analysis is about the benefits from shifting the negotiation approach from a country-specific to a core-periphery mode, explicitly recognizing the lender-borrower relation that characterizes the interaction between the union members. Regarding the actual policies proposed, the conditional debt relief and fiscal policy package is admittedly more unorthodox than the subsidized governmental loan. Indeed, the policy recommendation is counter-intuitive: while the tax on tradable consumption has some echoes of the narrative of fiscal austerity, the subsidy on nontradable consumption does not. Moreover, it raises fears of future asset price bubbles. Our model suggests that notwithstanding excessive asset price valuations in the Eurozone periphery before the shock of the global financial crisis, downward price flexibility in the nontradable sector is prolonging the Eurozone crisis. Regarding the possibility of institutionalizing Pareto-improving policies connected to governmental loans, one can imagine designing institutions like the IMF and the ESM which offer special loans in times of crisis. With the conditional debt relief and fiscal policy package, some degree of institutionalization may be possible, but additional care needs to be taken because the elements of the action space are more disparate, and the coalitions that need to be assembled are more diverse.

The remainder of this paper is organized as follows. Section 2 provides a literature review to complement the discussion above. Section 3 outlines the building blocks of our model and the definition of equilibrium. Section 4 collects the positive results of our environment without government intervention. Section 5 outlines the three policy interventions described above and characterizes the relevant Pareto sets where they exist. In each Section we provide analytical results for a Cobb-Douglas specification of the production function and numerical simulation for an alternative production function. Section 6 concludes with some thoughts on policy institutionalization, the role of extra-union countries, on comparisons of the interventions to each other and to some actual crisis-era policy measures, and finally on the ongoing discussion about fiscal and banking union. Proofs of Lemmas and Propositions are collected in the Appendix.

2. Related Literature

First, the question of how shocks affect the stability of integrated groups of countries was popularized by the optimal currency area literature. Given the post-War environment of expanding trade integration coupled with limited financial integration, this literature focused on the former. Mundell (1961) argued that countries which are susceptible to asymmetric shocks to export demand should not share membership of a common currency area. McKinnon (1963) and Kenen (1969) identified openness and output diversification as additional membership criteria. These considerations fit into contemporary debates on the role of exchange rate adjustment, for example Friedman (1953). We ask some similar questions to this literature, but we depart from their (explicit and implicit) assumptions that prices are sticky and that countries are identical before shocks strike. Instead of asymmetric shocks, we focus on the heterogeneity of countries before shocks strike—the core-periphery structure—as a primary determinant of post-shock outcomes.

After asymmetric shocks, Kenen (1969) calls for fiscal stimulus financed by inter-country transfers within the currency area. Beetsma and Jensen (2005) and Galí and Monacelli (2008) demonstrate that with sticky prices, country-level fiscal stimulus after shocks is necessary for macroeconomic stabilization. Farhi and Werning (2013a) develop a model with nontradable goods which nests results from the optimal currency area literature while identifying a novel externality: with sticky prices, private insurance is Pareto inefficient, and fiscal transfers may become necessary after asymmetric shocks. In our paper, we expand the set of policy interventions to include debt relief and budget-neutral tax/subsidy packages, and we attempt to generate Pareto improvements.

Second, we use borrowing constraints from the sudden stops literature for emerging markets. Dornbusch, Goldfajn and Valdés (1995) and Calvo (1998) characterize sudden stops and relate them to debt repayment problems. Caballero and Krishnamurthy (2001) develop a model of sudden stops with domestic and international collateral constraints, both based on tradable output. In order to match the output of calibrated models to the empirics of sudden stops, and to conduct policy analysis, Mendoza (2002, 2006), Mendoza and Smith (2006), Bianchi (2011), and Korinek (2011) have introduced a variety of borrowing constraints based on the value of tradable output, nontradable sector output and/or capital. We use in this paper a by-now standard borrowing constraint based on the market value of future tradable and nontradable endowments.

While these papers typically focus on taxes on capital inflows to limit debt before shocks, Jeanne and Korinek (2013) and a recent paper by Benigno, Chen, Otrok, Rebucci, and Young (2014) explore policy interventions both before and after shocks. In this paper, we mostly analyze policy interventions which are implemented after shocks. At a more fundamental level, while the sudden stop literature analyzes the impact of a binding constraint for a single borrower country, we develop a model with multiple borrower and lender countries with endogenous general equilibrium interest rates. The tax/subsidy packages we consider can be related to the fiscal instruments analyzed by Benigno et al. (2014), but the welfare results are turned around in our setting because interest rates adjust. Nontradable sector subsidies in borrower countries benefit lender countries instead of borrower countries, so the latter must be compensated through debt relief.

Third, our results on Pareto-improving governmental loans and conditional debt relief can be related to the literatures on debt overhang and on the multi-period version of Keynes (1929) and Ohlin's (1929) transfer problem. In his seminal paper on debt overhang, Krugman (1988) argued that debt relief may be optimal for lender countries when such relief increases future investment effort by the debtor countries. In our setting, debt relief is the inducement that lender countries must give borrower countries: in exchange, borrower countries implement tax/subsidy packages which benefit lender countries by raising general equilibrium interest rates. The subsidized governmental loans we consider also offer partial debt relief en exchange for a similar effect on interest rates.

On the multi-period transfer problem, Djajić, Lahiri and Raimondos-Møller (1998) and Cremers and Sen (2009) show that inter-country transfers affect interest rates when discount rates and intertemporal substitution elasticities vary across countries. In our paper, preferences are identical across countries, and transfers from lender countries to borrower countries increase general equilibrium interest rates after large shocks which cause borrowing constraints to bind.

Fourth, our work is related to the literature on the international transmission of debt deleveraging shocks. Benigno and Romei (2012) show that the interest rate falls substantially in response to an exogenous fall in the debt limit, and an extended period of easy monetary policy is their recommended response. Fornaro (2014) shows that a similar shock to some high-debt countries of a monetary union can push the whole union into a recession if nominal wages are sticky and the common central bank faces a zero lower bound for the policy rate. In this paper, we focus on different questions. We are interested in distributional concerns: i.e., why shocks that may originate in various different parts of a financial union ultimately always have their most severe effects on a subset of the union. We are also interested in designing Pareto-improving interventions when borrowing constraints are binding.

Finally, our paper is related to the literature on the evolution of capital flows in the Eurozone. Giavazzi and Spaventa (2010) present evidence that the Eurozone crisis was caused by the build-up of unsustainable external imbalances in the periphery. Decomposing net foreign asset flows during 2000-08 into transactions within the Eurozone and between the Eurozone and the rest of the world, Chen, Milesi-Ferretti and Tressel (2012) reveal the central role of intra-Eurozone debt flows: large net lending from the core to the periphery financed the current account deficits of the periphery with respect to the rest of the world. Merler and Pisani-Ferry (2012) document the reversal in private capital inflows into individual periphery economies and convincingly argue that they look much like sudden stops in emerging market economies. We present a model consistent with these observations, where a financial union endogenously assumes a core-periphery structure, and where shocks can force any financially integrated country with high levels of inherited debt and future borrowing needs into the inefficiently rapid deleveraging associated with a sudden stop.

Giavazzi and Spaventa (2010) were among the first to argue that Eurozone concerns should be broadened out from sovereign debt to external indebtedness in general. Shambaugh (2012) documents that overall external debt on the eve of the crisis is a better predictor of subsequent problems than just the public debt. Martin and Philippon (2014) calibrate a model to assess the macroeconomic effects of fiscal policy, private leverage and spreads on macroeconomic outcomes. For simplicity, in this paper we have only one variable—the external indebtedness of the private sector—to represent the net asset position of the entire economy, before policy interventions are considered. We include nontradable goods prices in the external borrowing constraint to deliver new positive and normative results.

3. Heterogeneous Countries in a Financial Union

Our goals are to represent an heterogeneous financial union in which countries are subject to borrowing constraints and to determine how a regional shock affects consumption, investment and welfare throughout the union. To do so, we use a multi-period model.

Its core is between periods $t = 0$ and $t = 1$. At $t = 0$ a shock hits a region and reduces the quantity of tradable goods that can be invested to produce nontradable goods in $t = 1$. The shock impacts borrowing/lending activities and investment at $t = 0$, and production and consumption at $t = 1$. There are no borrowing constraints at $t = 0$ so that the unconstrained effect of consumption on investment can be identified.

To this core we add two elements. On one side, to analyze how pre-existing heterogeneity within the union changes the effects of the shock, we add one period at the beginning, before the shock is observed, representing the past, i.e., $t = -1$. This allows us to break the implicit assumption in the optimal currency area literature that countries are identical before shocks strike. In our setting, countries start with different endowments of tradable goods, and, at $t = -1$, they consume, borrow and lend. The existence of nontradable goods is not necessary at $t = -1$. On the other side, to analyze the role of borrowing constraints, we add one period at the end, representing the future, i.e., $t = 2$. Consumers receive tradable and nontradable endowments in this future period. They can pledge some portion of these endowments to increase consumption earlier, in the core part of our model. How much can be pledged is related to the value of future nontradable output. To simplify matters, the quantity of this output is fixed so that the borrowing limit varies only with the future nontradable goods price.

Our orchestration of the precise sequence of investment and constraints is not accidental: we have tried to design a parsimonious model to produce results which are robust to more general environments. Our results are qualitatively unaltered even if—as is more plausible—there is a limited endowment in every period, consumption of tradable and nontradable goods in every period subject to constraints, and investment in each period subject to constraints to produce both tradable and nontradable goods in the next.

Nevertheless, we choose to reduce this more general model to a simpler setup, with investment to produce nontradable goods followed by consumption subject to borrowing constraints, in order to separately highlight some key relationships from the general model itself. First, relative to investment to produce tradable goods, investment to produce nontradable goods in country i is more sensitive to a constraint-induced collapse in country i's consumption demand, because nontradable output cannot be exported. Second, making investment also subject to constraints would mix the negative impacts of the constraints on credit demand and supply within every period. Instead, we highlight the impact on credit supply at $t = 1$ and on credit demand at $t = 0$.

3.1. Model

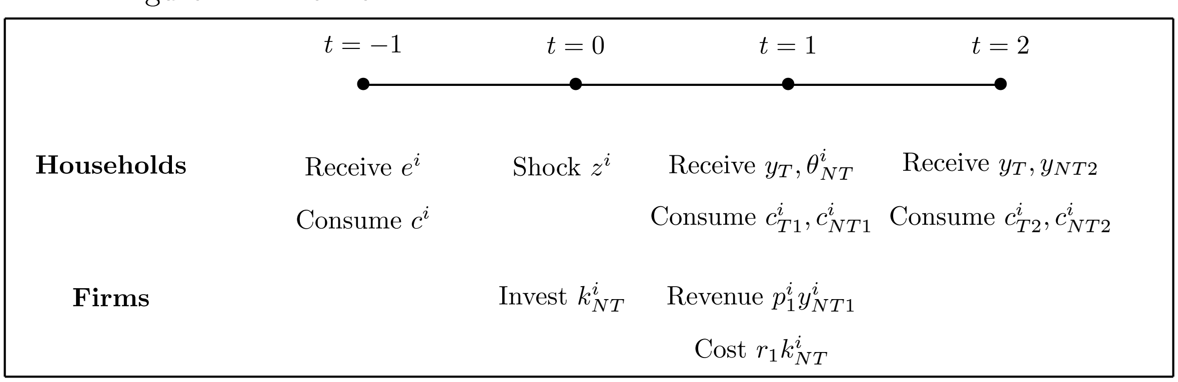
Countries, regions and the financial union. Each country i contains a unit measure of households who consume tradable and nontradable goods, and a measure μ of firms which invest tradable goods in order to produce nontradable goods. Households in country i have an initial endowment e^i of tradable goods. There are two regions $j \in \{N, S\}$ (North and South), each containing a unit measure of countries. Within each region, the initial endowments of households in each country are identical: $e^i = e^j$ for all $i \in j$. Regions are heterogeneous:

$$e^N \geq e^S, \text{ with } \sum_{j=N,S} e^j = e^T. \tag{1}$$

A financial union is a union of these two regions such that between each period and the next, individuals in all countries competitively trade noncontingent, one-period debt contracts in a single union-wide market.

Borrowing and lending. As shown in figure 1, the model has four periods. Goods can be stored one-for-one between $t = -1$ and $t = 0$, but not between the other periods.

Figure 1: Timeline



At $t = -1$, households in all countries receive their initial endowment of tradable goods, and they consume c^i of them. If they have goods left over, they store them; if they consume more goods than they own, they borrow goods to fill the gap. The possibility of storage fixes the union-wide gross interest rate between $t = -1$ and $t = 0$ at unity.

Households enter $t = 0$ with assets or debt. After this, there are two more cycles of borrowing and lending, and this paper is focused on the interaction between these cycles.

At $t = 0$, firms in country i borrow from households at union-wide interest rate r_1 in order to finance investment of k^i_{NT} units of the tradable good. At $t = 1$, this investment produces y^i_{NT1} units of the nontradable good, which is sold at price p^i_1. Households do not consume at $t = 0$ but do consume tradable goods c^i_{T1} and nontradable goods c^i_{NT1} at $t = 1$. Households have two sources of income at $t = 1$. First, households in all countries receive the same tradable endowment y_T. Second, households in each country own an equal share of all firms in the same country, and receive their profits θ^i_{NT}.

At $t = 1$, households can also borrow at union-wide interest rate r_2 in order to finance their consumption. However, there is a borrowing constraint: households can only pledge a fraction $\phi < 1$ of the net present value of the endowments that they **will** receive at $t = 2$. Households in all countries receive identical endowments of y_T units of tradable goods and y_{NT2} units of nontradable goods at $t = 2$. In this period, households consume tradable goods c^i_{T2} and nontradable goods c^i_{NT2}. The price of nontradable goods in country i at $t = 2$ is p^i_2. Therefore, the maximum borrowing at $t = 1$ is:

$$\frac{\phi\left[y_T + p^i_2 y_{NT2}\right]}{r_2}. \tag{2}$$

Shocks and resource constraints. At $t = 0$, there is a regional shock $z = \left(z^N, z^S\right)$: all households of countries i in region j must purchase $z^i = z^j$ units of the tradable good and burn it. The shock is designed to replicate a "financial shock" in the real world, where individuals need to put aside goods to compensate for past losses or to raise liquidity buffers. These goods do not contribute to household welfare. The shock has probability distribution:

$$\begin{array}{cc} z^N \sim U\left[0,\overline{z}\right], z^S = 0 & \text{with probability } \pi \\ z^N = 0, z^S \sim U\left[0,\overline{z}\right] & \text{with probability } \pi \\ z^N = z^S = 0 & \text{with probability } 1-2\pi \end{array} \tag{3}$$

Defining $k^j_{NT} = \int_{i\in j} k^i_{NT} di$ and $c^j = \int_{i\in j} c^i di$, the resource constraint at $t = 0$ is:

$$\sum_{j=N,S} k^j_{NT} \leq \sum_{j=N,S} \left(e^j - c^j - z^j\right). \tag{4}$$

Defining $c^j_{Tt} = \int_{i\in j} c^i_{Tt} di$, the resource constraints for tradable and nontradable goods at

$t = 1$ and $t = 2$ are:

$$\sum_{j=N,S} c_{Tt}^{j} \leq 2y_T \tag{5}$$

$$c_{NTt}^{i} \leq y_{NTt}^{i}. \tag{6}$$

Investment decision. Firms in each country i have a Cobb-Douglas production technology, which they use to convert investments k_{NT}^{i} of tradable goods at $t = 0$ into nontradable goods output y_{NT1}^{i} at $t = 1$:

$$y_{NT1}^{i} = \left(k_{NT}^{i}\right)^{\alpha}. \tag{7}$$

They maximize profits:

$$\theta_{NT}^{i} = p_1^i y_{NT1}^{i} - r_1 k_{NT}^{i}. \tag{8}$$

Therefore, the levels of investment, output and profits in country i are:

$$k_{NT}^{i} = \left(\frac{\alpha p_1^i}{r_1}\right)^{\frac{1}{1-\alpha}} \tag{9}$$

$$y_{NT1}^{i} = \left(\frac{\alpha p_1^i}{r_1}\right)^{\frac{\alpha}{1-\alpha}} \tag{10}$$

$$\theta_{NT}^{i} = \left(\frac{\alpha p_1^i}{r_1}\right)^{\frac{\alpha}{1-\alpha}} p_1^i \left(1-\alpha\right). \tag{11}$$

Consumption decision. Households in country i maximize expected utility:

$$\log\left(c^i\right) + \mathbb{E}_{t=-1}\left\{\begin{array}{c} \log\left(c_{T1}^{i}\right) + \nu\log\left(c_{NT1}^{i}\right) \\ +\log\left(c_{T2}^{i}\right) + \nu\log\left(c_{NT2}^{i}\right) \end{array}\right\} \tag{12}$$

subject to the budget constraint:

$$\left(c^i + z^i\right) + \frac{c_{T1}^{i} + p_1^i c_{NT1}^{i}}{r_1} + \frac{c_{T2}^{i} + p_2^i c_{NT2}^{i}}{r_1 r_2} \leq e^i + \frac{y_T + \theta_{NT}^{i}}{r_1} + \frac{y_T + p_2^i y_{NT2}}{r_1 r_2} \tag{13}$$

and the borrowing constraint:

$$r_1\left(c^i + z^i - e^i\right) + \left(c_{T1}^{i} + p_1^i c_{NT1}^{i}\right) - \left(y_T + \theta_{NT}^{i}\right) \leq \frac{\phi\left[y_T + p_2^i y_{NT2}\right]}{r_2}. \tag{14}$$

Household consumption is given by:

$$\frac{1}{c^i} = \mathbb{E}_{t=-1}\left\{\frac{r_1}{c_{T1}^{i}}\right\} \tag{15}$$

$$p_1^i c_{NT1}^{i} = \nu c_{T1}^{i} \tag{16}$$

$$p_2^i c_{NT2}^i = \nu c_{T2}^i. \tag{17}$$

When $\nu > 1$, there is *amplification*: spending on nontradable goods is more volatile than spending on tradable goods. When the borrowing constraint is not binding:

$$c_{T2}^i = r_2 c_{T1}^i, \tag{18}$$

and when the borrowing constraint is binding:

$$r_1 \left(c^i + z^i - e^i\right) + \left(c_{T1}^i + p_1^i c_{NT1}^i\right) - \left(y_T + \theta_{NT}^i\right) = \frac{\phi \left[y_T + p_2^i y_{NT2}\right]}{r_2}. \tag{19}$$

3.2. Equilibrium Definition

Definition 1 *A rational expectations equilibrium for this model is a set of interest rates* $\{r_1(z), r_2(z)\}$*, prices* $\{p_1^i(z), p_2^i(z)\}$ *and allocations* $\{c^i, k_{NT}^i(z), y_{NT1}^i(z), c_{T1}^i(z), c_{NT1}^i(z), c_{T2}^i(z), c_{NT2}^i(z)\}$ *which satisfy the optimality conditions of households and firms, and the financial union's resource constraints for tradable and nontradable goods.*

Lemma 1 *Households and firms of all countries* i *in the same region* j *have the same values of all equilibrium variables.*

In particular, all countries i in region j have identical asset positions in all periods. Following Lemma 1, we save on notation in the remainder of this paper by using the region identifier j to index all variables for country $i \in j$.

3.3. Competitive Equilibrium Allocations

Each country in region $j \in \{N, S\}$ is characterized by:

$$\frac{1}{c^j} = \beta \mathbb{E}_{t=-1} \left\{ \frac{r_1}{c_{T1}^j} \right\} \tag{20}$$

$$c_{T1}^j = y_T - r_1 \left(c^j + z^j + k_{NT}^j - e^j\right) + \frac{y_T - c_{T2}^j}{r_2} \tag{21}$$

$$k_{NT}^j = \frac{\alpha \nu c_{T1}^j}{r_1} \tag{22}$$

$$y_{NT1}^j = \left(\frac{\alpha \nu c_{T1}^j}{r_1}\right)^\alpha \tag{23}$$

$$p_1^j c_{NT1}^j = \nu c_{T1}^j \tag{24}$$

$$p_2^j c_{NT2}^j = \nu c_{T2}^j. \tag{25}$$

When the borrowing constraint is not binding:

$$c_{T2}^{j} = r_2 c_{T1}^{j}, \tag{26}$$

and when the borrowing constraint is binding:

$$c_{T1}^{j} = y_T - r_1 \left(c^j + z^j + k_{NT}^{j} - e^j\right) + \frac{\phi \left[y_T + p_2^j y_{NT2}\right]}{r_2}. \tag{27}$$

Notice that in equilibrium, the household's budget and borrowing constraints reduce to equations written purely in terms of tradable goods. Nontradable goods enter the equations only indirectly, in the form of interest payments $r_1 k_{NT}^{j}$ accrued on the investment k_{NT}^{j} used at $t = 0$ to produce the nontradable goods y_{NT1}^{j} at $t = 1$.

Finally, the resource constraints for tradable and nontradable goods are satisfied with equality:

$$\sum_{j=N,S} \left(c^j + z^j + k_{NT}^{j}\right) = e^T \tag{28}$$

$$\sum_{j=N,S} c_{T1}^{j} = 2y_T \tag{29}$$

$$c_{NTt}^{i} = y_{NTt}^{i} \text{ for } t = 1 \text{ and } t = 2. \tag{30}$$

Walras' Law allows us to ignore the union-level resource constraint for tradable goods at $t = 2$.

To facilitate the discussion later on in this paper, we define auxiliary variables measuring the equilibrium borrowing of countries in each region j in each period t:

$$b_{-1}^{j} = c^j - e^j \tag{31}$$

$$b_0^j = c^j + z^j + k_{NT}^{j} - e^j \tag{32}$$

$$b_1^j = c_{T1}^{j} + r_1 \left(c^j + z^j + k_{NT}^{j} - e^j\right) - y_T, \tag{33}$$

the equilibrium value of the borrowing limit of countries in each region j at $t = 1$:

$$B_1^j = \frac{\phi \left[y_T + p_2^j y_{NT2}\right]}{r_2}, \tag{34}$$

and the implied rate of return in region j, calculated by comparing the levels of tradable consumption at $t = 1$ and $t = 2$:

$$R_2^j = \frac{c_{T2}^{j}}{c_{T1}^{j}}. \tag{35}$$

Lemma 2 $R_2^j > r_2^j$ *when the borrowing constraint is binding in region* j*.*

Binding borrowing constraints in region j generate a wedge between the market interest rate r_2^j and the implied rate of return R_2^j.

4. Laissez-Faire Equilibrium

4.1. Heterogeneity and the Impact of Shocks

Lemma 3 *The shock z increases the union-wide interest rate r_1: $\frac{dr_1}{dz} > 0$.*

In partial equilibrium, the shock z has a direct negative effect on the consumption, investment and welfare of the region that it strikes. In general equilibrium, the shock increases the interest rate r_1, which has additional indirect effects on the consumption, investment and welfare of both regions. The direction of these indirect effects depends on the asset positions of the regions.

Definition 2 *The degree of heterogeneity H in the financial union is:*

$$H = e^N - e^S \in \left[0, e^T\right]. \tag{36}$$

We are going to identify several thresholds in terms of the degree of heterogeneity H that mark changes in the way in which a local shock affects the financial union.

Lemma 4 *For all H, $b_{-1}^N < 0$. There exists some $H^* \in \left(0, e^T\right)$, such that for all $H \in [0, H^*)$, $b_{-1}^S < 0$ and for all $H \in \left[H^*, e^T\right]$, $b_{-1}^S \geq 0$.*

Threshold H^* is defined independently of the shock size and location. When $H > H^*$, then the countries in the South borrow tradable goods from countries in the North at $t = -1$. This implies that, when $H > H^*$, even before the shock hits the union, the South holds debt.

Now we consider how the shock at $t = 0$ affects consumption, investment, and welfare levels in the union. We distinguish between a shock in the North and a shock in the South.

Proposition 1 (Consumption) *A shock $z > 0$ affects consumption of tradable goods at $t = 1$ as follows.*

(i) Shock in the North, $z^N > 0$. If $H < H^*$:

$$\frac{dc_{T1}^N}{dz^N} < 0 \text{ and } \frac{dc_{T1}^S}{dz^N} > 0,$$

while if $H > H^*$:

$$\frac{dc_{T1}^N}{dz^N} > 0 \text{ and } \frac{dc_{T1}^S}{dz^N} < 0.$$

(ii) Shock in the South, $z^S > 0$. For all $H \in \left[0, e^T\right]$:

$$\frac{dc_{T1}^N}{dz^S} > 0 \text{ and } \frac{dc_{T1}^S}{dz^S} < 0.$$

If the shock hits the South, the countries in the South suffer a decrease in consumption at $t = 1$, whereas the countries in the North experience an increase in consumption at $t = 1$. The outcome is more complicated if the shock hits the North. Indeed, in this case, the degree of heterogeneity H plays a crucial role. Independently of the shock size z^N, if the financial union has a degree of heterogeneity $H > H^*$, then the consumption in the South decreases and the consumption in the North increases.

Proposition 1 presents results about c_{T1}^N. The other consumption levels are easily derivable. For example, whenever the borrowing constraint is not binding, $c_{T1}^N = c_{T2}^N$ and $c_{T1}^S = c_{T2}^S$. In Section 4.2, we are going to study how the effects of the shock change depending on the borrowing constraint being binding or not. The consumption levels of nontradable goods c_{NT1}^N and c_{NT1}^S depend on the investment used to produce nontradable goods in $t = 1$: when k_{NT}^j decreases, then c_{NT}^j decreases. This is the focus of the next Proposition.

Proposition 2 (Investment) *Investment k_{NT}^j declines in both North and South after a shock $z > 0$ in either North or South.*

If $\nu > 1$, the decrease in spending on nontradable goods is an *amplification* of the decrease in spending on tradable goods. The adjustment in investment shown by the Proposition above is consistent with this amplification effect.

Proposition 3 (Welfare) *The higher is H, the more that any shock $z > 0$ hurts the South and the less the shock hurts the North.*

(i) Shock to the North, $z^N > 0$. There exists $\tilde{H}\left(z^N\right) \in [0, H^*)$, $\tilde{H}(0) = 0$, $\tilde{H}' > 0$, such that:

$$\frac{du^S}{dz^N} > 0 \text{ for all } H \in \left[0, \tilde{H}\left(z^N\right)\right), \text{ and } \frac{du^S}{dz^N} < 0 \text{ for all } H \in \left(\tilde{H}\left(z^N\right), e^T\right],$$

and there may exist some $\check{H}\left(z^N\right) \in \left(H^*, e^T\right]$ such that:

$$\frac{du^N}{dz^N} < 0 \text{ for all } H \in \left[0, \check{H}\left(z^N\right)\right), \text{ and } \frac{du^N}{dz^N} > 0 \text{ for all } H \in \left(\check{H}\left(z^N\right), e^T\right];$$

otherwise, if such $\check{H}$ does not exist, $\frac{du^N}{dz^N} < 0$ for all $H \in \left[0, e^T\right]$.

(ii) Shock to the South, $z^S > 0$. For all $H \in \left[0, e^T\right]$:

$$\frac{du^N}{dz^S} > 0 \text{ and } \frac{du^S}{dz^S} < 0.$$

The results on welfare level are similar to the ones about consumption. Whenever the shock hits the South, then the South is worse off and the North is better off. When the shock hits the North, then there exists a threshold $\tilde{H}\left(z^N\right)$ such that, if $H > \tilde{H}\left(z^N\right)$, the South is worse off. Moreover, there may be a different threshold $\check{H}\left(z^N\right)$ such that, when $H > \check{H}\left(z^N\right)$, then the North is better off. We can show that $\tilde{H}\left(z^N\right) < H^* < \check{H}\left(z^N\right)$. Therefore, the South can be worse off after a shock in the North even if, at $t = -1$, the South does not hold debt. On the contrary, the North can be better off after a shock in the North only if the South was already a debtor at $t = -1$.

Taken together, these results establish that in a financial union, the distributional effect of a shock depends on the degree of heterogeneity H that prevailed prior to the shock, and not just on which region j is hit by the shock. For a sufficiently high degree of heterogeneity H, the South is always disproportionately hurt in terms of consumption, investment and welfare—whether the shock strikes the North or the South. The shift in emphasis in this paper from the location of the shock to the heterogeneity prior to the shock breaks from the tradition followed in the optimal currency area literature.

4.1.1. Numerical Simulations

We use numerical simulations and graphs to illustrate the effects of local shock on consumption, investment, and welfare within the financial union when the specification of the production function is not Cobb-Douglas. We use a production function that is conceptually plausible, yet breaks clearly from the Cobb-Douglas framework and therefore does not yield clean linear relationships between consumption, investment and nontradable goods output. Our purpose is to show numerically that most of the results proven above still hold, but we can also derive some new and interesting results as well.

More specifically, we assume that firms can convert one unit of the tradable good at $t = 0$ into A units of nontradable goods at $t = 1$, earning profits:

$$p_1^i A - r_1. \tag{37}$$

Each country has firms of varying levels of productivity A, and the productivity distribution $A \sim U[0,1]$ is identical across countries. At $t = 0$, firms choose whether to operate or not. Only firms with sufficiently high productivity $A > \frac{r_1}{p_1^i}$ decide to operate. Now investment and output in country i are:

$$k_{NT}^i = \mu \left(1 - \frac{r_1}{p_1^i}\right), \tag{38}$$

$$y_{NT1}^i = \frac{\mu}{2}\left[1 - \left(\frac{r_1}{p_1^i}\right)^2\right]. \tag{39}$$

We will show which results derived under the Cobb-Douglas assumption are preserved in this different environment, and which are not. In general, the role of the degree of heterogeneity remains critical in determining the effects of the shock, but the specific thresholds for H do change when the production function changes.

Extreme cases. Figure 2 shows the effect of a shock in the North $z^N > 0$ when $H = 0$, i.e., the North and South have identical initial endowments:

$$e^N = e^S,$$

In this case, the location of the shock is the primary determinant of its effects.

The results are consistent with the discussion in subsection 4.1. The shock $z^N > 0$ decreases welfare u^N in the North and increases welfare u^S in the South. Households in the North and South enter $t = 0$ with identical and positive savings. The shock $z^N > 0$ induces the households of the North to borrow from the South, and at the same time increases the interest rate r_1 on this borrowing. The resulting higher burden of interest payments at $t = 1$ means that the shock decreases c^N_{T1} and increases c^S_{T1}. Anticipating lower spending by Northern households on nontradable goods at $t = 1$, and because of the higher interest rate r_1, firms in the North reduce investment k^N_{NT} at $t = 0$. Therefore, the shock decreases c^N_{NT1} too.

For $H = 0$, the effect of a shock in the South $z^S > 0$ is the same in magnitude as a shock in the North, but with the region labels reversed.

Figure 3 shows the effect of a shock in the North $z^N > 0$ when $H = e^T$, i.e., the North and South have very different initial endowments:

$$e^N = e^T \text{ and } e^S = 0.$$

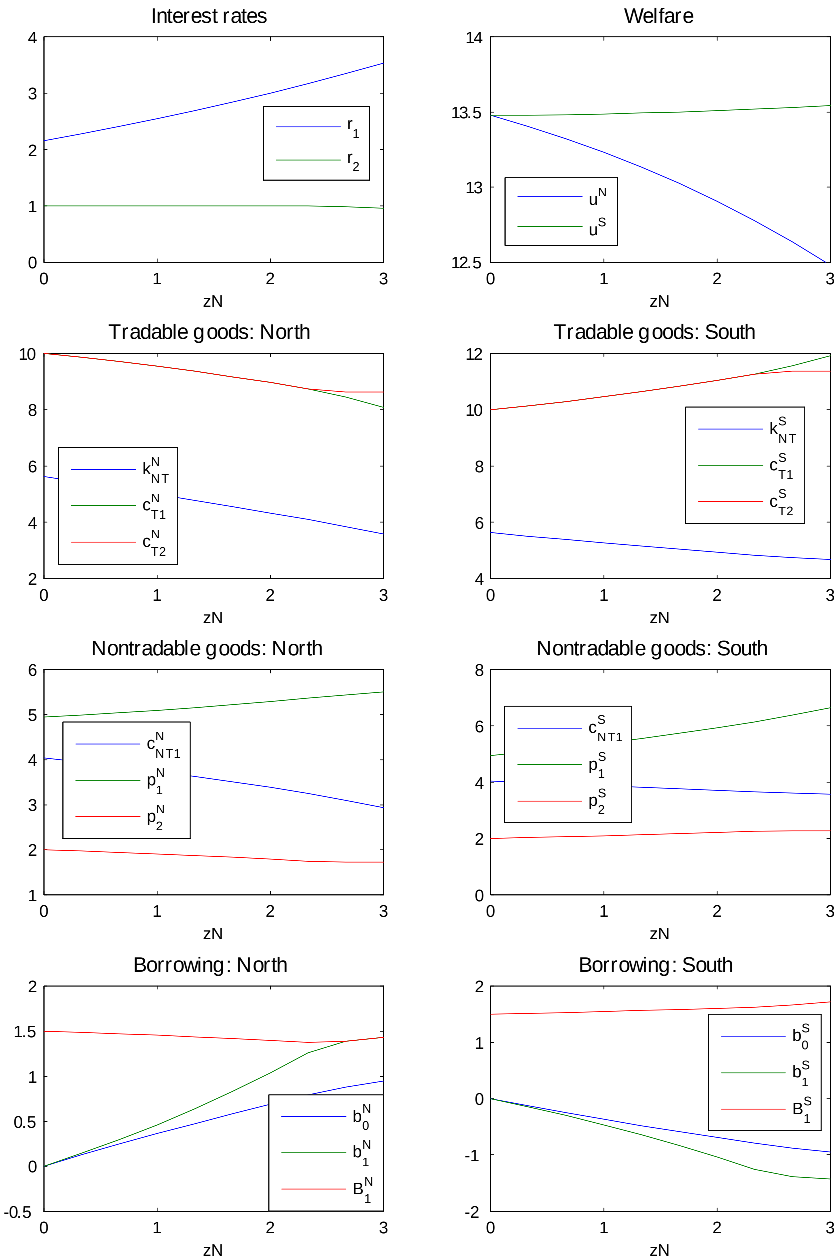
In this case, the degree of heterogeneity that prevailed prior to the shock is the primary determinant of the response to the shock.

The results are again consistent with the discussion in subsection 4.1. The shock decreases welfare in the South u^S steeply, and welfare in the North u^N gradually.

Households in the South enter $t = 0$ with debt $b^S_{-1} > 0$ while households in the North enter with assets $b^N_{-1} < 0$. Whichever value of the shock $z^N > 0$ is realized, the South must borrow more at $t = 0$: $b^S_0 > b^S_{-1}$. At the same time, the shock in the North increases the interest rate r_1, which benefits the lender North and hurts the borrower South. Therefore, the shock in the North increases c^N_{T1} and decreases c^S_{T1}. Within each region j, $c^j_{T1} = c^j_{T2}$ except when the borrowing constraint binds.[1] For $H = e^T$, it binds for the South after large shocks, not for the North. If the constraint binds, c^S_{T1} decreases

[1] Notice that for a given level of ϕ, the borrowing constraint is more likely to be binding for $H = e^T$ than for $H = 0$, because in the former case, one of the regions (the South) enters $t = 0$ with a higher level of debt b^j_0. Therefore, for presentational purposes, we set a higher value of ϕ for $H = e^T$.

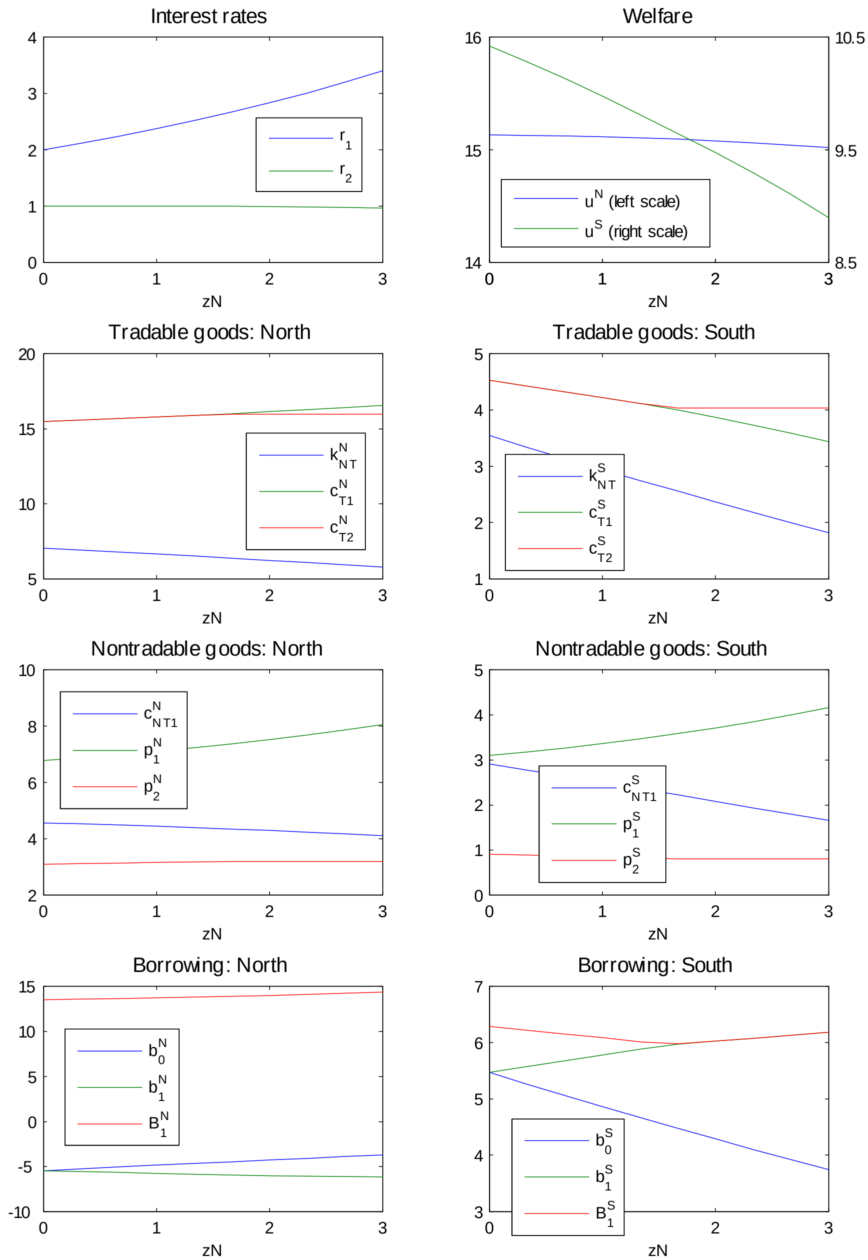
Figure 2: H = 0, Shock in the North z^N



$e^N = e^S = 10$, $\nu = 2$, $\phi = 0.05$, $\pi = 0.1$, $\mu = 10$, $z^{bar} = 3$, $y_T = y_{NT2} = 10$

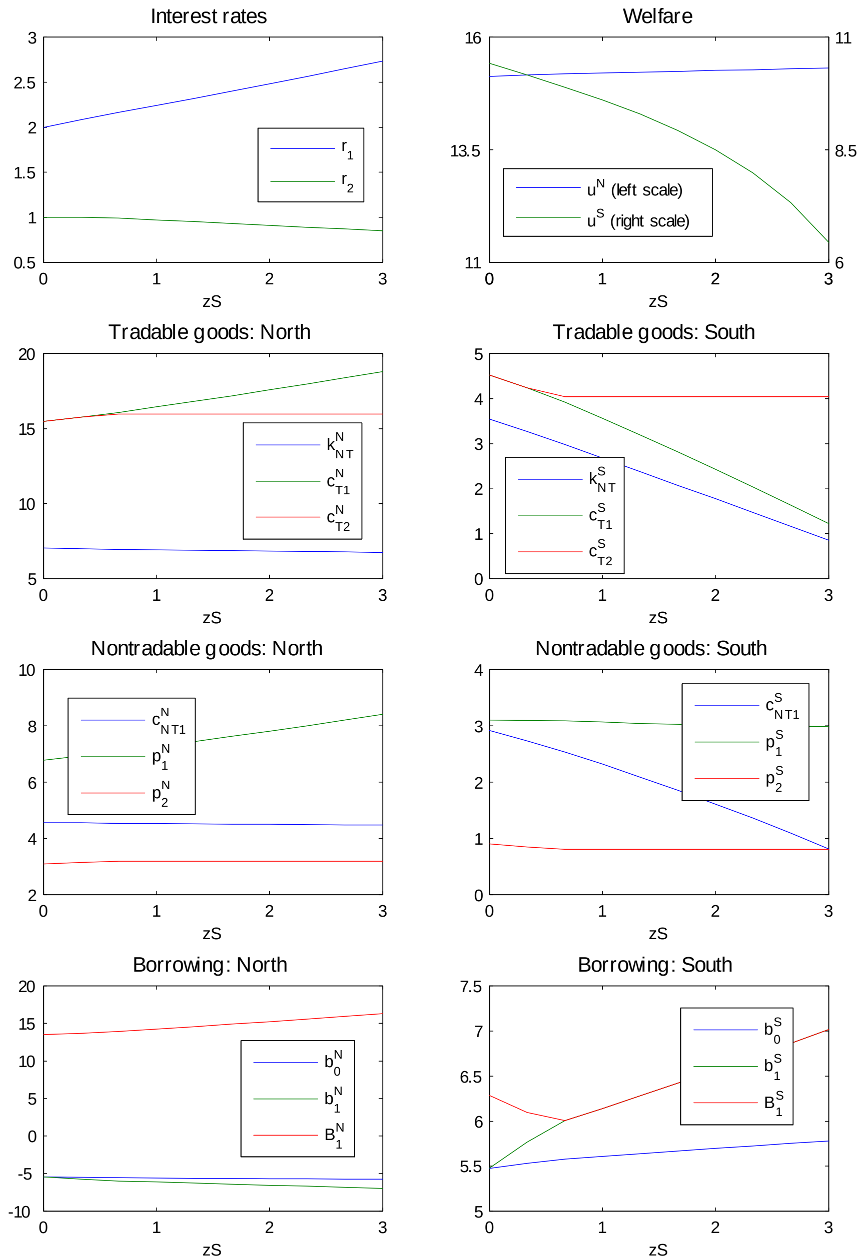
Period t=-1: $c^N = c^S = 4.4$, $b^N_{-1} = b^S_{-1} = -5.6$

Figure 3: $H = e^T$, Shock in the North z^N



$e^N = 20$, $e^S = 0$, $\nu = 2$, $\phi = 0.33$, $\pi = 0.1$, $\mu = 10$, $z^{bar} = 3$, $y_T = y_{NT2} = 10$
Period t = -1: $c^N = 7.5$, $b^N_{-1} = -12.5$, $c^S = 1.9$, $b^S_{-1} = 1.9$

Figure 4: $H = e^T$, Shock in the South z^S



$e^N = 20$, $e^S = 0$, $\nu = 2$, $\phi = 0.33$, $\pi = 0.1$, $\mu = 10$, $z^{bar} = 3$, $y_T = y_{NT2} = 10$

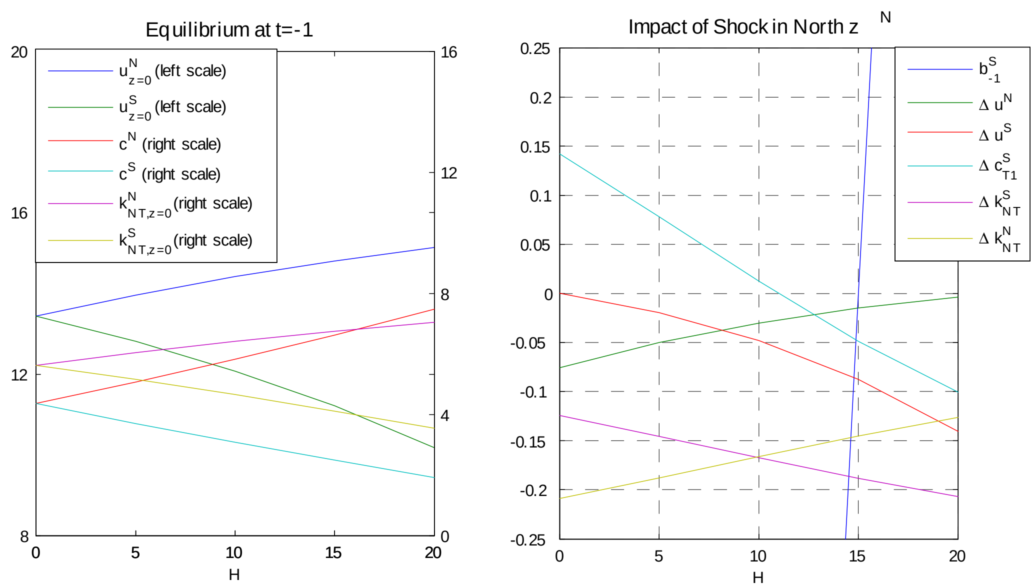
Period $t = -1$: $c^N = 7.5$, $b^N_{-1} = -12.5$, $c^S = 1.9$, $b^S_{-1} = 1.9$

even more steeply, c^S_{T2} remains at a fixed value, and the interest rate r_2 dips below unity. Firms decrease investment k^j_{NT} in both regions j, but this time more steeply in the South. If the borrowing constraint binds, k^S_{NT} decreases even more steeply.

Figure 4 shows the effect of a shock in the South $z^S > 0$ for $H = e^T$. The direct and indirect effects of the shock $z^S > 0$ now both fall on the South. The welfare of the South u^S decreases while the welfare of the North u^N increases.

Intermediate cases. For a sufficiently high degree of heterogeneity $H \in \left[0, e^T\right]$, the distributional dynamics of a financial union in response to a shock z switch qualitatively from the kind of response recorded above for $H = 0$, to the kind of response recorded above for $H = e^T$. Figure 5 illustrates propositions 1, 2 and 3 under the alternative specification for the production function.

Figure 5: $H \in [0,e^T]$, Shock in the North z^N



$e^T = 20$, $\nu = 2$, $\phi = 0.33$, $\pi = 0.1$, $\mu = 10$, $z^{bar} = 3$, $y_T = y_{NT2} = 10$

The left panel of figure 5 shows for different degrees of heterogeneity $H \in \left[0, e^T\right]$, the levels of consumption c^j and for $z = 0$, the levels of investment k^j_{NT} and welfare u^j. The higher is H, the higher are consumption, investment and welfare in the North and the lower they are in the South. The right panel of figure 5 illustrates how the qualitative switch from the shock response for $H = 0$ to the shock response for $H = e^T$ occurs. We consider a marginal shock to the North $z^N > 0$ in a financial union with $e^T = 20$. The debt of households in the South b^S_{-1} is shown as a function of H. In addition, the effect of the shock on a range of variables is illustrated, using the Δ prefix, for each value of H.

First, the higher is H, the more that any shock $z^N > 0$ hurts the South and the less the shock hurts the North:

$$\frac{d}{dH}\left(\Delta u^S\right) < 0 \text{ and } \frac{d}{dH}\left(\Delta u^N\right) > 0.$$

Second, we turn to the thresholds for $H \in \left[0, e^T\right]$ described in the propositions 1 and 3. Here the differences from the Cobb-Douglas case become visible. In comparison to the results presented in Section 4.1, when the production function is not Cobb-Douglas, we can identify a new threshold $\hat{H}\left(z^N\right) < H^*$. When $H > \hat{H}\left(z^N\right)$, after shock z^N occurs, c_{T1}^S decreases and c_{T1}^N increases. This implies that the South may see its consumption decrease even if, when the shock hits the North, the South is not a debtor to the North. The new ranking of critical thresholds become $\tilde{H}\left(z^N\right) < \hat{H}\left(z^N\right) < H^* < \check{H}\left(z^N\right)$.

In the simulation, for $H > H^* = 15.0$, which means $e^N > 17.5$ and $e^S < 2.5$, households in the South enter $t = 0$ with debt $b_{-1}^S > 0$. For $H > \hat{H} = 11.0$, which means $e^N > 15.5$ and $e^S < 4.5$, the shock in the North at $t = 0$ decreases c_{T1}^S and increases c_{T1}^N at $t = 1$. Investment k_{NT}^j, and therefore the consumption of nontradable goods c_{NT1}^j, decrease in both North and South for all $H \in \left[0, e^T\right]$. Since we consider a marginal shock $z^N > 0$, and we know that $\tilde{H}(0) = 0$, $\Delta u^S < 0$ for all $H \in \left(0, e^T\right]$. Finally, for the chosen parameters, $\check{H}$ does not exist so $\Delta u^N < 0$ for all $H \in \left[0, e^T\right]$.

4.2. Impact of Binding Borrowing Constraints

In the graphs of Section 4.1.1 it is apparent how the dynamics of the variables affected by the shock z change if borrowing constraints bind. This section analyze the role of borrowing constraints more in detail.

First, we hold the degree of heterogeneity H fixed and outline the effect when the borrowing constraint binds in a given region j. Then, we establish how the degree of heterogeneity H determines in which region j the borrowing constraint binds, after any shock $z \geq 0$.

In this subsection, any starred variable denotes the value of the variable for the same value of the shock z but without the borrowing constraint.

Proposition 4 (Consumption and Investment) *For a given value of the shock $z \geq 0$, when the borrowing constraint binds in region j:*

$$k_{NT}^j < k_{NT}^{j*},$$

$$c_{T1}^j < c_{T1}^{j*}, \quad c_{NT1}^j < c_{NT1}^{j*},$$

$$c_{T2}^j > c_{T2}^{j*},$$

The opposite inequalities hold for the other region $-j$.

Proposition 5 (Interest Rates) *For a given value of the shock $z \geq 0$, when the borrowing constraint binds in any region $j \in \{N, S\}$:*

$$r_1 = r_1^*, \quad r_2 < r_2^* = 1$$

$$R_2^j > R_2^{j*} = 1 \text{ and } R_2^{-j} = r_2 < R_2^{-j*} = 1.$$

Given the results presented in Proposition 4 and 5, we are in the position to derive conclusions in terms of the impact of a binding borrowing constraint on each region. Possibly surprisingly, a binding constraint benefits the region that is subject to it.

Proposition 6 (Welfare) *For all shocks $z \geq 0$, when the borrowing constraint binds in region j:*

$$u^{-j} < u^{-j*}.$$

There may exist some $\hat{z} \in [0, \overline{z})$ such that:

$$u^j > u^{j*} \text{ for all } z \in (0, \hat{z}), \text{ and } u^j < u^{j*} \text{ for all } z \in (\hat{z}, \overline{z}];$$

otherwise, if such $\hat{z}$ does not exist:

$$u^j > u^{j*} \text{ for all } z \in [0, \overline{z}].$$

Building upon the results from subsection 4.1, we can determine which region's borrowing constraint can become binding as a result of the occurrence of a shock.

Proposition 7 (Heterogeneity) *For all $H \in [0, H^*)$, if the borrowing constraint binds anywhere, it binds in the region j that suffers from the shock. For all $H \in \left(H^*, e^T\right]$, if the borrowing constraint binds anywhere, it binds in the South.*

Proposition 7 states that, if the South holds debt before the shock hits, then it is known in advance that the only borrowing constraint that can bind is the one of the South, no matter where the shock hits. Holding the endowments fixed, the constraint binds in the South when r_1, z^N and z^S are high, and ϕ and p_2^S are low. In particular, it is noteworthy to highlight the role of the price of nontradable goods p_2^S at $t = 2$. Through its effect on p_2^S, the shock decreases the borrowing limit B_1^S through a *pecuniary externality.*

The interaction between the two cycles of borrowing and lending at $t = 0$ at $t = 1$ explain our results. By Lemma 3, a shock z at $t = 0$ increases the interest rate r_1 between $t = 0$ and $t = 1$, which increases the cost of borrowing for the South between $t = 0$ and $t = 1$ and increases the desired borrowing of the South between $t = 1$ and $t = 2$. This in turn makes the borrowing constraint more likely to bind. Relative to the equilibrium without the constraint, a binding constraint decreases the interest rate

r_2 between $t = 1$ and $t = 2$. Consistent with Lemma 2, the constraint distorts the intertemporal substitution margin of the South R_2^S between $t = 1$ and $t = 2$ relative to the interest rate r_2, while $R_2^N = r_2$ continues to hold in the North.

When the value of ϕ is such that the borrowing constraint is just binding, the decrease in consumption c_{T1}^S has a second-order negative effect on welfare in the South. At the same time, the decrease in r_2 reduces the cost of all infra-marginal borrowing by the South, which has a first-order positive effect on welfare in the South. Therefore, in net terms, welfare in the South u^S increases. Intuitively, each country in the South is hurt by its own decrease in consumption at $t = 1$, but benefits from binding constraints in all other countries in the South, which reduce the interest rate r_2 and therefore repayments at $t = 2$. If the value of ϕ is such that the borrowing constraint is strongly binding, the marginal effect of the decrease in consumption at $t = 1$ may dominate the infra-marginal effect. Therefore, if the constraint is strongly binding, the welfare of the South u^S may decrease relative to the equilibrium without the constraint.

The North suffers both from a change in the time profile of consumption between $t = 1$ and $t = 2$, and from the decrease in the returns on its saving, r_2. Therefore, the binding constraint in the South decreases the welfare of the North u^N relative to the equilibrium without the constraint.

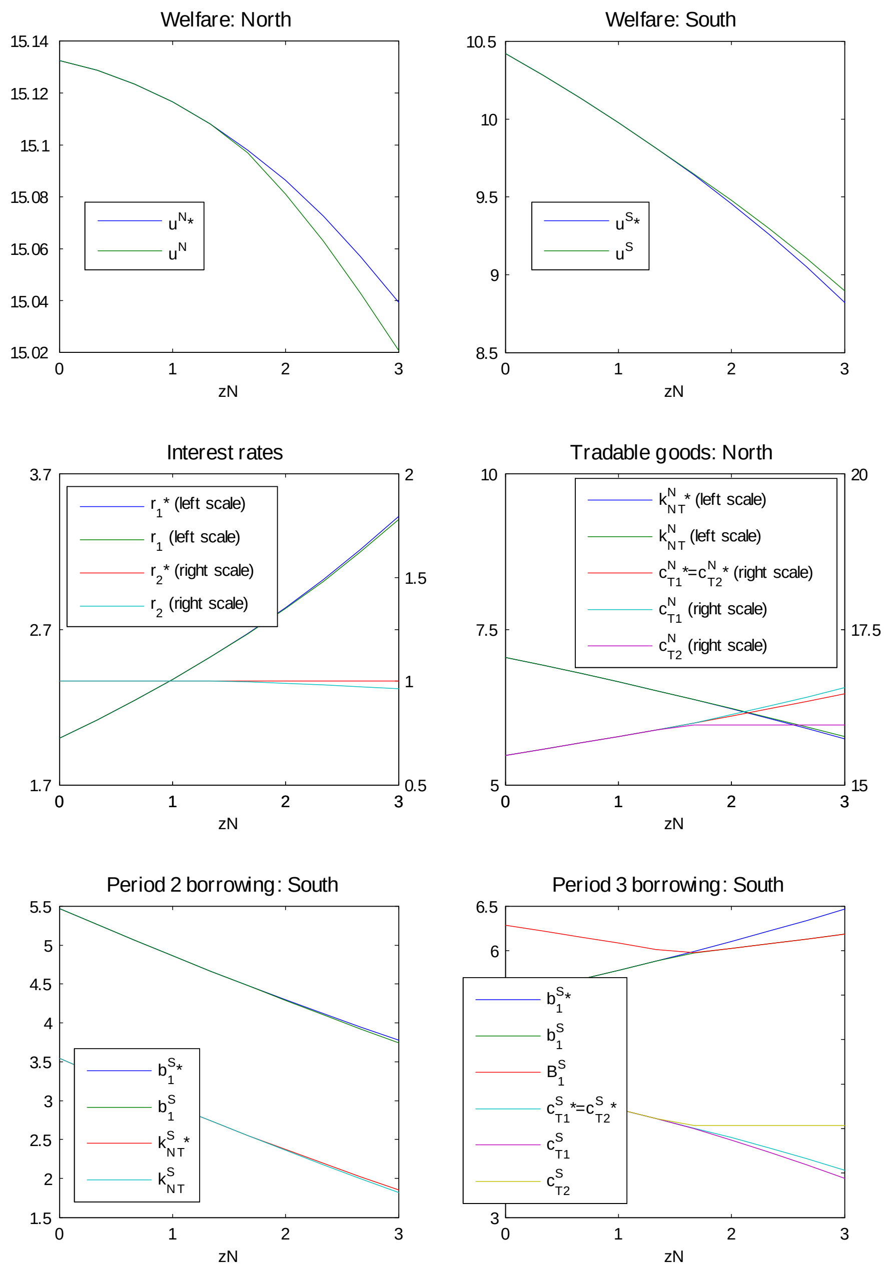
4.2.1. Numerical Simulations

Figures 6 and 7 are the analogs of figures 4 and 5. Instead of comparing the response to a shock for $H = 0$ and $H = e^T$, now we fix $H = e^T$. For any given value of the shock $z \geq 0$, we compare the equilibria with and without a binding borrowing constraint.

Once we consider a specification of the production function that is not Cobb-Douglas, then $r_1 < r_1^*$, which means that when the constraint binds in the South, it hurts the North in all prior periods, even in the absence of borrowing constraints in those prior periods. Credit supply constraints at $t = 1$ affect credit demand and the interest rate at $t = 0$.

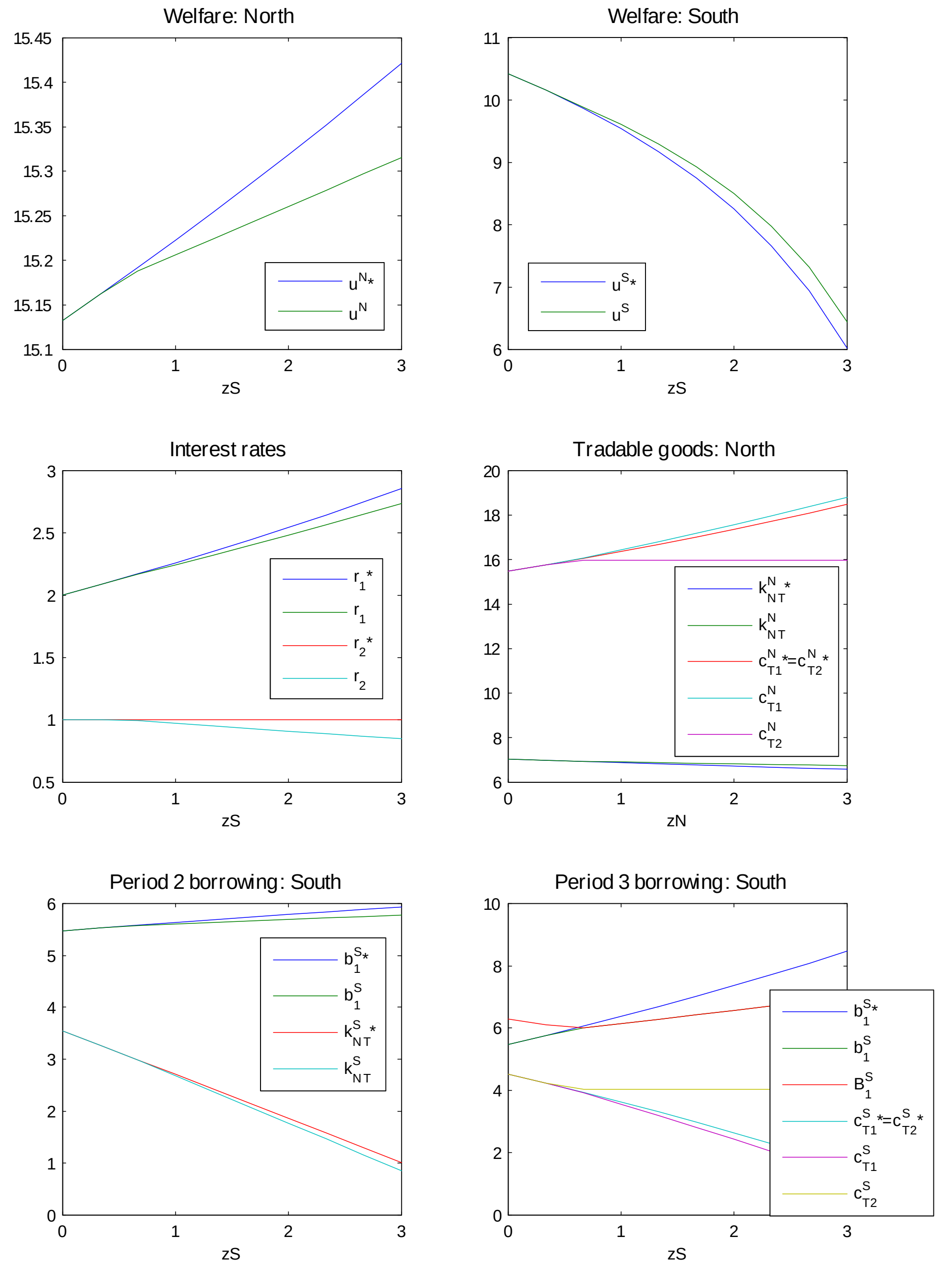
Figures 6 and 7 show that consistent with proposition 7, large shocks in both North and South cause the borrowing constraint to bind in the South. Consistent with propositions 4, the borrowing constraint decreases the level of investment k_{NT}^S at $t = 0$, and the levels of tradable and nontradable consumption c_{T1}^S and c_{NT1}^S at $t = 1$ in the South, relative to the equilibrium without the constraint. It also increases the level of tradable consumption c_{T2}^S at $t = 2$ in the South. Consistently with the results in subsection 4.2, a wedge opens up between the market interest rate $r_2^j < 1$ and the implied rate of return in the South $R_2^j > 1$. In addition, the binding borrowing constraint in the South makes the North worse off and the South better off. Differently from the results presented in Proposition 5, now interest rate r_1 decreases along with r_2.

Figure 6: Binding Constraint with $H = e^T$, Shock in the North z^N



$e^N = 20$, $e^S = 0$, $\nu = 2$, $\phi = 0.33$, $\pi = 0.1$, $\mu = 10$, $z^{bar} = 3$, $y_T = y_{NT2} = 10$
Period t = -1: $c^N = 7.5$, $b^N_{-1} = -12.5$, $c^S = 1.9$, $b^S_{-1} = 1.9$

Figure 7: Binding Constraint with $H = e^T$, Shock in the South z^S



$e^N = 20$, $e^S = 0$, $\nu = 2$, $\phi = 0.33$, $\pi = 0.1$, $\mu = 10$, $z^{bar} = 3$, $y_T = y_{NT2} = 10$
Period t = -1: $c^N = 7.5$, $b^N_{-1} = -12.5$, $c^S = 1.9$, $b^S_{-1} = 1.9$

5. Policy Interventions

5.1. Social Planner, Individual Countries and Coalitions

Our model features two market imperfections. First, debt between $t = -1$ and $t = 0$ is noncontingent. Second, there is a borrowing constraint, with an associated pecuniary externality, between $t = 1$ and $t = 2$.

The constrained social planner serves as an omnipotent benchmark in this paper. This planner maximizes a weighted sum of welfares of all countries i in both regions j, by making transfers of tradable goods between households in all countries in all periods contingent on the shock z; households and firms take the transfers as given and maximize utility and profits as in subsection 3.2. Households in country i receive x^i_{SPt} in period t.

Given the symmetry within each region, we can conclude that households in all countries i in the same region j receive the same transfers: $x^i_{SPt} = x^j_{SPt}$ for all $i \in j$ and for all periods t.

Proposition 8 (Social Planner) *The social planner fully mitigates both market imperfections through the following actions after a shock $z > 0$.*

(i) Noncontingent debt. If the shock z decreases c^j_{T1} in region j in the laissez-faire equilibrium:

$$\frac{dx^j_{SPt}}{dz} > 0, \frac{dx^{-j}_{SPt}}{dz} < 0 \text{ for some } t \geq 0.$$

(ii) Borrowing constraint. If the shock z causes the constraint to bind in region j in the laissez-faire equilibrium:

$$\frac{dx^j_{SPt}}{dz} > 0, \frac{dx^{-j}_{SPt}}{dz} < 0 \text{ for } t = 0 \text{ and/or } t = 1.$$

Combining Proposition 8 with the results derived in Section 4, we can determine how the social planner's action changes depending on the degree of heterogeneity in the financial union.

Corollary 1 (Heterogeneity) *For all $H \in [0, H^*)$, $\frac{dx^j_{SPt}}{dz} > 0, \frac{dx^{-j}_{SPt}}{dz} < 0$ for some $t \geq 0$ if region j is the region hit by the shock. For all $H \in \left(H^*, e^T\right]$, $\frac{dx^S_{SPt}}{dz} > 0, \frac{dx^N_{SPt}}{dz} < 0$ for some $t \geq 0$ whichever region j is hit by the shock.*

In other words, if $H > H^*$, then any shock triggers the same response from the Social Planner: transfer from the North to the South.

Next, we turn to what happens in a financial union without a social planner.

Each country i has a government, who maximizes the welfare of the households in country i and can act either individually or together with other governments from any subset of countries. The government's feasible actions vary in the following subsections, but in all cases, the action space faces a common limitation which ensures that governments cannot easily get around the noncontingent feature of debt: governments cannot commit at $t = -1$ to make its actions at $t \geq 0$ contingent on the shock z.

Proposition 9 (Pareto Implementation) *After a shock $z > 0$, unilateral actions by independent governments cannot implement all the available Pareto improvements from the laissez-faire equilibrium, and may not be able to implement any of them.*

We provide here the intuition behind the result presented in Proposition 9.

Conditional on the shock $z \geq 0$ already being realized, the equilibrium from $t = 0$ onward is Pareto efficient if no constraints bind. Unlike the social planner, governments do not individually extend assistance to countries in different regions in order to offset the income effects of a shock, because any such assistance can only make one country/region better off by making the other country/region worse off. On the other hand, if the borrowing constraint binds, then there may exist feasible actions which make both countries/regions better off.

Suppose that borrowing constraints bind in region j after a shock $z \geq 0$ and that governments in the union possess a non-empty action space. Each country i is small relative to the financial union. Therefore, acting individually, each government undertakes actions which relaxes the constraints of households in their country under the correct assumption that the interest rates r_1 and r_2 remain unchanged. But if a positive measure of the countries within the financial union undertakes these actions, then r_1 and r_2 do indeed change and the resulting first-order welfare loss for region j can wipe out any gains from relaxing the constraint. If the welfare gains are indeed wiped out, then Southern governments cannot agree in equilibrium to undertake actions that relax the borrowing constraints of their own households, unless Northern governments also agree to help.

If governments from a positive measure of countries act together, they internalize the impact of their actions on the interest rates r_1 and r_2. In the following subsections, we restrict attention to policy interventions whereby governments act within coalitions—in particular, regional coalitions of North and South.

Definition 3 *A coalition is a set of governments who undertake the same actions and who negotiate as a group with other governments.*

Definition 4 *A regional coalition j is a coalition of the governments from all countries in the same region j.*

From this point on, we also restrict attention to degrees of heterogeneity $H \in \left(H^*, e^T\right]$, such that in the absence of any interventions, if any region j suffers a binding borrowing constraint, it is the South. Since the most counter-intuitive case is when a shock in the

North causes the borrowing constraint to bind in the South, that is the case that we will illustrate in the simulations in the following subsections. The idea is that if our results are demonstrated for that case, it makes sense that they should hold for the less counter-intuitive cases also.

Since governments do not want to offset regional income effects but may attempt to generate Pareto improvements by mitigating regional borrowing constraints, without loss of generality we will consider interventions which alter the profile of consumption between periods $t = 1$ and $t = 2$.

First, we restrict attention to the equilibrium from $t = 0$ onward and identify conditions for ex-post Pareto improvements. Second, we outline the impact of the loan on equilibrium decisions at $t = -1$, and we state conditions for Pareto improvements when all four periods are taken into account. This latter issue relates to how Pareto improving policies, if available, should be institutionalized.

5.2. Unconditional Gifts

In this subsection, the government of country i is allowed to undertake the following actions at $t = 1$: lump-sum taxes and transfers on the tradable endowments of households in country i, and making and receiving transfers of tradable goods to and from governments in other countries $-i$, subject to the budget constraint that the total net transfer by each government is zero. All actions must be announced at $t = 0$, after the shock is realized. The transfers allowed between countries are unconditional gifts: they cannot be made contingent on any future actions, since no future actions are available.

The unconditional gift from the Northern coalition to the Southern coalition at $t = 1$ is x_{NS}. The lump-sum tax on households in the North and the lump-sum transfers to households in the South are both equal to x_{NS}. Households and firms take the transfers as given and maximize utility and profits as in subsection 3.2.

The unconditional gift x_{NS} changes the equilibrium equations in the North:

$$c_{T1}^N = y_T - x_{NS} - r_1 \left(c^N + z^N + k_{NT}^N - e^N \right) + \frac{y_T - c_{T2}^N}{r_2}, \tag{40}$$

$$b_1^N = c_{T1}^N + r_1 \left(c^N + z^N + k_{NT}^N - e^N \right) - y_T + x_{NS}, \tag{41}$$

and in the South:

$$c_{T1}^S = y_T + x_{NS} - r_1 \left(c^S + z^S + k_{NT}^S - e^S \right) + \frac{y_T - c_{T2}^S}{r_2}, \tag{42}$$

$$b_1^S = c_{T1}^S + r_1 \left(c^S + z^S + k_{NT}^S - e^S \right) - y_T - x_{NS} \text{ and } B_1^S = \frac{\phi \left[p_2^S y_{NT2} + y_T \right]}{r_2}. \tag{43}$$

As a benchmark, we allow the gift to affect equilibrium variables for $t \geq 0$, but we ignore the impact of the gift at $t = 1$ on equilibrium variables at $t = -1$, before the shock is realized.

Definition 5 *The ex post welfare u^i_{012} of country i is the welfare of country i allowing the values of equilibrium variables to vary for periods $t \geq 0$, while keeping all variables at $t = -1$ fixed at the values they take in the laissez-faire equilibrium.*

Lemma 5 *Suppose that in the laissez-faire equilibrium, the constraint binds in the South for $z \in [\underline{z}, \bar{z}]$. Then for $z \in (\underline{z}, \bar{z}]$, an unconditional gift $x_{NS} > 0$ from the Northern coalition to the Southern coalition at $t = 1$ has the following effects:*

$$\frac{dc^N_{T1}}{dx_{NS}} < 0, \frac{dc^N_{NT1}}{dx_{NS}} < 0, \frac{dp^N_1}{dx_{NS}} < 0, \frac{dk^N_{NT}}{dx_{NS}} < 0,$$

$$\frac{dc^S_{T1}}{dx_{NS}} > 0, \frac{dc^S_{NT1}}{dx_{NS}} > 0, \frac{dp^S_1}{dx_{NS}} > 0, \frac{dk^S_{NT}}{dx_{NS}} > 0,$$

$$\frac{dc^N_{T2}}{dx_{NS}} = \frac{dc^S_{T2}}{dx_{NS}} = 0, \text{ and } \frac{dr_1}{dx_{NS}} = 0, \frac{dr_2}{dx_{NS}} > 0, \frac{dR^S_2}{dx_{NS}} < 0.$$

Proposition 10 (Unconditional Gift) *An unconditional gift x_{NS} cannot generate a Pareto improvement. Whether or not the constraint binds in the South, the gift generates:*

$$\frac{du^N_{012}}{dx_{NS}} < 0 \text{ and } \frac{du^S_{012}}{dx_{NS}} > 0.$$

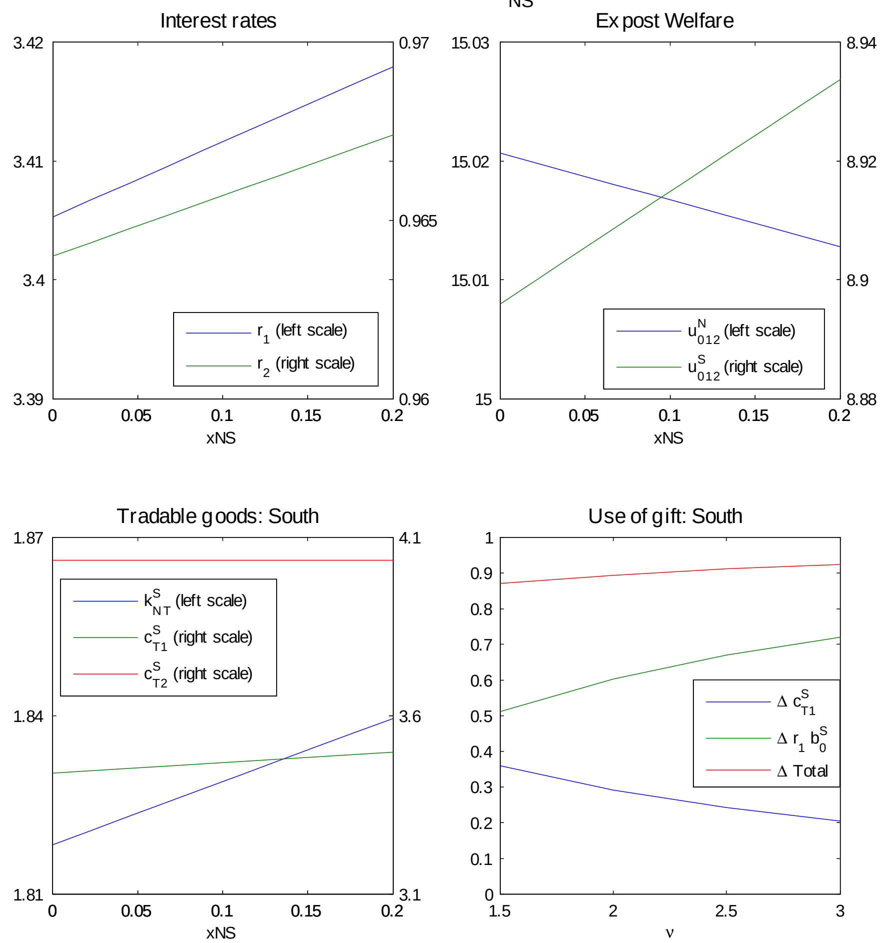
No Pareto improvement is possible using an unconditional gift $x_{NS} > 0$ from the Northern coalition to the Southern coalition. The ex post welfare of the North decreases and the ex post welfare of the South increases. The transfer of tradable goods at $t = 1$ generates a positive income effect in the South, and the level of tradable consumption c^S_{T1} in the South increases as its constraint is relaxed: the borrowing limit B^S_1 actually decreases owing to the increase in r_2, but the South's desired borrowing b^S_1 decreases as well. The resulting *amplified* increase in spending on nontradable goods generates an increase in the nontradable goods price p^S_1, which induces higher investment k^S_{NT}.

The fact that the unconditional gift does not change tradable consumption levels at $t = 2$, combined with the resource constraint at $t = 1$ which imposes that consumption in the North must fall when the borrowing constraint of the South is mitigated, means that the North must become worse off from the unconditional gift. Therefore, to go in the direction of a Pareto improvement, we need to offset the gift at $t = 1$ with some benefit for the North at $t = 2$.

5.2.1. Numerical Simulations

Figure 8 fixes the value of the shock to $\left(z^N = 3, z^S = 0\right)$, such that the constraint is binding in the South, and illustrates the impact of an unconditional gift $x_{NS} > 0$ for the alternative specification of the production function. Again, the major difference with the Cobb-Douglas case is that now $\frac{dr_1}{dx_{NS}} \neq 0$.

Figure 8: Unconditional Gift x_{NS} from North to South



$e^N = 20$, $e^S = 0$, $\phi = 0.33$, $\pi = 0.1$, $\mu = 10$, $z^{bar} = 3$, $y_T = y_{NT2} = 10$
Period t = -1: $c^N = 7.5$, $b^N_{-1} = -12.5$, $c^S = 1.9$, $b^S_{-1} = 1.9$. Shock: $z^N = 3$, $z^S = 0$. First 3 panels: $\nu = 2$

The bottom right panel of figure 8 shows that the increase in the consumption of tradable goods Δc^S_{T1} in the South at $t = 1$ is lower than the amount of the gift x_{NS}. There are two reasons for this result. First, the decline in the borrowing limit B^S_1 keeps the entire increase in borrowing by the South $\Delta Total$ at $t = 1$ lower than the amount of the gift x_{NS}. Second, the increase in the interest rate r_1 increases the interest repayments of the South at $t = 1$. The increase in interest repayments $\Delta r_1 b^S_0$ represents the portion of the gift x_{NS} which is endogenously recouped by the North.

For a given size of the unconditional gift x_{NS}, the North suffers in net terms, but it suffers less from providing the gift when the size of the nontradable sector is larger. This is because the higher is ν, the larger is the portion of the gift x_{NS} which is endogenously recouped by the North. Indeed, the higher is ν, the larger is the *amplification* from an increase in spending on tradable goods in the South to an increase in spending on nontradable goods at $t = 1$. This induces a larger increase in investment k_{NT}^S at $t = 0$ and a larger increase in the interest rate r_1. Therefore, the larger is the accumulated interest burden $r_1 b_0^S$ which is repaid by the South to the North at $t = 1$.

5.3. Subsidized Governmental Loans

In this subsection, the government of country i is allowed to undertake all the actions from the previous subsection at $t = 1$. It is also allowed to undertake the same actions at $t = 2$, up to a maximum lump-sum tax equal to $\bar{\tau}_{LS} \in (0, y_T)$. The budget constraint is that the total net transfer by each government (net over domestic households and foreign governments) is zero within each period. All actions must be announced at $t = 0$. In this subsection, the transfers between countries at $t = 1$ can be conditioned on transfers between countries at $t = 2$.

With this feasible action space, the government is now able to commit to repay loans up to a maximum debt limit. The government decides at $t = 0$, after the shock, to contract its own loans between $t = 1$ and $t = 2$ if such an action would increase their country's welfare. Such loans are purely intergovernmental, and parallel to the private market.

A governmental loan from the Northern coalition to the Southern coalition at interest rate r_{NS} entails a transfer from the North to the South of x_{NS} at $t = 1$ and a transfer in the opposite direction of $r_{NS} x_{NS}$ at $t = 2$, subject to $r_{NS} x_{NS} \leq \bar{\tau}_{LS}$. Households and firms take the transfers as given and maximize utility and profits as in subsection 3.2.

The governmental loan x_{NS} at interest rate r_{NS} changes the equilibrium equations in the North:

$$c_{T1}^N = y_T - x_{NS} - r_1 \left(c^N + z^N + k_{NT}^N - e^N \right) + \frac{y_T + r_{NS} x_{NS} - c_{T2}^N}{r_2}, \tag{44}$$

$$b_1^N = c_{T1}^N + r_1 \left(c^N + z^N + k_{NT}^N - e^N \right) - y_T + x_{NS}, \tag{45}$$

and in the South:

$$c_{T1}^S = y_T + x_{NS} - r_1 \left(c^S + z^S + k_{NT}^S - e^S \right) + \frac{y_T - r_{NS} x_{NS} - c_{T2}^S}{r_2}, \tag{46}$$

$$b_1^S = c_{T1}^S + r_1 \left(c^S + z^S + k_{NT}^S - e^S \right) - y_T - x_{NS}, \tag{47}$$

$$B_1^S = \frac{\phi\left[p_2^S y_{NT2} + y_T - r_{NS} x_{NS}\right]}{r_2}. \tag{48}$$

From the above equations, the equilibrium effect of governmental loans is to relax the borrowing constraints of households in the South. We assume that $\bar{\tau}_{LS}$ is sufficiently low that the Southern households' borrowing constraints cannot be fully relaxed.

In this subsection, we address the major policy questions in this environment. We characterize under which conditions the Northern and Southern coalitions decide to contract governmental loans, whether the interest rate on the loan must be subsidized, and the impact of the loans on consumption, investment and welfare. Notice that the governmental loan possesses some contingency, because it is announced after the shock $t = 0$.

The *ex post Pareto set* is written in reduced form as the set of *interest rates* r_{NS} on a marginal governmental loan between $t = 1$ and $t = 2$ such that both the North and the South have weakly higher ex post welfare u_{012}^j—and at least one region has strictly higher ex post welfare u_{012}^j—in the equilibrium with the governmental loan than in the laissez-faire equilibrium.

Definition 6 *We define $r_{\min}^N(z)$ as the minimum interest rate r_{NS} that the North coalition is willing to accept from the South to agree on a marginal transfer x_{NS} after a shock z. Similarly, $r_{\max}^S(z)$ is the maximum interest rate that the South is willing to pay to the North in exchange of a marginal transfer after shock z.*

Obviously, after a shock z, there are equilibria with an ex-post Pareto improving marginal transfer x_{NS} if and only if $r_{\max}^S(z) \geq r_{\min}^N(z)$. In that case, we indicate the set of interest rates $r_{NS} \in \left[r_{\min}^N(z), r_{\max}^S(z)\right]$ as ex-post Pareto set.

Proposition 11 (Subsidized Loan) *Suppose that in the laissez-faire equilibrium, the constraint binds in the South for $z \in [\underline{z}, \bar{z}]$. For $z \in [0, \underline{z}]$, the ex post Pareto set is empty. For each $z \in (\underline{z}, \bar{z}]$, there exists a non-empty ex post Pareto set $\left[r_{\min}^N(z), r_{\max}^S(z)\right]$ with the following characteristics:*

$$\lim_{z \to \underline{z}^+}\left\{r_{\min}^N(z)\right\} = \lim_{z \to \underline{z}^+}\left\{r_{\max}^S(z)\right\} < r_2 < 1,$$

$$\frac{dr_{\min}^N(z)}{dz} < 0 \text{ and } \frac{dr_{\max}^S(z)}{dz} > 0,$$

$$r_{\min}^N(z) < r_2 \text{ for all } z \in (\underline{z}, \bar{z}].$$

There may exist $\acute{z} \in [\underline{z}, \bar{z}]$ such that:

$$r_{\max}^S(z) < r_2 \text{ for all } z \in (\underline{z}, \acute{z}), \text{ and } r_{\max}^S(z) \geq r_2 \text{ for all } z \in [\acute{z}, \bar{z}];$$

otherwise, if such $\acute{z}$ does not exist:

$$r_{\max}^{S}(z) < r_2 \text{ for all } z \in (\underline{z}, \bar{z}].$$

When the shock is just of the sufficient size $\underline{z}$ for the constraint in the South to start binding, then $r_{\max}^{S}(z) = r_{\min}^{N}(z)$. As the shock value z increases, $r_{\min}^{N}(z)$ decreases and $r_{\max}^{S}(z)$ increases; this implies that the Pareto set of acceptable r_{NS} expands. Within the Pareto set, the higher is r_{NS}, the better off is the North and the worse off is the South.

Notice that, in some environments, when $\acute{z}$ exists and a shock $z \in [\acute{z}, \bar{z}]$ occurs, the Pareto-improving governmental loan does not need to be subsidized relative to the market interest rate. Indeed, the two coalitions may agree on an interest rate r_{NS} on the marginal transfer such that $r_{NS} \geq r_2$, as long as $r_{NS} \leq r_{\max}^{S}(z)$.

Lemma 6 *The following conditions hold within the Pareto set $\left[r_{\min}^{N}(z), r_{\max}^{S}(z)\right]$ for all $z \in (\underline{z}, \bar{z}]$:*

$$\Delta c_{T1}^{N} < 0, \Delta c_{NT1}^{N} < 0, \Delta p_1^{N} < 0, \Delta k_{NT}^{N} < 0,$$

$$\Delta c_{T2}^{N} > 0, \Delta p_2^{N} > 0$$

$$\Delta c_{T1}^{S} > 0, \Delta c_{NT1}^{S} > 0, \Delta p_1^{S} > 0, \Delta k_{NT}^{S} > 0,$$

$$\Delta c_{T2}^{S} < 0, \Delta p_2^{S} < 0, \Delta B_1^{S} < 0,$$

$$\Delta r_1 = 0, \Delta r_2 > 0, \Delta R_2^{S} < 0,$$

where the Δ prefix denotes the change in each variable in the equilibrium with the governmental loan relative to the laissez-faire equilibrium.

For an ex post Pareto improvement, a governmental loan succeeds where an unconditional gift fails. A governmental loan helps get around the borrowing constraint, not only by increasing the after-tax endowment of households in the South at $t = 1$ as the gift also achieves, but crucially by transferring some of the after-tax endowment of households in the South to households in the North at $t = 2$. This latter effect is needed for the North to gain from the intervention, relative to the laissez-faire equilibrium. Lemma 6 establishes that within the Pareto set, the nontradable goods price p_2^{S} and the borrowing limit B_1^{S} are lower after the intervention, because of the repayment promised by the South to the North at $t = 2$. Therefore, the borrowing constraint of the South is being mitigated by making Southern households need to borrow less, rather than making them able to borrow more, at $t = 1$.

The reason for a subsidy follows directly from the propositions in subsection 4.2. When the North eases the borrowing constraint of the South by providing the transfer $x_{NS} > 0$ at $t = 1$, the interest rate r_2 increases, which transfers welfare from the South to the North. The North is willing to offer a subsidized rate r_{NS} on the governmental loan because it anticipates that the governmental loan will increase the interest rates on private loans made by households in the North to the South. There is a strictly positive gap between $r^N_{\min}(z)$ and $r^S_{\max}(z)$ for $z \in (\underline{z}, \bar{z}]$ because when the constraint binds in the South, households in the South value the marginal unit of consumption at $t = 1$ more than do households in the North, which creates some scope for welfare gains.

Keynes-Ohlin transfer problem. Our results above represent a twist on the time-honored transfer problem of international economics. Keynes feared that inter-country transfers hurt the donor country both directly, as goods leave the country, and indirectly, as the price of the donor country's exports decrease. Following the logic of the original transfer problem, one might imagine that similar effects would occur for the Northern coalition because it provides a subsidized loan.

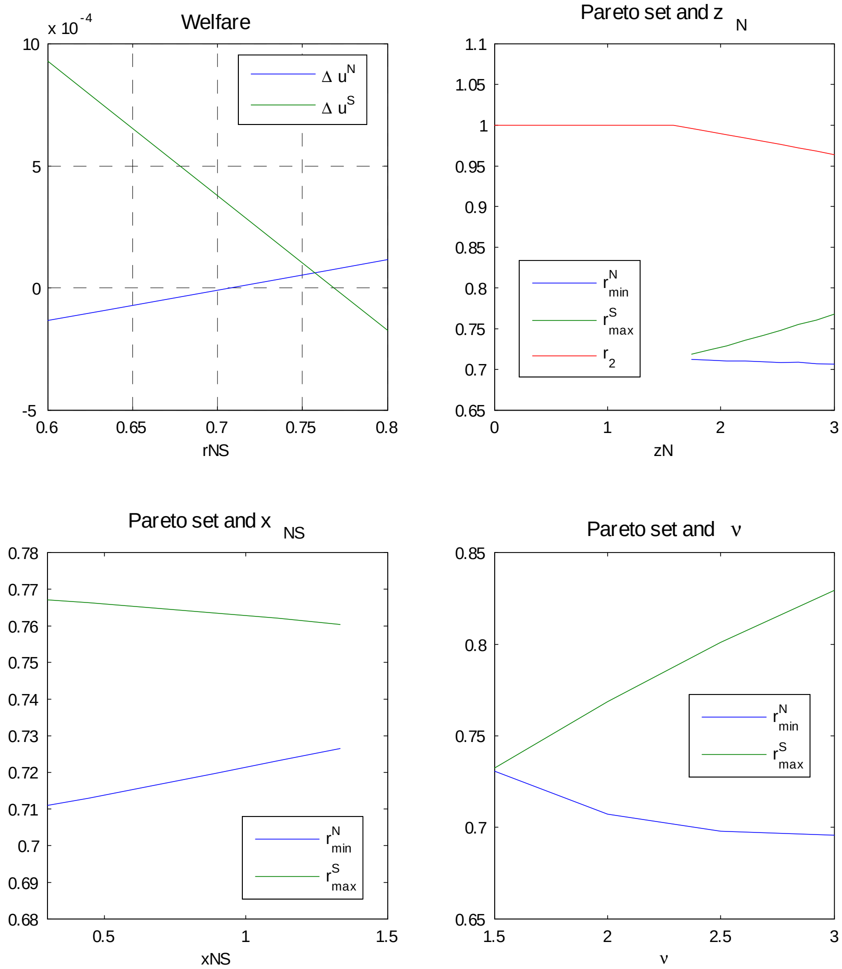
However, the original transfer problem does not consider inter-country transfers in a context where the private sector of the donor country has lent, or expects to lend, to the recipient country, and where borrowing constraints may bind. The single tradable good in our model means that there are no intratemporal terms of trade effects. In fact, the North even benefits from providing the subsidy, because when borrowing constraints bind in the South, subsidized governmental loans from the North to the South increase the market interest rate r_2. The North actually receives a higher return on its "export good"—a net supply of international loans. And far from exacerbating a terms of trade deterioration, a higher ν may even amplify the impact of the governmental loan on market interest rates, benefitting the North.

5.3.1. Numerical Simulations and Ex Ante Solution

Figure 9 illustrates the impact of a governmental loan $x_{NS} > 0$ at various interest rates r_{NS}, after shocks to the North $z^N > 0$, under the alternative specification for the production function. Because of the non Cobb-Douglas production function, r_1 is affected by the borrowing constraint, and the governmental loan increases not just r_2 through a credit supply effect, but also r_1 through a credit demand effect. These interest rate changes transfer ex post welfare from the South to the North in both periods.

The top left panel fixes the value of the shock at $\left(z^N = 3, z^S = 0\right)$, such that the constraint is binding in the South, and then considers various interest rates r_{NS} on a marginal governmental loan $x_{NS} > 0$. The ex post welfare of the North is increasing in the interest rate and the ex post welfare of the South is decreasing in the interest rate. For $r_{NS} \geq r^N_{\min} = 0.71$, the ex post welfare of the North is higher in the equilibrium with the loan than in the laissez-faire equilibrium. For $r_{NS} \leq r^S_{\max} = 0.77$, the ex post welfare of the South is higher in the equilibrium with the loan than in the laissez-faire equilibrium. Therefore, $r_{NS} \in [0.71, 0.77]$ is the ex-post Pareto set. The ex post Pareto set lies below the market interest rate $r_2 = 0.96$, which means that for the chosen value of the shock z, the governmental loan can only make an ex post Pareto improvement if r_{NS} is subsidized relative to r_2.

Figure 9: Loan from North to South



$e^N = 20$, $e^S = 0$, $\phi = 0.33$, $\pi = 0.1$, $\mu = 10$, $z^{bar} = 3$, $y_T = y_{NT2} = 10$
Period t = -1: $c^N = 7.5$, $b_1^N = -12.5$, $c^S = 1.9$, $b_1^S = 1.9$. Panels 1,3,4: $z^N = 3$, $z^S = 0$. First 3 panels: $\nu = 2$

The top right panel plots the ex post Pareto set for different values of the shock $z^N \in [0, \bar{z}]$. Consistent with proposition 11, there do not exist ex post Pareto-improving governmental loans for small shocks when the borrowing constraint is not binding. For large shocks when the borrowing constraint binds in the South, a non-empty Pareto set exists. When the constraint is just binding, $\lim_{z \to \underline{z}^+} \left\{ r^N_{\min} \left(z^N \right) \right\} = \lim_{z \to \underline{z}^+} \left\{ r^S_{\max} \left(z^N \right) \right\} = 0.72.$

The bottom left panel shows that as the size of the governmental loan x_{NS} increases, the ex post Pareto set shrinks because the constraint becomes less binding. We show the Pareto set up to a loan size of $x_{NS} = 1.4$ because above this size, the constraint is not binding and further increments in the loan size produce no additional Pareto gains.

Under the specification of parameters chosen for our graphs, the higher is ν, the larger is the ex post Pareto set $\left[r^N_{\min}(z), r^S_{\max}(z)\right]$ for any given value of the shock z:

$$\frac{dr^N_{\min}(z)}{d\nu} < 0 \text{ and } \frac{dr^S_{\max}(z)}{d\nu} > 0.$$

This result is illustrated in the bottom right panel of figure 9. As in the subsection 5.2.1, the higher is ν, the larger is the *amplification* effect, and therefore the larger is the increase in the interest rate r_1. Therefore, the more willing is the North to subsidize governmental loans to the South. In addition, the higher is ν, the more binding is the constraint in the South, so the higher the interest rate that the Southern government is willing to pay.

Ex ante solution. Next, we turn to the impact on equilibrium variables at $t = -1$ when both the North and the South expect at $t = -1$ that governmental loans will be announced at $t = 0$. For the remainder of this subsection, all variables at $t = -1$ are no longer fixed at the values they take in the laissez-faire equilibrium, but are allowed to adjust to take into account households' expectations at $t = -1$ regarding future governmental loans. In this vein, we also shift focus from the ex post welfare u^i_{012} to the total welfare u^i of each country i and region j.

If the households expect at $t = -1$ that governmental loans will be announced at $t = 0$ after large shocks z which make the constraint binding in the South, then $\Delta c^N < 0, \Delta c^S > 0$, and there is an ambiguous effect on the interest rate r_1 in the non Cobb-Douglas production function[2]. In partial equilibrium, the changes in consumption levels at $t = -1$ would hurt the North and benefit the South, which means that from the perspective of $t = -1$, the North would be willing to offer a lower subsidy on the governmental loan, while the South would be willing to pay a higher interest rate, relative to the ex post problem already analyzed above. However, the general equilibrium change in the interest rate r_1 complicates this reasoning. If r_1 increases, then the North may be willing to offer a higher subsidy and the South may be willing to pay less on the governmental loan.

At $t = -1$, which governmental loans are expected to be announced at $t = 0$? With non-empty Pareto sets, this is potentially a multi-dimensional problem because for each value of the shock z which makes the borrowing constraint of the South binding, there exist a continuum of ex post Pareto-improving loans. We can collapse the dimensionality of the problem by assuming that after any such shock, the interest rate r_{NS} on the governmental loan is determined according to a Nash bargaining game. We refine the notion of the Pareto set accordingly.

Definition 7 $\gamma \in [0, 1]$ *is the weight on the welfare of the North, and* $1 - \gamma$ *is the weight on the welfare of the South, in a Nash bargaining game that takes place at* $t = 0$ *after the*

[2]Under Cobb-Douglas specification for the production function, r_1 does not change.

shock z is realized:

$$r_{NS}(x_{NS}) = \arg\max_{r_{NS}} \left\{\Delta u^N(r_{NS}, x_{NS})\right\}^{\gamma} \left\{\Delta u^S(r_{NS}, x_{NS})\right\}^{1-\gamma}. \tag{49}$$

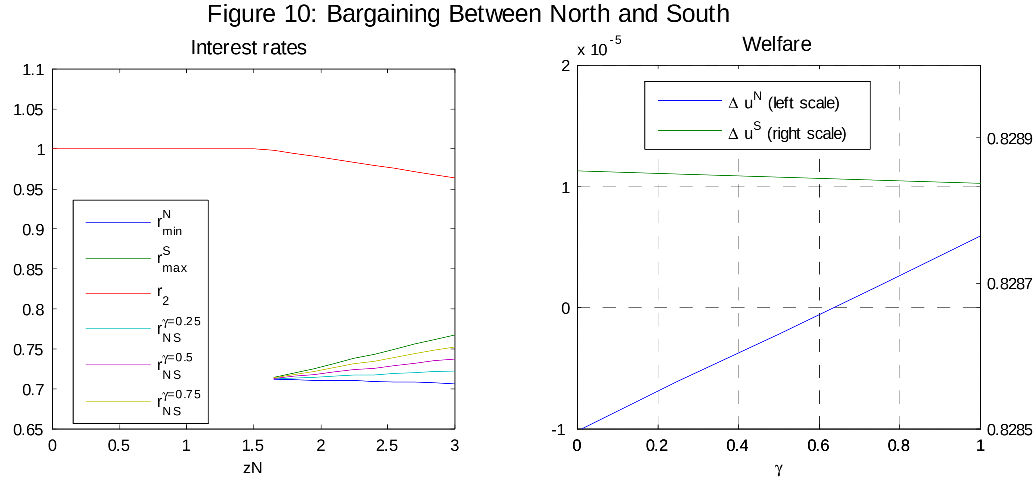
The *ex ante Pareto set* is written in reduced form as the set of *bargaining weights* γ for a governmental loan of marginal size, such that both the North and the South have weakly higher ex-ante welfare u^j—and at least one region has strictly higher welfare u^j—in the equilibrium with the loan than in the laissez-faire equilibrium.

Proposition 12 (Bargaining) *For some choices of parameters, there exists a ex-ante Pareto set $[\hat{\gamma}, 1]$ with $\hat{\gamma} \in [0, 1]$, such that the equilibrium with the governmental marginal loan is a Pareto improvement on the laissez-faire equilibrium.*

The proof of Proposition 12 is by example.

Figure 10 illustrates an example in which the choice of parameters is such that there exists a ex-ante Pareto set $[\hat{\gamma}, 1]$ for a marginal governmental loan $x_{NS} > 0$. The higher is γ, the closer is the interest rate r_{NS} of the governmental loan announced at $t = 0$ to the upper bound of the ex post Pareto set identified in proposition 11. Moreover, from the perspective of $t = -1$, the welfare of the North u^N is increasing in γ, while the welfare of the South u^S is decreasing in it. For the chosen parameters, there exists an ex ante Pareto set with bargaining: $\hat{\gamma} = 0.64 < 1$.

Figure 10: Bargaining Between North and South



$e^N = 20$, $e^S = 0$, $\nu = 2$, $\phi = 0.33$, $\pi = 0.1$, $\mu = 10$, $z^{bar} = 3$, $y_T = y_{NT2} = 10$
First panel has period t = -1: $c^N = 7.5$, $b_1^N = -12.5$, $c^S = 1.9$, $b_1^S = 1.9$.

There are two ways to interpret this result: positive and normative. The positive interpretation is that for some choices of parameters and bargaining weights γ, both the North and the South have an improvement in welfare from the perspective of $t = -1$ when they expect that subsidized governmental loans will be negotiated after the shock is realized at $t = 0$. The normative interpretation is that if the action space of Northern and

Southern governments at $t = -1$ includes the possibility of setting the bargaining weight γ, they will negotiate at $t = -1$ to set the weight within the set $[\hat{\gamma}, 1]$ shown above.

It is interesting to stress that, for Pareto-improving governmental loans to be institutionalized, the bargaining weight assigned ex-ante to the North has to be sufficiently high.

Fixing the interest rate and loan size ex ante. The above Nash bargaining game approach is our preferred perspective on the difference between ex ante and ex post welfares. Nevertheless, we recognize that in implementation terms, designing loan institutions such as the IMF or ESM requires general agreements on interest rates and available loan programs before any shocks strike. Therefore, a separate question is whether there exist Pareto-improving contracts when interest rates and loan sizes are fixed ex ante, although the actual loan disbursement is only made if the shock is binding (consistent with proposition 11). We define such a contract as a limited-contingency governmental loan.

Definition 8 *A limited-contingency governmental loan is a governmental loan of size x_{NS} at rate r_{NS} between $t = 1$ and $t = 2$ such that the loan is provided from the North to the South for all shocks $z \in [\underline{z}, \bar{z}]$ that make the borrowing constraint binding in the South, but such that neither x_{NS} nor r_{NS} are contingent on the shock z.*

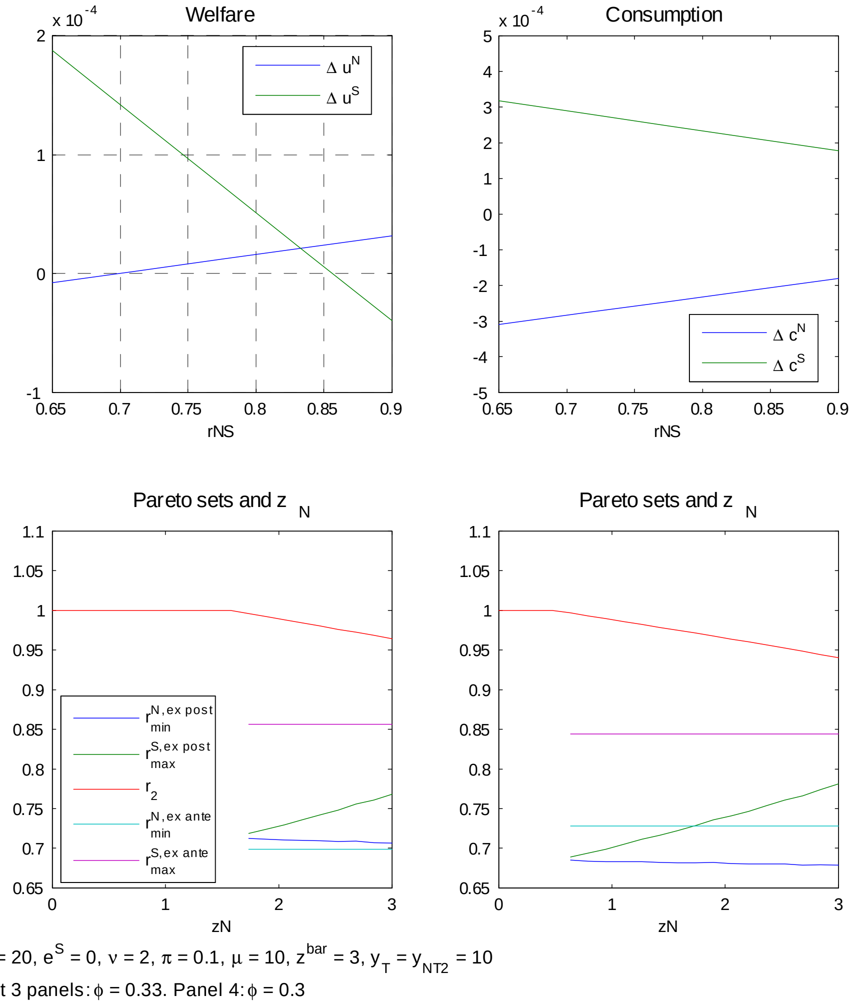
The *ex ante Pareto set* is written in reduced form as the set of interest rates r_{NS} on a limited-contingency governmental loan of marginal size, such that both the North and the South have weakly higher ex-ante welfare u^j—and at least one region has strictly higher welfare u^j—in the equilibrium with the loan than in the laissez-faire equilibrium.

Proposition 13 (Ex Ante Loan) *For some choices of parameters, if the borrowing constraint in the South binds for some values of the shock z, then there exists a non-empty ex ante Pareto set $\left[\tilde{r}^N_{\min}, \tilde{r}^S_{\max}\right]$.*

The proof of Proposition 13 is by example.

Figure 11 illustrates the ex ante Pareto set. The top panels fix the value of the shock at $\left(z^N = 3, z^S = 0\right)$, such that the constraint is binding in the South, and then considers various interest rates r_{NS} on a marginal limited-contingency governmental loan $x_{NS} > 0$. The Δ prefix denotes the change in each variable in the equilibrium with the loan relative to the laissez-faire equilibrium. As the interest rate r_{NS} on the limited-contingency governmental loan increases, the consumption levels and welfare u^N of the North increase and the consumption levels and welfare u^S of the South decrease. For $r_{NS} \in [0.70, 0.86]$, both the North and the South have weakly higher welfare, and at least one region has strictly higher welfare, in the equilibrium with the loan relative to the laissez-faire equilibrium.

Figure 11: Ex Ante and Ex Post Pareto Sets



The bottom panels of figure 11 show ex ante Pareto sets for different values of ϕ. We have included the corresponding ex post Pareto sets for comparison purposes. Since they do not always overlap, as in the bottom right panel, ex ante agreements may rule out interest rates which turn out to be Pareto-improving in the ex post problem.

5.4. Conditional Gift and Tax/Subsidy Package

The final policy intervention we consider in this paper is a combination of a gift and a fiscal policy package. Just like in subsection 5.2, gifts between countries i and regions j at $t = 1$ are feasible subject to each government's budget constraint on net transfers (net over domestic households and foreign governments). In addition, the government of country i is now allowed at $t = 2$ to impose taxes and subsidies on the consumption of both tradable and nontradable goods, subject to a balanced budget constraint within the same period. All actions must be announced at $t = 0$, and the gifts at $t = 1$ can be made conditional on a tax and subsidy package at $t = 2$.

Unlike in the previous subsection, the government is no longer able to commit at $t = 1$ to repay any loans at $t = 2$. By contrast, the government is able to commit at $t = 1$ to undertake a fiscal policy package at $t = 2$. The policy package does not entail governmental transfers between countries at $t = 2$ because the fiscal package is budget-neutral within the period $t = 2$.

The gift from the Northern coalition to the Southern coalition at $t = 1$ is g_{NS}. At $t = 2$, the ad valorem tax on tradable consumption in region j is $\tau^j \geq 0$ and the ad valorem subsidy on nontradable consumption in region j is $\eta^j \in [0,1]$. Households and firms take the transfers, taxes and subsidies as given and maximize utility and profits as in subsection 3.2.

The fiscal policy package (τ^j, η^j) in region j satisfies:

$$\tau^j c_{T2}^j = \eta^j p_2^j c_{NT2}^j. \tag{50}$$

The package can be indexed by τ^j alone, because $\eta^j = \eta^j(\tau^j)$ follows directly from the balanced budget constraint.

The conditional gift g_{NS} and fiscal policy packages τ^j change the equilibrium equations in the North:

$$c_{T1}^N = y_T - g_{NS} - r_1\left(c^N + z^N + k_{NT}^N - e^N\right) + \frac{y_T - c_{T2}^N}{r_2}, \tag{51}$$

$$b_1^N = c_{T1}^N + r_1\left(c^N + z^N + k_{NT}^N - e^N\right) - y_T + g_{NS}, \tag{52}$$

and in the South:

$$c_{T1}^S = y_T + g_{NS} - r_1\left(c^S + z^S + k_{NT}^S - e^S\right) + \frac{y_T - c_{T2}^S}{r_2}, \tag{53}$$

$$b_1^S = c_{T1}^S + r_1\left(c^S + z^S + k_{NT}^S - e^S\right) - y_T - g_{NS} \text{ and } B_1^S = \frac{\phi\left[p_2^S y_{NT2} + y_T\right]}{r_2}. \tag{54}$$

Finally, in both regions j:

$$\left(1 + \tau^j\right) c_{T2}^j = r_2 c_{T1}^j, \tag{55}$$

$$\left(\nu + \tau^j + \nu\tau^j\right) c_{T2}^j = p_2^j c_{NT2}^j. \tag{56}$$

In equilibrium, the tax and subsidy cancel in the budget constraints of the households, because of the balanced budget constraint of the government. Therefore, only the gift g_{NS} at $t = 1$ is visible in the budget constraints of the households, while the taxes τ^j at $t = 2$ are visible in the intertemporal consumption conditions between $t = 1$ and $t = 2$ and in the intratemporal consumption decisions at $t = 2$.

As in the previous subsection, we begin by focusing on the equilibrium from $t = 0$

onward, while assuming that all variables at $t=-1$ remain fixed at the values they take in the laissez-faire equilibrium.

We first solve for the fiscal policy package on its own, then we allow for a gift which is made conditional on the implementation of the fiscal policy package.

Fiscal policy package. In the previous subsection, the governmental loan from North to South relaxed the borrowing constraints of households in the South by raising their after-tax endowment at $t=1$, which decreased their desired borrowing b_1^S. Consumption in the South increased at $t=1$ despite some tightening of the borrowing limit B_1^S. The latter tightening, recorded in Lemma 6, was the result of a decrease in p_2^S as the South repaid the governmental loan at $t=2$ and an increase in the market interest rate r_2.

In the current subsection, the government is not able to commit to repay, so the only way to relax the borrowing constraints of households in the South is to increase B_2^S—for example, by increasing p_2^S. This increase allows the South to consume more of the tradable good at $t=1$. The benefit to the South from higher consumption at $t=1$ is straightforward. For such a change in the equilibrium to have any chance of also increasing the welfare of the North, it must be that the South consumes less of the tradable good at $t=2$. This conjecture is consistent with our earlier comparison between the unconditional gift and the loan.

A subsidy η^S on nontradable consumption, financed by the proceeds from a tax τ^S on tradable consumption, introduces a price wedge between nontradable and tradable consumption at $t=2$, and thereby succeeds in both increasing p_2^S and decreasing c_{T2}^S. By comparison, a transfer from the North to the South at $t=2$ would not succeed, because it would increase both p_2^S and c_{T2}^S.

Since we are primarily concerned with increasing p_2^S, we consider implementing the policy package only in the South.

Lemma 7 *When the constraint binds in the South, implementing the fiscal policy package $\tau^S \geq 0$ in the South has the following effects:*

$$\frac{dc_{T1}^N}{d\tau^S} < 0, \frac{dc_{NT1}^N}{d\tau^S} < 0, \frac{dp_1^N}{d\tau^S} < 0, \frac{dk_{NT}^N}{d\tau^S} < 0,$$

$$\frac{dc_{T1}^S}{d\tau^S} > 0, \frac{dc_{NT1}^S}{dx_{NS}} > 0, \frac{dp_1^S}{d\tau^S} > 0, \frac{dk_{NT}^S}{d\tau^S} > 0,$$

$$\frac{dc_{T2}^N}{d\tau^S} > 0, \frac{dp_2^N}{d\tau^S} > 0 \text{ and } \frac{dc_{T2}^S}{d\tau^S} < 0, \frac{dp_2^S}{d\tau^S} > 0, \frac{dB_1^S}{d\tau^S} > 0,$$

$$\frac{dr_1}{dx_{NS}} = 0, \frac{dr_2}{dx_{NS}} > 0, \frac{dR_2^S}{dx_{NS}} < 0.$$

The fiscal policy package increases the price of nontradable goods in the South p_2^S at $t=2$, and increases the borrowing limit B_1^S. Tradable and nontradable consumption levels c^S_{T1} and c^S_{NT1} in the South increase at $t = 1$, and the level of tradable consumption c^S_{T2} in the South decreases at $t = 2$. In general equilibrium, the additional borrowing capacity of the South also causes an increase in the market interest rate r_2.

Proposition 14 (Fiscal Package) *Suppose that in the laissez-faire equilibrium, the constraint binds in the South for $z \in [\underline{z}, \bar{z}]$. Implementing the fiscal policy package $\tau^S \geq 0$ in the South has the following effects:*

$$\frac{du^N_{012}}{d\tau^S} > 0 \text{ for all } z \in [0, \bar{z}] .$$

There exist some choice of parameters under which there is a thresholds $\grave{z} \in (\underline{z}, \bar{z}]$ such that:

$$\frac{du^S_{012}}{d\tau^S} < 0 \text{ for all } z \in [0, \grave{z})$$

$$\frac{du^S_{012}}{d\tau^S} > 0 \text{ for all } z \in (\grave{z}, \bar{z}] ,$$

and a fiscal policy package on its own can generate a Pareto improvement. Otherwise, if such $\grave{z}$ does not exist:

$$\frac{du^S_{012}}{d\tau^S} < 0 \text{ for all } z \in [0, \bar{z}] ,$$

and a fiscal policy package on its own cannot generate a Pareto improvement.

Notice that according to proposition 14, there exist parameter choices and very large shocks such that a fiscal policy package on its own can generate a Pareto improvement. However, this is only possible if the constraint is so strongly binding that the increase in the interest burden of the South generated by the increase in r_1 and r_2 is more than offset, in ex post welfare terms, by a very high marginal value of consumption by the South at $t = 1$. This result does not violate proposition 9 because it can be shown that while unilateral actions by Southern governments can achieve some Pareto improvements after very large shocks, these governments will not continue their unilateral actions until the interest rate r_2 rises back to one. And while the interest rate remains below one, Pareto improvements are possible. Therefore, there remain Pareto improvements on the table which will not be implementable through unilateral actions alone.

Unlike the unconditional gift and subsidized loan cases described in subsections 5.2 and 5.3, the fiscal policy package increases interest rate r_2 even when the constraint is not binding in the South at the laissez-faire equilibrium. The reason is that the tax τ^S enters the intertemporal decision of households in the South:

$$R^S_2 = \frac{c^S_{T2}}{c^S_{T1}} = \frac{r_2}{1 + \tau^S}. \tag{57}$$

The fiscal policy package increases c^S_{T1} relative to c^S_{T2}, generating upward pressure on interest rate r_2. When the constraint is binding, the fiscal policy package has an additional effect through the higher borrowing limit B^S_1.

Gift and fiscal policy package combination. Given the extreme conditions that are needed for the fiscal policy package to generate a Pareto improvement on its own, and given that such Pareto improvements remain incomplete under unilateral actions by the South, we now turn to the question of whether adding a conditional gift g_{NS} from the North to the South at $t = 1$ could be useful.

The ex post Pareto set is written in reduced form as the set of sizes of the conditional gift g_{NS} at $t = 1$ associated with a fiscal policy package at $t = 2$ of given marginal size τ^S, such that both the North and the South have weakly higher ex post welfare u^j_{012}—and at least one region has strictly higher ex post welfare u^j_{012}—in the equilibrium with the conditional gift and fiscal policy package than in the laissez-faire equilibrium.

Definition 9 *We define $g^S_{\min}(z)$ as the minimum gift that the Southern coalition would expect from the Northern coalition for undertaking fiscal policy with marginal tax τ^S after shock z occurs. Similarly, we define $g^N_{\max}(z^S)$ as the maximum gift that the Northern coalition would be willing to provide to the South if the South implements fiscal policy with marginal tax τ^S after shock z.*

There is an equilibrium with ex-post Pareto-improving conditional gift g_{NS} and associated fiscal policies indexed by τ^S if and only if $g^S_{\min}(z) \leq g^N_{\max}(z)$. In that case, we indicate the set of conditional gifts $g_{NS} \in \left[g^S_{\min}(z), g^N_{\max}(z)\right]$ as ex-post Pareto set.

Proposition 15 (Gift and Fiscal Package) *Suppose that in the laissez-faire equilibrium, the constraint binds in the South for $z \in [\underline{z}, \bar{z}]$. Consider a small fiscal policy package τ^S. Then for $z \in [0, \underline{z}]$, the ex post Pareto set is empty; for each $z \in (\underline{z}, \bar{z}]$, there exists a non-empty ex post Pareto set $\left[g^S_{\min}(z), g^N_{\max}(z)\right]$ with the following characteristics:*

$$\lim_{z \to \underline{z}^+} \left\{g^S_{\min}(z)\right\} = \lim_{z \to \underline{z}^+} \left\{g^N_{\max}(z)\right\} > 0,$$

$$\frac{dg^S_{\min}(z)}{dz} < \frac{dg^N_{\max}(z)}{dz} < 0 \textit{ for all } z \in (\underline{z}, \bar{z}].$$

$$g^N_{\max}(z) > 0 \textit{ for all } z \in (\underline{z}, \bar{z}].$$

For choices of parameters such that $\grave{z}$ and $\grave{\tau}^S(\grave{z})$ as defined in proposition 14 do exist:

$$g^S_{\min}(z) > 0 \textit{ for all } z \in (\underline{z}, \grave{z}), \textit{ and } g^S_{\min}(z) \leq 0 \textit{ for all } z \in [\grave{z}, \bar{z}];$$

otherwise, if such $\grave{z}$ does not exist:

$$g^S_{\min}(z) > 0 \textit{ for all } z \in (\underline{z}, \bar{z}].$$

Lemma 8 *The following conditions hold within the Pareto set* $\left[g^S_{\min}(z), g^N_{\max}(z)\right]$ *for all* $z \in (\underline{z}, \bar{z}]$*:*

$$\Delta c^N_{T1} < 0, \Delta c^N_{NT1} < 0, \Delta p^N_1 < 0, \Delta k^N_{NT} < 0,$$

$$\Delta c^N_{T2} > 0, \Delta p^N_2 > 0$$

$$\Delta c^S_{T1} > 0, \Delta c^S_{NT1} > 0, \Delta p^S_1 > 0, \Delta k^S_{NT} > 0,$$

$$\Delta c^S_{T2} < 0, \Delta p^S_2 > 0, \Delta B^S_1 > 0,$$

$$\Delta r_1 = 0, \Delta r_2 > 0, \Delta R^S_2 < 0,$$

where the Δ *prefix denotes the change in each variable in the equilibrium with the governmental loan relative to the laissez-faire equilibrium.*

When the constraint is just binding, $g^S_{\min}(z) = g^N_{\max}(z)$. Then as the shock value z^N increases, both $g^S_{\min}(z)$ and $g^N_{\max}(z)$ decrease. The Pareto set remains non-empty because the former decreases faster than the latter.

The reason for a positive gift is as follows. For a small fiscal policy package, the distortions to tradable and nontradable consumption c^S_{T2} and c^S_{NT2} in the South at $t = 2$ have a second-order negative effect on the ex post welfare of the South. At $t = 1$, the relaxation of the borrowing constraint generates a higher consumption level c^S_{T1} in the South. Finally, the increase in the interest rate r_2 generates a first-order welfare transfer from the South to the North. If the latter effect dominates the others, then the fiscal package cannot generate an ex post Pareto improvement on its own, and a positive gift $g_{NS} > 0$ must be provided to make the South willing to undertake the fiscal policy package.

There is a strictly positive gap between $g^S_{\min}(z)$ and $g^N_{\max}(z)$ for $z \in (\underline{z}, \bar{z}]$ because when the constraint binds in the South, households in the South value the marginal unit of consumption at $t = 1$ more than do households in the North, which creates some scope for welfare gains. Within the Pareto set, the higher is g_{NS}, the worse off is the North and the better off is the South.

The existence of the nontradable goods sector at $t = 2$ is absolutely necessary for our result. Without nontradable goods at $t = 2$, there is no way that the borrowing limit of the South B^S_1 can be increased. And without raising the borrowing limit, there is no way that consumption levels in the South can be increased at $t = 1$, because in this subsection, the government has no ability to commit to repay. Lemma 8 establishes that within the Pareto set, the nontradable goods price p^S_2 and the borrowing limit B^S_1 are higher after the intervention, because of the subsidy for nontradable goods consumption and the tax on tradable consumption at $t = 2$. Therefore, the borrowing constraint of the South is being mitigated by making Southern households able to borrow more, rather than making them need to borrow less, at $t = 1$.

Policy considerations. Two practical considerations apply when translating the results of this model into policy implementation. First, the conditional gift g_{NS} at $t = 1$ associated with an ex post Pareto improvement can be interpreted as conditional debt relief. It is always smaller than the desired borrowing of the South b_1^S, and it is conditioned on fiscal actions in the next period $t = 2$. Comparing subsection 5.2 to the current subsection, our model establishes that unconditional debt relief cannot generate an ex post Pareto improvement, but conditional debt relief can.

Second, remember that the purpose of the fiscal policy package at $t = 2$ is not just to reduce the South's tradable consumption at $t = 2$, but to increase the borrowing limit of the South between $t = 1$ and $t = 2$. In practice, unlike our model, not all nontradable goods can be used as collateral against borrowing. Therefore, our model recommends that subsidies are targeted to the prices of collateralizable nontradable goods and assets—such as housing and fixed capital—rather than equally on all nontradable goods. This crucial point distinguishes our recommendation from looking much like protectionism for the entire nontradable sector.

5.4.1. Numerical Simulations and Ex Ante Solution

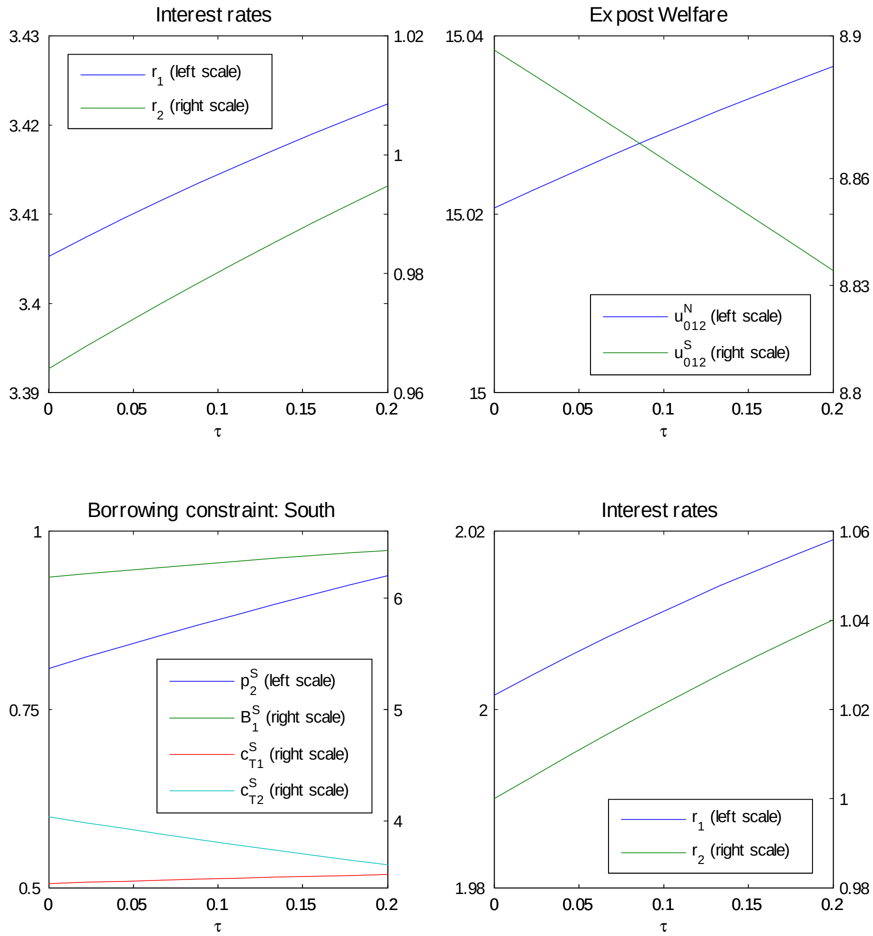
Fiscal policy package alone. Figure 12 fixes the value of the shock to $\left(z^N = 3, z^S = 0\right)$, such that the constraint is binding in the South, and illustrates the impact of the fiscal policy package being implemented in the South under the alternative (non Cobb-Douglas) production function.

Consistent with the results above, the fiscal policy package increases the price of nontradable goods in the South p_2^S at $t = 2$, and increases the borrowing limit B_1^S. Tradable and nontradable consumption levels c_{T1}^S and c_{NT1}^S in the South increase at $t = 1$, and the level of tradable consumption c_{T2}^S in the South decreases at $t = 2$. In general equilibrium, the additional borrowing capacity of the South also causes an increase in the market interest rates r_1 and r_2.

For the chosen parameters, the ex post welfare of the North u_{012}^N increases and the ex post welfare of the South u_{012}^S decreases, so no Pareto improvement is possible. The increase in consumption of the South at $t = 1$ is too small relative to the increase in the interest burden on the South's previous borrowing—in net terms, hurting the ex post welfare of the South u_{012}^S.

The bottom right panel of figure 12 shows the impact of a shock on r_1 and r_2 when the value of the shock is $z = 0$, i.e. the constraint does not bind. The tax still has an effect on allocations, as mentioned above.

Figure 12: Fiscal Policy Package τ^S in the South



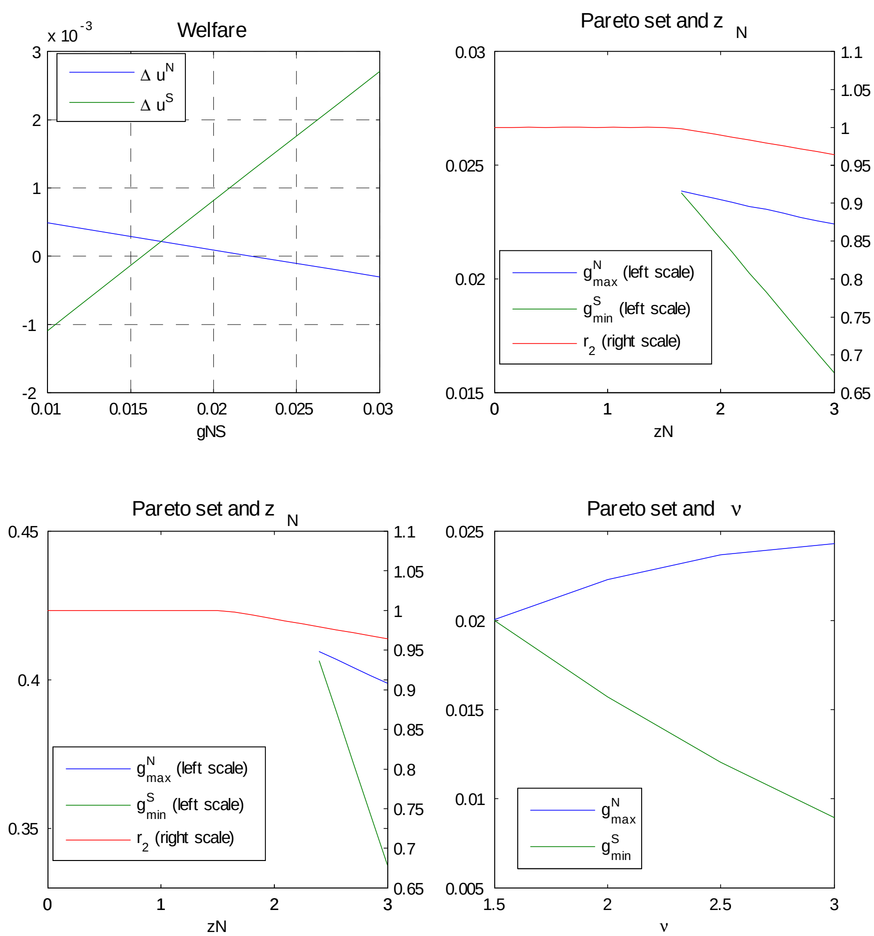
$e^N = 20$, $e^S = 0$, $\nu = 2$, $\phi = 0.33$, $\pi = 0.1$, $\mu = 10$, $z^{bar} = 3$, $y_T = y_{NT2} = 10$
Period t = -1: $c^N = 7.5$, $b^N_{-1} = -12.5$, $c^S = 1.9$, $b^S_{-1} = 1.9$. First 3 panels: $z^N = 3$, $z^S = 0$. Panel 4: $z^N = z^S = 0$.

Gift and fiscal policy package combination. Figure 13 illustrates the impact of a small fiscal policy package $\tau^S = 0.01$, with various sizes of the conditional gift g_{NS}, after shocks to the North $z^N > 0$, in the environment with the non Cobb-Douglas production function.

The top left panel fixes the value of the shock at $\left(z^N = 3, z^S = 0\right)$, such that the constraint is binding in the South, and then considers various sizes of the conditional gift g_{NS} at $t = 1$ to complement the fiscal policy package of fixed size at $t = 2$. The ex post welfare of the North is decreasing in the gift and the ex post welfare of the South is increasing in the gift. For $g_{NS} \leq g^N_{\max} = 0.022$, the ex post welfare of the North is higher in the equilibrium with the gift and policy package than in the laissez-faire equilibrium. For $g_{NS} \geq g^S_{\min} = 0.016$, the ex post welfare of the South is higher in the equilibrium with the gift and policy package than in the laissez-faire equilibrium. Therefore, $g_{NS} \in [0.016, 0.022]$ is an ex-post Pareto set. The ex post Pareto set lies above zero, which means that an ex post Pareto improvement requires a positive gift from the North to the South at $t = 1$.

The top right panel plots the ex post Pareto set for different values of the shock $z^N \in [0, \bar{z}]$. Consistent with proposition 15, there do not exist ex post Pareto-improving policy interventions for small shocks when the borrowing constraint is not binding. For large shocks when the borrowing constraint binds in the South, a non-empty Pareto set exists. When the constraint is just binding, $\lim_{z \to \underline{z}^+} \left\{ r^N_{\min}\left(z^N\right) \right\} = \lim_{z \to \underline{z}^+} \left\{ r^S_{\max}\left(z^N\right) \right\} = 0.024$.

Figure 13: Conditional Gift g_{NS} and Fiscal Policy Package τ^S



$e^N = 20$, $e^S = 0$, $\phi = 0.33$, $\pi = 0.1$, $\mu = 10$, $z^{bar} = 3$, $y_T = y_{NT2} = 10$. Panels 1,2,4: $\tau^S = 0.01$. Panel 3: $\tau^S = 0.1$
Period t = -1: $c^N = 7.5$, $b^N_1 = -12.5$, $c^S = 1.9$, $b^S_1 = 1.9$. Panels 1,3,4: $z^N = 3$, $z^S = 0$. First 3 panels: $\nu = 2$

The bottom left panel of figure 13 shows that for a larger fiscal policy package $\tau^S = 0.2$, the ex post Pareto set $\left[g^S_{\min}(z), g^N_{\max}(z)\right]$ shifts: it now only exists for higher values of the shock z^N, and a higher gift must be provided at $t = 1$. The higher gift compensates for the larger distortions to tradable and nontradable consumption c^S_{T2} and c^S_{NT2} in the South at $t = 2$.

The size of the ex post Pareto set $\left[g^S_{\min}(z), g^N_{\max}(z)\right]$ depends on ν. The higher is ν, the larger is the ex post Pareto set $\left[g_{\min}^S(z), g_{\max}^N(z)\right]$ for any given value of the shock z:

$$\frac{dg_{\min}^S(z)}{d\nu} < 0 \text{ and } \frac{dg_{\max}^N(z)}{d\nu} > 0.$$

Under a non Cobb-Douglas specification, there are two reasons for this result, which is illustrated in the bottom right panel of figure 13. First, as in the Cobb-Douglas case, the higher is ν, the larger is the impact of the fiscal policy package on the nontradable goods price at $t = 2$, and therefore the larger the increase in the borrowing limit B_1^S and interest rate r_2. Second, the higher is ν, the larger is the amplification effect from an increase in tradable spending onto nontradable spending at $t = 1$, and therefore the larger is the increase in the interest rate r_1. For both these reasons, the North is willing to provide a larger gift.

Ex ante solution. To complete this subsection, we allow all variables at $t = -1$ to adjust to take into account households' expectations at $t = -1$ regarding future conditional gifts and fiscal policy packages. We shift focus from the ex post welfare u_{012}^i to the total welfare u^i of each country i and region j.

Suppose that households expect at $t = -1$ that conditional gifts and fiscal packages will be announced at $t = 0$ after large shocks z which make the constraint binding in the South. Then: $\Delta c^N < 0, \Delta c^S > 0$, and there is an ambiguous effect on the interest rate r_1 in the non Cobb-Douglas production function[3].

Paralleling subsection 5.3.1, we first assume that after any shock which makes the borrowing constraint of the South binding, the conditional gift g_{NS} at $t = 1$ associated with the fixed-size fiscal policy package τ^S at $t = 2$ is determined according to a Nash bargaining game. We again refine the notion of the Pareto set accordingly.

Definition 10 $\gamma \in [0, 1]$ *is the weight on the welfare of the North, and* $1-\gamma$ *is the weight on the welfare of the South, in a Nash bargaining game that takes place at* $t = 0$ *after the shock* z *is realized:*

$$g_{NS}\left(\tau^S\right) = \arg\max_{g_{NS}} \left\{\Delta u^N\left(g_{NS}, \tau^S\right)\right\}^{\gamma} \left\{\Delta u^S\left(g_{NS}, \tau^S\right)\right\}^{1-\gamma}. \tag{58}$$

The *ex ante Pareto set* is written in reduced form as the set of bargaining weights γ for a fixed-size fiscal policy package τ^S, such that both the North and the South have weakly higher ex-ante welfare u^j—and at least one region has strictly higher ex-ante welfare u^j—in the equilibrium with the loan than in the laissez-faire equilibrium.

Proposition 16 (Bargaining) *For some choices of parameters, there exists a Pareto set* $[\hat{\gamma}, 1]$ *with* $\hat{\gamma} \in [0, 1]$*, such that the equilibrium with the conditional gift and fiscal policy package is a Pareto improvement on the laissez-faire equilibrium.*

[3]Under Cobb-Douglas specification for the production function, r_1 does not change.

Fixing the gift and fiscal package ex ante. Next, for implementation concerns, we allow the Northern and Southern coalitions to commit at $t = -1$ to limited-contingency gift and fiscal policy packages.

Definition 11 *A limited-contingency gift g_{NS} and fiscal policy package τ^S is a combination of the conditional gift and tax/subsidy package for all shocks $z \in [\underline{z}, \bar{z}]$ that make the borrowing constraint binding in the South, but such that neither g_{NS} nor τ^S are contingent on the shock z.*

The *ex ante Pareto set* is written in reduced form as the set of sizes of the conditional gift g_{NS} at $t = 1$ associated with a fiscal policy package at $t = 2$ of given size τ^S, such that both the North and the South have weakly higher ex-ante welfare u^j—and at least one region has strictly higher ex-ante welfare u^j—in the equilibrium with the conditional gift and fiscal policy package than in the laissez-faire equilibrium.

Proposition 17 (Ex Ante Gift and Package) *If the borrowing constraint in the South binds for some values of the shock z, then there exists a non-empty ex ante Pareto set $\left[\tilde{g}^S_{\min}(z), \tilde{g}^N_{\max}(z)\right]$.*

The normative interpretation of our results on the ex ante solution is that in general, the North and/or South would use any power that they possess at $t = -1$ to shape the bargaining between North and South after the shock z at $t = 0$. The North and/or South would try at $t = -1$ to rule out those combinations of gifts and fiscal policy packages that generate ex post but not ex ante Pareto improvements.

This normative interpretation is not without some tension. In the previous subsection, the features of governmental loans that we discussed institutionalizing at $t = -1$ were the interest rate r_{NS}, and the bargaining power γ over the interest rate. We could imagine designing special governmental loans which would be activated specifically for large shocks which cause constraints to bind in the South.

In the current subsection, the features that would need to be institutionalized are the haircuts for debt relief and the precise levels of fiscal taxes and subsidies. All of these governmental policy tools are used to address a wide variety of social ills, and therefore appear more difficult to commit to in advance. Nevertheless, if policymakers are to adopt the recommendations of our model, some work on institutionalization for even these more disparate tools is necessary.

6. Conclusion

Institutions for a financial union. Philosophically, our model and policy recommendations represent a departure from the optimal currency area literature—a literature which has shaped much thinking about supranational integration processes in general, and the European financial and sovereign debt crisis in particular. Instead, we design a model of supranational integration which builds on, and extends, fundamental market imperfections which have traditionally been found in the sudden stop literature for emerging market economies, and which have more recently been adapted in various forms for the analysis of individual economies in the European periphery.

We focus on noncontingent debt and borrowing constraints instead of the rigidity in price adjustment implicit or explicit within the optimal currency area literature. As a result, our conception of a union of countries is different. First, it is defined in a financial sense, with a free flow of debt contracts. Second, instead of dividing the countries within a union into those which have not suffered from shocks and those which have, we divide the countries into those with high assets—the Northern "core"—and those with high debt—the Southern "periphery". In a heterogeneous union, our policy recommendations are not that countries who have suffered from shocks should obtain support from the rest, but rather that the South should receive support from the North, irrespective of where the shock materialized. In addition, implementation is simplified because the support can be conditioned only on how binding the constraint is in the South, instead of on the specific size and location of the shock.

Bargaining between regional coalitions of countries is needed in order to overcome a collective action problem. If the Northern countries do not act in a coalition, then support will be under-provided. The government of each country in the North ignores their own impact on union-wide interest rates, and hopes instead that the borrowing constraint in the South is relaxed by the actions of the rest of the governments of the North. The government of each country in the South also ignores their own impact on union-wide interest rates and is willing to undertake any actions that help relax their borrowing constraint, even if such actions when undertaken by all the Southern countries ends up hurting all of them. Pareto improvements are made possible through bargaining at the regional level.

When governmental actions are available, whether in the form of governmental loans, conditional gifts and/or taxes and subsidies, some degree of institutionalization of such actions is desirable. Actions which all governments will agree to once the shock has been realized may or may not be consistent with Pareto improvements from the perspective of countries before the shock has been realized. In general, before any shocks occur, some restrictions should be placed on the actions available in future regional bargaining.

Comparing policy interventions. In our model, unconditional gifts from North to South cannot alter interest rates sufficiently to compensate the North, which means that such redistributive measures will be resisted. Governmental loans from North to South can succeed in generating Pareto improvements, even when they appear to carry a subsidy, because the general equilibrium effects of such lending benefit the North. However, notice that governmental loans only have potency if governments in the South have some debt capacity independent of the private sector.

If fiscal capacity is used up, attention must turn to other governmental actions that are available to raise borrowing limits in the South. In our model, conditional debt relief tied to a tax/subsidy package is feasible without violating any constraints, and can generate Pareto improvements. So we face a puzzle: in positive terms, what makes such policies difficult?

We can think of three reasons. First, the coalitions needed to agree on simultaneous debt relief and fiscal policies are highly diverse. In some countries, debt levels were initially high in the private sector, while in others the public sector was first constrained. And taxes and subsidies are fraught with rigidities owing to political economy considerations. Second, different governmental tools are controlled by different institutions. For example, targeted quantitative easing by authorities could help increase the prices of collateralizable nontradable assets in the South, as our model recommends, but will be opposed by the North without coordinated fiscal policies that limit tradable consumption in the South. Third, the policy package is counter-intuitive. While austerity across the board is easily supported by a narrative of fiscal profligacy and asset price bubbles in the South, a proposal to impose taxes on tradable consumption and subsidies on collateralizable nontradable assets is not. Our model suggests that irrespective of past valuations before the shock was realized, excessive downward price flexibility in the nontradable sector can prolong crises.

Countries outside the financial union. How are our results changed by the existence of other countries outside the financial union? Our result is not dependent on the current account of the financial union being closed, but it is related to financial integration being stronger within the financial union than between the financial union and other countries. In particular, if the North has an advantage relative to the rest of the world in intermediating funds to the South, then Northern households can earn a rent from lending to the South. Our Pareto-improving interventions still work: for example, the tax/subsidy policy in the South pushes up the rent of the North, and in exchange the North can provide the South with debt relief.

Fiscal and banking union. There remains to be solved a complex implementation problem to breathe life into any proposed policy interventions. At the moment, the terminology of fiscal and banking union remains in considerable flux. The main lesson from our model is that the borrowing constraints of constrained countries need to be relaxed in ways that also benefit the lender countries. In practice, governmental loans can be helpful, and to the extent that they must be specifically designed and made available in advance, institutionalizing them in the form of a fiscal union and/or IMF membership is recommended. Moreover, governments are best suited to impose taxes and subsidies, and to provide targeted tax cuts to reach borrowing-constrained agents who have no access to the banking system.

On the other hand, banks are best suited to identify which households are in fact constrained, and therefore may be an essential vehicle for Southern governments who are figuring out how to most efficiently use the funds they have received from governments in the North. The North is willing to offer a higher subsidy if more of the governmental loan is forcibly channeled to constrained households in the South, rather than to generalized spending on public goods—which means that subsidized supranational loans may be easier to operationalize when the loans are earmarked for banking sector resolution. Once fiscal capacity is used up, bank participation is needed to enact debt relief and to provide support to the prices of collateralizable nontradable assets.

7. References

Aoki, Kosuke, Gianluca Benigno, and Nobuhiro Kiyotaki. (2009). "Capital Flows and Asset Prices," in *NBER International Seminar on Macroeconomics 2007*, Richard Clarida and Francesco Giavazzi (eds.). Chicago: University of Chicago Press.

Beetsma, Roel M. W. J., and Henrik Jensen. (2005). "Monetary and Fiscal Policy Interactions in a Micro-Founded Model of a Monetary Union," *Journal of International Economics*, Vol. 67 (2), 320-352.

Benigno, Gianluca, Huigang Chen, Christopher Otrok, Alessandro Rebucci, and Eric R. Young. (2014). "Optimal Capital Controls and Real Exchange Rate Policies: A Pecuniary Externality Perspective," CEPR Discussion Paper No. 9936.

Bianchi, Javier. (2011). "Overborrowing and Systematic Externalities in the Business Cycle," *The American Economic Review*, Vol. 101 (7), 3400-3426.

Blanchard, Olivier, and Francesco Giavazzi. (2002). "Current Account Deficits in the Euro Area: The End of the Feldstein-Horioka Puzzle?," *Brookings Papers on Economic Activity*, Vol. 33 (2), 147-210.

Caballero, Ricardo J., and Arvind Krishnamurthy. (2001). "International and Domestic Collateral Constraints in a Model of Emerging Market Crises," *Journal of Monetary Economics*, Vol. 48 (3), 513-548.

Calvo, Guillermo A. (1998). "Capital Flows and Capital Market Crises: The Simple Economics of Sudden Stops," *Journal of Applied Economics*, Vol. 1 (1), 35-54.

Chari, Varadarajan V., and Patrick J. Kehoe. (2007). "On the Need for Fiscal Constraints in a Monetary Union," *Journal of Monetary Economics*, Vol. 54 (8), 2399-2408.

Chari, Varadarajan V., and Patrick J. Kehoe. (2008). "Time Inconsistency and Free-Riding in a Monetary Union," *Journal of Money, Credit and Banking*, Vol. 40 (7), 1329-1356.

Chen, Ruo, Gian Maria Milesi-Ferretti, and Thierry Tressel. (2012). "External Imbalances in the Euro Area," IMF Working Paper No. 12/236.

Cremers, Emily T., and Partha Sen. (2009). "Transfers, the Terms of Trade, and Capital Accumulation," *The Canadian Journal of Economics*, Vol. 42 (4), 1599-1616.

Dornbusch, Rudiger, Ilan Goldfajn, and Rodrigo O. Valdés. (1995). "Currency Crises and Collapses," *Brookings Papers on Economic Activity*, 219-293.

Djajić, Slobodan, Sajal Lahiri, and Pascalis Raimondos-Møller. (1998). "The Transfer Problem and the Intertemporal Terms of Trade," *The Canadian Journal of Economics*, Vol. 31 (2), 427-436.

Farhi, Emmanuel, and Iván Werning. (2013a). "Fiscal Unions," NBER Working Paper No. 18280.

Farhi, Emmanuel, and Iván Werning. (2013b). "Fiscal Multipliers: Liquidity Traps and Currency Unions," NBER Working Paper No. 18381.

Friedman, Milton. (1953). "The Case for Flexible Exchange Rates," in *Essays in Positive Economics*, Milton Friedman, Chicago: University of Chicago Press.

Galí, Jordi, and Tommaso Monacelli. (2008). "Optimal Monetary and Fiscal Policy in a Currency Union," *Journal of International Economics*, Vol. 76 (1), 116-132.

Giavazzi, Francesco, and Luigi Spaventa. (2010). "Why the Current Account May Matter in a Monetary Union: Lessons from the Financial Crisis in the Euro Area," CEPR Discussion Paper No. 8008.

Goyal, Rishi, Petya Koeva Brooks, Mahmood Pradhan, Thierry Tressel, Giovanni Dell'Ariccia, Ross Leckow, and Ceyla Pazarbasioglu. (2013). "A Banking Union for the Euro Area," IMF Staff Discussion Note No. 13/01.

Jeanne, Olivier, and Anton Korinek. (2013). "Macroprudential Regulation Versus Mopping Up After the Crash," NBER Working Paper No. 18675.

Kenen, Peter. (1969). "The Theory of Optimum Currency Areas: An Eclectic View," in *Monetary Problems of the International Economy*, Robert A. Mundell and Alexander K. Swoboda (eds.), Chicago: University of Chicago Press.

Keynes, John M. (1929). "The German Transfer Problem," *The Economic Journal*, Vol. 39, 1-7.

Kiyotaki, Nobuhiro, and John Moore. (1997). "Credit Cycles," *Journal of Political Economy*, Vol. 105 (2), 211–48.

Korinek, Anton. (2011). "The New Economics of Prudential Capital Controls," *IMF Economic Review*, Vol. 59 (3), 523-561.

Krugman, Paul. (1988). "Financing versus Forgiving a Debt Overhang," *Journal of Development Economics*, Vol. 29, 253-268.

Martin, Philippe, and Thomas Philippon. (2014). "Inspecting the Mechanism: Leverage and the Great Recession in the Eurozone," Mimeo, New York University.

McKinnon, Ronald. (1963). "Optimum Currency Areas," *The American Economic Review*, Vol. 53, 717-724.

Mendoza, Enrique G. (2002). "Credit, Prices and Crashes: Business Cycles with a Sudden Stop," in *Preventing Currency Crises in Emerging Markets*, S. Edwards and J. Frankel (eds.), Chicago: University of Chicago Press.

Mendoza, Enrique G. (2006). "Lessons from the Debt-Deflation Theory of Sudden Stops," *The American Economic Review*, Vol. 96 (2), 411-416.

Mendoza, Enrique G. and Katherine A. Smith. (2006). "Quantitative Implications of a Debt-Deflation Theory of Sudden Stops and Asset Prices," *Journal of International Economics*, Vol. 70 (1), 82-114.

Merler, Silvia, and Jean Pisani-Ferry. (2012). "Sudden Stops in the Euro Area," Bruegel Policy Contribution No. 2012/06.

Mundell, Robert A. (1961). "A Theory of Optimum Currency Areas," *The American Economic Review*, Vol. 51 (4), 657-665.

Mundell, Robert A. (1973). "Uncommon Arguments for Common Currencies," in *The Economics of Common Currencies*, H. G. Johnson and A. K. Swoboda (eds.), London: Allen and Unwin Limited.

Ohlin, Bertil. (1929). "The Reparation Problem: A Discussion," *The Economic Journal*, Vol. 39, 172-173.

Reis, Ricardo. (2013). "The Portuguese Slump and Crash and the Euro Crisis," *Brookings Papers on Economic Activity*, Vol. 46 (1), 143-210.

Shambaugh, Jay C. (2012). "The Euro's Three Crises," *Brookings Papers on Economic Activity*, Spring 2012, 157-231.

Appendix

Proof of Lemmas 1 and 2. By inspection. ■

Proof of Lemma 3. Substituting the investment formula (22) into the $t = 0$ union-wide resource constraint (28), and then applying the $t = 1$ union-wide resource constraint (29):

$$r_1 = \frac{2\alpha\nu y_T}{e^T - \sum\limits_{j=N,S} \left(c^j + z^j\right)}, \tag{59}$$

which establishes the result. ■

Proof of Lemma 4. First, assume the borrowing constraints are never binding. Substituting the investment formula (22) and the Euler condition (26) into the budget constraint (21):

$$c^j_{T1} = \frac{1}{2+\alpha\nu}\left\{y_T\left(1+\frac{1}{r_2}\right) + r_1\left(e^j - c^j - z^j\right)\right\}. \tag{60}$$

Applying the $t = 1$ union-wide resource constraint (28):

$$r_2 = 1 \tag{61}$$

$$c^j_{T1} = \frac{1}{2+\alpha\nu}\left\{2y_T + r_1\left(e^j - c^j - z^j\right)\right\}. \tag{62}$$

Substituting equations (59) and (62) into the Euler condition (20):

$$1 = \mathbb{E}_{t=-1}\left\{\frac{\alpha\nu\left(2+\alpha\nu\right)c^j}{\sum\limits_{j=N,S}\left(e^j - c^j - z^j\right) + \alpha\nu\left(e^j - c^j - z^j\right)}\right\}. \tag{63}$$

This equation can be manipulated to show that as the initial tradable endowment e^j increases, the consumption level c^j and the level of savings $\left(e^j - c^j\right) = -b^j_{-1}$ both increase.

Therefore, the level of savings of Northern households is always positive (and the level of their debt always negative) for all $H \in \left[0, e^T\right]$.

The level of savings of Southern households is positive when $H = 0$ but negative when $H = e^T$, the latter because of the infinite marginal utility of consumption when consumption is near zero. Therefore, by continuity, there exists $H^* \in \left(0, e^T\right)$ such that:

$$\begin{array}{ll} e^S - c^S > 0 & \text{for } H \in [0, H^*) \\ e^S - c^S = 0 & \text{for } H = H^* \\ e^S - c^S < 0 & \text{for } H \in \left(H^*, e^T\right]. \end{array} \tag{64}$$

Next, it can be shown that the lemma holds, with some changes to the proof, when the borrowing constraints bind after large shocks. ■

Proof of Proposition 1. First, assume the borrowing constraints are never binding. Subsituting equation (59) into equation (62) and rearranging, we obtain the general equilibrium equation for c_{T1}^j:

$$c_{T1}^j = \frac{2y_T}{2+\alpha\nu}\left\{1+\alpha\nu\frac{e^j - c^j - z^j}{\sum\limits_{j=N,S}\left(e^j - c^j - z^j\right)}\right\}. \tag{65}$$

Taking the derivative with respect to a shock in the North z^N:

$$\frac{dc_{T1}^N}{dz^N} = \frac{r_1}{2+\alpha\nu}\left\{\frac{e^N - c^N - z^N}{\sum\limits_{j=N,S}\left(e^j - c^j - z^j\right)} - 1\right\} \begin{matrix} <0 & \text{for } H\in[0,H^*) \\ =0 & \text{for } H=H^* \\ >0 & \text{for } H\in\left(H^*,e^T\right] \end{matrix} \tag{66}$$

$$\frac{dc_{T1}^S}{dz^N} = \frac{r_1\left(e^S - c^S\right)}{(2+\alpha\nu)\sum\limits_{j=N,S}\left(e^j - c^j - z^j\right)} \begin{matrix} >0 & \text{for } H\in[0,H^*) \\ =0 & \text{for } H=H^* \\ <0 & \text{for } H\in\left(H^*,e^T\right]. \end{matrix} \tag{67}$$

Taking the derivative with respect to a shock in the South z^S:

$$\frac{dc_{T1}^N}{dz^S} = \frac{r_1\left(e^N - c^N\right)}{(2+\alpha\nu)\sum\limits_{j=N,S}\left(e^j - c^j - z^j\right)} > 0 \quad \text{for } H\in\left[0,e^T\right] \tag{68}$$

$$\frac{dc_{T1}^S}{dz^S} = \frac{r_1}{2+\alpha\nu}\left\{\frac{e^S - c^S - z^S}{\sum\limits_{j=N,S}\left(e^j - c^j - z^j\right)} - 1\right\} < 0 \quad \text{for } H\in\left[0,e^T\right]. \tag{69}$$

Next, it can be shown that the proposition holds, with some changes to the proof, when the borrowing constraints bind after large shocks. ■

Proof of Proposition 2. First, assume the borrowing constraints are never binding. Substituting equations (59) and (62) into the investment condition (22):

$$k_{NT}^j = \frac{1}{2+\alpha\nu}\left\{\sum_{j=N,S}\left(e^j - c^j - z^j\right) + \alpha\nu\left(e^j - c^j - z^j\right)\right\}, \tag{70}$$

which is declining in shocks whether they strike in region j or region $-j$. In particular, with respect to a shock in the North z^N:

$$\frac{dk_{NT}^N}{dz^N} = -\frac{1+\alpha\nu}{2+\alpha\nu} < 0, \ \frac{dk_{NT}^S}{dz^N} = -\frac{1}{2+\alpha\nu} < 0. \tag{71}$$

With respect to a shock to the South z^S:

$$\frac{dk_{NT}^N}{dz^S} = -\frac{1}{2+\alpha\nu} < 0, \ \frac{dk_{NT}^S}{dz^S} = -\frac{1+\alpha\nu}{2+\alpha\nu} < 0. \tag{72}$$

Next, it can be shown that the proposition holds, with some changes to the proof, when the borrowing constraints bind after large shocks. ■

Proof of Proposition 3. First, assume the borrowing constraints are never binding. Using equations (22), (23), (66), (67) and (71), we can derive the following with respect to a shock in the North z^N:

$$\frac{du^N}{dz^N} = \frac{r_1}{(2+\alpha\nu)\, c_{T1}^N} \left\{ \frac{2\left(e^N - c^N - z^N\right)}{\sum\limits_{j=N,S} \left(e^j - c^j - z^j\right)} - (3+\alpha\nu) \right\} \begin{matrix} < 0 & \text{for } H \in \left[0, \check{H}\left(z^N\right)\right) \\ = 0 & \text{for } H = \check{H}\left(z^N\right) \\ > 0 & \text{for } H \in \left(\check{H}\left(z^N\right), e^T\right] \end{matrix}, \tag{73}$$

where $\check{H}\left(z^N\right)$, if it exists, is the degree of heterogeneity such that $c^S - e^S = \frac{1+\alpha\nu}{3+\alpha\nu}\left(e^N - c^N - z^N\right)$. We can also derive:

$$\frac{du^S}{dz^N} = \frac{r_1}{(2+\alpha\nu)\, c_{T1}^S} \left\{ \frac{2\left(e^S - c^S\right)}{\sum\limits_{j=N,S} \left(e^j - c^j - z^j\right)} - 1 \right\} \begin{matrix} > 0 & \text{for } H \in \left[0, \tilde{H}\left(z^N\right)\right) \\ = 0 & \text{for } H = \tilde{H}\left(z^N\right) \\ < 0 & \text{for } H \in \left(\tilde{H}\left(z^N\right), e^T\right] \end{matrix}, \tag{74}$$

where $\tilde{H}\left(z^N\right) < H^*$ always exists and represents the degree of heterogeneity for which the shock changes the ordering between the North and the South: $e^S - c^S = e^N - c^N - z^N > 0$.

Using equations (22), (23), (68), (69) and (72), we can derive the following with respect to a shock in the South z^S:

$$\frac{du^N}{dz^S} = \frac{r_1}{(2+\alpha\nu)\, c_{T1}^N} \left\{ \frac{2\left(e^N - c^N\right)}{\sum\limits_{j=N,S} \left(e^j - c^j - z^j\right)} - 1 \right\} > 0 \quad \text{for } H \in \left[0, e^T\right] \tag{75}$$

$$\frac{du^S}{dz^S} = \frac{r_1}{(2+\alpha\nu)\, c_{T1}^S} \left\{ \frac{2\left(e^S - c^S - z^S\right)}{\sum\limits_{j=N,S} \left(e^j - c^j - z^j\right)} - (3+\alpha\nu) \right\} < 0 \quad \text{for } H \in \left[0, e^T\right] . \tag{76}$$

Next, it can be shown that the proposition holds, with some changes to the proof, when the borrowing constraints bind after large shocks. ■

Proof of Propositions 4 and 5. Take the shock $z \geq 0$ as given. From equation (59), r_1 is unaffected by whether borrowing constraints are binding or not:

$$r_1 = r_1^*. \tag{77}$$

When the borrowing constraint is not binding in any region, equations (61) and (62) hold. Therefore, the borrowing constraint is binding in region j if and only if:

$$\frac{1}{2+\alpha\nu}\left\{2y_T + r_1^*\left(e^j - c^j - z^j\right)\right\} > \frac{1}{1+\alpha\nu}\left\{y_T + r_1^*\left(e^j - c^j - z^j\right) + \phi\left[y_T + p_2^{j*} y_{NT2}\right]\right\}, \tag{78}$$

which is true when ϕ is low. Let ϕ^{crit} be the value of ϕ such that the borrowing constraint is just binding in region j. By conducting the thought experiment of reducing ϕ below ϕ^{crit}, we can derive the impact of binding borrowing constraints on the competitive equilibrium allocation.

If the borrowing constraint is binding in region j, the system of equations to solve is:

$$(2+\alpha\nu)\, c_{T1}^{-j} = y_T\left(1+\frac{1}{r_2}\right) + r_1\left(e^{-j} - c^{-j} - z^{-j}\right) \tag{79}$$

$$(1+\alpha\nu)\, c_{T1}^{j} + \frac{c_{T2}^j}{r_2} = y_T\left(1+\frac{1}{r_2}\right) + r_1\left(e^{j} - c^{j} - z^{j}\right) \tag{80}$$

$$(1+\alpha\nu)\, c_{T1}^{j} = y_T + r_1\left(e^{j} - c^{j} - z^{j}\right) + \frac{\phi\left[y_T + p_2^j y_{NT2}\right]}{r_2} \tag{81}$$

$$\nu c_{T2}^j = p_2^j y_{NT2} \tag{82}$$

$$c_{T1}^N + c_{T1}^S = 2y_T. \tag{83}$$

Equations (80), (81) and (82) imply:

$$c_{T2}^j = \frac{1-\phi}{1+\phi\nu} y_T. \tag{84}$$

Therefore, starting from ϕ^{crit}, $d\phi < 0$ implies $dc_{T2}^j > 0$. From equations (79), (80) and (83):

$$(2+\alpha\nu)\, dc_{T1}^j = \frac{y_T}{r_2^2} dr_2 \tag{85}$$

$$(1+\alpha\nu)\, dc_{T1}^j + \frac{1}{r_2} dc_{T2}^j + \frac{y_T - c_{T2}^j}{r_2^2} dr_2 = 0 \tag{86}$$

Combining these equations, $dc^j_{T2} > 0$ implies $dc^j_{T1} < 0$ and $dr_2 < 0$.

From the investment formula (22), $dc^j_{T1} < 0$ implies $dk^j_{NT} < 0$ and therefore, $dc^j_{NT1} < 0$. Finally, by definition, $dR^j_2 > 0$.

The above results establish the results in the proposition for union-wide variables and variables for region j. The results for region $-j$ follow from general equilibrium. ■

Proof of Proposition 6. Combining equations (85) and (86):

$$dc^j_{T1}\left\{(1+\alpha\nu)+(2+\alpha\nu)\frac{y_T-c^j_{T2}}{y_T}\right\} = -\frac{1}{r_2}dc^j_{T2}. \tag{87}$$

Using the above equation, the change in the welfare levels of the two regions can be written as:

$$du^{-j} = \frac{dc^j_{T1}}{c^{-j}_{T1}}(2+\alpha\nu)\frac{y_T-c^j_{T2}}{y_T} < 0 \tag{88}$$

$$du^j = \frac{dc^j_{T1}}{c^j_{T1}}\left\{(1+\alpha\nu)-\frac{c^j_{T1}}{c^j_{T2}}r_2\left[(1+\alpha\nu)+(2+\alpha\nu)\frac{y_T-c^j_{T2}}{y_T}\right]\right\}, \tag{89}$$

so the change in the welfare of households in region $-j$ from imposing the borrowing constraint is ambiguous, but the change in welfare of households in region j is ambiguous. If the shock is small and the constraint is just binding, $\frac{c^j_{T1}}{c^j_{T2}}r_2 = 1$ and:

$$du^j = -\frac{dc^j_{T1}}{c^j_{T1}}(2+\alpha\nu)\frac{y_T-c^j_{T2}}{y_T} > 0. \tag{90}$$

If the shock is very large and the constraint is very binding so $\frac{c^j_{T1}}{c^j_{T2}}r_2$ is far below 1, then it is possible that $du^j < 0$. Then, the constraint makes households in region j worse off overall. ■

Proof of Proposition 7. The proof is divided into three parts.

The first part of the proof establishes that the constraint is binding in equilibrium if and only if the assumption of a binding constraint produces an interest rate r_2 below 1 according to the resource constraint. When the constraint is binding in region j in equilibrium:

$$(1+\alpha\nu)c^{j*}_{T1} > y_T + r_1\left(e^j - c^j - z^j\right) + \frac{\phi\left[y_T + p^j_2 y_{NT2}\right]}{r_2}, \tag{91}$$

which implies that:

$$\frac{1}{1+\alpha\nu}\left\{y_T + r_1\left(e^j - c^j - z^j\right) + \frac{\phi\left[y_T + p_2^j y_{NT2}\right]}{r_2}\right\}$$
$$< \frac{1}{2+\alpha\nu}\left\{y_T\left(1+\frac{1}{r_2}\right) + r_1\left(e^j - c^j - z^j\right)\right\}. \tag{92}$$

Turning to the resource constraint under the assumption of a binding constraint in region j:

$$\frac{1}{1+\alpha\nu}\left\{y_T + r_1\left(e^j - c^j - z^j\right) + \frac{\phi\left[y_T + p_2^j y_{NT2}\right]}{r_2}\right\}$$
$$+\frac{1}{2+\alpha\nu}\left\{y_T\left(1+\frac{1}{r_2}\right) + r_1\left(e^{-j} - c^{-j} - z^{-j}\right)\right\} = 2y_T. \tag{93}$$

Combining equations (59) and (93):

$$\frac{1}{1+\alpha\nu}\left\{y_T + r_1\left(e^j - c^j - z^j\right) + \frac{\phi\left[y_T + p_2^j y_{NT2}\right]}{r_2}\right\}$$
$$= \frac{1}{2+\alpha\nu}\left\{y_T\left(1+\frac{1}{r_2}\right) + r_1\left(e^j - c^j - z^j\right)\right\} + 2y_T\left(1-\frac{1}{r_2}\right). \tag{94}$$

It is straightforward to use equations (92) and (94) to finish the first part of the proof.

Notice that when the assumption of a binding constraint produces an interest rate r_2 above 1, the constraint does not in fact bind at equilibrium. Therefore, equation (94) does not apply, and the equilibrium interest rate is $r_2 = r_2^* = 1$.

The second part of the proof establishes that under the maintained assumption that the constraint is binding in region j, the interest rate r_2 moves together with c_{T1}^j in response to shocks z. Whether the borrowing constraint is binding or not in region j, the Euler condition (26) for households in region $-j$ can be rearranged as follows:

$$r_2 = \frac{2y_T - c_{T2}^j}{2y_T - c_{T1}^j}. \tag{95}$$

From equation (84), c_{T2}^j does not change with the shock z once the borrowing constraint is binding. Therefore, r_2 moves together with c_{T1}^j under the maintained assumption that the constraint is binding in region j.

The third part of the proof establishes the thresholds for H. From the above two parts of the proof, region j is susceptible to binding borrowing constraints when the shock z reduces consumption levels in region j. Suppose that the shock reduces c_{T1}^j when the constraint is not binding. In that case, $r_2 = r_2^* = 1$ continues to prevail. The interest rate

r_2 according to equation (94) is above 1 but is reduced by the shock. As the shock gets larger, c_{T1}^{j} is reduced further and the interest rate r_2 according to equation (94) eventually falls below 1. From that point onward, the constraint becomes binding in region j.

Proposition 1 is therefore crucial here. It shows that when $H \in [0, H^{*})$, a shock in the North z^N reduces c_{T1}^{N} and a shock in the South z^S reduces c_{T1}^{S}. Therefore, a shock in region j makes region j susceptible to binding borrowing constraints. When $H \in \left(H^{*}, e^{T}\right]$, shocks in both the North and the South reduce c_{T1}^{S}. Therefore, a shock in either region j or $-j$ makes the South susceptible to binding borrowing constraints. ■

Proof of Lemma 5. Take the shock z as given, such that the constraint is binding in the South. Then r_1 is fixed, and the system of equations to solve with $dx_{NS} > 0$ is:

$$(2+\alpha\nu)\, c_{T1}^{N} = y_T \left(1 + \frac{1}{r_2}\right) + r_1 \left(e^{N} - c^{N} - z^{N}\right) - x_{NS} \tag{96}$$

$$(1+\alpha\nu)\, c_{T1}^{S} + \frac{c_{T2}^{S}}{r_2} = y_T \left(1 + \frac{1}{r_2}\right) + r_1 \left(e^{S} - c^{S} - z^{S}\right) + x_{NS} \tag{97}$$

$$(1+\alpha\nu)\, c_{T1}^{S} = y_T + r_1 \left(e^{S} - c^{S} - z^{S}\right) + x_{NS} + \frac{\phi \left[y_T + p_2^{S} y_{NT2}\right]}{r_2} \tag{98}$$

$$\nu c_{T2}^{S} = p_2^{S} y_{NT2} \tag{99}$$

$$c_{T1}^{N} + c_{T1}^{S} = 2 y_T. \tag{100}$$

Equation (84) still holds, so the gift does not change c_{T2}^{N}, c_{T2}^{S} or the prices of nontradable goods at $t = 2$. This is crucial. Next, take derivatives of each of the above equations and impose $dc_{T2}^{S} = 0$, $dp_2^{S} = 0$:

$$(2+\alpha\nu)\, dc_{T1}^{N} = -\frac{y_T}{r_2^2} dr_2 - dx_{NS} \tag{101}$$

$$(1+\alpha\nu)\, dc_{T1}^{S} + \frac{y_T - c_{T2}^{S}}{r_2^2} dr_2 - dx_{NS} = 0 \tag{102}$$

$$(1+\alpha\nu)\, dc_{T1}^{S} = dx_{NS} - \frac{\phi \left[y_T + p_2^{S} y_{NT2}\right]}{r_2^2} dr_2 \tag{103}$$

$$dc_{T1}^{N} = -dc_{T1}^{S}. \tag{104}$$

Substitute equations (101) and (104) into equation (102):

$$dc_{T1}^{S} \left\{(1+\alpha\nu) + (2+\alpha\nu)\, \frac{y_T - c_{T2}^{S}}{y_T}\right\} = dx_{NS} \left\{1 + \frac{y_T - c_{T2}^{S}}{y_T}\right\} \tag{105}$$

which means that $dc_{T1}^S > 0$. Therefore, $dk_{NT}^S > 0$, $dp_1^S > 0$ and $dc_{NT1}^S > 0$, while by definition, $dR_2^S < 0$. The corollary results for the North are $dc_{T1}^N < 0$, $dk_{NT}^N < 0$, $dp_1^N < 0$ and $dc_{NT1}^N < 0$. Finally, substituting equations (104) and (105) into equation (101):

$$\left\{1 + \frac{y_T - c_{T2}^S}{y_T}\right\} \frac{y_T}{r_2^2} dr_2 = dc_{T1}^S, \tag{106}$$

so $dr_2 > 0$. ■

Proof of Proposition 10. Using the formulae from the above lemma, the change in the ex post welfare levels of the North and the South can be written as:

$$du_{012}^N = -\frac{1+\alpha\nu}{c_{T1}^N} dc_{T1}^S < 0 \tag{107}$$

$$du_{012}^S = \frac{1+\alpha\nu}{c_{T1}^S} dc_{T1}^S > 0, \tag{108}$$

which establishes the desired result. ■

Proof of Proposition 11. Take the shock z as given, such that the constraint is binding in the South. Then r_1 is fixed, and the system of equations to solve with $dx_{NS} > 0$ is:

$$(2+\alpha\nu)\, c_{T1}^N = y_T \left(1 + \frac{1}{r_2}\right) + r_1 \left(e^N - c^N - z^N\right) - x_{NS} \left(1 - \frac{r_{NS}}{r_2}\right) \tag{109}$$

$$(1+\alpha\nu)\, c_{T1}^S + \frac{c_{T2}^S}{r_2} = y_T \left(1 + \frac{1}{r_2}\right) + r_1 \left(e^S - c^S - z^S\right) + x_{NS} \left(1 - \frac{r_{NS}}{r_2}\right) \tag{110}$$

$$(1+\alpha\nu)\, c_{T1}^S = y_T + r_1 \left(e^S - c^S - z^S\right) + x_{NS} + \frac{\phi \left[y_T + p_2^S y_{NT2} - r_{NS} x_{NS}\right]}{r_2} \tag{111}$$

$$\nu c_{T2}^S = p_2^S y_{NT2} \tag{112}$$

$$c_{T1}^N + c_{T1}^S = 2 y_T. \tag{113}$$

Equations (110), (111) and (112) imply:

$$c_{T2}^S = \frac{1-\phi}{1+\phi\nu} \left[y_T - r_{NS} x_{NS}\right], \tag{114}$$

which implies that when $dx_{NS} > 0$ and $r_{NS} > 0$, we obtain $dc_{T2}^S < 0$. This is crucial. Next, take derivatives of equations (109) to (113):

$$(2+\alpha\nu)\, dc_{T1}^N = -\frac{y_T + r_{NS} x_{NS}}{r_2^2} dr_2 - \left(1 - \frac{r_{NS}}{r_2}\right) dx_{NS} \tag{115}$$

$$(1+\alpha\nu)\,dc_{T1}^{S}+\frac{y_T-r_{NS}x_{NS}-c_{T2}^{S}}{r_2^2}dr_2-\left(1-\frac{r_{NS}}{r_2}\right)dx_{NS}=-\frac{1}{r_2}dc_{T2}^{S} \tag{116}$$

$$(1+\alpha\nu)\,dc_{T1}^{S}=\left(1-\phi\frac{r_{NS}}{r_2}\right)dx_{NS}+\frac{\phi y_{NT2}}{r_2}dp_2^S-\frac{\phi\left[y_T+p_2^S y_{NT2}-r_{NS}x_{NS}\right]}{r_2^2}dr_2 \tag{117}$$

$$\nu dc_{T2}^{S}=y_{NT2}dp_2^S \tag{118}$$

$$dc_{T1}^{N}=-dc_{T1}^{S}. \tag{119}$$

Substituting equations (115) and (119) into equation (116):

$$dc_{T1}^{S}\left\{(1+\alpha\nu)+(2+\alpha\nu)\beta\right\}-dx_{NS}\left(1-\frac{r_{NS}}{r_2}\right)\{1+\beta\}=-\frac{1}{r_2}dc_{T2}^{S}, \tag{120}$$

where $\beta=\frac{y_T-r_{NS}x_{NS}-c_{T2}^{S}}{y_T+r_{NS}x_{NS}}>0$. Notice that for $dx_{NS}>0$ and r_{NS} below an upper bound (which is higher than r_2), we have $dc_{T1}^{S}>0$, which will be useful in what follows.

Using the above formula, the change in the welfare levels of the North and the South can be written as:

$$du_{012}^{N}=\frac{1}{c_{T1}^{N}}\left[dc_{T1}^{S}(2+\alpha\nu)\beta-dx_{NS}\left(1-\frac{r_{NS}}{r_2}\right)\{1+\beta\}\right] \tag{121}$$

$$\begin{aligned} du_{012}^{S} &= \frac{1}{c_{T1}^{S}}\left[dc_{T1}^{S}\left\{(1+\alpha\nu)-\frac{c_{T1}^{S}}{c_{T2}^{S}}r_2\left[(1+\alpha\nu)+(2+\alpha\nu)\beta\right]\right\}\right. \\ &\left.\quad +dx_{NS}\frac{c_{T1}^{S}}{c_{T2}^{S}}r_2\left(1-\frac{r_{NS}}{r_2}\right)\{1+\beta\}\right]. \end{aligned} \tag{122}$$

If the constraint is binding, then $\frac{c_{T1}^{S}}{c_{T2}^{S}}r_2<1$.

Let $r_{\min}^{N}(z)$ be the lowest interest rate that the North is willing to accept, i.e. $du_{012}^{N}=0$:

$$dc_{T1}^{S}(2+\alpha\nu)\beta=dx_{NS}\left(1-\frac{r_{\min}^{N}(z)}{r_2}\right)\{1+\beta\}. \tag{123}$$

When this is true, equation (120) indicates that:

$$dc_{T1}^{S}(1+\alpha\nu)=-\frac{1}{r_2}dc_{T2}^{S}>0. \tag{124}$$

Therefore, equation (123) is only consistent with $r_{\min}^{N}(z)<r_2$. Substituting equation (123) into equation (122), the South is strictly better off if the constraint is binding:

$$du_{012}^{S}=\frac{(1+\alpha\nu)}{c_{T1}^{S}}\left\{1-\frac{c_{T1}^{S}}{c_{T2}^{S}}r_2\right\}dc_{T1}^{S}>0. \tag{125}$$

Therefore, a Pareto improvement is possible with $r_{NS} = r^N_{\min}(z)$.

Let $r^S_{\max}(z)$ be the highest interest rate that the South is willing to pay, i.e. $du^S_{012} = 0$:

$$dc^S_{T1}\left\{\frac{c^S_{T1}}{c^S_{T2}}r_2\left[(1+\alpha\nu)+(2+\alpha\nu)\beta\right]-(1+\alpha\nu)\right\} = dx_{NS}\frac{c^S_{T1}}{c^S_{T2}}r_2\left(1-\frac{r^S_{\max}(z)}{r_2}\right)\{1+\beta\}. \tag{126}$$

When this is true, equation (120) indicates that:

$$dc^S_{T1}\frac{c^S_{T2}}{c^S_{T1}}\frac{1}{r_2}(1+\alpha\nu) = -\frac{1}{r_2}dc^S_{T2} > 0. \tag{127}$$

Substituting equation (126) into equation (121), the North is strictly better off if the constraint is binding:

$$du^N_{012} = \frac{(1+\alpha\nu)}{c^N_{T1}}\left[\frac{c^S_{T2}}{c^S_{T1}}\frac{1}{r_2}-1\right]dc^S_{T1} > 0. \tag{128}$$

Therefore, a Pareto improvement is possible with $r_{NS} = r^S_{\max}(z)$.

From equations (123) and (126), $r^N_{\min}(z) = r^S_{\max}(z) < r_2$ when z is such that the constraint is just binding. When the constraint is strictly binding, $\frac{c^S_{T1}}{c^S_{T2}}r_2 < 1$ and $r^N_{\min}(z) < r^S_{\max}(z)$. Finally, from equation (126), $r^S_{\max}(z) > r_2$ is possible when the constraint is very binding and $\frac{c^S_{T1}}{c^S_{T2}}r_2$ is far below 1. ■

Proof of Lemma 6. From equations (112) and (114), when $dx_{NS} > 0$ and $r_{NS} > 0$, we obtain $dc^S_{T2} < 0$ and $dp^S_2 < 0$. From the union-wide resource constraint, $dc^N_{T2} > 0$ and $dp^N_2 > 0$. These hold throughout the Pareto set.

Equations (120), (123) and (126) indicate that within the Pareto set:

$$dc^S_{T1}\{(1+\alpha\nu)+X\} = -\frac{1}{r_2}dc^S_{T2} > 0, \tag{129}$$

where $X \in \left[0, (1+\alpha\nu)\left\{\frac{c^S_{T2}}{c^S_{T1}}\frac{1}{r_2}-1\right\}\right]$. Therefore, $dc^S_{T1} > 0$. As a corollary, $dk^S_{NT} > 0$, $dp^S_1 > 0$ and $dc^S_{NT1} > 0$, while by definition, $dR^S_2 < 0$. The corollary results for the North are $dc^N_{T1} < 0$, $dk^N_{NT} < 0$, $dp^N_1 < 0$ and $dc^N_{NT1} < 0$.

Finally, substituting equations (119), (123) and (126) into equation (115):

$$\left\{1+\frac{y_T-c^S_{T2}}{y_T}\right\}\frac{y_T+r_{NS}x_{NS}}{r_2^2}dr_2 = dc^S_{T1}\{(2+\alpha\nu)+X\}. \tag{130}$$

Therefore, $dr_2 > 0$. ■

Proof of Propositions 12 and 13. By numerical example. ■

Proof of Lemma 7. Take the shock z as given, such that the constraint is binding in the South. Then r_1 is fixed, and the system of equations to solve with $d\tau^S > 0$ is:

$$(2+\alpha\nu)\, c_{T1}^N = y_T \left(1 + \frac{1}{r_2}\right) + r_1 \left(e^N - c^N - z^N\right) \tag{131}$$

$$(1+\alpha\nu)\, c_{T1}^S + \frac{c_{T2}^S}{r_2} = y_T \left(1 + \frac{1}{r_2}\right) + r_1 \left(e^S - c^S - z^S\right) \tag{132}$$

$$(1+\alpha\nu)\, c_{T1}^S = y_T + r_1 \left(e^S - c^S - z^S\right) + \frac{\phi\left[y_T + p_2^S y_{NT2}\right]}{r_2} \tag{133}$$

$$c_{T2}^S \left\{\nu + \tau^S + \nu\tau^S\right\} = p_2^S y_{NT2} \tag{134}$$

$$c_{T1}^N + c_{T1}^S = 2y_T. \tag{135}$$

Equations (132), (133) and (134) imply:

$$c_{T2}^S \left[1 + \phi\left\{\nu + \tau^S + \nu\tau^S\right\}\right] = y_T\,(1-\phi)\,, \tag{136}$$

so when $d\tau^S > 0$, we obtain $dc_{T2}^S < 0$. This is crucial. Next, take derivatives of equations (131) to (135):

$$(2+\alpha\nu)\, dc_{T1}^N = -\frac{y_T}{r_2^2} dr_2 \tag{137}$$

$$(1+\alpha\nu)\, dc_{T1}^S + \frac{y_T - c_{T2}^S}{r_2^2} dr_2 = -\frac{1}{r_2} dc_{T2}^S \tag{138}$$

$$(1+\alpha\nu)\, dc_{T1}^S = \frac{\phi y_{NT2}}{r_2} dp_2^S - \frac{\phi\left[y_T + p_2^S y_{NT2}\right]}{r_2^2} dr_2 \tag{139}$$

$$dc_{T2}^S \left\{\nu + \tau^S + \nu\tau^S\right\} + c_{T2}^S\,(1+\nu)\, d\tau^S = y_{NT2}\, dp_2^S \tag{140}$$

$$dc_{T1}^N = -dc_{T1}^S. \tag{141}$$

Substituting equations (137) and (141) into equations (138) and (139):

$$dc_{T1}^S \left\{(1+\alpha\nu) + (2+\alpha\nu)\, \frac{y_T - c_{T2}^S}{y_T}\right\} = -\frac{1}{r_2} dc_{T2}^S \tag{142}$$

$$dc_{T1}^S \left\{(1+\alpha\nu) + (2+\alpha\nu)\, \phi \frac{y_T + p_2^S y_{NT2}}{y_T}\right\} = \phi \frac{y_{NT2}}{r_2} dp_2^S. \tag{143}$$

Therefore, $dc_{T1}^S > 0$, which implies that $dB_2^S > 0$, and $dp_2^S > 0$. In addition, $dk_{NT}^S > 0$, $dp_1^S > 0$ and $dc_{NT1}^S > 0$, while by definition, $dR_2^S < 0$. The corollary results for the North are $dc_{T1}^N < 0$, $dk_{NT}^N < 0$, $dp_1^N < 0$ and $dc_{NT1}^N < 0$, together with $dc_{T2}^N > 0$ and $dp_2^N > 0$. At the union-wide level, equation (137) implies that $dr_2 > 0$. ■

Proof of Proposition 14. Using the formulae from the above lemma, the change in the welfare levels of the North and the South can be written as:

$$du_{012}^N = \frac{dc_{T1}^S}{c_{T1}^N}(2+\alpha\nu)\frac{y_T - c_{T2}^S}{y_T} > 0 \tag{144}$$

$$du_{012}^S = \frac{dc_{T1}^S}{c_{T1}^S}\left\{(1+\alpha\nu) - \frac{c_{T1}^S}{c_{T2}^S}r_2\left[(1+\alpha\nu) + (2+\alpha\nu)\frac{y_T - c_{T2}^S}{y_T}\right]\right\}, \tag{145}$$

so the change in the welfare of Southern households is ambiguous. If the constraint is just binding, $\frac{c_{T1}^S}{c_{T2}^S}r_2 = 1$ and:

$$du_{012}^S = -\frac{dc_{T1}^S}{c_{T1}^S}(2+\alpha\nu)\frac{y_T - c_{T2}^S}{y_T} < 0.$$

If the constraint is very binding and $\frac{c_{T1}^S}{c_{T2}^S}r_2$ is far below 1, then it is possible that $du_{012}^S > 0$. ■.

Proof of Proposition 15. Take the shock z as given, such that the constraint is binding in the South. Then r_1 is fixed, and the system of equations to solve with $d\tau^S > 0$ and $dg_{NS} > 0$ is:

$$(2+\alpha\nu)c_{T1}^N = y_T\left(1+\frac{1}{r_2}\right) + r_1\left(e^N - c^N - z^N\right) - g_{NS} \tag{146}$$

$$(1+\alpha\nu)c_{T1}^S + \frac{c_{T2}^S}{r_2} = y_T\left(1+\frac{1}{r_2}\right) + r_1\left(e^S - c^S - z^S\right) + g_{NS} \tag{147}$$

$$(1+\alpha\nu)c_{T1}^S = y_T + r_1\left(e^S - c^S - z^S\right) + g_{NS} + \frac{\phi\left[y_T + p_2^S y_{NT2}\right]}{r_2} \tag{148}$$

$$c_{T2}^S\left\{\nu + \tau^S + \nu\tau^S\right\} = p_2^S y_{NT2} \tag{149}$$

$$c_{T1}^N + c_{T1}^S = 2y_T. \tag{150}$$

Equation (136) still applies, so when $d\tau^S > 0$, we obtain $dc_{T2}^S < 0$. Take derivatives of equations (146) to (150):

$$(2+\alpha\nu)dc_{T1}^N = -\frac{y_T}{r_2^2}dr_2 - dg_{NS} \tag{151}$$

$$(1+\alpha\nu)dc_{T1}^S + \frac{y_T - c_{T2}^S}{r_2^2}dr_2 - dg_{NS} = -\frac{1}{r_2}dc_{T2}^S \tag{152}$$

$$(1+\alpha\nu)dc_{T1}^S = dg_{NS} + \frac{\phi y_{NT2}}{r_2}dp_2^S - \frac{\phi\left[y_T + p_2^S y_{NT2}\right]}{r_2^2}dr_2 \tag{153}$$

$$dc_{T2}^{S}\left\{\nu+\tau^{S}+\nu\tau^{S}\right\}+c_{T2}^{S}\left(1+\nu\right)d\tau^{S}=y_{NT2}dp_{2}^{S} \tag{154}$$

$$dc_{T1}^{N}=-dc_{T1}^{S}. \tag{155}$$

Substituting equations (151) and (155) into equation (152):

$$dc_{T1}^{S}\left\{(1+\alpha\nu)+(2+\alpha\nu)\frac{y_{T}-c_{T2}^{S}}{y_{T}}\right\}-dg_{NS}\left\{1+\frac{y_{T}-c_{T2}^{S}}{y_{T}}\right\}=-\frac{1}{r_{2}}dc_{T2}^{S} \tag{156}$$

Notice that for $dg_{NS}>0$, we have $dc_{T1}^{S}>0$, which will be useful in what follows. Using the above formula, the change in the welfare levels of the North and the South can be written as:

$$du_{012}^{N}=\frac{1}{c_{T1}^{N}}\left[dc_{T1}^{S}\left(2+\alpha\nu\right)\frac{y_{T}-c_{T2}^{S}}{y_{T}}-dg_{NS}\left\{1+\frac{y_{T}-c_{T2}^{S}}{y_{T}}\right\}\right] \tag{157}$$

$$\begin{aligned}du_{012}^{S} &= \frac{1}{c_{T1}^{S}}\left[dc_{T1}^{S}\left\{(1+\alpha\nu)-\frac{c_{T1}^{S}}{c_{T2}^{S}}r_{2}\left[(1+\alpha\nu)+(2+\alpha\nu)\frac{y_{T}-c_{T2}^{S}}{y_{T}}\right]\right\}\right.\\ &\left.+dg_{NS}\frac{c_{T1}^{S}}{c_{T2}^{S}}r_{2}\left\{1+\frac{y_{T}-c_{T2}^{S}}{y_{T}}\right\}\right].\end{aligned} \tag{158}$$

If the constraint is binding, then $\frac{c_{T1}^{S}}{c_{T2}^{S}}r_{2}<1$.

Let $g_{\max}^{N}(z)$ be the largest gift that the North is willing to give, i.e. $du_{012}^{N}=0$:

$$dc_{T1}^{S}\left(2+\alpha\nu\right)\frac{y_{T}-c_{T2}^{S}}{y_{T}}=g_{\max}^{N}(z)\left\{1+\frac{y_{T}-c_{T2}^{S}}{y_{T}}\right\}. \tag{159}$$

This equation establishes that $g_{\max}^{N}(z)>0$. For this gift size, the South is strictly better off if the constraint is binding:

$$du_{012}^{S}=\frac{(1+\alpha\nu)}{c_{T1}^{S}}\left\{1-\frac{c_{T1}^{S}}{c_{T2}^{S}}r_{2}\right\}dc_{T1}^{S}>0. \tag{160}$$

Therefore, a Pareto improvement is possible with $g_{NS}=g_{\max}^{N}(z)$.

Let $g_{\min}^{S}(z)$ be the smallest gift that the South is willing to accept, i.e. $du_{012}^{S}=0$:

$$\begin{aligned}&dc_{T1}^{S}\left\{\frac{c_{T1}^{S}}{c_{T2}^{S}}r_{2}\left[(1+\alpha\nu)+(2+\alpha\nu)\frac{y_{T}-c_{T2}^{S}}{y_{T}}\right]-(1+\alpha\nu)\right\}\\ =\;& g_{\min}^{S}(z)\frac{c_{T1}^{S}}{c_{T2}^{S}}r_{2}\left\{1+\frac{y_{T}-c_{T2}^{S}}{y_{T}}\right\}.\end{aligned} \tag{161}$$

For this gift size, the North is strictly better off if the constraint is binding:

$$du_{012}^{N}=\frac{(1+\alpha\nu)}{c_{T1}^{N}}\left[\frac{c_{T2}^{S}}{c_{T1}^{S}}\frac{1}{r_2}-1\right]dc_{T1}^{S}>0. \tag{162}$$

Therefore, a Pareto improvement is possible with $g_{NS}=g_{\min}^{S}(z)$.

From equations (159) and (161), $g_{\max}^{N}(z)=g_{\min}^{S}(z)>0$ when z is such that the constraint is just binding and $\frac{c_{T2}^{S}}{c_{T1}^{S}}\frac{1}{r_2}=1$. When the constraint is strictly binding, $\frac{c_{T1}^{S}}{c_{T2}^{S}}r_2<1$ and $g_{\min}^{S}(z)<g_{\max}^{N}(z)$. Finally, from equation (161), $g_{\min}^{S}(z)<0$ is possible when the constraint is very binding and $\frac{c_{T1}^{S}}{c_{T2}^{S}}r_2$ is far below 1. ■

Proof of Lemma 8. Taking the derivative of equation (136), which still applies:

$$dc_{T2}^{S}\left[1+\phi\left\{\nu+\tau^{S}+\nu\tau^{S}\right\}\right]+c_{T2}^{S}\phi(1+\nu)\,d\tau^{S}=0. \tag{163}$$

Therefore, when $d\tau^{S}>0$, we obtain $dc_{T2}^{S}<0$. Substituting into equation (140):

$$y_{NT2}dp_2^{S}=-\frac{1}{\phi}dc_{T2}^{S}>0,$$

which establishes that $dp_2^{S}>0$. From the union-wide resource constraint, $dc_{T2}^{N}>0$ and $dp_2^{N}>0$. All of these results hold throughout the Pareto set.

Equations (156), (159) and (161) indicate that within the Pareto set:

$$dc_{T1}^{S}\left\{(1+\alpha\nu)+Y\right\}=-\frac{1}{r_2}dc_{T2}^{S}>0, \tag{164}$$

where $Y\in\left[0,(1+\alpha\nu)\left\{\frac{c_{T2}^{S}}{c_{T1}^{S}}\frac{1}{r_2}-1\right\}\right]$. Therefore, $dc_{T1}^{S}>0$. As a corollary, $dk_{NT}^{S}>0$, $dp_1^{S}>0$ and $dc_{NT1}^{S}>0$, while by definition, $dR_2^{S}<0$. The corollary results for the North are $dc_{T1}^{N}<0$, $dk_{NT}^{N}<0$, $dp_1^{N}<0$ and $dc_{NT1}^{N}<0$.

Finally, substituting equations (155), (159) and (161) into equation (151):

$$\left\{1+\frac{y_T-c_{T2}^{S}}{y_T}\right\}\frac{y_T}{r_2^2}dr_2=dc_{T1}^{S}\left\{(2+\alpha\nu)+Y\right\}. \tag{165}$$

Therefore, $dr_2>0$. ■