Asset Pricing with Horizon-Dependent Risk Aversion

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First version: August 2014

This version: January 5, 2015

Abstract

We study general equilibrium asset prices in a multi-period endowment economy when agents' risk aversion is allowed to depend on the maturity of the risk. In our pseudo-recursive preference framework, agents are time inconsistent for their intra-temporal decision making, though time consistent for inter-temporal decisions. We find, in the absence of jumps and under log-normal consumption growth, horizon-dependent risk aversion preferences affect the term structure of risk premia if and only if volatility is stochastic. When risk aversion decreases with the horizon (as lab experiments indicate), and the elasticity of intertemporal substitution is greater than one, our model results in a downward slopping (in absolute value) pricing of volatility risk, which, in turns, can explain the recent empirical results on the term structure of risky asset returns. We confirm this prediction using index options data.

JEL Classification: D03, D90, G02, G12

Keywords: risk aversion, term structure, volatility risk

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1 Introduction

Most of the literature on general equilibrium asset pricing theory is premised on the assumption that risk aversion is constant across maturities. We investigate whether the standard tool box of asset pricing can be generalized to accommodate risk preferences that differ across temporal horizons, and whether such a generalization has the potential to address observed patterns in asset prices.

Inspired by ample experimental evidence that subjects are more risk averse to immediate than to delayed risks,[1] Eisenbach and Schmalz (2014) introduce a two-period model with horizon-dependent risk aversion and show it is conceptually orthogonal to other non-standard preferences such as non-exponential time discounting (Phelps and Pollak, 1968; Laibson, 1997), time-varying risk aversion (Constantinides, 1990; Campbell and Cochrane, 1999), and a preference for the timing of resolution of uncertainty (Kreps and Porteus, 1978; Epstein and Zin, 1989, (EZ)). In the present paper, we investigate the impact of horizon-dependent risk aversion preferences on asset prices in a fully dynamic framework. The conceptual difficulties of solving a multi-period model with dynamically inconsistent preferences are numerous. To start, the commonly used recursive techniques in finance and macroeconomics only apply to dynamically consistent preferences. At the same time, the only dynamically consistent time-separable expected utility preference is the special case in which risk aversion is constant across horizons.[2] In an effort to overcome these difficulties, we use techniques in the spirit of Strotz (1955) to solve the problem of a rational agent with horizon-dependent risk aversion preferences in a setting without time separability. Such an agent is dynamically consistent for deterministic payoffs, so that only uncertain payoffs induce time inconsistency. Unable to commit to future behavior but being aware of her preferences and perfectly rational, the agent optimizes today, taking into account reoptimization in future periods. Solving for the subgame-perfect equilibrium of the intra-personal game yields a stochastic discount factor (SDF) that nests the standard Epstein and Zin (1989) case, with an new multiplicative term representing the discrepancy between the continuation value used for optimization at any period t versus the actual valuation at $t+1$.

[1] See, e.g., Jones and Johnson (1973); Onculer (2000); Sagristano et al. (2002); Noussair and Wu (2006); Coble and Lusk (2010); Baucells and Heukamp (2010); Abdellaoui et al. (2011). See Eisenbach and Schmalz (2014) for a more thorough review.

[2] As a result, combining time-separability with horizon-dependent risk aversion in a dynamic model necessarily introduces inconsistent time preferences, which precludes isolating the effect on asset prices of horizon-dependent risk preferences.

We investigate the implications of horizon-dependent risk aversion on both the level and on the term structure of risk premia. We find the model can match risk prices in levels, very much in line with the long-run risk literature (Bansal and Yaron, 2004; Bansal et al., 2013) based on standard Epstein and Zin (1989) preferences. Further, we find that the term structure of equity risk premia is non-trivial if and only if the economy features stochastic volatility. In such a setting, the horizon dependent risk aversion model can explain a downward-sloping term structure of equity risk premia, as documented empirically (see the literature review below). Interestingly, this effect is solely driven by a downward-sloping term structure of the price of volatility risk, which is a testable prediction.

We test the key predictions of our model using index options. We estimate the price of volatility risk at different maturities, using both a parametric GMM approach, based on the option pricing model of Heston (1993), as well as a model-free approach that measures Sharpe ratios of straddle returns at various maturities. Both approaches confirm the horizon-dependent risk aversion model's predictions that the price of volatility risk is negative and that its term structure is decreasing in absolute value. Specifically, the GMM estimate of the price of volatility risk is strongly negative for short maturities but close to zero for longer maturities with the term structure flattening out beyond a maturity of 150 days: volatility risk is priced, but mainly at short maturities. The same result arises in the non-parametric estimation: we show the Sharpe ratios of at-the-money straddles are strongly negative for straddles with short maturities, but close to zero for maturities beyond six months.

The paper proceeds as follows. Section 2 is a review of the existing literature. Section 3 presents a two-period model that illustrates the intuition of some of our result. Section 4 presents the dynamic model, Section 5 our formal results for the pricing of risk and its term-structure. The empirical results are in Section 6. Section 7 concludes.

2 Related Literature

This paper is the first to solve for equilibrium asset prices in an economy populated by agents with dynamically inconsistent *risk* preferences. It complements Luttmer and Mariotti (2003), who show that dynamically inconsistent *time* preferences of the kind examined by Harris and Laibson (2001); Luttmer and Mariotti (2003) have little power to explain cross-sectional variation in asset returns. (Given cross-sectional asset pricing involves intra-period risk-return tradeoffs, it is indeed quite intuitive that horizon-dependent *time* preferences are not suitable to address puzzles related to cross-sectional variation in returns.)

Our formal results on the term structure of risk pricing are consistent with patterns uncovered by a recent empirical literature. Van Binsbergen et al. (2012) show the Sharpe ratios for short-term dividend strips are higher than for long-term dividend strips; see also van den Steen, 2004; van Binsbergen and Koijen, 2011; Boguth et al., 2012.[3] These empirical findings have led to a vigorous debate, because they appear to be inconsistent with traditional asset pricing models.

Our micro-founded model of preferences implies a downward slopping pricing of risk, in a simple endowment economy. By contrast, other approaches typically generate the desired implications by making structural assumptions about the economy or about the priced shocks driving the stochastic discount factor directly. For example, in a model with financial intermediaries, Muir (2013) uses time-variation in institutional frictions to explain why the term structure of risky asset returns changes over time. Ai et al. (2013) derive similar results in a production-based RBC model in which capital vintages face heterogeneous shocks to aggregate productivity; Zhang (2005) explains the value premium with costly reversibility and a countercyclical price of risk. Other production-based models with implications for the term structure of equity risk are, e.g. Kogan and Papanikolaou (2010); Gârleanu et al. (2012); Kogan and Papanikolaou (2014); Favilukis and Lin (2013). Similarly, Belo et al. (2013) offer an explanation why risk levels and thus risk premia could be higher at short horizons; by contrast, our contribution is about risk prices. Croce et al. (2007) use informational frictions to generate a downward-sloping equity term structure.

The predictions of our model do not rely on the possibility of rare disasters, which is an assumption that some have argued may be more difficult to verify empirically. Also, our results are furthermore distinguishable from several alternative explanations for a downward-sloping term structure of equity risk premia: in our model, volatility risk is the only driver for a downward-sloping term structure of equity risk. Our empirical results linking the value premium to exposure to volatility risk is a first step in testing the strict, and unique, predictions of our general equilibrium model, which does not rely on exogenously varying parameters. Our predictions for the risk-pricing *levels* are also consistent with Campbell et al. (2012), who show that volatility risk is an important

[3]Giglio et al. (2013) show a similar pattern exists for discount rates over much longer horizons using real estate markets. Lustig et al. (2013) document a downward-sloping term structure of currency carry trade risk premia.

driver of asset returns in a CAPM framework, and Ang et al. (2006); Adrian and Rosenberg (2008); Bollerslev and Todorov (2011); Menkhoff et al. (2012); Boguth and Kuehn (2013), who examine the relation between volatility risk and returns.

Our empirical analysis of index option returns may be the most direct evidence of a downward-sloping term structure of the *unit price* of volatility risk, as opposed to the term structure of risk premia, which are the product of price and quantity of risk at different horizons. Recent papers by Dew-Becker et al. (2014) and Cheng (2014) also point to a decreasing term structure for the pricing of volatility risk, yet, they do so over longer horizons, using different data sources or different methodologies than the present study. These results supplement earlier studies of volatility risk *premia*, such as those by Amengual (2009) and Ait-Sahalia et al. (2012).[4]

3 Static Model

As discussed in Eisenbach and Schmalz (2014), introducing horizon-dependent risk aversion into a time separable expected utility model with more than two periods necessarily introduces horizon-dependent inter-temporal tradeoffs similar to quasi-hyperbolic discounting. This is undesirable, as we want to study the effects of horizon-dependent risk aversion in isolation. Our general model in Section 4 solves this problem by dropping time separability, though this comes at the cost of more analytical complexity and less intuitive clarity. Here we present a simple two-period model with time separability and uncertainty both in the immediate and proximate future to illustrate the effect of horizon-dependent risk aversion on risk pricing.

Consider a two-period model with uncertainty in both periods. The agent has time separable expected utility U_t in period $t = 0, 1$ given by

$$U_0 = E\big[v(c_0) + \delta u(c_1)\big] \quad \text{and} \quad U_1 = E\big[v(c_1)\big],$$

where v and u are von Neumann-Morgenstern utility indexes and v is more risk averse than u. At the beginning of period 0 the agent forms a portfolio of two risky assets and a risk free bond. Asset 0 is a claim on consumption in period 0 while asset 1 is a claim on consumption in period 1. Consumption in the two periods is i.i.d. Denoting the prices of

[4]We omit a review of the large literature on variance risk premia more generally, including the seminal works by Coval and Shumway (2001); Carr and Wu (2009), and the link to political uncertainty (Amengual and Xiu, 2013; Kelly et al., 2014).

the two assets by p_0 and p_1, respectively, the first-order conditions for the agent's portfolio choice yield:

$$\begin{aligned} & E\left[v'(c_0)\left(c_0 - p_0\right)\right] = 0 \\ \text{and} \quad & E\left[\delta u'(c_1)\left(c_1 - (1+r)\, p_1\right)\right] = 0. \end{aligned}$$

Eisenbach and Schmalz (2014) show the equilibrium prices p_0, p_1 and r satisfy:

$$p_0 < (1+r)\, p_1.$$

In this two-period setting, horizon-dependent risk aversion therefore leads to an equilibrium term-structure of risk premia that is downward sloping.

This simple example illustrates how horizon-dependent risk aversion can affect the pricing of risk at different horizons as intuition would suggest. There are, however, important limitations to this example. The setting is subtly different from standard asset pricing models, even with only two periods $t = 0, 1$: there is uncertainty in both periods and a period's decision is taken before the period's uncertainty resolves. This allows for horizon-dependent risk aversion to have a term-structure effect without worrying about inconsistent inter-temporal tradeoffs, since only one such tradeoff arises. However, the period-0 portfolio choice problem above implicitly assumes that the agent has no opportunity to re-trade the claim to period-1 consumption at the beginning of period 1.

To generalize this setting, the next section presents our fully dynamic model, which allows for re-trading every-period.

4 Dynamic Model

Our approach is to generalize the model of Epstein and Zin (1989) (hereafter EZ) to allow for horizon-dependent risk aversion without affecting intertemporal substitution.

4.1 Preferences

Let $\{\gamma_h\}_{h\geq 0}$ be a decreasing sequence representing risk aversion at horizon h. In period t, the agent evaluates a consumption stream starting in period $t+h$ by:

$$V_{t,t+h} = \left((1-\beta)\, C_{t+h}^{1-\rho} + \beta E_{t+h}\left[V_{t,t+h+1}^{1-\gamma_h}\right]^{\frac{1-\rho}{1-\gamma_h}}\right)^{\frac{1}{1-\rho}} \quad \text{for all} \quad h \geq 0. \tag{1}$$

The agent's utility in period t is given by setting $h = 0$ in (1) which we denote by $V_t \equiv V_{t,t}$ for all t:

$$V_t = \left((1-\beta)\, C_t^{1-\rho} + \beta E_t\big[V_{t,t+1}^{1-\gamma_0} \big]^{\frac{1-\rho}{1-\gamma_0}} \right)^{\frac{1}{1-\rho}} .$$

As in the EZ model, utility V_t depends on (deterministic) current consumption C_t and a certainty equivalent $E_t\big[V_{t,t+1}^{1-\gamma_0}\big]^{\frac{1}{1-\gamma_0}}$ of (uncertain) continuation values $V_{t,t+1}$, where the aggregation of the two periods occurs with constant elasticity of intertemporal substitution given by $1/\rho$. However, in contrast to the EZ model, the continuation value $V_{t,t+1}$ is *not* the same as the agent's utility V_{t+1} in period $t+1$:

$$\begin{aligned} V_{t,t+1} &= \left((1-\beta)\, C_{t+1}^{1-\rho} + \beta E_{t+1}\big[V_{t,t+2}^{1-\gamma_1} \big]^{\frac{1-\rho}{1-\gamma_1}} \right)^{\frac{1}{1-\rho}} \\ &\neq \left((1-\beta)\, C_{t+1}^{1-\rho} + \beta E_{t+1}\big[V_{t+1,t+2}^{1-\gamma_0} \big]^{\frac{1-\rho}{1-\gamma_0}} \right)^{\frac{1}{1-\rho}} = V_{t+1} \end{aligned}$$

The key feature of the definition (1) is that certainty equivalents at different horizons h are formed with different levels of risk aversion γ_h. Imminent uncertainty is treated with risk aversion γ_0, uncertainty one period ahead is treated with γ_1 and so on.

In contrast to EZ, the preference of our model captured by $V_t \equiv V_{t,t}$ is *not* recursive since $V_{t+1} \equiv V_{t+1,t+1}$ does not recur in the definition of V_t.[5] This non-recursiveness is a direct implication of the horizon-dependent risk aversion, in which uncertain consumption streams starting in $t+1$ are evaluated differently by the agent's selves at t and $t+1$. Crucially, this disagreement arises only for *uncertain* consumption streams as for any *deterministic* consumption stream the horizon-dependence in (1) becomes irrelevant and we have $V_{t,t+1} = V_{t+1}$. Our model therefore implies dynamically inconsistent risk preferences while maintaining dynamically consistent time preferences.[6]

To solve our model, we follow the tradition of Strotz (1955), assuming that the agent

[5] Only the definition of $V_{t,t+h}$ for different h in (1) is recursive since the object for $h+1$ recurs in the definition of the object for h.

[6] An interesting question is the possibility to axiomatize the horizon-dependent risk aversion preferences we propose. The static model in Section 3 could be axiomatized as a special version of the temptation preferences of Gul and Pesendorfer (2001). Their preferences deal with general disagreements in preferences at a period 0 and a period 1. In our case, the disagreement is about the risk aversion so an axiomatization would require adding a corresponding axiom to the set of axioms in Gul and Pesendorfer (2001). Our dynamic model builds on the functional form of Epstein and Zin (1989) which capture non-time-separable preferences of the form axiomatized by Kreps and Porteus (1978). However, our generalization of Epstein and Zin (1989) explicitly violates Axiom 3.1 (temporal consistency) of Kreps and Porteus (1978) which is necessary for the recursive structure.

is fully rational when making choices in period t to maximize V_t. This means self t realizes that its evaluation of future consumption given by $V_{t,t+1}$ differs from the objective function V_{t+1} which self $t+1$ will maximize. The solution then corresponds to the subgame-perfect equilibrium in the sequential game played among the agent's different selves (Luttmer and Mariotti, 2003).[7]

4.2 Stochastic Discount Factor

For asset pricing purposes, the object of interest is the stochastic discount factor (SDF) resulting from the preferences in equation (1). We can arrive at the SDF intuitively using a derivation based on the intertemporal marginal rate of substitution:[8]

$$\frac{\Pi_{t+1}}{\Pi_t} = \frac{dV_t/dW_{t+1}}{dV_t/dC_t}.$$

The derivative of current utility V_t with respect to current consumption C_t is standard and given by:

$$\frac{dV_t}{dC_t} = V_t^{\rho}\,(1-\beta)\,C_t^{-\rho}. \tag{2}$$

The derivative of current utility V_t with respect to next-period wealth W_{t+1} is almost standard:

$$\frac{dV_t}{dW_{t+1}} = \frac{dV_t}{dV_{t,t+1}} \times \frac{dV_{t,t+1}}{dW_{t+1}} = V_t^{\rho}\beta E_t\big[V_{t,t+1}^{1-\gamma_0}\big]^{\frac{\gamma_0-\rho}{1-\gamma_0}}\,V_{t,t+1}^{-\gamma_0} \times \frac{dV_{t,t+1}}{dW_{t+1}}. \tag{3}$$

At this point, however, we cannot appeal to the envelope condition at $t+1$ to replace the term $dV_{t,t+1}/dW_{t+1}$ by $dV_{t,t+1}/dC_{t+1}$. This is because $V_{t,t+1}$ is the value self t attaches to all future consumption while the envelope condition at $t+1$ is in terms of the objective function of self $t+1$ which is given by V_{t+1}:

$$\frac{dV_{t+1}}{dW_{t+1}} = \frac{dV_{t+1}}{dC_{t+1}} = V_{t+1}^{\rho}\,(1-\beta)\,C_{t+1}^{-\rho}. \tag{4}$$

The disagreement between selves t and $t+1$ requires us to take an extra step. Due to the homotheticity of our preferences, we can rely on the fact that both $V_{t,t+1}$ and V_{t+1} are

[7] See Appendix A for more details.
[8] See Appendix A for a more rigorous derivation.

homogeneous of degree one which implies that

$$\frac{dV_{t,t+1}/dW_{t+1}}{dV_{t+1}/dW_{t+1}} = \frac{V_{t,t+1}}{V_{t+1}}. \tag{5}$$

This relationship captures a key element of our model: The marginal benefit of an extra unit of wealth in period $t+1$ differs whether evaluated by self t (the numerator on the left hand side) or by self $t+1$ (the denominator on the right hand side).

To arrive at the stochastic discount factor, we first combine the relationship (5) with the envelope condition (4) to eliminate the term $dV_{t,t+1}/dW_{t+1}$ in the derivative of current utility with respect to next-period wealth (3). Then we can combine (3) with (2) to form the marginal rate of substitution and arrive at:

$$\frac{\Pi_{t+1}}{\Pi_t} = \underbrace{\beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho}}_{\text{(I)}} \times \underbrace{\left(\frac{V_{t,t+1}}{E_t\left[V_{t,t+1}^{1-\gamma_0}\right]^{\frac{1}{1-\gamma_0}}}\right)^{\rho-\gamma_0}}_{\text{(II)}} \times \underbrace{\left(\frac{V_{t,t+1}}{V_{t+1}}\right)^{1-\rho}}_{\text{(III)}}.$$

The SDF consists of three multiplicative parts:

(I) The first term is that of the standard time-separable CRRA model with discount factor β and constant relative risk aversion ρ.

(II) The second part originates from the wedge between the risk aversion and the inverse of the elasticity of intertemporal substitution, i.e. from the non time separable framework. It is similar to the standard EZ model, taking risk aversion as the immediate one, γ_0.

(III) The third part is unique to our model and originates from the fact that, with horizon-dependent risk aversion, different selves disagree about the evaluation of a given consumption stream, depending on their relative horizon. Since the SDF Π_{t+1}/Π_t captures trade-offs between periods t and $t+1$, the key disagreement is how selves t and $t+1$ evaluate consumption starting in period $t+1$.

If we set $\gamma_h = \gamma$ for all horizons h, our SDF for horizon-dependent risk aversion preferences simplifies to the standard SDF for recursive preferences: it nests the model of EZ which, in turn, nests the standard time-separable model for $\gamma = \rho$.

5 Pricing of Risk and the Term Structure

To derive the pricing of risk under horizon-dependent risk aversion preferences, we consider a simplified version of the model where risk aversion for immediate risk is given by γ, and by $\tilde{\gamma}$ for all future risks. This framework, and our derivations for risk pricing, easily extends to a case where risk aversion is decreasing up to a given horizon, after which, for risks beyond, it remains constant ($\tilde{\gamma}$).

Our general model (1) thus becomes:

$$V_t = \left((1-\beta)\, C_t^{1-\rho} + \beta E_t \left[\tilde{V}_{t+1}^{1-\gamma} \right]^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1}{1-\rho}}$$
$$\tilde{V}_t = \left((1-\beta)\, C_t^{1-\rho} + \beta E_t \left[\tilde{V}_{t+1}^{1-\tilde{\gamma}} \right]^{\frac{1-\rho}{1-\tilde{\gamma}}} \right)^{\frac{1}{1-\rho}}.$$

The second equation is simply the standard EZ framework with risk aversion $\tilde{\gamma}$. If solutions for the recursion on the continuation value $\tilde{V}$ are derived, the value function V is automatically obtained from the first equation. The simplified version of the SDF is:

$$\frac{\Pi_{t+1}}{\Pi_t} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \times \left(\frac{\tilde{V}_{t+1}}{E_t \left[\tilde{V}_{t+1}^{1-\gamma} \right]^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma} \times \left(\frac{\tilde{V}_{t+1}}{V_{t+1}} \right)^{1-\rho} \tag{6}$$

As in the standard EZ framework, closed-form solutions for $\tilde{V}$ (and thus for V) obtain only for the knife-edge case of a unit elasticity of intertemporal substitution (EIS), $\rho = 1$. To aid in the comparison of the standard EZ framework with horizon-independent risk aversion and our generalization to horizon dependence, we therefore start by analyzing the case of unit-EIS in Section 5.1. We then go beyond the special case of $\rho = 1$ in Section 5.2 by studying solutions for V and $\tilde{V}$ under the approximation of a discount factor close to unity, $\beta \approx 1$.

5.1 Closed-Form Solutions under Unit EIS

To determine the pricing implications of our model, we analyze the wedge between the continuation value $\tilde{V}_{t+1}$ and the valuation V_{t+1}, which is the key difference between the SDF in our framework (6) and in the standard EZ framework. Denoting logs by lowercase letters, we consider a Lucas-tree economy with an exogenous endowment process given by

$$c_{t+1} - c_t = \mu + \phi_c x_t + \alpha_c \sigma_t W_{t+1},$$

where the time varying drift, x_t, and the time varying volatility, σ_t, have evolutions

$$\begin{aligned} x_{t+1} &= \nu_x x_t + \alpha_x \sigma_t W_{t+1} \\ \sigma_{t+1}^2 - \sigma^2 &= \nu_\sigma \left(\sigma_t^2 - \sigma^2\right) + \alpha_\sigma \sigma_t W_{t+1}. \end{aligned}$$

Both state variables are stationary (ν_x and ν_σ are contracting), and for simplicity, we assume the three shocks are orthogonal.

Lemma 1. *Under these specifications for the endowment economy, and $\rho = 1$, we find:*

$$v_t - \tilde{v}_t = -\frac{1}{2}\beta\left(\gamma - \tilde{\gamma}\right)\left(\alpha_c^2 + \phi_v^2\alpha_x^2 + \psi_v^2\alpha_\sigma^2\right)\sigma_t^2, \tag{7}$$

where ϕ_v and ψ_v are constant functions of the model parameters such that:

$$\phi_v = \beta\phi_c\left(I - \nu_x\right)^{-1}$$

and

$$\psi_v = \frac{1}{2}\frac{\beta\left(1-\tilde{\gamma}\right)}{1-\beta\nu_\sigma}\left(\alpha_c^2 + \phi_v^2\alpha_x^2 + \psi_v^2\alpha_\sigma^2\right).$$

Observe from equation (7) that $v_t < \tilde{v}_t$ at all times, as should be expected. Indeed, $\tilde{v}_t$ is derived from the standard EZ model with risk-aversion $\tilde{\gamma}$, whereas v_t is derived from our horizon-dependent risk aversion model with a higher risk-aversion $\gamma > \tilde{\gamma}$ for immediate risk. More striking, however, is that the difference in the valuations under the two models is constant when volatility σ_t is constant.

Corollary 1. *Under constant volatility in the consumption process, the ratios $\tilde{V}/V$ are constant and therefore do not affect excess returns, both in levels and in the term-structure.*

This is one of the central results of our paper and, as shown below, is not limited to the special case of $\rho = 1$. The intuition is that when our time-inconsistent representative agent is aware prices will be set, the following period, by her next-period self, then the term-structure of prices is affected by risk-horizon dependent risk-aversion only if unexpected shocks to volatility can occur.

This result makes clear that the intuition from the simple two-period horizon-dependent risk aversion model of Section 3 does not trivially extend to the dynamic model and that there is no tautological relationship between horizon-dependent preferences and horizon-dependent risk pricing. It makes also clear, however, why the generalized EZ preferences we employ in this paper are necessary to derive interesting predictions. Before we make use of that feature, we derive one more result under the $\rho = 1$ case.

Proposition 1. *In the knife-edge case $\rho = 1$, the stochastic discount factor satisfies:*

$$\frac{\Pi_{t+1} C_{t+1}}{\Pi_t C_t} = \beta \underbrace{\left(\frac{\tilde{V}_{t+1}^{1-\gamma}}{E_t\left[\tilde{V}_{t+1}^{1-\gamma}\right]} \right)}_{\textit{multiplicative martingale}}$$

Borovicka et al. (2011) show the pricing of consumption risk, at time t, and for horizon h is determined by $E_t[\Pi_{t+h} C_{t+h}]$. Under the $\rho = 1$ case, the evolution of the risk adjusted payoffs as multiplicative martingales, yields $E_t[\Pi_{t+h} C_{t+h}]$ independent of the horizon h and thus a flat term-structure of risk prices, even under stochastic consumption volatility.

In the following section, we relax the assumption $\rho = 1$, and analyze the term-structure impact of our horizon-dependent risk-aversion model, under stochastic volatility.

5.2 General Case and Role of Volatility Risk

We consider the general case $\rho > 0$, $\rho \neq 1$, and we approximate the two relations

$$V_t = \left((1-\beta)\, C_t^{1-\rho} + \beta E_t \left[\tilde{V}_{t+1}^{1-\gamma} \right]^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1}{1-\rho}}$$
$$\tilde{V}_t = \left((1-\beta)\, C_t^{1-\rho} + \beta E_t \left[\tilde{V}_{t+1}^{1-\tilde{\gamma}} \right]^{\frac{1-\rho}{1-\tilde{\gamma}}} \right)^{\frac{1}{1-\rho}}.$$

under $\beta \approx 1$.

When the coefficient of time discounting β approaches 1, the recursion in $\tilde{V}$ can be re-written as

$$E_t \left[\left(\frac{\tilde{V}_{t+1}}{C_{t+1}} \right)^{1-\tilde{\gamma}} \left(\frac{C_{t+1}}{C_t} \right)^{1-\tilde{\gamma}} \right] \approx \beta^{-\frac{1-\rho}{1-\tilde{\gamma}}} \left(\frac{\tilde{V}_t}{C_t} \right)^{1-\tilde{\gamma}},$$

an eigenfunction problem, in which $\beta^{-\frac{1-\rho}{1-\tilde{\gamma}}}$ is an eigenvalue.

Lemma 2. *Under the Lucas-tree endowment process considered in the previous section, this eigenfunction problem admits a unique eigenvalue, and eigenfunction (up to a scalar multiplier):*

$$\tilde{v}_t - c_t = \tilde{\mu} + \phi_v x_t + \psi_v \sigma_t^2,$$

where

$$\begin{aligned}
\phi_v &= \phi_c \left(I - \nu_x\right)^{-1}, \\
\psi_v &= \frac{1}{2} \frac{1-\tilde{\gamma}}{1-\nu_\sigma} \left(\alpha_c^2 + \phi_v^2 \alpha_x^2 + \psi_v^2 \alpha_\sigma^2\right) < 0,
\end{aligned}$$

and

$$\log \beta = -\left(1-\rho\right)\left(\mu + \psi_v \sigma^2 \left(1-\nu_\sigma\right)\right).$$

Note the eigenvalue solution for β yields $\beta < 1$, as desired, for $\rho < 1$.[9] It also makes valid the approximation around 1: Using the calibration of Bansal and Yaron (2004) for the consumption process, we obtain solutions for β above 0.998, for any values of ρ between 0.1 and 1, and $\tilde{\gamma}$ between 1 and 10.

To derive the pricing equations, we use the approximation, valid for β close to 1:

$$\frac{V_t}{\tilde{V}_t} \approx \frac{E_t\left[\tilde{V}_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}{E_t\left[\tilde{V}_{t+1}^{1-\tilde{\gamma}}\right]^{\frac{1}{1-\tilde{\gamma}}}}.$$

Theorem 1. *Under the Lucas-tree endowment process and the $\beta \approx 1$ approximation:*

$$v_t - \tilde{v}_t = -\left(\gamma - \tilde{\gamma}\right) \frac{1-\nu_\sigma}{1-\tilde{\gamma}} \psi_v \sigma_t^2 < 0,$$

and the stochastic discount factor satisfies:

$$\begin{aligned}
\pi_{t,t+1} = \bar{\pi}_t &- \gamma \alpha_c \sigma_t W_{t+1} + \left(\rho - \gamma\right) \phi_v \alpha_x \sigma_t W_{t+1} \\
&+ \left(\left(\rho - \gamma\right) + \left(1-\rho\right)\left(\gamma - \tilde{\gamma}\right) \frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right) \psi_v \alpha_\sigma \sigma_t W_{t+1},
\end{aligned}$$

where

[9] Even though $\psi_v < 0$, the term $\left(\mu + \psi_v \sigma^2 \left(1-\nu_\sigma\right)\right)$ remains positive for all reasonable parameter values.

$$\bar{\pi}_t = -\mu - \rho\phi_c x_t - (1-\rho)\,\psi_v \sigma^2 (1-\nu_\sigma)\left(1 - (\gamma - \tilde{\gamma})\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)$$
$$- \Big((\rho - \gamma)(1-\gamma) - (1-\rho)(\gamma - \tilde{\gamma})\,\nu_\sigma \Big)\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v \sigma_t^2.$$

Our model yields a negative price for volatility shocks, consistent with the existing long-run risk literature, and the observed data for one-period returns. Observe further that the SDF loading on the drift shocks $\alpha_x \sigma_t W_{t+1}$ is unaffected by the specificities of our horizon-dependent risk-aversion model – it is exactly the same as in the standard EZ model. Any novel pricing effects we obtain – both in level and in the term-structure – derive from the volatility shocks. For this reason, we shut down the drift shocks in the part that follows, and assume $x_t = 0$ and $\alpha_x = 0$ in the remainder of the paper.

We now analyze the pricing of volatility risk in the term-structure. Denote by $P_{t,h}$ the price at time t for a claim to the endowment consumption at horizon h, and let $P_{t,0} = C_t$ for all t. The one-period holding returns for such assets are determined by

$$R_{t\to t+1,h} = \frac{P_{t+1,h-1} - P_{t,h}}{P_{t,h}},$$

and we denote by $\mathrm{SR}_{t,h}$ the conditional sharpe ratio for the one-period holding return at time t for a claim to consumption in period $t+h$.

Theorem 2. *Pricing in the term-structure is given by:*

$$\frac{P_{t,h}}{C_t} = \exp\left(a_h + A_h \sigma_t^2\right),$$

and the conditional Sharpe ratios are given by:

$$\mathrm{SR}_{t,h} = \frac{1 - \exp\left[-\left(\bar{r} + A\sigma_t^2 - \left(\rho - \gamma + (1-\rho)(\gamma - \tilde{\gamma})\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v A_{h-1}\alpha_\sigma^2 \sigma_t^2\right)\right]}{\sqrt{\exp\left(\left(\alpha_c^2 + A_{h-1}^2 \alpha_\sigma^2\right)\sigma_t^2\right) - 1}},$$

where $\bar{r}$ and A are constant (independent of t and h) and A_h is determined by the initial condition $A_0 = 0$ and the recursion:

$$A_{h-1}\nu_\sigma - A_h + \frac{1}{2}\left(\alpha_c^2 + A_{h-1}^2 \alpha_\sigma^2\right) = A - \left(\rho - \gamma + (1-\rho)(\gamma - \tilde{\gamma})\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v A_{h-1}\alpha_\sigma^2.$$

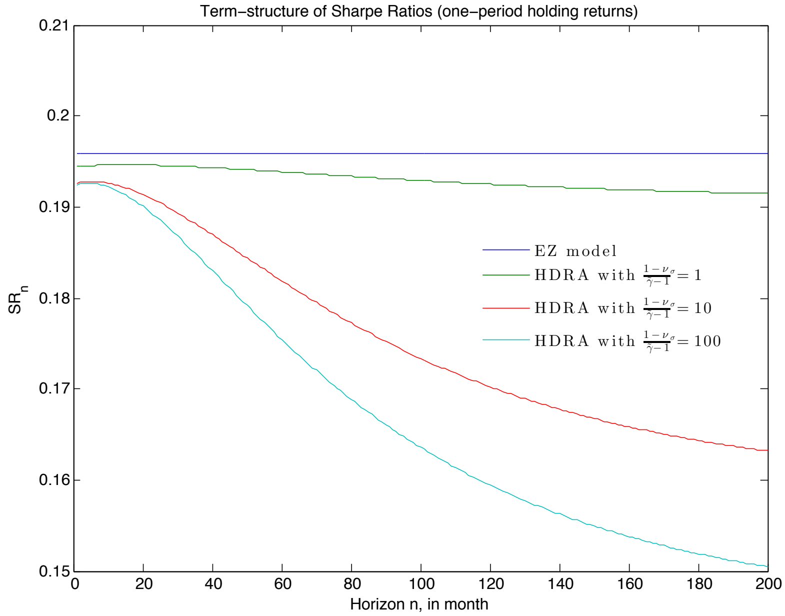

Figure 1: Calibrated term-structure. We use the parameters from Bansal and Yaron (2004) for μ, ν_σ, α_c and α_σ and $\rho = 1/1.5$. HDRA stands for "horizon-dependent risk aversion."

From Theorem 1 and Theorem 2, observe that both the pricing of volatility risk and the term-structure of Sharpe ratios for one-period returns on the consumption claims at various horizons depend mostly on a term of the model parameters:

$$\left(\rho - \gamma + (1-\rho)(\gamma - \tilde{\gamma})\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v \tag{8}$$

The first term in this expression, $\rho - \gamma$, is a standard EZ term while the second term, $(1-\rho)(\gamma-\tilde{\gamma})\frac{1-\nu_\sigma}{1-\tilde{\gamma}}$, is new and originates in the horizon-dependent risk aversion. If the novel term is dominated by the standard EZ term then our model of preferences has no significant impact on the pricing of volatility risk in either its level or its term-structure.

Corollary 2. *Horizon-dependent risk aversion affects risk prices and their term-structure if and only if $\tilde{\gamma}$ is close to* 1. *The effect is stronger, the more persistent the volatility risk.*

In Figure 1, we plot the term-structure of the Sharpe ratios of one-period holding returns on horizon-dependent consumption claims, for various values of the ratio $\frac{1-\nu_\sigma}{1-\tilde{\gamma}}$, which determines how much impact the variations in risk-aversion across horizons have. For each value of $\frac{1-\nu_\sigma}{1-\tilde{\gamma}}$, the immediate risk aversion γ is chosen such that the pricing of volatility risk, given by the whole term in (8), always remains the same as in the standard EZ model with risk aversion $\gamma = 10$. Figure 1 shows clearly that our horizon-dependent risk-aversion model can generate a downward sloping term-structure for the Sharpe ratios of one-period holding returns of consumption claims. This is in contrast to the standard EZ model which generates a flat term-structure – a result that has been highlighted in the literature starting with van Binsbergen et al. (2012), most recently by Dew-Becker et al. (2014). Observe, however, that for such a term-structure effect to be notable quantitatively, the long-horizon risk aversion $\tilde{\gamma}$ must be very close to one, i.e. approximate log utility. To match the pricing of volatility risk of the standard EZ model, the difference in risk aversion between the short horizon and the long horizon must become very large, unrealistically so under the very persistent volatility calibration of Bansal and Yaron (2004). However, this calibration problem can be largely avoided by making the time-varying volatility less persistent (without changing the volatility's stationary distribution).

To summarize, in a simple endowment economy, our model with horizon-dependent risk aversion has very specific implications for the level and the term-structure of the pricing of risk. If volatility is constant over time, our model does not affect the pricing of risk relative to the standard EZ model. Even with time-varying volatility, our model affects solely the loading of the stochastic discount factor on the shocks to volatility. The pricing of shocks to immediate consumption and of shocks to the consumption drift are unchanged from the standard EZ model. On the other hand, the pricing of the shocks to volatility presents a clear downward slopping term-structure, in contrast to the standard EZ model. A testable implication of our model is thus that there is a term-structure in the pricing of volatility risk. Section 6 analyzes how well this implication holds in the data, and provides supportive evidence.

6 Empirical Analysis of Volatility Risk Pricing

Our empirical analysis is motivated by the key predictions of our horizon-dependent risk aversion model. This section presents empirical results from index option returns: we provide estimates of the term structure of the (unit) price of volatility risk. We do so in two alternative ways: first with a parametric approach, using Efficient Generalized Method of Moments (GMM), and then with a model-free approach, using short-horizon Sharpe ratios of at-the-money straddles. A benefit of the former approach is to avoid using Sharpe ratios as a proxy for the price of risk, which can be problematic when returns are not normally distributed, and jumps occur (Broadie et al., 2009). A drawback is that a model for option pricing must be assumed, which, as with any model, comes with limitations.

For both approaches, we test two hypotheses. The first is that the price of volatility risk is negative: are investors willing to pay a premium to hold an insurance claim against increases in volatility? In contrast to previous papers that have tested the hypothesis jointly across several maturities or for specific maturities alone (Carr and Wu, 2009; Coval and Shumway, 2001), we investigate whether the hypothesis holds at different maturities independently. The second hypothesis is that the term structure for the price of volatility risk is downward-sloping (in absolute value).

We next present a theoretical derivation of the tests we use to investigate the above hypotheses.

6.1 Theory

We base our empirical analysis on the option pricing model of Heston (1993) which extends the classic model of Black and Scholes (1973) by adding stochastic volatility. Specifically, we assume the stock price S and volatility V satisfy

$$\begin{aligned} dS &= S\mu\, dt + S\sqrt{V}\, dW_1, \\ dV &= \kappa\,(\theta - V)\, dt + \sigma\sqrt{V}\, dW_2, \\ dW_1\, dW_2 &= \rho\, dt, \end{aligned}$$

where μ denotes the return drift, κ the speed of mean reversion, θ the level to which volatility reverts, σ the volatility of volatility, dW_1 and dW_2 are Brownian Motions, and ρ denotes the correlation between shocks to the return and volatility processes.

The no-arbitrage price X of any option satisfies the partial differential equation[10]

$$X_t + \frac{1}{2}X_{ss}VS^2 + \frac{1}{2}X_{vv}\sigma^2 V + X_{sv}\rho\sigma SV - rX + rX_sS + X_v\left[\kappa\,(\theta - V) - \lambda\right] = 0,$$

where λ denotes the volatility risk premium. This total risk premium λ can then be

[10] Subscripts denote partial derivatives.

Table 1: Summary Statistics for Option Data

Maturity	Number of Observations	Average Mid-Price	Average Bid-Ask Spread
0 - 30	8394	15.62	1.38
30 - 60	7716	31.06	1.90
60 - 90	6682	40.96	2.11
90 - 120	3436	51.83	2.37
120 - 150	2566	57.59	2.29
150 - 180	2370	64.25	2.38
180 - 210	2186	70.32	2.39
210 - 240	2460	75.30	2.43
240 - 270	2270	80.69	2.54
270 - 300	2178	85.18	2.56
300 - 330	2294	89.49	2.63
330 - 360	2312	93.36	2.82
Total	44864	61.03	2.45

The table summarizes for each segment of the term structure the number of observations, average mid-price, and average bid-ask spread We use option data from February 1996 to April 2013. Mid-price is the average of bid-price and ask-price.

decomposed into the unit price of volatility risk λ^* and the amount of volatility V:

$$\lambda = \lambda^* \times V$$

In contrast to existing studies that measure the total variance risk premium λ, our interest is in measuring λ^*, the unit price of volatility risk, at different horizons. We estimate λ^* using GMM, and we do so separately at different horizons using only options that mature at the respective horizon.

6.2 Data Sources and Summary Statistics

We use daily quotes of S&P 500 index options from 1996 to 2011 from OptionMetrics, and the mid closing price (average of bid and ask) for each day as the option price. The 3-month Treasury yield proxies for the risk-free rate. Table 1 gives summary statistics. The average mid-price increases with maturity. The average bid-ask spread in dollar terms also increases almost monotonically with maturity, but not if measured as a percentage of the mid-price, indicating good liquidity of options up to 360 days maturity. We discard observations for maturities less than 30 days, because of microstructure noise and jump risks concerns. Because liquidity drops significantly beyond 360 days maturity, we drop such observations. After filtering out options with maturities shorter than 30 days and longer than 360 days, we are left with 36,470 observations.

6.3 Parametric Estimation Using GMM

To estimate the pricing of risk, we use the relation, derived from the no-arbitrage pricing of X:

$$\lambda^* \frac{\sqrt{V\Delta t}}{\sigma} + \varepsilon = \frac{\Delta(X - X_s S) - r\,(X - X_s S)\,\Delta t}{X_v \sigma \sqrt{V\Delta t}},$$

where $\varepsilon = \Delta W_2/\sqrt{\Delta t}$.

To price X, we use straddles, i.e. combinations of calls and puts with the same strike prices and maturities. Denote the call price by C and the put price by P, then $X = C + P$, $X_s = C_s + P_s$ and $X_v = C_v + P_v$. Note that C_s and C_v, and P_s and P_v denote the "Delta" and "Vega" of the call and the put, respectively, and can be calculated numerically.[11]

While σ can be estimated, V cannot be directly measured or calculated. However, we can calibrate the model locally to estimate the initial volatility V_0; when the period is short enough, we can then assume $V \approx V_0$. Empirically, we use option prices in the neighboring 10 days for calibration. For example, for a call at time t, we calibrate the model using $C(t+i)$ for $i \in \{-5, \dots, 5\}$. The objective is to minimize the mean absolute deviation of the theoretical straddle prices from their empirically observed values:

$$\min \sum_{i=-5}^{i=5} \frac{1}{11} \left| \big(C_{\text{theo}}(t+i) + P_{\text{theo}}(t+i)\big) - \big(C_{\text{mkt}}(t+i) + P_{\text{mkt}}(t+i)\big) \right|$$

We impose the following constraints to eliminate noise from the observations: $0 < \kappa < 5$, $0 < \theta < 1$, $0.01 < \sigma \leq 1$, $0.01 < V_0 < 1$.[12]

We denote the pricing of volatility risk as a function of maturity by λ^*_τ, where τ is the maturity. For each τ, we use options with maturities in the neighboring 10 days. For example, to estimate λ^*_τ at a horizon of 30 days, options with maturities ranging from 25 to 35 days are used. The reason for this procedure is to smooth out microstructure noise arising from possible illiquidity of one option or another. To mitigate the influence of outliers, we truncate the data used for all estimations at the 1 percent level with respect to λ^*_τ. To accommodate for the non-linear nature of the term-structure for the pricing of

[11] The formulas are derived and presented in Appendix D.

[12] See Appendix D for details on the estimation.

volatility risk, we fit a logarithmic function through the pricing λ^*_τ estimates.

The results are given in Figure 2. The pricing of volatility risk is strongly negative for maturities shorter than 150 days. Beyond 150 days maturity, the price is approximately 0 and the term structure flattens out. We conclude that both null hypotheses are rejected: the price of volatility risk is negative – though only at short maturities – and the term structure is downward-sloping (in absolute value).

6.4 Model-Free Estimation with Straddle Returns

As the parametric estimation may be biased by model misspecification and the estimation procedure, we also proceed with an estimation of the sign of the pricing of volatility risk and of the slope of its term structure – but not a precise estimate of the level of either – that does not depend on any specific econometric model.[13]

In the Heston (1993) model above, note λ^* can be approximated with the Sharpe ratio of an at-the-money straddle, up to a factor $\sqrt{V\Delta t}/\sigma$:

$$\lambda^* \frac{\sqrt{V\Delta t}}{\sigma} \approx \mathrm{SR}(C+P).$$

The factor $\sqrt{V\Delta t}/\sigma$ is not measurable without making further assumptions, however, it is guaranteed to be positive. Moreover, it is the same across maturities, and therefore does not affect the sign of the slope of the term structure of volatility risk pricing. The Sharpe ratios of at-the-money straddles provide a qualitative measure for the prices of volatility risk, in the term structure, though they are not quantitatively comparable to the results from the parametric estimation of Section 6.3.

We use option straddles with maturities ranging from 30 days to 360 days, and regress the Sharpe ratio estimates on maturity τ to gauge the overall shape of the term structure. The regression model is

$$\mathrm{SR}^*_\tau = \beta_2 + \beta_3\tau + \varepsilon_2,$$

where β_2 and β_3 are the intercept and slope coefficients and ε_2 is the error term.[14]

Recall the estimates are not quantitatively comparable to the parametric estimation, but the sign is guaranteed to conform. The empirical results are illustrated in Figure 3. Confirming the previous results from the parametric estimation, the pricing of volatility

[13]We thank Ralph Koijen for suggesting this measure of volatility risk pricing using at-the-money straddle returns.

[14]See Appendix D for details on the estimation.

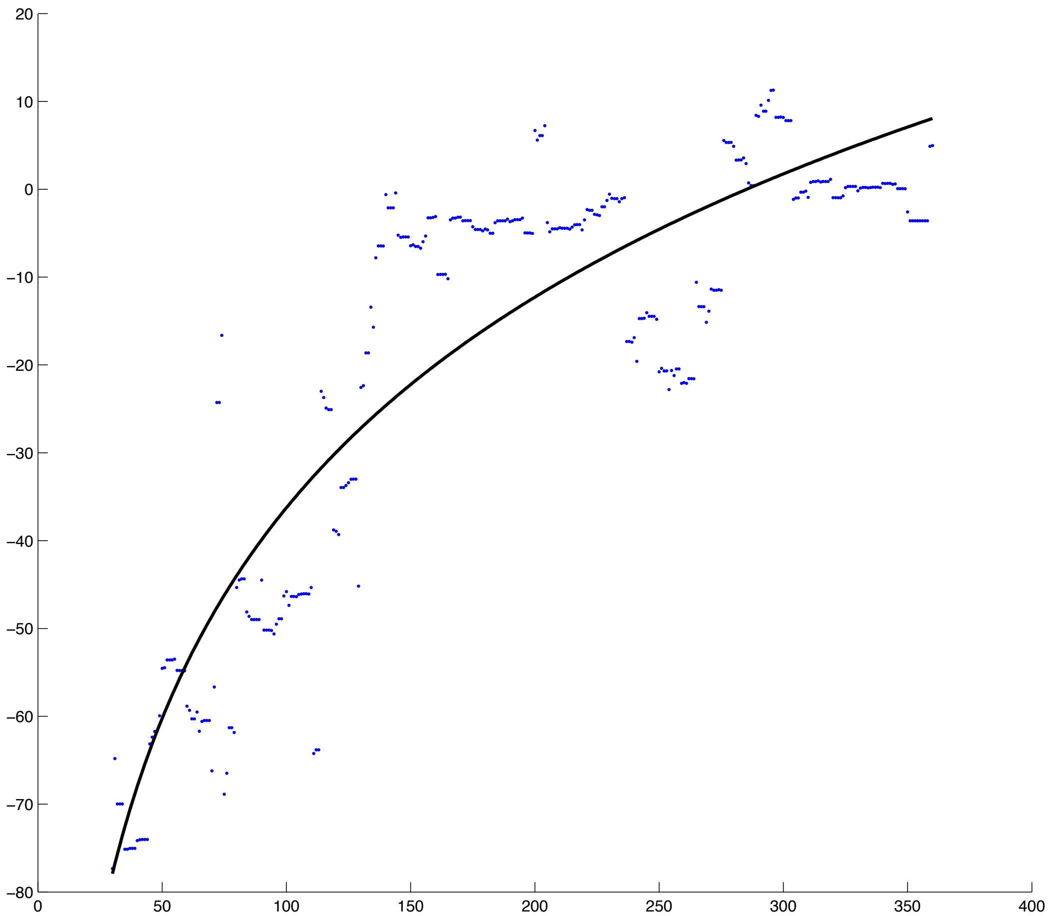

Figure 2: Parametric estimation of the term structure of the price of volatility risk.

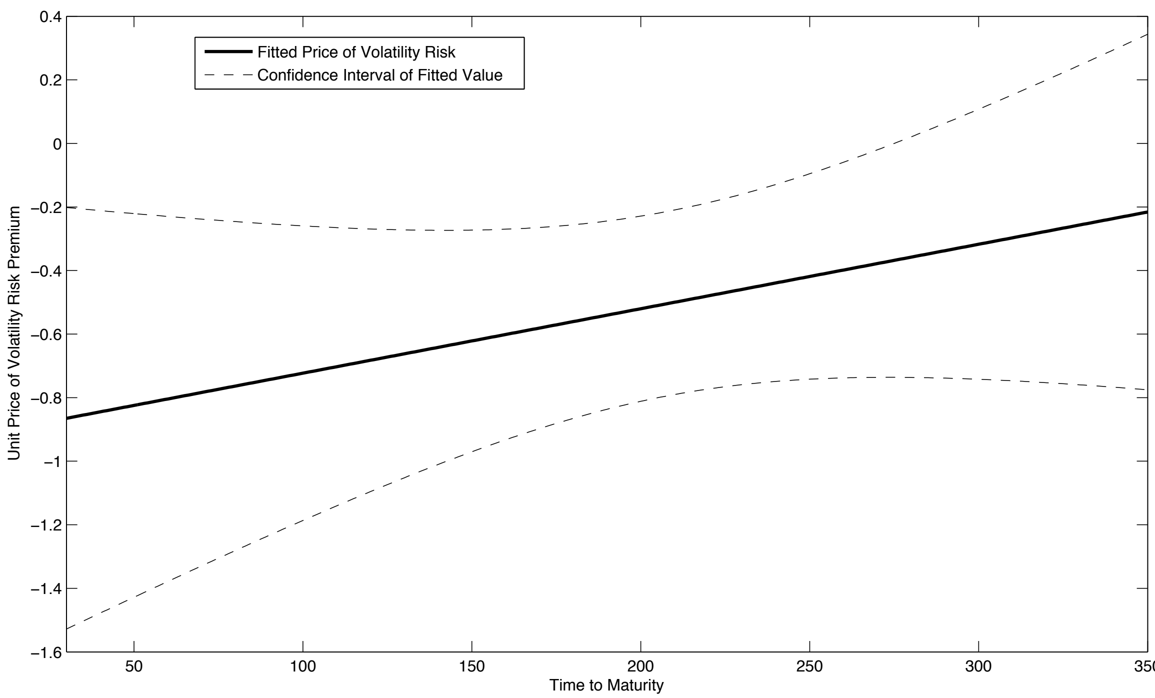

Figure 3: Model-free estimation of the term structure of the price of volatility risk.

risk estimated with the model-free approach is negative for short maturities. The point estimate is -0.86 at the very short end (30 days), and the 95 percent confidence bounds are -0.2 and -1.53. At and beyond a maturity of 300 days, the pricing of volatility risk becomes statistically indistinguishable from 0.

In sum, both the parametric estimation and the model-free approach indicate that the well-known result that the price of volatility risk is negative is solely driven by short maturities. Investors are not willing to pay for insurance against increases in volatility risk beyond 150 days. The model-free results suggest that although any model misspecification in the parametric estimation part may introduce a quantitative bias, it does not affect the negativeness and upward slope of the term structure of the price of volatility risk.

7 Conclusion

We solve for general equilibrium asset prices in an endowment economy in which assets are priced by an agent who can have different levels of risk aversion for risks at different maturities. Such preferences are dynamically inconsistent with respect to risk-return tradeoffs. We find horizon-dependent risk aversion preferences have a meaningful impact on asset prices, and have the ability to address recent puzzles in general equilibrium asset pricing unaccounted for by the standard models. In particular, we show the price of risk depends on the horizon, but only if volatility is stochastic. This insight leads to several testable predictions. The prediction speaking most directly to the proposed channel is that the price of volatility risk is negative and its term structure is downward-sloping in absolute value. The prediction finds support in the data.

We are not aware of competing mainstream general equilibrium endowment models that can predict a downward-sloping term structure and that make similarly detailed and empirically valid predictions for its driver. Relaxing the common assumption that risk preferences are constant across maturities – and specifically, replacing it with the no more flexible assumption that short-horizon risk aversion is higher than long-horizon risk aversion – may thus be a useful tool in different subfields of asset pricing research.

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Appendix

A Derivation of the Stochastic Discount Factor

This appendix derives the stochastic discount factor of our dynamic model using an approach similar to the one used by Luttmer and Mariotti (2003) for non-geometric discounting. In every period t the agent chooses consumption C_t for the current period and state-contingent levels of wealth $\{W_{t+1,s}\}$ for the next period to maximize current utility V_t subject to a budget constraint *and* anticipating optimal choice C^*_{t+h} in all following periods ($h \geq 1$):

$$\max_{C_t,\{W_{t+1}\}} \left((1-\beta) C_t^{1-\rho} + \beta E_t \left[\left(V^*_{t,t+1} \right)^{1-\gamma_0} \right]^{\frac{1-\rho}{1-\gamma_0}} \right)^{\frac{1}{1-\rho}}$$

$$\text{s.t.} \quad \Pi_t C_t + E_t[\Pi_{t+1} W_{t+1}] \leq \Pi_t W_t$$

$$V^*_{t,t+h} = \left((1-\beta) \left(C^*_{t+h} \right)^{1-\rho} + \beta E_{t+h} \left[\left(V^*_{t,t+h+1} \right)^{1-\gamma_h} \right]^{\frac{1-\rho}{1-\gamma_h}} \right)^{\frac{1}{1-\rho}} \quad \text{for all} \quad h \geq 1.$$

Denoting by λ_t the Lagrange multiplier on the budget constraint for the period-t problem, the first order conditions are:[15]

- For C_t:

$$\left((1-\beta) C_t^{1-\rho} + \beta E_t \left[V_{t,t+1}^{1-\gamma_0} \right]^{\frac{1-\rho}{1-\gamma_0}} \right)^{\frac{1}{1-\rho}-1} (1-\beta) C_t^{-\rho} = \lambda_t.$$

- For each $W_{t+1,s}$:

$$\frac{1}{1-\rho} \left((1-\beta) C_t^{1-\rho} + \beta E_t \left[V_{t,t+1}^{1-\gamma_0} \right]^{\frac{1-\rho}{1-\gamma_0}} \right)^{\frac{1}{1-\rho}-1} \beta \frac{d}{dW_{t+1,s}} \beta E_t \left[V_{t,t+1}^{1-\gamma_0} \right]^{\frac{1-\rho}{1-\gamma_0}}$$
$$= \Pr[t+1,s] \frac{\Pi_{t+1,s}}{\Pi_t} \lambda_t.$$

Combining the two, we get an initial equation for the SDF:

$$\frac{\Pi_{t+1,s}}{\Pi_t} = \beta \, \frac{\frac{1}{1-\rho} \frac{1}{\Pr[t+1,s]} \frac{d}{dW_{t+1,s}} E_t \left[V_{t,t+1}^{1-\gamma_0} \right]^{\frac{1-\rho}{1-\gamma_0}}}{1} \frac{1}{(1-\beta) C_t^{-\rho}}. \tag{9}$$

[15] For notational ease we drop the star from all Cs and Vs in the following optimality conditions but it should be kept in mind that all consumption values are the ones optimally chosen by the corresponding self.

The agent in state s at $t+1$ maximizes

$$\left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}\Big[\big(V^{*}_{t+1,s,t+2}\big)^{1-\gamma_0}\Big]^{\frac{1-\rho}{1-\gamma_0}}\right)^{\frac{1}{1-\rho}}$$

and has the analogous first order condition for $C_{t+1,s}$:

$$\left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}\big[V_{t+1,s,t+2}^{1-\gamma_0}\big]^{\frac{1-\rho}{1-\gamma_0}}\right)^{\frac{1}{1-\rho}-1} (1-\beta)\, C_{t+1,s}^{-\rho} = \lambda_{t+1,s}.$$

The Lagrange multiplier $\lambda_{t+1,s}$ is equal to the marginal utility of an extra unit of wealth in state $t+1, s$:

$$\begin{aligned}\lambda_{t+1,s} = \frac{1}{1-\rho}&\left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}\big[V_{t+1,s,t+2}^{1-\gamma_0}\big]^{\frac{1-\rho}{1-\gamma_0}}\right)^{\frac{1}{1-\rho}-1} \\ &\times \frac{d}{dW_{t+1,s}}\left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}\big[V_{t+1,s,t+2}^{1-\gamma_0}\big]^{\frac{1-\rho}{1-\gamma_0}}\right).\end{aligned}$$

Eliminating the Lagrange multiplier $\lambda_{t+1,s}$ and combining with the initial equation (9) for the SDF, we get:

$$\frac{\Pi_{t+1,s}}{\Pi_t} = \beta\, \frac{\frac{1}{\Pr[t+1,s]}\frac{d}{dW_{t+1,s}} E_t\big[V_{t,t+1}^{1-\gamma_0}\big]^{\frac{1-\rho}{1-\gamma_0}}}{\frac{d}{dW_{t+1,s}}\left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}\big[V_{t+1,s,t+2}^{1-\gamma_0}\big]^{\frac{1-\rho}{1-\gamma_0}}\right)}\left(\frac{C_{t+1,s}}{C_t}\right)^{-\rho}$$

Expanding the V expressions, we can proceed with the differentiation in the numerator:

$$\begin{aligned}\frac{\Pi_{t+1,s}}{\Pi_t} = E_t&\left[\left((1-\beta)\, C_{t+1}^{1-\rho} + \beta E_{t+1}[\dots]^{\frac{1-\rho}{1-\gamma_1}}\right)^{\frac{1-\gamma_0}{1-\rho}}\right]^{\frac{1-\rho}{1-\gamma_0}-1} \\ &\times \left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}[\dots]^{\frac{1-\rho}{1-\gamma_1}}\right)^{\frac{1-\gamma_0}{1-\rho}-1} \\ &\quad\times \beta\, \frac{\frac{d}{dW_{t+1,s}}\left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}[\dots]^{\frac{1-\rho}{1-\gamma_1}}\right)}{\frac{d}{dW_{t+1,s}}\left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}[\dots]^{\frac{1-\rho}{1-\gamma_0}}\right)}\left(\frac{C_{t+1,s}}{C_t}\right)^{-\rho}.\end{aligned} \qquad (10)$$

For Markov consumption $C = \phi W$, we can divide by $C_{t+1,s}$ and solve both differentiations:

- For the numerator:

$$\frac{d}{dW_{t+1,s}}\left((1-\beta)\,C_{t+1,s}^{1-\rho}+\beta E_{t+1,s}\left[\left((1-\beta)\,C_{t+2}^{1-\rho}+\beta E_{t+2}[\dots]^{\frac{1-\rho}{1-\gamma_2}}\right)^{\frac{1-\gamma_1}{1-\rho}}\right]^{\frac{1-\rho}{1-\gamma_1}}\right)$$

$$=\left((1-\beta)\,1+\beta E_{t+1,s}\left[\left((1-\beta)\left(\frac{C_{t+2}}{C_{t+1,s}}\right)^{1-\rho}+\beta E_{t+2}[\dots]^{\frac{1-\rho}{1-\gamma_2}}\right)^{\frac{1-\gamma_1}{1-\rho}}\right]^{\frac{1-\rho}{1-\gamma_1}}\right)$$

$$\times\,\phi_{t+1,s}^{1-\rho}W_{t+1,s}^{-\rho}.$$

- For the denominator:

$$\frac{d}{dW_{t+1,s}}\left((1-\beta)\,C_{t+1,s}^{1-\rho}+\beta E_{t+1,s}\left[\left((1-\beta)\,C_{t+2}^{1-\rho}+\beta E_{t+2}[\dots]^{\frac{1-\rho}{1-\gamma_1}}\right)^{\frac{1-\gamma_0}{1-\rho}}\right]^{\frac{1-\rho}{1-\gamma_0}}\right)$$

$$=\left((1-\beta)\,1+\beta E_{t+1,s}\left[\left((1-\beta)\left(\frac{C_{t+2}}{C_{t+1,s}}\right)^{1-\rho}+\beta E_{t+2}[\dots]^{\frac{1-\rho}{1-\gamma_1}}\right)^{\frac{1-\gamma_0}{1-\rho}}\right]^{\frac{1-\rho}{1-\gamma_0}}\right)$$

$$\times\,\phi_{t+1,s}^{1-\rho}W_{t+1,s}^{-\rho}.$$

Substituting these into equation (10) and canceling we get:

$$\frac{\Pi_{t+1,s}}{\Pi_t}=\frac{(1-\beta)\,C_{t+1,s}^{1-\rho}+\beta E_{t+1,s}\left[\left((1-\beta)\,C_{t+2}^{1-\rho}+\beta E_{t+2}[\dots]^{\frac{1-\rho}{1-\gamma_2}}\right)^{\frac{1-\gamma_1}{1-\rho}}\right]^{\frac{1-\rho}{1-\gamma_1}}}{(1-\beta)\,C_{t+1,s}^{1-\rho}+\beta E_{t+1,s}\left[\left((1-\beta)\,C_{t+2}^{1-\rho}+\beta E_{t+2}[\dots]^{\frac{1-\rho}{1-\gamma_1}}\right)^{\frac{1-\gamma_0}{1-\rho}}\right]^{\frac{1-\rho}{1-\gamma_0}}}$$

$$\times\,\beta\left(\frac{C_{t+1,s}}{C_t}\right)^{-\rho}\left(\frac{(1-\beta)\,C_{t+1,s}^{1-\rho}+\beta E_{t+1,s}[\dots]^{\frac{1-\rho}{1-\gamma_1}}}{E_t\left[\left((1-\beta)\,C_{t+1,s}^{1-\rho}+\beta E_{t+1}[\dots]^{\frac{1-\rho}{1-\gamma_1}}\right)^{\frac{1-\gamma_0}{1-\rho}}\right]}\right)^{\rho-\gamma_0}.$$

Simplifying and cleaning up notation, we arrive at the same SDF as in the main text:

$$\frac{\Pi_{t+1}}{\Pi_t}=\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\left(\frac{V_{t,t+1}}{E_t\left[V_{t,t+1}^{1-\gamma_0}\right]^{\frac{1}{1-\gamma_0}}}\right)^{\rho-\gamma_0}\left(\frac{V_{t,t+1}}{V_{t+1}}\right)^{1-\rho}.$$

B Exact solutions for $\rho = 1$

Suppose the risk aversion parameter differs only for immediate risk shocks: between t and $t+1$, risk aversion is γ, for all shocks further down, risk aversion is $\tilde{\gamma}$. The model simplifies to:

$$V_t = \left[(1-\beta) C_t^{1-\rho} + \beta \left(\mathcal{R}_{t,\gamma} \left(\widetilde{V}_{t+1} \right) \right)^{1-\rho} \right]^{\frac{1}{1-\rho}},$$
$$\tilde{V}_t = \left[(1-\beta) C_t^{1-\rho} + \beta \left(\mathcal{R}_{t,\tilde{\gamma}} \left(\widetilde{V}_{t+1} \right) \right)^{1-\rho} \right]^{\frac{1}{1-\rho}},$$

where

$$\mathcal{R}_{t,\lambda}(X) = \left(E_t \left(X^{1-\lambda} \right) \right)^{\frac{1}{1-\lambda}}.$$

Take the evolutions

$$c_{t+1} - c_t = \mu + \phi_c x_t + \alpha_c \sigma_t W_{t+1},$$
$$x_{t+1} = \nu_x x_t + \alpha_x \sigma_t W_{t+1},$$
$$\sigma_{t+1}^2 - \sigma^2 = \nu_\sigma \left(\sigma_t^2 - \sigma^2 \right) + \alpha_\sigma \sigma_t W_{t+1},$$

and suppose the three shocks are independent. (We can relax this assumption.)

If $\rho = 1$, then the recursion for $\tilde{V}$ becomes:

$$\frac{\widetilde{V}_t}{C_t} = \left(\mathcal{R}_{t,\tilde{\gamma}} \left(\frac{\widetilde{V}_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t} \right) \right)^{\beta}$$

Assume that

$$\tilde{v}_t - c_t = \tilde{\mu_v} + \phi_v x_t + \psi_v \sigma_t^2.$$

Then the solution to the recursion yields

$$\phi_v = \beta \phi_c \left(I - \nu_x \right)^{-1},$$

and

$$\psi_v = \frac{1}{2} \frac{\beta \left(1 - \tilde{\gamma} \right)}{1 - \beta \nu_\sigma} \left(\alpha_c^2 + \phi_v^2 \alpha_x^2 + \psi_v^2 \alpha_\sigma^2 \right) < 0.$$

Because

$$\frac{V_t}{C_t} = \left(\mathcal{R}_{t,\gamma}\left(\frac{\widetilde{V}_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)\right)^{\beta},$$

we have

$$\frac{V_t}{\tilde{V}_t} = \left[\frac{\mathcal{R}_{t,\gamma}\left(\widetilde{V}_{t+1}\right)}{\mathcal{R}_{t,\tilde{\gamma}}\left(\widetilde{V}_{t+1}\right)}\right]^{\beta},$$

which yields

$$\begin{aligned} & v_t - \tilde{v}_t = -\frac{1}{2}\beta\left(\gamma-\tilde{\gamma}\right)\left(\alpha_c^2+\phi_v^2\alpha_x^2+\psi_v^2\alpha_\sigma^2\right)\sigma_t^2, \\ \Rightarrow \quad & v_t - \tilde{v}_t = -\left(\gamma-\tilde{\gamma}\right)\frac{1-\beta\nu_\sigma}{1-\tilde{\gamma}}\psi_v\sigma_t^2 < 0. \end{aligned}$$

C Approximation for $\beta \approx 1$

As in Appendix B, consider the simplified model with only two levels of risk aversion:

$$\begin{aligned} V_t &= \left[(1-\beta)C_t^{1-\rho} + \beta\left(\mathcal{R}_{t,\gamma}\left(\widetilde{V}_{t+1}\right)\right)^{1-\rho}\right]^{\frac{1}{1-\rho}}, \\ \tilde{V}_t &= \left[(1-\beta)C_t^{1-\rho} + \beta\left(\mathcal{R}_{t,\tilde{\gamma}}\left(\widetilde{V}_{t+1}\right)\right)^{1-\rho}\right]^{\frac{1}{1-\rho}}, \end{aligned}$$

where

$$\mathcal{R}_{t,\lambda}\left(X\right) = \left(E_t\left(X^{1-\lambda}\right)\right)^{\frac{1}{1-\lambda}}.$$

Also, as in Appendix B, take the evolutions:

$$\begin{aligned} c_{t+1} - c_t &= \mu + \phi_c x_t + \alpha_c\sigma_t W_{t+1}, \\ x_{t+1} &= \nu_x x_t + \alpha_x\sigma_t W_{t+1}, \\ \sigma_{t+1}^2 - \sigma^2 &= \nu_\sigma\left(\sigma_t^2-\sigma^2\right) + \alpha_\sigma\sigma_t W_{t+1}, \end{aligned}$$

and suppose the three shocks are independent. (We can relax this assumption.)

Now, for β close to 1, we have:

$$\left(\frac{\tilde{V}_t}{C_t}\right)^{1-\tilde{\gamma}} = \beta^{\frac{1-\tilde{\gamma}}{1-\rho}}E_t\left[\left(\frac{\tilde{V}_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)^{1-\tilde{\gamma}}\right].$$

This is an eigenfunction problem with eigenvalue $\beta^{-\frac{1-\tilde{\gamma}}{1-\rho}}$ and eigenfunction $\left(\tilde{V}/C\right)^{1-\tilde{\gamma}}$ known up to a multiplier. Let's assume:

$$\tilde{v}_t - c_t = \tilde{\mu} + \phi_v x_t + \psi_v \sigma_t^2.$$

Then we have:

- Terms in x_t (standard formula with $\beta = 1$):

$$\phi_v = \phi_c \left(I - \nu_x\right)^{-1}$$

- Terms in σ_t^2:

$$\psi_v = \frac{1}{2} \frac{(1-\tilde{\gamma})}{1-\nu_\sigma} \left(\alpha_c^2 + \phi_v^2 \alpha_x^2 + \psi_v^2 \alpha_\sigma^2\right) < 0$$

- Constant terms:

$$\log \beta = -\left(1-\rho\right)\left(\mu + \psi_v \sigma^2 \left(1-\nu_\sigma\right)\right)$$

For β close to 1, we have:

$$\frac{V_t}{\tilde{V}_t} \approx \frac{\mathcal{R}_{t,\gamma}\left(\widetilde{V}_{t+1}\right)}{\mathcal{R}_{t,\tilde{\gamma}}\left(\widetilde{V}_{t+1}\right)} = \frac{\left(E_t\left[\left(\frac{\tilde{V}_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}}{\left(E_t\left[\left(\frac{\tilde{V}_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)^{1-\tilde{\gamma}}\right]\right)^{\frac{1}{1-\tilde{\gamma}}}},$$

and therefore:

$$\begin{aligned} & v_t - \tilde{v}_t = -\frac{1}{2}\left(\gamma - \tilde{\gamma}\right)\left(\alpha_c^2 + \phi_v^2 \alpha_x^2 + \psi_v^2 \alpha_\sigma^2\right)\sigma_t^2, \\ \Rightarrow \quad & v_t - \tilde{v}_t = -\left(\gamma - \tilde{\gamma}\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}} \psi_v \sigma_t^2 < 0. \end{aligned}$$

The stochastic discount factor becomes:

$$\begin{aligned} \pi_{t,t+1} = \bar{\pi}_t &- \gamma \alpha_c \sigma_t W_{t+1} + \left(\rho - \gamma\right)\phi_v \alpha_x \sigma_t W_{t+1} \\ &+ \left(\left(\rho - \gamma\right) + \left(1-\rho\right)\left(\gamma - \tilde{\gamma}\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v \alpha_\sigma \sigma_t W_{t+1}, \end{aligned}$$

where

$$\bar{\pi}_t = -\mu - \rho\phi_c x_t - (1-\rho)\,\psi_v\sigma^2\,(1-\nu_\sigma)\left(1-(\gamma-\tilde{\gamma})\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)$$
$$-\left((\rho-\gamma)\,(1-\gamma)-(1-\rho)\,(\gamma-\tilde{\gamma})\,\nu_\sigma\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v\sigma_t^2.$$

Observe that in all the analysis the impact and the pricing of the state variable x_t is unaffected by the horizon dependent model. We can therefore simplify the analysis by setting $x_t = 0$ for all t. Going forward, take the evolutions:

$$c_{t+1} - c_t = \mu + \alpha_c\sigma_t W_{t+1},$$
$$\sigma_{t+1}^2 - \sigma^2 = \nu_\sigma\left(\sigma_t^2 - \sigma^2\right) + \alpha_\sigma\sigma_t W_{t+1},$$

and suppose the two shocks are independent.

We have

$$\tilde{v}_t - c_t = \tilde{\mu} + \psi_v\sigma_t^2,$$

where

$$\psi_v = \frac{1}{2}\frac{(1-\tilde{\gamma})}{1-\nu_\sigma}\left(\alpha_c^2 + \psi_v^2\alpha_\sigma^2\right) < 0,$$

and

$$\log\beta = -(1-\rho)\left(\mu + \psi_v\sigma^2\,(1-\nu_\sigma)\right)$$

$$v_t - \tilde{v}_t = -(\gamma-\tilde{\gamma})\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v\sigma_t^2 < 0$$
$$v_t - c_t = \tilde{\mu} + \psi_v\sigma_t^2\left(1-(\gamma-\tilde{\gamma})\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)$$

The stochastic discount factor becomes:

$$\pi_{t,t+1} = \bar{\pi}_t - \gamma\alpha_c\sigma_t W_{t+1} + \left((\rho-\gamma) + (1-\rho)\,(\gamma-\tilde{\gamma})\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v\alpha_\sigma\sigma_t W_{t+1}$$

where

$$\bar{\pi}_t = -\mu - (1-\gamma)^2\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v\sigma^2$$
$$-\left((1-\gamma)^2 - (1-\rho)\,(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma)\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v\left(\sigma_t^2 - \sigma^2\right).$$

Term structure: Now let's look a the term structure for endowment consumption strips. Let the period-t price for the endowment consumption in h periods be $P_{t,h}$. For $h=0$, we have $P_{t,0}=C_t$. For $h\geq 1$ we have:

$$\frac{P_{t,h}}{C_t}=E_t\left(\Pi_{t,t+1}\frac{C_{t+1}}{C_t}\frac{P_{t+1,h-1}}{C_{t+1}}\right).$$

We can guess that

$$\frac{P_{t,h}}{C_t}=\exp\left(a_h+A_h\sigma_t^2\right),$$

with $a_0=0$ and $A_0=0$. Suppose $h\geq 1$, then:

$$\begin{aligned}\log\Pi_{t,t+1}\frac{C_{t+1}}{C_t}\frac{P_{t+1,h-1}}{C_{t+1}}=&-(1-\rho)\left(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v\sigma^2\\&-\left((1-\gamma)^2-(1-\rho)\left(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma\right)\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v\sigma_t^2\\&+a_{h-1}+A_{h-1}\sigma^2\left(1-\nu_\sigma\right)+A_{h-1}\nu_\sigma\sigma_t^2\\&+(1-\gamma)\,\alpha_c\sigma_tW_{t+1}\\&+\left(\left((\rho-\gamma)+(1-\rho)(\gamma-\tilde{\gamma})\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v+A_{h-1}\right)\alpha_\sigma\sigma_tW_{t+1}.\end{aligned}$$

We find the recursion

$$\begin{aligned}A_h=&-\left((1-\gamma)^2-(1-\rho)\left(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma\right)\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v+A_{h-1}\nu_\sigma\\&+\frac{1}{2}\left(\left(\rho-\gamma+(1-\rho)(\gamma-\tilde{\gamma})\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v+A_{h-1}\right)^2\alpha_\sigma^2+\frac{1}{2}(1-\gamma)^2\alpha_c^2,\end{aligned}$$

and

$$a_h=-(1-\rho)\left(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v\sigma^2+a_{h-1}+A_{h-1}\sigma^2\left(1-\nu_\sigma\right).$$

The one-period excess returns on the dividend strips are given by:

$$R_{t+1}^h=\frac{P_{t+1,h-1}-P_{t,h}}{P_{t,h}}=\frac{\frac{P_{t+1,h-1}}{C_{t+1}}\frac{C_{t+1}}{C_t}}{\frac{P_{t,h}}{C_t}}-1.$$

We have:

$$\log\left(R_{t+1}^h+1\right)=\overbrace{\mu+(1-\rho)\left(1-\gamma+(\gamma-\tilde{\gamma})\nu_\sigma\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v\sigma^2}^{\equiv\bar{r}}$$
$$+\left(A_{h-1}\nu_\sigma-A_h\right)\sigma_t^2+\left(\alpha_c+A_{h-1}\alpha_\sigma\right)\sigma_t.$$

So the conditional Sharpe ratio term structure is given by:

$$\begin{aligned}\mathrm{SR}_t\left(R_{t+1}^h\right)&=\frac{\exp\left(\bar{r}+\left(A_{h-1}\nu_\sigma-A_h+\frac{1}{2}\left(\alpha_c^2+A_{h-1}^2\alpha_\sigma^2\right)\right)\sigma_t^2\right)-1}{\sqrt{\begin{cases}\exp\left(2\bar{r}+2\left(A_{h-1}\nu_\sigma-A_h+\left(\alpha_c^2+A_{h-1}^2\alpha_\sigma^2\right)\right)\sigma_t^2\right)\\-\left[\exp\left(\bar{r}+\left(A_{h-1}\nu_\sigma-A_h+\frac{1}{2}\left(\alpha_c^2+A_{h-1}^2\alpha_\sigma^2\right)\right)\sigma_t^2\right)\right]^2\end{cases}}}\\&=\frac{1-\exp\left[-\left(\bar{r}+\left(A_{h-1}\nu_\sigma-A_h+\frac{1}{2}\left(\alpha_c^2+A_{h-1}^2\alpha_\sigma^2\right)\right)\sigma_t^2\right)\right]}{\sqrt{\exp\left(\left(\alpha_c^2+A_{h-1}^2\alpha_\sigma^2\right)\sigma_t^2\right)-1}}\end{aligned}$$

Observe that:

$$\begin{aligned}&A_{h-1}\nu_\sigma-A_h+\frac{1}{2}\left(\alpha_c^2+A_{h-1}\alpha_\sigma^2\right)\\&\quad=\frac{(1-\rho)}{1-\tilde{\gamma}}\left(1-\gamma+(\gamma-\tilde{\gamma})\nu_\sigma\right)\left(\rho-\gamma+(\gamma-\tilde{\gamma})\left((1-\rho)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}-1\right)\right)\frac{1}{2}\psi_v^2\alpha_\sigma^2\\&\qquad+\left(1-(1-\rho)\left(1-\gamma+(\gamma-\tilde{\gamma})\nu_\sigma\right)\right)\frac{1}{2}\alpha_c^2\\&\qquad-\left(\rho-\gamma+(1-\rho)(\gamma-\tilde{\gamma})\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_vA_{h-1}\alpha_\sigma^2,\end{aligned}$$

which we can re-write as:

$$A_{h-1}\nu_\sigma-A_h+\frac{1}{2}\left(\alpha_c^2+A_{h-1}^2\alpha_\sigma^2\right)=A-\left(\rho-\gamma+(1-\rho)(\gamma-\tilde{\gamma})\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_vA_{h-1}\alpha_\sigma^2,$$

where

$$\begin{aligned}A=&\frac{(1-\rho)}{1-\tilde{\gamma}}\left(1-\gamma+(\gamma-\tilde{\gamma})\nu_\sigma\right)\left(\rho-\gamma+(\gamma-\tilde{\gamma})\left((1-\rho)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}-1\right)\right)\frac{1}{2}\psi_v^2\alpha_\sigma^2\\&+\left(1-(1-\rho)\left(1-\gamma+(\gamma-\tilde{\gamma})\nu_\sigma\right)\right)\frac{1}{2}\alpha_c^2.\end{aligned}$$

We therefore have:

$$\mathrm{SR}_t\left(R_{t+1}^h\right)=\frac{1-\exp\left[-\left(\bar{r}+A\sigma_t^2-\left(\rho-\gamma+(1-\rho)\left(\gamma-\tilde{\gamma}\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v A_{h-1}\alpha_\sigma^2\sigma_t^2\right)\right]}{\sqrt{\exp\left(\left(\alpha_c^2+A_{h-1}^2\alpha_\sigma^2\right)\sigma_t^2\right)-1}}.$$

D Details on Estimation of PVR

Closed-form Solution of Price, Delta, and Theta

The closed-form solution to the model is as follows. Denote by P_1 and P_2 pseudo-probabilities. The integration needs to be solved numerically.

$$C=SP_1-Ke^{-r(T-t)}P_2$$

$$P=S(P_1-1)-Ke^{-r(T-t)}(P_2-1)$$

$$\begin{aligned}
P_j &= \tfrac{1}{2}+\tfrac{1}{\pi}\int_0^\infty \Re\left[\tfrac{e^{-i\phi\ln(K)}f_j}{i\phi}\right]d\phi\\
f_j &= \exp\left(C_j+D_jv+i\phi\ln(S)\right)\\
C_j &= r\phi i(T-t)+\tfrac{a}{\sigma^2}\left[\left(b_j-\rho\sigma\phi i+d_j\right)(T-t)-2\ln\left(\tfrac{1-g_je^{d_j(T-t)}}{1-g_j}\right)\right]\\
D_j &= \tfrac{b_j-\rho\sigma\phi i+d_j}{\sigma^2}\left(\tfrac{1-e^{d_j(T-t)}}{1-g_je^{d_j(T-t)}}\right)\\
g_j &= \tfrac{b_j-\rho\sigma\phi i+d_j}{b_j-\rho\sigma\phi i-d_j}\\
d_j &= \sqrt{\left(\rho\sigma\phi i-b_j\right)^2-\sigma^2\left(2\mu_j\phi i-\phi^2\right)}\\
\mu_1 &= \tfrac{1}{2},\mu_2=-\tfrac{1}{2},a=\kappa\theta,b_1=\kappa+\lambda-\rho\sigma,b_2=\kappa+\lambda
\end{aligned}$$

Delta is, respectively,

$$\begin{aligned}
C_s &= P_1+S\tfrac{\partial P_1}{\partial S}-Ke^{-r(T-t)}\tfrac{\partial P_2}{\partial S},\\
P_s &= C_s-1,
\end{aligned}$$

in which

$$\begin{aligned}
\frac{\partial P_1}{\partial S}&=\frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-i\phi\ln(K)+C_1+D_1v+i\phi\ln(S)}}{i\phi}i\phi\frac{1}{S}\right]d\phi=\frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{C_1+D_1v+i\phi\ln\left(\frac{S}{K}\right)}}{S}\right]d\phi\\
&=\frac{1}{S\pi}\int_0^\infty \Re\left(e^{C_1+D_1v+i\phi\ln\left(\frac{S}{K}\right)}\right)d\phi.
\end{aligned}$$

Similarly,

$$\frac{\partial P_2}{\partial S} = \frac{1}{S\pi}\int_0^\infty \left(e^{C_2+D_2 v+i\phi\ln\left(\frac{S}{K}\right)}\right) d\phi.$$

For Vega, it is worth noting that v is the variance, not standard variation. With $z=\sqrt{v}$,

$$\begin{aligned} C_z &= S\tfrac{\partial P_1}{\partial z} - Ke^{-r(T-t)}\tfrac{\partial P_2}{\partial z}, \\ P_z &= C_z. \end{aligned}$$

in which

$$\frac{\partial P_1}{\partial z} = \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-i\phi\ln(K)+C_1+D_1 z^2+i\phi\ln(S)}}{i\phi}2D_1 z\right] d\phi = \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{C_1+D_1 v+i\phi\ln\left(\frac{S}{K}\right)}}{i\phi}2D_1\sqrt{v}\right] d\phi$$

and

$$\frac{\partial P_2}{\partial z} = \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-i\phi\ln(K)+C_2+D_2 z^2+i\phi\ln(S)}}{i\phi}2D_2 z\right] d\phi = \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{C_2+D_2 v+i\phi\ln\left(\frac{S}{K}\right)}}{i\phi}2D_2\sqrt{v}\right] d\phi.$$

GMM Estimation of PVR

Let $y = \frac{\Delta[C+P-(C_s+P_s)S]-r[C+P-(C_s+P_s)S]\Delta t}{(C_v+P_v)\sigma\sqrt{V\Delta t}}$ and $x=\frac{\sqrt{V\Delta t}}{\sigma}$. Assuming the sample size is N, the instrumental variable is $z=\begin{bmatrix} x & 1\end{bmatrix}^T$ and the kernel is $\hat{W}$.

$$\begin{aligned} \hat{\lambda}^* &= \left(S_{zx}^T\hat{W}S_{zx}\right)^{-1} S_{zx}^T\hat{W}S_{zy} \\ \mathrm{Var}(\hat{\lambda}^*) &= \frac{1}{N}\left(S_{zx}^T\hat{W}S_{zx}\right)^{-1} S_{zx}^T\hat{W}\hat{S}\hat{W}S_{zx}\left(S_{zx}^T\hat{W}S_{zx}\right)^{-1}. \end{aligned}$$

Using the efficient GMM estimator, $\hat{W}=\hat{S}^{-1}=\left(Z^TZ/N\right)^{-1}$. Then

$$\begin{aligned} \hat{\lambda}^* &= \left(S_{zx}^T\hat{S}^{-1}S_{zx}\right)^{-1} S_{zx}^T\hat{S}^{-1}S_{zy} \\ \mathrm{Var}(\hat{\lambda}^*) &= \frac{1}{N}\left(S_{zx}^T\hat{S}^{-1}S_{zx}\right)^{-1}. \end{aligned}$$

where $S_{zx}=E\left(zx\right)=Z^TX/N$ and $S_{zy}=E\left(zy\right)=Z^TY/N$. $S=E\left(zz^T\xi^2\right)$.

To determine the parametric constraints, we start from their empirically feasible ranges. Because the model is highly nonlinear, the calibration is potentially sensitive to such constraints. Therefore, we perform sensitivity analysis by slightly adjusting the lower and upper bounds, and find the constraints above yield the smallest total mean absolute deviation. After the calibration, we obtain an estimated parameter value for every observation. We can then estimate the pricing of volatility risk using GMM methods, for options with different maturities ranging from 30 days to 360 days, separately.