Optimal Redistribution with a Shadow Economy

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WORK IN PROGRESS

Abstract

We examine a constrained efficient allocation in the Mirrlees [1971] model with an informal labor market. There is a distribution of individuals with different earning potential and a social planner that wants to redistribute income. Individuals can supply labor to two labor markets: the formal and the informal one. Suppose that each agent is more productive in the formal labor market, yet the income from the informal market is not observable by the planner. In this framework, the highly productive agents should never work informally. However, sometimes the constrained efficient allocation has positive levels of informal labor supplied by the less productive agents. Specifically, the shadow economy may allow for high marginal tax rates at the low levels of income, which extract high revenue from the productive types. In the economy without informal sector such a solution may not be desirable, since it deters the labor supply of the less productive agents. The shadow economy is likely to exist in the constrained efficient allocation when the productivity loss from moving to the informal sector is sufficiently small for agents that are less productive formally. We estimate the productivity differences across sectors for Colombia, an economy with a large shadow economy, and quantify the optimal size of the shadow economy implied by the mechanism proposed. We find that at the constrained efficient allocation the shadow economy employs 30% of workers, which is a half of the actual size of the informal sector in Colombia. We also find that the optimal tax scheme is far more progressive than the Colombian tax policy.

1 Introduction

Informal activity, defined broadly as any endeavor which is not necessarily illegal but evades taxation, accounts for a large fraction of economic activity in both developing and developed economies. According to the OECD report (Jutting et al. [2009]), more then a half of jobs in the non-agricultural sector worldwide can be considered informal. The share of informal production in the GDP of high income OECD countries in the years 1999-2007 was estimated as 13.5% (Schneider et al. [2011]). As

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such, the informal sector should be considered in the design of fiscal policy. This paper attempts to shed light on the implications of informal labor markets in the classic model of income redistribution by Mirrlees [1971]. The planner wants to extract resources from the productive, rich agents and transfer them to the unproductive poor. Individuals can substitute formal, taxable income with hidden income from the less efficient informal employment. If the planner knows the individual productivities, lump sum taxation is used and no agent supplies the less productive, informal labor. When the planner does not know the individual productivities, distortionary taxation is part of the optimal allocation and informal labor might emerge.

The welfare consequences of the existence of a shadow economy are not obvious. The shadow economy distorts incentives of workers, by providing an alternative income source that affects the outside option for taxable income. This is an additional layer over the information rents considered in the standard Mirrlees model. Intuitively, if the shadow economy strengthens the outside option of the productive agents, it constrains redistribution even more than the standard Mirrlees model. On the other hand, the shadow economy may weaken the outside option of the productive agents, allowing for more redistribution.

To illustrate the novel channel that the existence of the shadow economy brings to the optimal taxation analysis, consider a reform of an increase in the average tax paid at high levels of income. Such a policy can make the very productive workers work less, have low income (with a lower average tax rate) and more leisure. This reaction of workers to the change in taxes might even lower revenues. One way to ameliorate such reaction is to raise the marginal tax rates at the low rather than high levels of income. The high marginal tax rates at the low income levels motivate the productive agents to work hard, but they also discourage the labor supply of the less productive workers. This is the famous efficiency-equity trade-off of the Mirrlees model.

The shadow economy modifies this trade-off in two ways. First, it changes the efficiency cost of discouraging the *formal* work of the *formally* less productive agents. If these workers lose little earning potential by moving to the shadows, the efficiency cost is going to be low. Second, the shadow economy changes the utility that workers which are highly productive *formally* can obtain at the low levels of *formal* income. If they are also very productive in the shadow economy, they can complement their lower formal income with the shadow earnings. The relative strength of these two effects determines whether the shadow economy limits or improves the possible redistribution.

We explain the intuition behind our main findings with a simple model of two types and a Rawlsian planner. We derive a sharp condition for the optimality of the shadow economy. The informal sector can be a part of the optimum only if the type less productive formally has a *comparative* advantage in shadow labor. When this type has an *absolute* advantage in shadow labor over the other type and is sufficiently scarce, placing these agents in the shadow economy will lead to the welfare improvement in comparison to the optimum of the standard Mirrlees model.

It is hard to match the diversity of the actual economies with just two types. In order to make our model applicable in policy making, we explain how to solve the model with a continuum of types. We derive the full set of optimality conditions, including the conditions for optimal bunching of types at the kinks of the tax schedule. Perhaps surprisingly, we find that the condition for a given type to optimally work in the shadow economy is almost identical to the analogous condition from the simple model. It optimal for a given type to work in the shadow economy if the productivity loss from doing so is low, the type has low density in comparison to the fraction of types above and the planner wants to efficiently tax the agents with higher income.

Finally we take our model to the data. Using Colombian data the three key objects of the theoretical model are estimated: formal productivity, the informal productivity and the probability distribution of workers at the different productivity levels. Then we proceed to quantify the optimal size of the shadow economy implied by the model. We find that at the constrained efficient allocation the shadow economy employs 30% of workers. It is a sizable number, but still a half of the actualsize of the informal sector we measure in the data. We also find that the optimal tax scheme is far more progressive than the Colombian tax policy.

Below we discuss the related literature. On the next few pages we illustrate the main ideas of the paper with the simple model. In Section 3 we explain how to solve the model with a large number of types and general social preferences. In the next two sections we investigate the empirical shape of productivity distributions of Colombia and derive the optimal Colombian tax schedule.

Related literature Tax avoidance has been studied at least since Allingham and Sandmo [1972]. For us, the most relevant paper from this literature was written by Kopczuk [2001]. He considers the general model with two-dimensional heterogeneity in productivity and tax avoidance skills. He shows that optimally no one evades taxes if all agents have the same tax avoidance ability. Moreover, he gives an example of tax avoidance at the optimum, when the less productive agents are better in avoiding taxes than others. The general model of Kopczuk [2001] includes our shadow economy model as a special case. The tax avoidance skill is captured by the ratio of the shadow and formal productivity. The quick implication is that when no agent has a comparative advantage in the shadow employment, optimally no agent will work informally. In contrast to Kopczuk [2001], we provide a sharp characterization of when it is optimal to use the shadow economy at the optimum. We also show that this theory is relevant quantitatively in the case of the Colombian personal income tax.

Ales and Maziero [2009] describe the dynamic Mirrlees model in which the agents face a sequence of stochastic productivity shocks and engage in employment with non-exclusive contracts. The authors show that possibility of having a job on the side erodes firms' ability to insure their employees - the equilibrium is a self-insurance allocation. Their model is equivalent to the limiting case of the model with shadow economy, in which the agents are equally productive in formal and informal employment. Then the planner cannot extract any resources from the productive types, since those agents can costlessly mimic formal market behavior of the less productive agents. Alvarez-Parra and Sánchez [2009] study the optimal unemployment insurance with moral hazard in search effort and informal labor market. Hidden employment limits the ability of the planner to use a decreasing future consumption in order to motivate agents to search hard for a job. After a sufficiently long unemployment spell, it is impossible to provide incentives for search, so agents are allowed to stay in the shadow economy. It is another environment with information frictions, in which the informal employment is utilized in the optimal allocation. Stantcheva [2014] presents another framework in which labor market frictions improve the ability of the planner to redistribute income. In her setting, in addition to the Mirrlees model, employers do not know workers' productivities. Workers are screened by both firms and the planner, which restricts their ability to mimic other types. In comparison to Stantcheva [2014], in our setting workers are screened only once, but the planner can choose whether to screen them with formal, or informal productivities.

2 Simple model

Imagine an economy inhabited by people that share preferences but differ in productivity. There are two types of individuals, indexed by letters l and h, with population shares μ_l and μ_h. They all care about consumption c and labor supply n according to the utility function

$$U(c,n) = c - v(n). \tag{1}$$

We assume that v is increasing, strictly convex and twice differentiable. We also impose that $v'(0) = 0$ and $\lim_{n\to\infty} v'(n) = \infty$.

There are two labor markets and, correspondingly, each agent is equipped with two linear production technologies. An agent $i \in \{l, h\}$ produces with productivity ϕ_i in a formal labor market, and with productivity ψ_i in an informal labor market. Type h is more productive in the formal market than type l: $\phi_h > \phi_l$. Moreover, in this section we assume that each type's informal productivity is lower than formal productivity. We relax this assumption when we consider the full model.

$$\forall_i \ \phi_i > \psi_i. \tag{2}$$

Any agent may work formally, informally, or in both markets simultaneously. An agent of type i works n_i hours in total, which is the sum of n_i^f hours at the formal job and n_i^s hours in the shadow economy. The formal and the informal income, denoted by y_i^f and y_i^s respectively, is a product of the relevant productivity and the relevant labor supply. The allocation of resources may involve transfers across types, so one's consumption may be different than the sum of formal and informal income. In order to capture these flows of resources, we introduce a tax T_i, equal to the gap between total income and total consumption

$$T_i \equiv y_i^f + y_i^s - c_i. \tag{3}$$

A negative tax is called a transfer, and we are going to use these terms interchangeably.

The social planner follows John Rawls' theory of justice and wants to improve the well-being of the least well-off agents,[1] but is limited by imperfect knowledge. The planner knows the structure and parameters of the economy, but, as in the standard Mirrlees model, does not observe the type of any individual. In addition, the informal income and the informal labor are unobserved by the planner as well. Consequently, the only individual variables the planner sees and can directly verify

[1]We pick this particular point of the Pareto frontier because it allows us to show the interesting features of the model with relatively easy derivations. At the end of this section we discuss how other constrained efficient allocations look like.

are the formal income y_i^f and the tax T_i. We can think about y_i^f and $y_i^f - T_i$ as a pre-tax and an after-tax reported income.

The planner maximizes the Rawlsian social welfare function, given by a utility level of the worst-off agent

$$\max_{\left\{T_i, n_i^f, n_i^s\right\}_{l,h} \in \mathbb{R}_+^6} \min\left\{U\left(c_l, n_l\right), U\left(c_h, n_h\right)\right\}, \tag{4}$$

subject to a resource constraint

$$\sum_{\{l,h\}} \mu_i \left(y_i^f + y_i^s - c_i\right) \geq 0 \tag{5}$$

and incentive-compatibility constraints

$$\forall_{i,i' \in \{l,h\}}\; U\left(c_i, n_i\right) \geq U\left(y_{i'}^f - T_{i'} + \psi_i n_{i,i'}^s, \frac{y_{i'}^f}{\phi_i} + n_{i,i'}^s\right), \tag{6}$$

$$\text{where } n_{i,i'}^s \equiv \arg\max_{n^s} U\left(y_{i'}^f - T_{i'} + \psi_i n^s, \frac{y_{i'}^f}{\phi_i} + n^s\right).$$

We denote this generic incentive constraint by $IC_{i,i'}$. When $i = i'$, it means that an agent i cannot be better off by changing only the informal labor. When $i \neq i'$, it means that an agent i cannot be better off by changing formal income to $y_{i'}^f$ and simultaneously adjusting the informal labor. All four incentive constraints $\{IC_{i,i'}\}_{i,i' \in \{l,h\}}$ guarantee that no agent will deviate in any way from the planner's allocation. However, it is easy to show that both $IC_{l,l}$ and $IC_{h,h}$ are redundant. What matters for the planner is that agents pay taxes according to their true type. If they also work informally on the side, it will not reduce the social welfare, since the agents are individually rational.

Lemma 1. *Incentive constraints* $\{IC_{i,i}\}_{i \in \{l,h\}}$ *never bind.*

2.1 First-best

What if the planner is omniscient and directly observes all variables? The planner knows types and can choose the shadow labor supply directly. The optimal allocation is a solution to the welfare maximization problem (4) subject only to the resource constraint (5). All types are more productive in the formal sector than in the shadow economy, so no agent will work informally. Each agent will supply the formal labor efficiently, equalizing the marginal social cost and benefit of working. Moreover, the planner redistributes income from h to l in order to achieve the equality of well-being.

Proposition 1. *In the first-best both types work only formally and supply an efficient amount of labor:* $\forall_i v'\left(n_i\right) = \phi_i$. *Utility levels of the two types are equal:* $U\left(c_l, n_l\right) = U\left(c_h, n_h\right)$.

We can slightly restrict the amount of information available to the planner without affecting the optimal allocation. Suppose that the planner still observes the formal productivity, but informal labor supply and informal income are hidden. The only friction comes from the existence of the informal sector. The optimal allocation is a solution to (4) subject to the resource constraint (5) and two of the incentive compatibility constraints from (6), $IC_{l,l}$ and $IC_{h,h}$. However, Lemma (1) tells us that these incentive constraints do not matter.

Corollary 1. *If the planner knows types, but does not observe shadow economy variables, the optimal allocation is the first-best.*

When the types are known, lump-sum taxation is feasible and the planner can implement the first-best. Without additional frictions, the shadow economy does not constrain the social planner.

2.2 Second-best

Let's consider the problem in which neither type nor informal activity is observed. The planner solves (4) subject to the resource constraint (5) and all the incentive constraints (6). We call the solution to this problem the second-best or simply an optimum.

In the first-best, both types work only on the formal market and their utilities are equal. If h could mimic the other type, higher formal productivity would allow h to increase utility. Hence, the first-best does not satisfy $IC_{h,l}$ and this constraint limits the welfare at the optimum. On the other hand, $IC_{l,h}$ never binds at the optimum. It would require the redistribution of resources from l to h, which is clearly suboptimal.

Proposition 2. *The optimum exists and is not the first-best. $IC_{h,l}$ is binding, while $IC_{l,h}$ is slack.*

Let's introduce marginal tax rate t_i, which describes the slope of the tax schedule at the level of formal income y_i^f.[2] By varying the marginal tax, the planner controls the formal labor supply according to an individual optimality condition

$$\min \left\{ v'\left(n_i^f + n_i^s\right) - (1 - t_i)\,\phi_i, n_i^f \right\} = 0. \tag{7}$$

This condition says that the marginal cost of working $v'(n_i)$ equals the marginal benefit from working formally $(1 - t_i)\,\phi_i$. Equation (7) also allows for $v'(n_i) > (1 - t_i)\,\phi_i$ and n_i^f equal zero. This can happen only if i works in the shadow economy. Note that we can derive similar condition for the shadow labor

$$\min \left\{ v'\left(n_i^f + n_i^s\right) - \psi_i, n_i^s \right\} = 0. \tag{8}$$

This condition is not affected by the tax schedule directly, since informal labor cannot be taxed. However, the marginal tax rate affects formal labor n_i^f according to (7), and consequently changes the marginal disutility of labor. When t_i is smaller than $\frac{\phi_i - \psi_i}{\phi_i}$, the marginal benefit from working formally $(1 - t_i)\,\phi_i$ is higher than the marginal benefit from working informally ψ_i. The agent will not supply any informal labor. In other words, the marginal tax rates have to be high enough for agents to work in the shadow economy.

[2] We use terminology of marginal tax rates rather than write about the abstract wedges, because in this model the decentralization of the planner's solution with the labor income tax is unambiguous.

Marginal rates different than zero distort the choice of formal labor away from the efficient level. By Proposition 1, the first-best is free of such distortions and is implemented with flat, type-specific tax schedules. In the second-best the tax schedule cannot be type specific, since the types are unknown. In this case, differential treatment of different types requires that the total tax T is changing with formal income. As a result, the second-best tax schedule exhibits some non-zero marginal rates and is distortionary.

We can also think about the marginal tax rates as instruments to manage incentives. If type h is tempted to mimic l, the planner makes such a deviation less appealing by decreasing the formal labor of type l. Since the advantage of h comes from higher formal productivity, lower formal labor of l reduces to opportunity for h to use this advantage and shrinks gains from the deviation. Proposition 2 tells us that no agent wants to mimic type h, hence the planner has no reason to distort the labor choice of this type. The marginal tax of h is zero, which means that the net wage in the formal sector is greater than the return from the shadow labor. The classic result of zero marginal tax rate at the top implies here that h will work only formally.

Corollary 2. *Type h faces no distortions and never works in the shadow economy.*

On the other hand, the planner can improve social welfare by introducing a positive marginal tax of type l. The higher marginal tax rate of l implies the higher absolute tax paid by type h, resulting in more resources for redistribution. The cost of such reform consists in the reduced labor supply of l. In the standard Mirrlees model the planner sets the marginal rate on the level which equalizes the benefit of redistribution and the cost of distortions. Note that in the model with the shadow economy, agents can avoid tax distortions by moving to the informal sector. In other words, pushing type l to the shadow economy may reduce the cost of redistribution. In the following propositions we investigate when it is optimal to do so.

Proposition 3. *At the optimum, no agent works in the shadow economy if $\Delta\frac{\psi}{\phi} \equiv \frac{\psi_h}{\phi_h} - \frac{\psi_l}{\phi_l} \geq 0$.*

The necessary condition for l to work in the shadow economy is a comparative advantage of type l over h in informal employment: $\Delta\frac{\psi}{\phi} < 0$. If h is relatively better than l in informal labor, then sending l to the shadow economy will make it easier for h to deviate.[3] We will see that the comparative advantage condition is crucial in the full model.

In the proofs of the two propositions below we exploit the optimality condition that is derived in the Appendix 2 (see Lemma 6). In order to make sure that this condition is well behaved, we require that v'' is nondecreasing.[4]

Proposition 4. *Suppose that v'' is nondecreasing. Define $\bar{n}$ by $v'(\bar{n}) = \psi_l$.*

1. *Suppose that $\psi_h \geq v'\left(\frac{\phi_l}{\phi_h}\bar{n}\right)$. Then l optimally works in the shadow economy* ***if and only if***

$$1 \geq \frac{\phi_l - \psi_l}{\phi_l}\left(-\Delta\frac{\psi}{\phi}\right)^{-1}\frac{\mu_l}{\mu_h}. \tag{9}$$

[3] We can think about the mechanism design interpretation of the planner's problem. The planner uses shadow economy to facilitate screening of types. If the relative differences between types are not magnified in the shadow sector, there is no point in using the inferior, informal production.

[4] In the canonical case of isoelastic utility, it means that the elasticity of the labor supply is not greater than 1.

2. *Suppose that* $\psi_h < v'\left(\frac{\phi_l}{\phi_h}\bar{n}\right)$. *Then* l *optimally works in the shadow economy* ***if***

$$\frac{\psi_l}{\phi_l} - \frac{v'\left(\frac{\phi_l}{\phi_h}\bar{n}\right)}{\phi_h} \geq \frac{\phi_l - \psi_l}{\phi_l}\frac{\mu_l}{\mu_h}. \tag{10}$$

When ψ_h is greater or not much lower than ψ_l, the inequality (9) tells us when exactly it is optimal to push l into informal employment. This inequality is satisfied if the comparative advantage of l over h in informal employment is higher than the product of two ratios on the right-hand side. The first ratio is a relative productivity loss of l from moving to informal employment. If it is small, then even a modest marginal tax rate is enough to push l to the informal sector. The second is a ratio of population shares. It allows us to compare the aggregate cost of tax distortions with the aggregate benefit from redistribution. If $\frac{\mu_l}{\mu_h}$ is high, the planner would have to distort many to tax only a few, so it is unlikely that l will be employed informally.

If ψ_h is substantially lower than ψ_l, the informal employment of l is optimal when (10) is satisfied. Notice that the informal productivity of h no longer matters. If h mimics l and n_l^s is small, h will not engage in the shadow employment. Hence, the planner's decision whether to marginally increase n_l^s above zero is not influenced by ψ_h. Instead, it depends on a marginal rate of substitution of h, evaluated at the level of formal income of l: $v'\left(\frac{\phi_l}{\phi_h}\bar{n}\right)$. Note that (10) is not a necessary condition for l to work in the shadow economy.

In the Proposition 5 we conduct a welfare comparison of the optimal allocation in the model with and without the informal sector. We can think about the standard Mirrlees model as a special case of our model, in which both ψ_l and ψ_h are equal 0.

Proposition 5. *The optimum welfare is weakly lower than in the standard Mirrlees model if optimally* $n_l^s = 0$.

Suppose that v'' *is nondecreasing.*

1. *The optimum welfare is strictly lower than in the standard Mirrlees model if either* $\psi_h \geq \phi_l$ *or* $\frac{\phi_h - \phi_l}{\phi_l - \psi_h}\frac{\psi_h}{\phi_h} \geq \frac{\mu_l}{\mu_h}$.
2. *The optimum welfare is strictly higher than in the standard Mirrlees model if* $\psi_l > \psi_h$ *and* μ_l *is low enough.*

Intuitively, the shadow economy can be welfare deteriorating, if it enhances the ability of the actual taxpayers to evade taxation. It happens when ψ_h is sufficiently high. More interestingly, the shadow economy may improve welfare. Suppose that μ_l is low, which means that the planner will strongly distort the formal labor supply of l. In the model without shadow economy, type l will be mostly idle. However, if l can work in the shadow sector, this type will seek informal employment. When ψ_l is greater than ψ_h, it is l type that gains the most from the existence of the shadow economy. Consequently, the utility l type receives in the model with shadow economy is greater than in the standard Mirrlees model. This utility gain comes from greater redistribution. It means that the shadow economy can be welfare improving, however it will never be Pareto improving.

3 Full model

In this section we describe how to find an optimal tax schedule in an economy with a large number of types. We are going to focus on the preferences without the wealth effect. Agents' types are distributed on the segment $[0,1]$ according to a density μ_i and a cumulative density M_i. The density μ_i is atomless. We assume that formal and informal productivities are differentiable functions of type and denote these derivatives by $\dot{\phi}_i$ and $\dot{\psi}_i$. We sort types such that the formal productivity is increasing in i.

Agent's preferences are described by the utility function $U(c,n) = c - v(n)$, where v is increasing, strictly convex and twice differentiable. Let $V_i\left(y^f, T\right)$ be an indirect utility function of an agent of type i whose declared income is y^f and who pays a tax T:

$$V_i\left(y^f, T\right) \equiv \max_{y^s \geq 0} y^f + y^s - T - v\left(\frac{y^f}{\phi_i} + \frac{y^s}{\psi_i}\right). \tag{11}$$

We denote the partial derivatives of this function by V_y and V_T. We can write V in a more explicit manner

$$V_i\left(y^f, T\right) = y^f + \psi_i n_i^s - T - v\left(\frac{y^f}{\phi_i} + n_i^s\right), \ \min\left\{v'(n_i) - \psi_i, n_i^s\right\} = 0. \tag{12}$$

Whenever distortions imposed on some agent are sufficiently small, no informal labor is supplied. However, when $v'\left(n_i^f\right) < \phi_i$, agent seeks informal employment.

The planner maximizes a weighted, concave transformation of agent's utilities

$$\max_{\left(y_i^f, T_i\right)_{i=0}^1} \int_0^1 \lambda_i G\left(V_i\left(y_i^f, T_i\right)\right) d\mu_i, \tag{13}$$

where G is increasing, concave and differentiable, while the welfare weights $(\lambda_i)_{i=0}^1$ integrate to 1. The distribution of welfare weights is atomless.[5]

The planner is constrained by available resources, where E denotes some fixed expenditures

$$\int_0^1 T_i d\mu_i \geq E. \tag{14}$$

Moreover, the tax schedule has to satisfy the incentive-compatibility

$$\forall_{i,i' \in [0,1]} V_i\left(y_i^f, T_i\right) \geq V_i\left(y_{i'}^f, T_{i'}\right), \tag{15}$$

which means that no agent can gain by mimicking any other type. The allocation which solves (13) subject to (14) and (15) is called the second-best or the optimum.

[5] It's easy to relax this assumption and we are going to do it in the quantitative exercise, where we consider the Rawlsian planner.

3.1 Incentive-compatibility

In the standard Mirrlees model, a single crossing property of agents' preferences allows the planner to focus only on local incentive compatibility constraint. The single-crossing means that if we fix the marginal tax rate, then a higher type is willing to work more than the lower type. In the context of our model it implies that the formal income selected by an agent is a nondecreasing function of type.

Lemma 2. *The indirect utility function V satisfies the single crossing condition if* $\frac{d}{dt}\left(\frac{\psi_t}{\phi_t}\right) < 0$.

The single-crossing holds when the agents with lower formal productivity have a comparative advantage in working in the informal sector. From now onwards we are going to assume that it is the case. It allows us to replace the complicated incentive compatibility condition 15 with two simpler requirements.

Proposition 6. *The allocation* $\left(y_i^f, T_i\right)_{i=0}^1$ *is incentive-compatible if and only if* y_i^f *is nondecreasing in type and* $\frac{d}{dj}V_i\left(y_j^f, T_j\right)\Big|_{j=1} = 0$ *whenever* $\frac{d}{dj}y_j^f\Big|_{j=1}$ *exists. The utility schedule* $V_i \equiv V_i\left(y_i^f, T_i\right)$ *of the incentive compatible allocation is continuous everywhere, differentiable almost everywhere and for any* $i < 1$ *can be expressed as*

$$V_i\left(y_i^f, T_i\right) = V_0\left(y_0^f, T_0\right) + \int_0^i \dot{V}_j dj, \tag{16}$$

where

$$\dot{V}_i \equiv \left(\frac{\dot{\phi}_i}{\phi_i} n_i^f + \frac{\dot{\psi}_i}{\psi_i} n_i^s\right) v'\left(n_i\right). \tag{17}$$

The single crossing implies that higher types choose higher formal income, hence assigning a lower income to a higher type would violate incentive compatibility. Moreover, it is enough to focus just on local deviations: no agent should be able to improve utility by marginally changing the formal income. Note that the formal income may be, and often will be, discontinuous in type. Nevertheless, the indirect utility function preserves some smoothness and can be expressed as an integral of its marginal increments, which will prove useful soon.

Let's call $\dot{V}_i$ the *marginal information rent* of type i. It tells us how the utility level changes with type. The higher is the average rate of productivity growth, weighted by the labor inputs in two sectors, the faster utility increases with type. We will use perturbations in $\dot{V}$ to derive optimality conditions.

3.2 Optimality conditions

Doligalski [2015] derives the full set of optimality conditions in the standard Mirrlees model, including the optimal income tax kinks.[6] We extend his approach to the model with the shadow

[6] The latest version of the paper can be found here: https://dl.dropboxusercontent.com/u/19338650/papers/optimal_kinks.pdf

economy. The shadow economy makes it much more likely that the optimal allocation will involve bunching of different types at the kinks of the income tax schedule.

We obtain the interior optimality conditions by assuring that the social welfare cannot be improved by perturbing the marginal information rent of any type.[7] Decreasing the slope of the utility schedule at type i results in an increase of tax distortions of this type, which is costly. On the other hand, by (16) the perturbation shift downwards the entire utility schedule above type i, which is equivalent to increasing a nondistortionary tax. The optimality condition balances the cost of distortions with the benefit of efficient taxation. The shadow economy enters the picture by affecting the cost of increasing distortions of agents that work informally. If the income schedule implied by the interior optimality conditions is nondecreasing, then, under the mild sufficiency conditions listed in Theorem 1, the interior allocation is optimal.

On the other hand, if the interior optimality conditions result in the locally decreasing income schedule, the interior conditions fail to be even necessary for the optimum, since they lead to the allocation that is not incentive-compatible. The optimal allocation involves bunching different types at the kinks of the tax schedule. The optimal kinks balance the cost of distortions of bunched agents with the benefit of nondistortionary taxation of all agents above the kink. The shadow economy makes the interior optimality conditions more likely to imply a locally decreasing income schedule. Whenever it is optimal for a given agent to supply some informal labor, interior conditions imply that the formal income of this agent should be reduced to zero. Since it can easily happen for agents in the middle of the formal income distribution, the shadow economy is likely to produce kinks in the optimal income tax.

Interior optimality conditions The benefit of shifting the utility schedule of type j without affecting its slope is given by the standard expression

$$N_j \equiv (1 - w_j)\,\mu_j, \text{ where } w_j = \frac{\lambda_j}{\eta} G'\left(V_j\left(y_j^f, T_j\right)\right). \tag{18}$$

A marginal increase of nondistortionary taxation leads on the one hand to one-to-one increase of tax revenue. On the other hand, it reduces the social welfare, since the utility of type j falls. This impact is captured by the marginal welfare weight w_j. We assumed that there are no wealth effects, so the nondistortionary tax does not affect the labor choice of agents. It means that this term does not depend on whether type j is working informally.

The cost of distorting some agent's allocation depends on the involvement of this agent in the shadow activity. Let's group types into three sets:

$$\begin{array}{lll} \text{formal workers} & \mathcal{F} = \left\{i \in [0,1] : v'\left(n_i^f\right) > \psi_i\right\} & t_i < \frac{\phi_i - \psi_i}{\phi_i} \\ \text{marginal workers} & \mathcal{M} = \left\{i \in [0,1] : v'\left(n_i^f\right) = \psi_i\right\} & t_i = \frac{\phi_i - \psi_i}{\phi_i} \\ \text{shadow workers} & \mathcal{S} = \left\{i \in [0,1] : v'\left(n_i^f\right) < \psi_i\right\} & t_i \geq \frac{\phi_i - \psi_i}{\phi_i}. \end{array}$$

[7]Brendon [2013] was the first to use this approach in the Mirrlees model.

The formal workers supply only the formal labor and their net marginal wage $(1 - t_i)\,\phi_i$ is strictly higher than what they could get in the shadow economy. The marginal workers also supply only the formal labor, but their net marginal wage is equal to their informal productivity. Finally, the shadow workers are employed informally, although they can also supply some formal labor.

The formal workers act exactly as agents in the standard Mirrlees model. The cost of increasing their distortions is given by

$$D_i^f \equiv \frac{t_i}{1 - t_i} \left(\frac{\dot{\phi}_i}{\phi_i} \left(1 + \frac{1}{\zeta_i} \right) \right)^{-1} \mu_i. \tag{19}$$

The cost of increasing distortions depends positively on the marginal tax rate of this type. The marginal tax rate tell us how strongly the reduction of formal income of a given type influences the tax revenue. Moreover, the cost increases with the elasticity of labor supply and is proportional to the density of the distorted type.

The perturbation of the marginal information rent works differently for the shadow workers. They supply shadow labor in the quantity that satisfies $v'\left(n_i^f + n_i^s\right) = \psi_i$, which means that their total labor supply n_i is constant. By distorting the formal income, the planner just shift the labor of these types from the formal to the informal sector. As a result, the cost of increasing distortions does not depend on the elasticity of labor supply, but rather on the sectoral productivity differences

$$D_i^s \equiv \frac{\phi_i - \psi_i}{\psi_i} \left(\frac{\dot{\phi}_i}{\phi_i} - \frac{\dot{\psi}_i}{\psi_i} \right)^{-1} \mu_i. \tag{20}$$

The first term, the relative productivity difference between formal and informal sector, is just $\frac{t_i}{1-t_i}$ evaluated the the marginal tax rate of the shadow worker. The second term describes how effective the planner can manipulate the agent's marginal information rent by discouraging the formal labor supply. By the single-crossing assumption, this term is always positive. The density μ_i aggregates the expression to include all agents of type i. Note that this term depends only on the fundamentals of the economy.

The marginal workers are at a tightrope between their formal and shadow colleagues. If the planner marginally reduces their income, they become the shadow workers. The the planner lifts distortions, they join the formal workers. The cost of changing distortions of these types depends on the direction of perturbation and is equal to either D_i^f or D_i^s.

Having all the cost and benefit terms ready, we can derive the interior optimality conditions. In the optimum, the planner cannot increase the social welfare by varying the marginal information rents of any type. For the formal workers, it means that

$$\forall_{i \in \mathcal{F}}\ D_i^f = \int_i^1 N_j dj. \tag{21}$$

It is the standard optimal tax rate formula derived by Diamond [1998].

For the marginal workers it must be the case that increasing tax distortions is beneficial as long as they work only formally, but it is too costly when they start to supply the shadow labor. Their marginal tax rate has to equalize the return to labor in the two sectors:

$$\forall_{i\in\mathcal{M}}\ D_i^s > \int_i^1 N_j dj \geq D_i^f. \tag{22}$$

Recall that the cost of distorting the shadow worker is fixed. Moreover, the benefit of distorting one particular worker, given by (18), is fixed as well, since the perturbation of the marginal information rent of i has an infinitesimal effect on the allocation of types above. Hence, if the planner finds it optimal to decrease the formal income of agent i so much that i starts supplying informal labor, it will be optimal to decrease the formal income even further, until i works only in the shadow economy:

$$\forall_{i\in\mathcal{S}}\ \int_i^1 N_j dj \geq D_i^s,\ y_i^f = 0. \tag{23}$$

The conditions above determine the slope of the utility schedule at each type. What is left is finding the optimal level. Suppose that the planner vary the tax paid by the lowest type, while keeping all the marginal rates fixed. Optimum requires that such perturbation cannot bring welfare improvement:

$$\int_0^1 N_j dj = 0. \tag{24}$$

The conditions we derived above are meaningful only if they do not violate the incentive-compatibility, i.e. lead to income schedule which is never decreasing. They become sufficient, if they pin down a unique allocation. This happens when the cost of distortions is increasing in the amount of distortions imposed, which is implied by the regularity conditions listed in the Theorem 1.

Theorem 1. *If the conditions (21)-(24) always imply the formal income schedule that is nondecreasing in type, they are the necessary optimality conditions. If in addition the elasticity of labor supply is nonincreasing in labor and* $\forall_i \frac{\dot{\phi}_i}{\phi_i} + \frac{\dot{\psi}_i}{\psi_i}\zeta_i \geq 0$, *they are the sufficient optimality conditions.*

When is the implied formal income schedule nondecreasing? We cannot say much about the income schedule of the formal workers. If the distortion cost (19) falls rapidly relative to the benefit, the interior optimality condition can imply a fall in formal income. But we have the sharp characterization of other types of workers. The marginal workers face the tax rate which equates the return to labor in the two sectors. If the shadow productivity does not fall too rapidly in type, the formal income of these agents will be weakly increasing. Finally, by (23) the shadow workers should not work formally at all. Hence, if some shadow worker have higher type than any formal or marginal worker, the income schedule will be decreasing.

Lemma 3. *The formal income schedule* y^f *implied by conditions (21)-(24) is nondecreasing on set* M *if and only if* $\forall_i \frac{\dot{\phi}_i}{\phi_i} + \frac{\dot{\psi}_i}{\psi_i}\zeta_i \geq 0$. *The schedule* y^f *is nondecreasing on the closure of set* $\mathcal{S}$ *if and only if* $\mathcal{S}$ *is convex (possibly empty) and located at the bottom of the type space.*

Optimal kinks Whenever conditions (21)-(24) prescribe the formal income schedule that is decreasing for some type, they fail to be even necessary. Under the single-crossing assumption we have made, the decreasing formal income is a violation of the incentive compatibility. The condition fail to be necessary not only for the set of types for which the formal income is decreasing, but also for types *above* and *below* this set.

Suppose that (21)-(24) imply an income schedule $\bar{y}^f$, which is decreasing on the set of types $\left[\bar{a}, \bar{b}\right]$ and nondecreasing elsewhere. Since the allocation of types $\left(\bar{a}, \bar{b}\right]$ violates the incentive compatibility, it has to be changed. We can simply lift the formal income schedule such that it becomes flat at $\left[\bar{a}, \bar{b}\right]$. Note that then we have to lift the formal income of some types above $\bar{b}$, in order to make sure that the schedule is not decreasing there. Let's say that $\bar{c}$ is the first type above $\bar{b}$ for which $\bar{y}^f_{\bar{c}} = \bar{y}^f_{\bar{a}}$. Then this correction of the income schedule involves all types $(\bar{a}, \bar{c})$. After the correction, the allocation of these types clearly does not satisfy conditions (21)-(24). Let's call the "corrected" formal income schedule $\tilde{y}^f$. Note that the types $[\bar{a}, \bar{c}]$ are bunch together at the kink of the income schedule corresponding to $\tilde{y}^f$. In other words, the marginal tax rate increases discontinuously at the income level $\tilde{y}^f_{\bar{a}}$.

The new schedule $\tilde{y}^f$ is incentive compatible. However, in general it is not optimal. The formal income of types $[0, \bar{a}]$ is set according to conditions (21)-(24). These conditions do not take into an account the restriction of the formal income being nondecreasing which is binding for types directly above. Marginally decreasing $\bar{y}^f_{\bar{a}}$ allows the planner to decrease the formal income of all the constrained types $(\bar{a}, \bar{c})$, which is what the planner wanted to do in in the first place. An infinitesimal cost of distorting $\bar{a}$ results in a noninfinitesimal welfare by decreasing the gap between cost and benefit for the whole segment $(\bar{a}, \bar{c})$. It suggests that the planner will be willing to sacrifice the local optimality conditions of some types below (and including) $\bar{a}$ as well.

Below we apply the optimality condition with respect to the distortions at he kink, derived by Doligalski [2015]. As in the interior case, it balances the cost of distortions of agents at the given income level with the benefits of taxing efficiently agents above. However, there is a whole interval of types that we distort at the same time. Since we change their allocation in the identical way, we will not be able to keep the utility level of all of them constant. As a result, increasing distortions is costly both in terms of fiscal revenue and in terms of welfare.

Suppose that an interval of agents $[a, c]$ is bunched at the kink. Let's marginally decrease the formal income of agents $[a, c)$ these agents and adjust their total tax paid such that the utility of type a is unchanged. In this way we preserve the continuity of the utility schedule at a. This perturbation will decrease the utility level of all the other types at the kink. We will normalize the perturbation such that we obtain a unit change of the utility of the last agent at the kink. The total cost of this perturbation is given by

$$D^K_{a,c} \equiv \frac{M_c - M_a}{t_c - t_a} \left(t_a + \mathbb{E}\left\{\Delta MRS_j w_j \middle| c > j \geq a\right\}\right), \tag{25}$$

$$\text{where } \Delta MRS_j = \frac{v'(n_a)}{\phi_a} - \frac{v'(n_j)}{\phi_j}.$$

The factor outside the brackets is a normalization term. A unit decrease in the formal income of the bunched agents reduces the utility of the highest bunched type by $t_c - t_a$. We can interpret the expression within the brackets as an average impact of a unit perturbation in the formal income. The brackets contain two components: fiscal and welfare loss. The fiscal loss from reducing the formal income of each bunched agents is just the marginal tax rate below the kink. The welfare loss is an average marginal welfare weight corrected by a distance from type a in terms of a marginal rate of substitution. The larger is ΔMRS_j, the more type j suffers from the perturbation. If the kink appears because the bunched agents work in the shadow economy, the difference in rates of substitution becomes a difference in comparative advantages in shadow labor $\frac{\psi}{\phi}$. By the single crossing condition, this ΔMRS_j is increasing in j. Note that ΔMRS_c is just equal $t_c - t_a$, which explains the normalization term in front of the brackets. In order to aggregate this average effect, we multiply it by the fraction of types bunched at the kink.

The benefit of this perturbation comes from the efficient taxation of types above the kink and is the same as in the interior case. The optimality requires that

$$D^K_{a,c} = \int_c^1 N_j dj \tag{26}$$

holds, whenever types $[a, c]$ are bunched together at some positive formal income level.

Theorem 2. *The optimal allocation satisfies (24) and at every formal income level* $\bar{y}^f$

- *if there is a unique type* i *s.t.* $y_i^f = \bar{y}^f$*, then the relevant interior condition (21)-(23) holds for* i*,*
- *if there is a segment* I *such that* $\forall_{i \in I} y_i^f = \bar{y}^f$*, then (26) holds for* I*.*

Although we managed to characterize the full set of necessary optimality conditions, in general it is easier to understand the interior conditions. Below we show that the interior allocation, even if not incentive compatible, may be a good predictor of which agents will optimally work in the shadow economy.

Lemma 4. *The set of shadow workers from the interior allocation is a subset of* $\mathcal{S}$ *from the optimal allocation.*

3.3 Interpretation

Which agents should work in the shadow economy?

Lemma 5. *Type* i *optimally works in the shadow economy if*

$$\mathbb{E}\left\{ \bar{w} - w_j | \, j > i \right\} \geq \frac{\phi_i - \psi_i}{\phi_i} \left(-\frac{d}{di} \left(\frac{\psi_i}{\phi_i} \right) \right)^{-1} \frac{\mu_i}{1 - M_i}. \tag{27}$$

This condition is both necessary and sufficient if the interior allocation is incentive-compatible.

Inequality (27) compares the gains from efficient taxation of all types above i with the cost of sending type i to the shadow economy. Some type i is likely to optimally work in the shadow economy, if the planner puts a low marginal weight on the utility of types above, the relative productivity loss from moving to informal employment is low and the density of distorted types is low in comparison to the fraction of types above. Finally, the shadow employment is more likely if the comparative advantage of working in the shadow sector $\frac{\psi_i}{\phi_i}$ is quickly decreasing with type. It means that higher types have less incentives to follow type i into the shadow economy. Note that if the planner is Rawlsian and type i is higher then (or equal to) the worst-off type, then the left hand side equals 1 and the condition (27) is just a continuum equivalent of the analogous condition (9) from the simple model.

The optimal tax rates Let's focus on agents that are not bunched at the kinks of the tax schedule. These types never supply informal labor. The optimal tax formula is

$$\frac{t_i}{1-t_i} = \min\left\{\frac{\dot{\phi}_i}{\phi_i}\left(1+\frac{1}{\zeta_i}\right)\frac{1-M_i}{\mu_i}\mathbb{E}\left(1-\frac{\lambda_j}{\eta}G'\left(V_j\right)\middle| j>i\right), \frac{\phi_i-\psi_i}{\psi_i}\right\}. \tag{28}$$

Suppose that the productivity loss from joining the shadow economy $\frac{\phi_i-\psi_i}{\psi_i}$ is high enough. Then the tax rate should be set according to the formula derived by Diamond [1998]. The expectations operator describes average social preferences towards all types above. In general, the less the planner cares about increasing utility of the types above i, the higher t_i will be. Note that if the welfare weights increase with type or G is a strictly convex function, this term may become negative, leading to negative marginal tax rates, as explained by Choné and Laroque [2010]. Since the sign of the tax rate is ambiguous, below we write how the other terms influence the absolute value of tax rate. The optimal tax rate increases in absolute value when the growth rate of formal productivity with respect to type is high. If the planner is redistributive and types above i are much more productive than types below, it is optimal to set a high tax rate. The tax rate decreases with elasticity of labor supply, since it makes the affected agents more responsive to the tax changes. The ratio $\frac{1-M_i}{\mu_i}$ tells us how many agents will be taxed in a nondistortionary manner relative to the density of distorted agents. If this ratio is high, gains from increasing tax rates will be high as well, relative to the cost.

If the Diamond formula prescribes tax rates which are too high, the optimal tax rate will be set in order to equalize the return from formal and informal labor. This is the highest tax rate consistent with agents working in the formal sector.

Optimal bunching Bunching may arise at the bottom of the formal income distribution, resulting in de facto exclusion from the formal labor market. Bunching may also appear at a positive level of formal income, which implies a kink in a tax schedule. All the shadow workers are subject to bunching, yet some formal workers can also be found at the kinks. The formal income schedule at which the kink is located is determined by

$$\frac{t_a}{t_c-t_a} = \frac{1-M_c}{M_c-M_a}\mathbb{E}\left\{\bar{w}-w_j\middle| j\geq c\right\} - \mathbb{E}\left\{\Delta MRS_j w_j\middle| c>j\geq a\right\}, \tag{29}$$

where a and c are respectively the lowest and the highest type bunched at the kink. Note that both t_a and t_c are set according to (28). The location of the kink is determined by the trade-off between tax and welfare losses from the bunched agents and the tax revenue gains from the efficient taxation of agents above the kink.

4 Shadow and formal productivities in the data

In this section we estimate the empirical counterparts of the three key objects in the model: formal productivity (ϕ_i), the informal productivity (ψ_i) and the distribution of types (μ_i). The estimation is conducted with data from Colombia, an economy with a large shadow economy[8].

In the model, ϕ_i and ψ_i correspond to the pre-tax (real) income for one unit of labor for individual i in each sector. In the data this corresponds to the wage that a given individual can get in each sector of the economy[9]. If the wage every individual can get in each sector were observable then it would be enough to order individuals according to their formal (pre-tax) wage, such that ϕ_i is increasing in i. With this order of individuals and the observed wages it is straightforward to compute ϕ_i, ψ_i and μ_i.

The difficulty of extracting these theoretical objects from the data arises from the fact that we do not observe the wages of each individual in both sectors. For some agents we only observe the informal wage and for some others only the formal wage. Therefore we cannot order individuals according to their formal wage and, furthermore, in the subset of individuals that we can order we do not observe the informal wage for most of them[10]. On top of this, it is important to note that we cannot rely on the idea that we are sampling over the population distribution of formal and informal wages, because workers endogenously sort themselves into each sector. This implies that it is not enough to assume that informal productivity and formal productivity increase with type to identify the productivities and distribution of types.

Our approach to solve these difficulties consists in estimating a model with a factor that can explain most of the variability of wages in both sectors and use that factor to order individuals in the population. The factor we use is a linear combination of worker characteristics and job characteristics; the education level and the task done in the job are examples of such characteristics. The estimated formal and shadow productivities correspond to the predicted wage in each sector using the value of the factor for each worker.

For top earners (top 0.8%) the factor cannot account for their income dispersion and the gap with respect to the rest of the population. We extend our identification strategy estimating a Pareto distribution for the wages of top earners in the formal sector. We do not need to recover in this case the informal productivity of top earners and only assume that if the single crossing property holds for non-top earners then it also holds for top earners.

[8] 58% of the workers are part of the shadow economy according to our estimates. Estimates by the official statistical agency in Colombia of the size of the shadow economy are close to ours but the definition is different and uses the scale of the firm or business to classify workers instead of the taxation criteria we use here.

[9] Or equivalently income if she is a self-account worker.

[10] Some workers work have a second job that potentially can be in the other sector, so for them there is an observation of both wages.

We find that both productivity estimates are increasing with type and that the single crossing property is satisfied (ϕ_i/ψ_i is increasing in type). The types with the lower formal productivity tend to have the same productivity in both sectors and as we move to types with higher formal productivity the wedge between productivities widens fast enough to have the single crossing property satisfied. At the top the Pareto parameter is 1.8 that is close to what was found for the income distribution in the US by Saez [2001].

The main novelty of this section is the assessment of the differences between the formal and the shadow economy at the worker level controlling for the sorting of workers. Productivity as measured in Porta and Shleifer [2008] can come also from the worker characteristics and not only from the type of firms or jobs in each sector. With our approach we are able to discuss the wage differential across sectors for a given worker and job. On the other hand we cannot interpret our results as direct measures of productivity, as those other studies do, since wages are not only determined by the worker productivity but also on the structure of the labor and goods market, for example, it depends in the bargaining power of workers when the wage is decided[11]. For our purposes in this paper this is not important since our object of interest is the income of the worker in each sector, but for a study of the productive structure of sectors our results can only be interpreted with the specification of the link between wages and productivity[12].

The remaining of the section is organized as follows: first, we present the data used and how we identify if a worker is formal or not. Second, the empirical specification is presented and last, the results are shown and discussed.

4.1 Data

All the information we use in this section is obtained from the household survey done by the official statistical agency in Colombia (DANE). Our sample is for the year 2013 and we have 170.000 observations of workers. The sample includes personal information such as age, gender, years of education and also labor market related variables including hours worked, number of jobs, type of job, income sources and social security affiliation. All of the information is reported by the worker.

The variables we use from the survey can be grouped in 4 categories: worker characteristics, job characteristics, worker-firm relationship and social security status. A linear combination of the variables in the first three categories are used to construct a factor that captures the variability of wages . The fourth is used to classify workers between the formal and the shadow economy.

Next we provide a brief description of the variables included in each category, for more detailed information see the Data appendix.

-Worker characteristics: these variables capture the type of worker irrespectively of the job he is currently doing. Here the variables included are age, gender, education level and work experience in previous jobs.

[11]Although this particular dimension is partially taken into account in our analysis since we use variables related to the bargaining power of the worker such as union membership and if the employer is an agency and not directly the firm in which the worker provides his services.

[12]For example if it is assumed perfect competition in the labor market then our measure corresponds directly to the worker marginal productivity. If a production function with constant returns to scale is also assumed then our measure also reflects the average productivity of the worker.

-Job characteristics: these variables characterize the type of job and task that the worker does irrespectively of the characteristics of the worker. The variables included are: firm or production unit number of workers (size), type of industry to which the firm belongs, location of the firm and the task the worker has to do.

-Worker-firm relationship: these variables characterize the information about the type of match worker-production unit. The variables included here are: The wage of the worker, number of working hours, the length of the match, if the worker is hired trough and intermediary firm and if the worker belongs to a union.

-Social security status: these variables determine if the worker is affiliated to social security in its different dimensions and the type of affiliation. The variables included are: affiliation to the health system, the pension system and the labor accidents insurance; also who pays for the affiliation to each component.

Every observation has assigned a sampling weight that is designed to compute all the statistics of the whole population; workers in Colombia. We use this weights in the estimation of productivities and the density of types to make it representative of the Colombian economy.

Classification of workers into formal and shadow workers

Workers are not directly asked whether they belong to the formal or shadow economy, or alternatively whether they pay or not the labor income taxes. Therefore, to identify if the worker belongs to the formal economy or not we have to rely on indirect measures. The measures we use are based on the survey questions related to the compliance with the labor regulation. Specifically, the affiliation to the health security system, the pension system and the accidents insurance policy. These three components of the social security system are expected to be highly effective to identify which workers do not comply with labor taxes since their payments are linked administratively with the payment of the payroll tax for employees and the income tax for self account workers. The criteria we use to identify a formal worker is the affiliation to the three components through his work (not as a beneficiary of other worker).

Gross wage of workers

Shadow workers do not pay taxes and therefore their reported labor income corresponds to the gross wage. Therefore, the gross hourly wage is computed as the ratio of the gross wage to the working hours.

On the other hand, formal workers do pay taxes; payroll taxes with monthly frequency and income taxes paid annually. The reported labor income is before discounts before the payroll taxes and social security contributions paid by the worker but it does not account for those directly paid by the employer. The gross wage is recovered adding up the reported income with the payments done by the employer. The hourly wage is computed dividing by hours worked.

The gross wage measure we use (noted as w) corresponds to the income a worker would get if he worked full time for one year; where full time is taken to be the maximum of 140 hours per week.

4.2 Empirical specification

The pre-tax income per unit of labor in each sector (ϕ_i and ψ_i) corresponds to the gross hourly wage. The logarithm of both productivities can be written as a function of a single factor F_i and noise as follows

$$\ln \phi_i = \gamma_0^f + \gamma_1^f F_i \tag{30}$$

$$\ln \psi_i = \gamma_0^s + F_i \tag{31}$$

where β^j characterize the linear function in sector $j \in \{f, s\}$. We have set $\gamma_1^s = 1$ without loss of generality, given that this will just rescale the factor.

The factor is a linear combination of a set of n variables contained in vector X_i with weights given by the vector β. Then we have that

$$F_i = \beta X_i \tag{32}$$

The proxy we have for the model productivities are the wages of workers w_i^j in each sector j, then we have that[13]

$$\ln w_i^f = \ln \phi_i + u_i^f \tag{33}$$

$$\ln w_i^s = \ln \psi_i + u_i^s \tag{34}$$

where u_i^f and u_i^s are the measurement errors. These terms capture the fact that in the model we consider a single formal and shadow productivity for each type and in reality the wage in each sector can be considered to be drawn from a probability distribution. The theoretical concepts in our analysis (ϕ_i and ψ_i) are then the location parameters of such distributions and in the theoretical analysis we abstract from the underlying variance of the distribution.

Combining equations (30), (31), (32), (33) and (34) we get the specification of the empirical model that corresponds to

$$\ln w_i = \gamma_0^s + I_i\left(\gamma_0^f - \gamma_0^s\right) + \left(1 + I_i\left(\gamma_1^f - 1\right)\right)\beta_i X_i + u_i \tag{35}$$

where I_i is an indicator function that takes the value of 1 if type i works in the formal economy and $u_i = u_i^s + I_i u_i^f$. Our estimators of parameters γ and β are the functions of the data that minimize the variance of the unexplained variation in wages u_i.

Ordering of agents and estimated productivities

Note the estimate of parameter a as $\hat{a}$. We proceed to order the individuals in our sample with indexes $i \in [0, 1]$ such that $i < i' \iff \hat{\beta} X_i < \hat{\beta} X_{i'}$. We compute the index of each individual

[13] Note that, as discussed earlier, w_i^j is only observed if type i works in sector j.

using the following formula

$$i = \frac{\hat{\beta}X_i - min_{i'}\{\hat{\beta}X_{i'}\}}{max_{i'}\{\hat{\beta}X_{i'}\}}$$

that is just rescaling the factor using the minimum and the maximum values it takes in the sample. The estimated productivities of each type i then correspond to

$$\hat{\phi}_i = \exp\left\{\hat{\gamma}_0^f + \hat{\gamma}_1^f \hat{\beta}X_i\right\} \quad (36)$$

$$\hat{\psi}_i = \exp\left\{\hat{\gamma}_0^s + \hat{\beta}X_i\right\} \quad (37)$$

Top income earners

Note that since w_i is in units of year income for full time work, then $\hat{\phi}_i$ corresponds (on average) to the maximum income that type i can achieve. That is not a tight bound considering we allow for a high maximum of 140 hours of work per week. Nevertheless, it is possible that workers could achieve income levels above what the worker with the highest index ($i' = 1$) in our sample could achieve working full time. That is, there could be yearly labor income observations y_i that satisfy

$$y_i > max_{i'}\{\hat{\phi}_{i'}\} = \hat{\phi}_1 \quad (38)$$

these are atypical observations if the deviation is big relative to variance of $\hat{u}$ controlling for the reported number of hours worked. We classify the individuals that satisfy this criterion as top earners. These are individuals with a very large wage premium that cannot be accounted for with our benchmark specification and for which their wage does not seem to have the same relationship with the factor as the rest of the population.

To characterize with more accuracy this behavior at the top of the income distribution we estimate the upper tail of the productivity distribution by fitting a Type I Pareto distribution for the gross wage w of top earners. The minimum of the support (or scale parameter) of the distribution is given by $max_{i'}\{\hat{\phi}_{i'}\}$ and the shape parameter is estimated by maximum likelihood.

A final adjustment has to be made to the index of agents, to fit the top earners in the type space $[0, 1]$ we compress the indexes on non-top earners to the interval $[0, k]$ and top earners are assigned to $[k, 1]$ and ordered by their gross wage.

Distribution of types

The assignment of indexes for each observation and their corresponding sampling weights implies a given discrete distribution of workers (non-top earners). The continuous distribution of types is obtained by a kernel density estimation with a linear interpolation at the evaluation points. The estimated kernel distribution gives us the distribution of types in the interval $[0, k]$.

There is not a unique probability distribution and formal productivity profile For the types in the interval $[k, 1]$ (top earners). For top earners we have a Pareto distribution for productivities with the support $[max_{i'}\{\hat{\phi}_{i'}\}, \infty)$ but this distribution can be replicated by different types distributions in $[k,1]$ at the types space, provided that the formal productivities ϕ_i for $i \in [k,1]$ are adjusted accordingly. This phenomenon does not occur with non-top earners because their productivity profiles are given by our parametric model.

The only two requirements that every distribution of types and productivity profiles of top earners have to satisfy always are that the total mass of the distribution has to coincide with the mass of workers that satisfy (38) and that $\lim_{i \to 1} \phi_i = \infty$.

4.3 Estimation results

Here we discuss the results of the estimation of the formal productivity (ϕ_i), the informal productivity (ψ_i) and the distribution of types (μ_i). Parameter estimates for β and the detailed description of the variables included in X_i are presented in the appendix.

Figure (1) presents the estimated productivities and the types distribution for non-top earners. The estimated values of γ_0^f and γ_0^s are almost identical with $\hat{\gamma}_0^s$ slightly greater so type 0 is slightly more productive in the shadow economy. We see that both productivities are increasing with type and furthermore that formal productivity grows faster than informal productivity. This pattern satisfies the single crossing property of the theoretical model, ϕ_i/ψ_i is increasing in i. The most productive individual among non-top earners is almost three times more productive in the formal economy than in the shadow economy.

Top earners are assigned to the set $[0.98, 1]$, the estimated value of the shape parameter of the Pareto distribution is 1.81 and comprise a mass of about 1% of the total population (details of the estimation are presented in the appendix). The shaded region in Figure (1) corresponds to the top earners. We do not plot their productivity profiles and density. Recall that what is identified is the distribution of formal productivities at the top with support $[max_{i'}\{\hat{\phi}_{i'}\}, \infty)$ and this can be matched with many different combinations of formal productivity and probability density specifications in the types space. The shadow productivity for top earners is not specified, the only condition imposed on it is that the single crossing property has to hold also at the top.

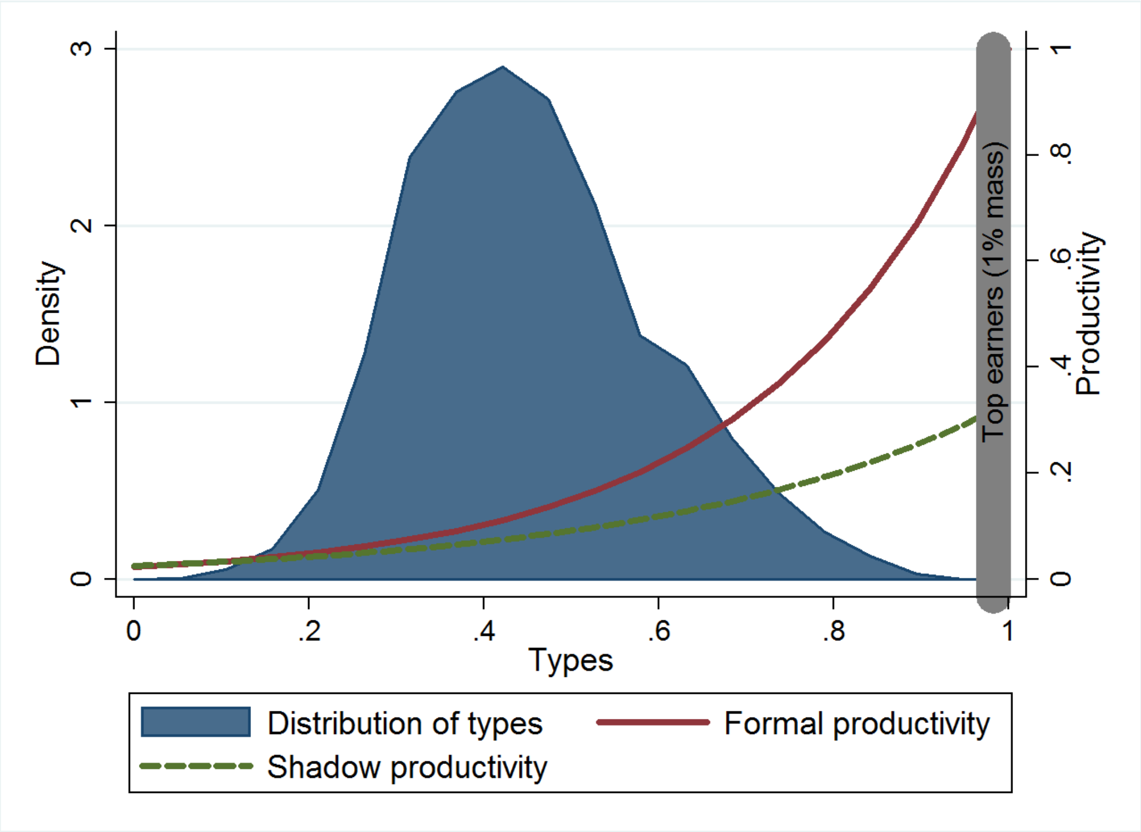
5 Calibrated exercise

We assume that the agents' utility function is

$$U(c,n) = c - \Gamma \frac{n^{1+\gamma}}{1+\gamma}.$$

We set the elasticity of labor supply $\frac{1}{\gamma}$ to 0.5, which is a relatively high value in the literature. In this way we take a conservative stance on the amount of redistribution the planner can conduct. We set Γ in order to match the average hours worked by workers in the formal economy. We find the optimum according to the Rawlsian welfare criterion. We set require the planner to obtain the same tax revenue as the actual tax schedule.

Figure 1: Estimated productivities and types distributions



In the actual Colombian economy 58% of workers are engaged in the shadow employment. We find that in the Rawlsian optimum half that many should work in the shadow economy. The bottom 30% of working age population optimally works only in the informal sector and collects a generous transfer.

Figure 2: The Rawlsian optimum

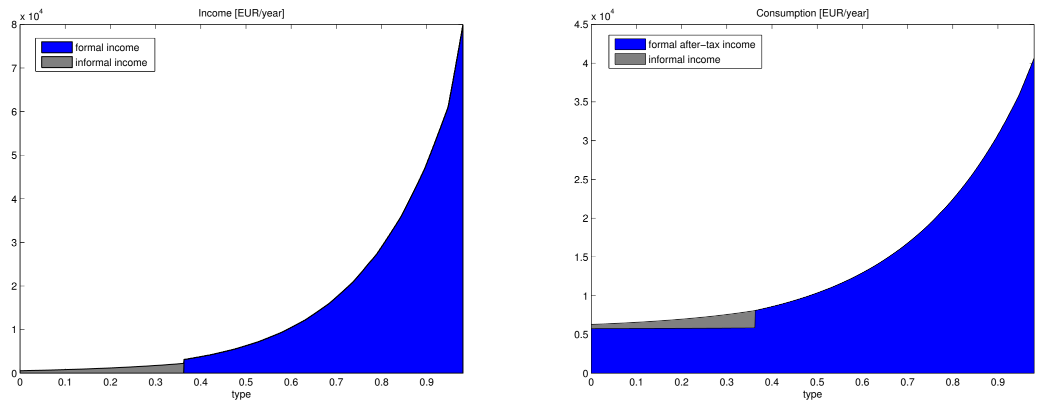
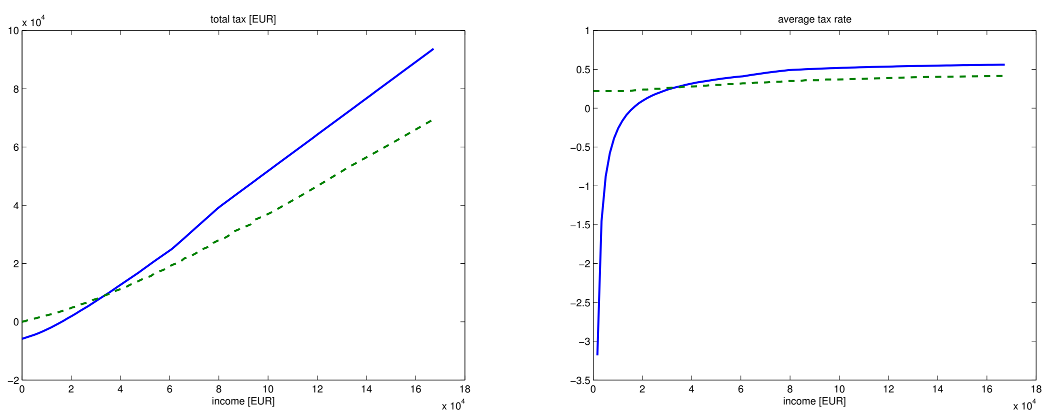


Figure 3 presents the comparison of the actual and the optimal Rawlsian tax schedule. The optimal tax schedule is much more redistributive.

Figure 3: Comparison of the actual (dashed line) and the optimal (solid line) tax schedule



6 Conclusions

A large fraction of the economic activity in most countries is carried out in the shadow sector. Our analysis shows that this fact pose an additional challenge for the design of the tax policy. The existence of a shadow economy generates new tradeoffs in the optimal design of the tax system, as agents can move across sectors in response to a tax policy change.

The shadow economy can limit redistribution if the difference in productivities across sectors is low, in particular for the highly productive agents. If this is not the case, we show that the shadow economy can be used as an additional instrument of redistribution and may lead to an increase in welfare. The optimal tax scheme in this case is characterised by the high marginal tax rates at the low levels of income which increase the revenues from the workers with high productivity. Such a policy can improve welfare, since the high marginal tax at low levels of reported income does not distort the labor decision of the less productive workers if they work in the shadow economy.

Three objects are key to determine the optimal size of the shadow economy from our optimal taxation perspective: formal productivity, the informal productivity and the probability distribution of workers at the different productivity levels. The mechanism proposed has a quantitatively sizable effect. In the case of Colombia we find that the analysis provides a rationale for a large shadow economy. Nevertheless, the observed levels of informality are even higher and double the optimal level.

With respect to the social security system, this paper suggests that, under some circumstances, allowing less productive people to collect welfare benefits and simultaneously work in the shadow economy is desirable. Furthermore, policies that are designed to deter the creation of shadow jobs should aim at those that are taken by very productive agents in the formal sector. It is important to stress that the way the shadow economy is modeled in this paper abstracts from many issues, such as competition between formal and informal firms, lack of regulation and law enforcement, as well as potential negative externalities caused by the informal activity. All those phenomena are likely to reduce the potential welfare gains from exploiting the shadow economy.

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Appendix

A Proofs from Section 2

Lemma 1.

Proof. Suppose that $IC_{i,i}$ is binding. It means that, conditional on chosen y_i^f, the agent would like to choose different shadow labor than the planner. However, either the planner wants to maximize the utility of the agent and both choose the unique, utility maximizing n_i^s, or the planner doesn't care, because the social preferences focus on the other type. Either way, distorting the choice of n_i^s can never improve social welfare, so $IC_{i,i}$ cannot bind.

□

Proposition 2.

Proof. Existence follows from the Weierstrass theorem. The objective function is continuous. All constraints consist of weak inequalities, so the choice set is a closed set. We can ensure boundedness of the choice set by considering the allocations that yield welfare not lower than the allocation without transfers, *laissez-faire*. Since *laissez-faire* satisfies constraints (4) and (6), the optimum cannot be worse. Note the welfare of laissez-faire by $W_{lf} > -\infty$. Suppose that the set of allocations with welfare greater than W_{lf} is unbounded. By the resource constraint, it means that the labor supply of some type in some allocations has to diverge to infinity. By $\lim_{n\to\infty} v'(n) = \infty$, it implies that disutility from labor of this type, as well as welfare of the whole allocation, goes to $-\infty$. It contradicts the assumption that the allocation is superior than *laissez-faire*. Finally, notice that our restricted choice set is always non-empty, since it contains *laissez-faire*.

In the first-best, $U(c_l, n_l) \geq U(c_h, n_h)$. By assumption of $v'(0) = 0$, we know that $n_l^f > 0$. Then the utility of h mimicking l is $U\left(c_l, \frac{\phi_l}{\phi_h} n_l^f\right) > U\left(c_l, n_l^f\right) \geq U(c_h, n_h)$, which violates $IC_{h,l}$. Hence, the optimum is not the first-best.

Suppose that at the optimum $IC_{h,l}$ does not bind. First, let's consider the case in which $U(c_h, n_h) > U(c_l, n_l)$. Since $IC_{h,l}$ is slack, the planner may increase transfers from h to l, which raises welfare, so it could not be the optimum in the first place. Second, suppose that $U(c_l, n_l) \geq U(c_h, n_h)$. It can happen only if $n_l^s > 0$. Otherwise, as we have shown above, $IC_{h,l}$ is violated. If $n_l^s > 0$ and $IC_{h,l}$ is slack, the planner can marginally decrease n_l^s and increase n_l^f, which generates free resources. Hence, at the optimum $IC_{h,l}$ has to bind.

Suppose that $IC_{l,h}$ binds. If the resource constraint is satisfied as equality, it may happen only if l type is paying a positive tax, while h type receives a transfer. Then the planner can improve welfare by canceling the redistribution altogether and reverting to *laissez-fare*, where none of the incentive constraints bind.

□

Lemma 6. *At the optimum either $U(c_l, n_l) = U(c_h, n_h)$ and $n_l^s > 0$, or the following optimality condition holds*

$$\min\left\{\frac{v'(n_l)}{\phi_l} - \left(\mu_l + \mu_h \frac{v'(n_{h,l})}{\phi_h}\right), n_l^f\right\} = 0. \tag{39}$$

Suppose that v'' is nondecreasing. Define $\bar{n}$ by $v'(\bar{n}) = \psi_l$. If $v'\left(\frac{\phi_l}{\phi_h}\bar{n}\right) \leq \psi_h$, then this optimality condition is sufficient for the optimum.

Proof. If $U(c_l, n_l) = U(c_h, n_h)$ and $n_l^s = 0$, then such allocation is not incentive compatible. The proof is identical as the proof of the claim that the first-best is not incentive compatible in Proposition 2. Hence, if $U(c_l, n_l) = U(c_h, n_h)$, then $n_l^s > 0$.

Let's consider the case, in which $U(c_h, n_h)$ is always greater than $U(c_l, n_l)$. In this case $IC_{h,l}$ has to bind, otherwise the planner could equalize utilities of both types. Consider changing n_l^f by a small amount and adjusting T_l such that $IC_{h,l}$ is satisfied as equality. It means that

$$\frac{\partial T_l}{\partial n_l^f} = \phi_l \mu_h \left(1 - \frac{v'(n_{h,l})}{\phi_h}\right).$$

This perturbation affects social welfare by

$$\frac{dU(c_l, n_l)}{dn_l^f} = \phi_l - \frac{\partial T_l}{\partial n_l^f} - v'(n_l) = \phi_l\left(\mu_l + \mu_h \frac{v'(n_{h,l})}{\phi_h}\right) - v'(n_l).$$

Optimum requires that either $\frac{dU(c_l,n_l)}{dn_l^f} = 0$ or $\frac{dU(c_l,n_l)}{dn_l^f} < 0$ and $n_l^f = 0$, which results in (39). Sufficiency of this first order condition depends on the second order derivative of welfare with respect to the perturbation. Suppose that v'' is nondecreasing. Define $\bar{n}$ by $v'(\bar{n}) = \psi_l$. If v'' is nondecreasing and $\psi_h \geq v'\left(\frac{\phi_l}{\phi_h}\bar{n}\right)$, then

$$\frac{d^2U(c_l, n_l)}{dn_l^{f2}} \begin{cases} = 0 & \text{for } n_l^f < \bar{n} \\ \leq 0 & \text{for } n_l^f > \bar{n}. \end{cases}$$

In this case the optimality condition (39) is sufficient. In the other case, when $v'\left(\frac{\phi_l}{\phi_h}\bar{n}\right) > \psi_h$, then $\frac{dU(c_l,n_l)}{dn_l^f}$ may change the sign twice:

$$\frac{d^2U(c_l, n_l)}{dn_l^{f2}} \begin{cases} = 0 & \text{for } n_l^f < \underline{n} \\ \geq 0 & \text{for } \bar{n} > n_l^f > \underline{n} \\ \leq 0 & \text{for } n_l^f > \bar{n}. \end{cases}$$

In such situation the optimality condition may point at two different local maxima and a local minimum.

□

Proposition 3.

Proof. Suppose that $\frac{\psi_h}{\phi_h} \geq \frac{\psi_l}{\phi_l}$. It means that $\psi_h > \psi_l$. $IC_{h,l}$ implies that at the optimum $U(c_h, n_h) > U(c_l, n_l)$. By Lemma 6 above the condition (39) is necessary for the optimum. We will show that l working informally violates this condition. Suppose that $n_l^s > 0$. The condition (39) becomes

$$\frac{\psi_l}{\phi_l} \geq \left(\mu_l + \mu_h \frac{\psi_h}{\phi_h}\right) \Leftrightarrow \frac{\psi_l}{\phi_l} - \frac{\psi_h}{\phi_h} \geq \frac{\mu_l}{\mu_h}\left(1 - \frac{\psi_l}{\phi_l}\right).$$

It is a contradiction: the left-hand side is negative, while the right-hand side is positive.

□

Proposition 4.

Proof. In the proof of Lemma 6 above we described the impact of changing formal labor of l on the social welfare, $\frac{dU(c_l,n_l)}{dn_l^f}$. Suppose that $\psi_h \geq v'\left(\frac{\phi_l}{\phi_h}\bar{n}\right)$. It is easy to see that (9) implies that $\frac{dU(c_l,n_l)}{dn_l^f}$ is always non-positive, so it is optimal to reduce n_l^f as long as $U(c_h, n_h) > U(c_l, n_l)$. From Lemma 6 we know also that $U(c_h, n_h) > U(c_l, n_l)$ if l works only formally, so it is optimal to push this type to the shadow economy. Now suppose that (9) does not hold. Then the optimality condition (39) will be satisfied for $n_l^f > \bar{n}$ and l will optimally work only formally.

Now suppose that $\psi_h < v'\left(\frac{\phi_l}{\phi_h}\bar{n}\right)$. (10) implies that $\frac{dU(c_l,n_l)}{dn_l^f}$ is always non-positive. Therefore, it is optimal to reduce n_l^f until utilities of both types are equalized, which can happen only when l works in the shadow economy. This condition is not necessary for l to work in the shadow economy, because the social welfare changes in a non-monotone way with n_l^f. If (10) is not satisfied, marginally increasing n_l^s from 0 is in this case bad for welfare, but increasing it further may eventually lead to welfare gains, and the total effect on welfare is ambiguous.

□

Proposition 5.

Proof. If there is an incentive-feasible allocation in which no agent works in the shadow economy, this allocation is incentive-feasible also in the standard Mirrlees model. It means that the optimum of the Mirrlees model is not worse than this allocation.

Now we will show when the optimum welfare is strictly lower than in the standard Mirrlees model. Following Lemma 6, it is easy to derive the optimality condition with respect to labor of type l in the Mirrlees model:

$$\frac{v'\left(n_l^M\right)}{\phi_l} = \mu_l + \mu_h \frac{v'\left(\frac{\phi_l}{\phi_h} n_l^M\right)}{\phi_h}. \tag{40}$$

This condition is sufficient for the optimum if v'' is nondecreasing. The shadow economy makes the incentive constraint $IC_{h,l}$ more stringent than in the standard Mirrlees model if type h while deviating wants to supplying some informal labor. It happens when $\psi_h > v'\left(\frac{\phi_l}{\phi_h} n_l^M\right)$. Define $\underline{n}$ by $v'\left(\frac{\phi_l}{\phi_h}\underline{n}\right) = \psi_h$. Whenever $n_l^M < \underline{n}$, h type works informally when mimicking the other type. According to the optimality condition, it when

$$\frac{v'(\underline{n})}{\phi_l} - \frac{v'\left(\frac{\phi_l}{\phi_h}\underline{n}\right)}{\phi_h} > \frac{\mu_l}{\mu_h}\left(1 - \frac{v'(\underline{n})}{\phi_l}\right).$$

Suppose that $\psi_h \geq \phi_l$. It means that the right-hand side is negative and the condition hold. Suppose that $\phi_l > \psi_h$. Then we can bound $v'(\underline{n})$ from below by $v'\left(\frac{\phi_l}{\phi_h}\underline{n}\right) = \psi_h$. Then the inequality above holds when

$$\frac{\phi_h - \phi_l}{\phi_l - \psi_h}\frac{\psi_h}{\phi_h} \geq \frac{\mu_l}{\mu_h}.$$

In order to examine when the optimum welfare is strictly higher than in the standard Mirrlees model, we will compare utility of type l in the standard Mirrlees model $\left(U\left(c_l^M, n_l^M\right)\right)$ and in the shadow economy model, when l is working only in the shadow economy $\left(U\left(c_l^{SE}, n_l^{SE}\right)\right)$. Clearly, when the second scenario yields higher utility, the existence of the shadow economy is welfare improving.

In the standard Mirrlees model, the binding constraint is $U\left(\phi_h n_h^M, n_h^M\right) - T_h^M = U\left(\phi_l n_l^M, \frac{\phi_l}{\phi_h} n_l^M\right) - T_l^M$. The resource constraint implies that $T_h^M = -\frac{\mu_l}{\mu_h} T_l^M$ and we get $T_l^M = \mu_h \left(U\left(\phi_l n_l^M, \frac{\phi_l}{\phi_h} n_l^M\right) - U\left(\phi_h n_h^M, n_h^M\right)\right)$. Now, the utility of l type is

$$U\left(c_l^M, n_l^M\right) = U\left(\phi_l n_l^M, n_l^M\right) - T_l = U\left(\phi_l n_l^M, n_l^M\right) - \mu_h \left(U\left(\phi_l n_l^M, \frac{\phi_l}{\phi_h} n_l^M\right) - U\left(\phi_h n_h^M, n_h^M\right)\right).$$

Using the same steps, we can express the utility of l working only in the shadow economy

$$U\left(c_l^{SE}, n_l^{SE}\right) = U\left(\psi_l n_l^{SE}, n_l^{SE}\right) - \mu_h \left(U\left(\psi_h n_{h,l}^{SE}, n_{h,l}^{SE}\right) - U\left(\phi_h n_h^{SE}, n_h^{SE}\right)\right).$$

Since there are no wealth effects, $n_h^M = n_h^{SE}$. The shadow economy is welfare improving if

$$U\left(\psi_l n_l^{SE}, n_l^{SE}\right) - \mu_h U\left(\psi_h n_{h,l}^{SE}, n_{h,l}^{SE}\right) > U\left(\phi_l n_l^M, n_l^M\right) - \mu_h U\left(\phi_l n_l^M, \frac{\phi_l}{\phi_h} n_l^M\right).$$

The left-hand side is always positive when $\psi_l > \psi_h$. The right side is negative for $\mu_h > \frac{U\left(\phi_l n_l^M, n_l^M\right)}{U\left(\phi_l n_l^M, \frac{\phi_l}{\phi_h} n_l^M\right)} \in (0,1)$

□

B Proofs from Section 3

Lemma 2.

Proof. The single-crossing requires that $\frac{\partial}{\partial i}\left(\frac{V_y\left(y^f, T, i\right)}{V_T\left(y^f, T, i\right)}\right) < 0$. Suppose that $v'\left(\frac{y^f}{\phi_i}\right) < \psi_i$. Then the agent supplies no informal labor and the indirect utility function V is just U evaluated at the given allocation. Since v' is increasing, the single crossing is holds. When $v'\left(\frac{y^f}{\phi_i}\right) \geq \psi_i$, then the optimal provision of informal labor means that $v'\left(n_i\right) = \psi_i$, which implies $\frac{V_y}{V_T} = \frac{\psi_t}{\phi_t}$. Therefore the single crossing condition requires that $\frac{d}{dt}\left(\frac{\psi_t}{\phi_t}\right) < 0$.

□

Proposition 6.

Proof. The proof follows from the analogous proof in Doligalski [2015] (Proposition 1) by simply replacing the income schedule y with the formal income schedule y^f. The only significant difference comes in the definition of $\dot{V}_i$. In order to show that if $y_a^f > y_b^f$, then $\dot{V}_i\left(y_a^f, T_a\right) > \dot{V}_i\left(y_b^f, T_b\right)$, note that we can write $\dot{V}$ as

$$\dot{V}_i\left(y^f, T\right) = \left(\frac{\dot{\phi}_i}{\phi_i}\frac{y^f}{\phi_i} + \frac{\dot{\psi}_i}{\psi_i}\max\left\{g\left(\psi_i\right) - \frac{y^f}{\phi_i}, 0\right\}\right) v'\left(n_i\right).$$

The single-crossing implies that $\frac{\dot{\phi}_i}{\phi_i} > \frac{\dot{\psi}_i}{\psi_i}$, so $\dot{V}_i\left(y^f, T\right)$ is increasing in y^f.

□

Theorem 1.

Proof. First we will derive formally the term D_i^s. Then we will show that conditions from the theorem are necessary. Finally we will prove sufficiency. The interior optimality conditions for the formal type are derived in Doligalski [2015].

Suppose that $i \in \mathcal{S}$. The individual rationality implies that $v'(n_i) = \psi_i \implies n_i = g(\psi_i)$, where g is an inverse function of v'. The marginal information rent can be expressed as

$$\dot{V}_i = \left(\frac{\dot{\phi}_i}{\phi_i} n_i^f + \frac{\dot{\psi}_i}{\psi_i} \left(g(\psi_i) - n_i^f \right) \right) \psi_i. \tag{41}$$

We marginally perturb the formal income of i. In order to keep the utility schedule continuous at i, the perturbation has to be accompanied by a change in total tax such that the utility of this type is unchanged. The required change of the tax paid is $dT = 1 + \frac{dV_i}{dy^f}$, which for the shadow worker equals $\frac{\phi_i - \psi_i}{\phi_i}$. By multiplying the tax revenue change with μ_i and normalizing it with $\frac{d\dot{V}_i}{dy^f}$, we obtain the tax revenue cost of decreasing the marginal information rent of type i, given by (20).

If the formal income is nondecreasing in type, the allocation implied by the conditions (21), (22), (23) and (24) is incentive-compatible. The necessity of these conditions was demonstrated in the main text. They are sufficient when the cost of decreasing the marginal information rent of a given type is nondecreasing. It happens when for each type ζ_i is nonincreasing in the labor supply (then D_i^f is nondecreasing in the formal labor supply) and when $\frac{\dot{\phi}_i}{\phi_i} + \frac{\dot{\psi}_i}{\psi_i}\zeta_i \geq 0$ holds (then for each type $D_i^f \leq D_i^s$).

□

Lemma 3.

Proof. Suppose set $\mathcal{S}$ is nonempty. If $\mathcal{S}$ either not convex or not at the bottom of the type space, then there is a left limit point of some subset of $\mathcal{S}$, denoted by i, which does not belong to $\mathcal{S}$. Since i is to the left of some subset of $\mathcal{S}$, there is type $j > i$ which belongs to $\mathcal{S}$. But $y_i^f > y_j^f = 0$, so the formal income is decreasing.

For any type $i \in \mathcal{M}$ it is true that $v'\left(n_i^f\right) = \psi_i$. Denote the inverse function of v' by g, then $n_i^f = g(\psi_i)$. The derivative of formal income with respect to type is

$$\dot{y}_i^f = \frac{d\phi_i g(\psi_i)}{di} = \dot{\phi}_i g(\psi_i) + \phi_i \dot{\psi}_i g'(\psi_i) = \frac{1}{\phi_i g(\psi_i)} \left(\frac{\dot{\phi}_i}{\phi_i} + \frac{\dot{\psi}_i}{\psi_i} \frac{\psi_i g'(\psi_i)}{g(\psi_i)} \right).$$

Finally, notice that $\frac{\psi_i g'(\psi_i)}{g(\psi_i)} = \frac{v'\left(n_i^f\right)}{n_i^f v''\left(n_i^f\right)} = \zeta_i$.

□

Theorem 2.

Proof. There are three cases we should consider. In the first one, the interior formal income schedule, as implied by conditions (21)-(23), is increasing in type. In such case, by Theorem 1, the interior allocation is optimal. Second, the interior income schedule may be decreasing. In this case, we will derive formally the necessary optimality condition (26). Finally, the interior formal income schedule may be locally flat. In the last part of this proof we will show that in this case interior optimality conditions (21)-(23) and the optimal kink formula (26) are equivalent.

If the interior income schedule is at some point decreasing, (21)-(23) are no longer necessary for the optimum - they lead to a violation of the incentive-compatibility. The optimal formal income schedule will involve bunching some types, and the planner may perform a perturbation of the formal income level at which the types are bunched. No such perturbation is available, if (26) holds. Now we will derive formally this optimality condition. The derivations follow the same steps as Doligalski [2015], we provide them here for the sake of completeness. Suppose that the formal income schedule y^f is constant at the segment of types $[a, c]$. Let's marginally decrease the formal income of types $[a, c)$. Since we don't change the allocation of types below a, we have to make sure that V_a is unchanged - otherwise the utility schedule would become discontinuous. Together with the cut of the formal income, we have to introduce a change in the total tax paid at this income level $dT_a = 1 + \frac{\partial V_a}{\partial y^f} = t_a$. Since all types $[a, c)$ are affected, the tax revenue loss is equal

$$t_a \left(M_c - M_a\right). \tag{42}$$

Although this perturbation does not affect the utility of type a, it does affect the utility of all other types at the kink. The utility impact of the perturbation of some type $i \in (a, c)$ equals $dU_i = 1 + \frac{\partial V_i}{\partial y^f} - dT_a = \frac{v'(n_a)}{\phi_a} - \frac{v'(n_i)}{\phi_i}$. The welfare loss due to this utility change is

$$\int_a^c \Delta MRS_i w_i d\mu, \tag{43}$$

where $\Delta MRS_i = \frac{v'(n_a)}{\phi_a} - \frac{v'(n_i)}{\phi_i}$ is the distance in terms of the marginal rate of substitution from a, the initial type at the kink. Having the fiscal and welfare loss at the kink, we can add them into a cost of increasing distortions at the kink, $D^K_{a,c}$. The definition (25) involves also a normalization term $\frac{1}{t_c - t_a,}$ which makes sure that the perturbation results in the unit change of the utility of type c.

The change of allocation at the kink results in a shift in the utility level of all types above the kink. Since we normalized the perturbation such that the utility level of type c falls by a unit, the benefit of perturbation is the familiar term $\int_c^1 N_j dj$, which captures the gains from nondistortionary taxation of types above (and including) c. By equalizing the cost and the benefit of the perturbation, we arrive at the optimality condition (26).

Finally, let's consider the case when the interior income schedule is flat on the segment $[a, c]$. The proof of equivalence of the interior optimality conditions and the optimality condition at the kink follows the same steps as the proof of Theorem 2 in Doligalski [2015], with one qualification. The proof involves integrating $\frac{\partial \dot{V}_i}{\partial y^f} = -\dot{MRS}_i$ over the segment $[a, c]$. As we demonstrated before, the expression for $\frac{\partial \dot{V}_i}{\partial y^f}$ of marginal workers in general depends on the direction of change of formal income. However, in this specific case, when the interior formal income is flat, the expression $\frac{\partial \dot{V}_i}{\partial y^f}$ is well defined for the marginal workers. By Lemma 3, formal income is constant for the marginal workers if $\frac{\dot{\phi}_i}{\phi_i} + \frac{\dot{\psi}_i}{\psi_i}\zeta_i = 0$. It is easy to see that it implies

$$\frac{\dot{\phi}_i}{\phi_i}\left(1 + \frac{1}{\zeta_i}\right)\frac{\psi_i}{\phi_i} = \left(\frac{\dot{\phi}_i}{\phi_i} - \frac{\dot{\psi}_i}{\psi_i}\right)\frac{\psi_i}{\phi_i}. \tag{44}$$

The left-hand side is equal to $\frac{\partial \dot{V}_i}{\partial y^f}$ evaluated as if i was a formal worker, while the right-hand side is $\frac{\partial \dot{V}_i}{\partial y^f}$ as if i was a shadow worker. The two expressions coincide, so they determine the derivative of the marginal information rent of a marginal worker with respect to formal income. Therefore, the integral of $\frac{\partial \dot{V}_i}{\partial y^f}$ over the set $[a, c]$ is well defined.

□

Lemma 4.

Proof. Suppose that the interior allocation is not incentive compatible. Let's first consider the case in which the function G in the social welfare is affine. In this case term $N_j = (1 - w_j)\mu_j$ does not depend on the allocation. Whenever there is a kink, it is optimal to increase the marginal tax rate below, but the other tax rates above remain unchanged. Hence, the assignment to sets from the interior allocation remains the same in the optimum for the types above the highest kink (of the interior allocation). For the other types, the marginal tax rate increases or stays constant, so the incentives to join the shadow economy are strengthened.

Consider the general case of G concave. Then the large changes of the marginal tax rates below a given type i affect N_j for $j > i$. When the interior allocation is not incentive compatible, the optimal allocation will involve different marginal tax rates for the mass of agents at the kink. It means that the terms N_j will be different for all types above the kink, which changes the tax rate trade-off for all types. The tax rates on the entire type space (apart from the top) will differ between the optimum and the interior allocation. However, as before, the tax rates will be always higher in the optimum. The incentive compatibility in the optimum prevents the planner from reducing the information rents of the top types as much as in the interior allocation. Incentives to increase tax rates are higher everywhere because the benefit $\int_i^1 N_j dj$ is higher for every i.

□

Lemma 5.

Proof. It is just an interior optimality condition for the shadow worker (23). By Lemma 4, all the shadow workers from the interior allocation are shadow workers in the optimum.

□

C Estimation results of the factor F_i and top earners Pareto distribution.

Here we present the variables included in the vector X_i and the parameter estimates of β and γ obtained from the specification given by (35).

Table (1) lists the variables included in X_i with its corresponding description and associated category. The parameter estimates are presented in Table (2).

Finally Table XX presents the estimate of the Pareto distribution for top earners. The table contains the scale parameter estimates and the quantiles of formal productivity of top earners.

Table 1: Variables included in X_i

Variable	Description	Values
Worker characteristics		
Gender	Dummy variable equal to 1 for women	0-1
Age	Age of the worker	16-90
Age^2	Age squared	
Ed years	Number of education years	0-26
Degree	Highest degree achieved	1-5 1 - no degree 5 - postgraduate degree
Y work	Number of months worked in the last year	1-12
Experience	Number of months worked in the last job	0-720
First job	Dummy for the first job (1 if it is the first job)	0-1
Production unit (firm) characteristics		
Sector Man	Dummy for the manufacturing sector	0-1
Sector Fin	Dummy for financial intermediation	0-1
Sector ret	Dummy for the sales and retailers sector	0-1
Big city	Dummy for a firm in one of the two largest cities	0-1
Size	Categories for the number of workers	1-9 1 - One worker 9 - More than 101 workers
Production unit (Type of job) characteristics		
Lib	Dummy for a liberal occupation	0-1
Admin	Dummy for an administrative task	0-1
Seller	Dummy for sellers and related	0-1
Services	Dummy for a service task (bartender ..)	0-1
Worker-firm relationship		
Union	Dummy for labor union affiliation (1 if yes)	0-1
Agency	Dummy for agency hiring (1 if yes)	0-1
Seniority	Number of months of the worker in the firm	0-720

Table 2: Estimation results

Parameter	Point estimate	std. error	t-statistic	95% conf. interval	
γ_0^s	6.96	0.033	211.9	6.89	7.02
$\gamma_0^f - \gamma_0^s$	-0.10	0.032	-3.2	-0.16	-0.04
$\gamma_1^f - 1$	0.47	0.037	12.6	0.39	0.54
β-Gender	-0.05	0.005	-11.6	-0.06	-0.04
β-Age	0.02	0.001	13.1	0.01	0.02
β-Age^2	0.00	0.000	-8.8	0.00	0.00
β-Ed years	0.03	0.002	15.4	0.02	0.03
β-Degree	0.11	0.005	21.1	0.10	0.12
β-Sector Man	-0.07	0.006	-11.9	-0.08	-0.06
β-Sector Fin	0.11	0.015	6.9	0.08	0.14
β-Sector ret	-0.10	0.006	-16.9	-0.11	-0.09
β-Big city	0.01	0.007	1.0	-0.01	0.02
β-Size	0.02	0.001	18.7	0.02	0.02
β-Union	0.09	0.010	8.3	0.07	0.11
β-Agency	-0.10	0.005	-18.3	-0.11	-0.09
β-Seniority	0.00	0.000	17.9	0.00	0.00
β-Y work	0.02	0.001	18.4	0.02	0.02
β-First job	-0.04	0.008	-4.7	-0.05	-0.02
β-Experience	0.00	0.000	5.3	0.00	0.00
β-Lib	0.05	0.013	3.9	0.03	0.08
β-Admin	-0.19	0.009	-19.9	-0.20	-0.17
β-Seller	-0.13	0.014	-9.2	-0.15	-0.10
β-Services	-0.18	0.009	-19.3	-0.20	-0.16

Table 3: Pareto distribution estimates

Parameter	Point estimate	std. error	z-statistic	95% conf. interval	
Shape parameter	1.81	0.0018	953.34	1.806	1.833
Quantile	Formal productivity ϕ_i				
1%	1.006				
5%	1.029				
10%	1.060				
20%	1.131				
25%	1.173				
30%	1.218				
40%	1.327				
50%	1.467				
60%	1.659				
70%	1.945				
75%	2.152				
80%	2.434				
90%	3.569				
95%	5.235				
99%	12.737				