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An Envelope Approach to Tournament Design

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Abstract Optimal rank-order tournaments have traditionally been studied using a first-order approach. The present analysis relies instead on the construction of an "upper envelope" over all incentive compatibility conditions. It turns out that the first-order approach is *not* innocuous. For example, in contrast to the traditional understanding, tournaments may be dominated by piece rates even if workers are risk-neutral. The paper also offers a strikingly simple characterization of the optimal tournament for quadratic costs and CARA utility, as well as an extension to large tournaments.

Keyword. Rank-order tournaments · First-order approach · Envelope theorem

JEL-Codes C62, D86, L23

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1 Introduction

The economic analysis of rank-order tournaments present itself today as a tremendously successful research area that has experienced a steady increase in interest since its very beginnings.[1] On the theoretical front, it has often been crucial to characterize the optimal tournament (Lazear and Rosen, 1981; Nalebuff and Stiglitz, 1983; Akerlof and Holden, 2012). This task has most commonly been accomplished using the so-called first-order approach, i.e., by replacing a continuum of incentive compatibility conditions in the firm's design problem with a single marginal condition. However, the first-order approach is not generally valid, and as a consequence, the properties of the optimal tournament have sometimes been discussed under somewhat restrictive or even indistinct conditions.[2]

In this paper, an alternative route to the analysis of optimal rank-order tournaments is taken. The approach entails the construction of an "upper envelope" over all incentive compatibility conditions, which is then added as an inequality constraint to the relaxed problem. Thereby, the optimal tournament may be characterized as the solution of an optimization problem with a finite number of constraints. Of course, the thereby reformulated problem remains difficult. However, in contrast to the original problem, techniques from Milgrom and Segal (2002) may be applied to derive key properties of the optimal tournament even if the first-order approach is invalid or difficult to justify.

[1] See, e.g., the evidence provided by Connelly et al. (2014). For an introduction to the economics of tournaments, see Prendergast (1999, Sec. 2.3).

[2] Useful discussion of the scope and limitations of the first-order approach in tournament theory can be found in McLaughlin (1988) and Gürtler (2011).

The main result of this paper is that the first-order approach to tournament design is *not* innocuous. Specifically, it is found that traditional conclusions regarding the efficiency of rank-order tournaments are not universally valid and sometimes too optimistic. In fact, tournaments may be substantially less efficient than suggested by the existing literature. Further, with additional structure imposed on the cost and utility functions, the optimal tournament may be characterized in explicit terms even if the first-order approach is invalid. The paper also considers an extension to tournaments with many contestants and a single winner, which may be seen as an equilibrium analysis complementing prior work.

The observation that the first-order approach is not generally valid in a moral hazard setting is due to Mirrlees (1975). Subsequent research on the first-order approach may be roughly divided into two strands. A first strand of literature is concerned with formulating sufficient conditions for the first-order approach (Rogerson, 1985; Jewitt, 1988; Sinclair-Desgagné, 1994; Conlon, 2009; Ke, 2013). A second strand of literature has aimed at eliminating restrictive assumptions from the standard model of moral hazard (Mirrlees, 1986; Araujo and Moreira, 2001; Ke, 2012; Kadan and Swinkels, 2013). The present paper differentiates itself from these contributions already by its focus on rank-order tournaments. However, also the approach is different. For example, the present paper does not employ a Lagrangian function. Some implications of this point will be discussed in the conclusion.

The remainder of this paper is structured as follows. Section 2 introduces the set-up, and discusses existence. The envelope approach is presented in Section 3. A characterization of the optimal tournament is presented in

Section 4. Section 5 discusses the case of more than two contestants. Section 6 concludes. All proofs have been relegated to an Appendix.

2 Set-up and existence

Considered is a market environment in which risk-neutral firms hire workers to produce output of per-unit value $V > 0$. Given a wage W and an effort level $\mu \geq 0$, a worker's utility is defined as $U(W) - C(\mu)$, where U is twice differentiable with $U' > 0$, $U'' \leq 0$, and C is four times differentiable with $C' \geq 0$, $C'' > 0$, $C'(0) = 0$, and $C'(\mu) \to \infty$ as $\mu \to \infty$.[3] Worker j's output ($j = 1, 2$) is the sum of his effort μ_j and a random component ε_j, i.e., $q_j = \mu_j + \varepsilon_j$.[4] For contestants $k \neq j$, it will be assumed that the distribution function G of the differential error term $\xi \equiv \varepsilon_k - \varepsilon_j$ is symmetric with respect to the origin and allows a twice differentiable density $g = G' > 0$ such that g' and g'' are bounded. Given prizes $W_1 \geq W_2$, worker j's expected utility is then given as

$$\begin{aligned} & U(W_1)\text{prob}[q_j > q_k] + U(W_2)(1 - \text{prob}[q_j > q_k]) - C(\mu_j) \\ & \quad = U(W_2) + (U(W_1) - U(W_2))G(\mu_j - \mu_k) - C(\mu_j). \end{aligned} \tag{1}$$

In the usual dual formulation, firms choose prizes and an effort level so as to maximize a worker's expected utility subject to zero-profit and incentive

[3] The additively separable form of the utility function ensures tractability. As discussed in McLaughlin (1988), alternative specifications of the worker's utility function tend to produce similar conclusions under the first-order approach. It is conjectured that the same is true for the additional settings considered in the present paper.

[4] The introduction of a common additive shock would not change the conclusions.

compatibility conditions:

$$\max_{\substack{W_1 \geq W_2 \\ \mu \geq 0}} \qquad \frac{U(W_1) + U(W_2)}{2} - C(\mu) \tag{2}$$

s.t.

$$\mu V = \frac{W_1 + W_2}{2} \tag{3}$$

$$(U(W_1) - U(W_2))G(\widehat{\mu} - \mu) - C(\widehat{\mu}) \tag{4}$$

$$\leq (U(W_1) - U(W_2))G(0) - C(\mu) \qquad (\widehat{\mu} \geq 0)$$

A solution (W_1^*, W_2^*, μ^*) of problem (2-4) will be referred to as an *optimal tournament* associated with G.

Under the *first-order approach* (FOA), the continuum of incentive compatibility conditions (4) is replaced by the necessary marginal condition

$$g(0)(U(W_1) - U(W_2)) = C'(\mu). \tag{5}$$

The relaxed problem is known to allow a solution $(W_1^{\text{FOA}}, W_2^{\text{FOA}}, \mu^{\text{FOA}})$ that can be approximated by replacing utility and cost functions with their respective second-order Taylor expansions. For example, the effort level and the prize spread may be approximated by

$$C'(\mu^{\text{FOA}}) \approx \frac{V}{1 + sC''/4g(0)^2} \tag{6}$$

and

$$W_1^{\text{FOA}} - W_2^{\text{FOA}} \approx \frac{g(0)V}{g(0)^2 + sC''/4}, \tag{7}$$

respectively, where $s = -U''/U'$ denotes the worker's Arrow-Pratt coefficient of absolute risk aversion, and marginal utility is normalized to unity at mean income.[5] Moreover, if the worker's expected utility function in the

[5]For further details including proofs of equations (6) and (7), see McLaughlin (1988, p. 231).

corresponding tournament is, say, strictly concave, then $(W_1^{\mathrm{FOA}}, W_2^{\mathrm{FOA}}, \mu^{\mathrm{FOA}})$ solves the unrelaxed program. In particular, in the risk-neutral case, $C'(\mu^{\mathrm{FOA}}) = V$, and the resulting allocation of resources is efficient.

When the worker's expected utility function is not strictly concave, however, then condition (5) need not be sufficient for incentive compatibility. In that case, the optimal effort level μ^{FOA} in the relaxed program may be merely a local maximum of the worker's expected utility function.[6] In other words, the tournament that uses the prize structure $(W_1^{\mathrm{FOA}}, W_2^{\mathrm{FOA}})$ need not possess a symmetric pure-strategy Nash equilibrium. In particular, $\mu^* \neq \mu^{\mathrm{FOA}}$, i.e., the first-order approach is not justified. Notwithstanding, as pointed out by Green and Stokey (1983, fn. 3), it may still be feasible for the firm to design the tournament in such a way that a symmetric pure-strategy Nash equilibrium exists. In fact, as shown in the Appendix, this can always be done in an optimal way.

Proposition 1. *An optimal tournament exists (i.e., even if the first-order approach is not justified).*

The proposition raises the question of how the optimal tournament looks like in settings not traditionally considered. This question is addressed in the following sections.

3 Side-stepping the first-order approach

This section describes the envelope approach to rank-order tournaments that has been outlined in the Introduction. Note first that one may add the

[6] Indeed, checking the local second-order condition shows that μ^{FOA} cannot be, say, a local minimum or a saddle point.

equality constraint

$$U(W_1) - U(W_2) = \Delta(\mu) \equiv \frac{C'(\mu)}{g(0)} \tag{8}$$

to problem (2-4) without affecting the solution. Incentive compatibility (4) then becomes equivalent to

$$\Delta(\mu)(G(\widehat{\mu} - \mu) - G(0)) + C(\mu) - C(\widehat{\mu}) \leq 0 \qquad (\widehat{\mu} \geq 0). \tag{9}$$

Consider now the "upper envelope" of the individual constraints in (9), i.e.,

$$\varphi(\mu) = \max_{\widehat{\mu} \geq 0} \{(G(\widehat{\mu} - \mu) - G(0))\Delta(\mu) + C(\mu) - C(\widehat{\mu})\}. \tag{10}$$

By the Inada conditions, the maximum in (10) is indeed attained. Problem (2-4) may now be reformulated as

$$\max_{\mu \geq 0} \quad \overline{U}(\mu) \tag{11}$$

$$\text{s.t.} \quad \varphi(\mu) \leq 0, \tag{12}$$

where $\overline{U}(\mu)$ denotes indirect utility, i.e., the value of the firm's objective function (2) under the condition that the prize structure (W_1, W_2) is defined implicitely through (3) and (5).[7] The reformulated problem (11-12) is still not standard, because φ may have kinks. However, using the tools provided by Segal and Milgrom (2002), it can be shown that φ is monotone increasing

[7] It is not hard to check that $\overline{U}(\mu)$ is well-defined for any $\mu \geq 0$. Indeed, using (3) to eliminate W_2 in (5), one obtains

$$U(W_1) - U(2V\mu - W_1) = \frac{C'(\mu)}{g(0)}.$$

Differentiating the left-hand side with respect to W_1, and noting that $U' > 0$, shows that there is at most one solution. Further, since $U'' \leq 0$, the left-hand side approaches $\pm\infty$ as $W_1 \to \pm\infty$. By continuity, there is a unique solution.

provided that marginal costs are logconcave.[8] Moreover, since $\varphi(0) = 0$, monotonicity implies that the feasible set in problem (11-12) is a closed interval whose left endpoint is zero. Hence, the following result is obtained.

Proposition 2. *Suppose that C' is logconcave. Then $\mu^* \leq \mu^{\mathrm{FOA}}$. In particular, if the first-order approach is not justified, then $\mu^* < \mu^{\mathrm{FOA}}$.*

Proposition 2 shows that the first-order approach to tournament design is not innocuous, in the sense that it has the potential to cause a bias in the level of effort considered to be implementable. Indeed, when the usual equilibrium is disrupted, then there necessarily exists at least one effort level $\mu_* \neq \mu^*$ such that the worker's expected utility from choosing μ_* equals the equilibrium utility.[9] This type of constraint tends to make it harder for the firm to elicit a high level of effort from the worker, and thereby lowers the optimally implemented level of effort relative to the solution obtained through the first-order approach.

To understand why an assumption on costs is needed, note that raising μ has altogether three effects on the envelope constraint (12). First, $C(\mu)$ increases, which tightens the constraint. Second, $U(W_1) - U(W_2)$ increases, which loosens the constraint. Finally, deviations become less likely to win, which also loosens (12). However, if costs are not excessively convex then the change to the prize structure remains sufficiently moderate compared

[8]It should be noted that logconcavity of marginal costs is a very mild assumption that is consistent with both convex marginal costs (Chan et al., 2009) and concave marginal costs (Akerlof and Holden, 2012). Also, marginal costs cannot be globally logconvex under the Inada conditions imposed. Still, it remains an assumption, of course.

[9]If g is strictly unimodal, then the second-order condition at μ_* implies $\mu_* < \mu^*$, as one would expect.

to the differential of the other two effects, tipping the balance in favor of a tightening constraint.

The size of the potential welfare loss captured by Proposition 2 is not negligible. To the contrary, as will become clear below, tournaments may be quite ineffective as an incentive device.[10]

4 An explicit characterization

This section presents a complete characterization of the optimal tournament in a standard setting. Specifically, it will be assumed that costs are quadratic, i.e., that $C(\mu) = c\mu^2/2$ for some $c > 0$, and that workers exhibit a constant absolute risk aversion, i.e., that either $U(W) = -e^{-sW}/s$ for $s > 0$ or $U(W) = W$. These assumptions are made for tractability and can be relaxed. Indeed, as discussed below, the main features of the optimal tournament do not depend on these assumptions.

To describe the equilibrium in cases where the first-order approach is not valid, it proves useful to take a comparative statics perspective with respect to the dispersion of the differential error term. Thus, for a given distribution function G and an arbitrary parameter $\sigma > 0$, one defines a new distribution function $G_\sigma(z) \equiv G(z/\sigma)$, where a larger σ corresponds to a more dispersed distribution of the differential error term. E.g., if G is standard normal, then $G_\sigma(z)$ is normal with mean zero and standard deviation σ.

Denote by $\mu^{\mathrm{FOA}}(\sigma)$ the optimal effort level in the relaxed problem associated with G_σ. As discussed in Section 2, this solution can be approximated

[10] To mitigate the welfare loss, firms might decide to use deliberately inaccurate performance measures (O'Keeffe et al., 1984), or to induce mixed-strategy equilibria (Nalebuff and Stiglitz, 1983, Appendix). Both options are excluded here, however.

in the case of risk aversion and fully solved in the case of risk neutrality. The optimally implemented effort $\mu^*(\sigma)$ may now be characterized as follows.

Proposition 3. *Suppose that costs are quadratic and that workers have CARA utility (which includes the case of risk-neutrality as a limit case). Then, there is a threshold value $\sigma^* > 0$ such that, for any $\sigma > 0$, the optimal tournament associated with G_σ implements the effort level*

$$\mu^*(\sigma) = \min\{\frac{\sigma}{\sigma^*} \cdot \mu^{\mathrm{FOA}}(\sigma^*), \mu^{\mathrm{FOA}}(\sigma)\}. \tag{13}$$

As the proposition shows, the optimal tournament will be shaped by the envelope constraint (12) once the level of individual-specific uncertainty falls below a certain level. In particular, the usual comparative statics result that effort is decreasing in σ (Lazear and Rosen, 1981, p. 853; McLaughlin, 1988, fn. 5) breaks down. Instead, the optimally implemented effort level $\mu^*(\sigma)$ is strictly unimodal in the case of risk aversion, and piecewise linear in the case of risk neutrality where $\mu^{\mathrm{FOA}}(\sigma)$ is a constant.

Denote by $W_1^{\mathrm{FOA}}(\sigma)$ and $W_2^{\mathrm{FOA}}(\sigma)$ the optimal prizes for the relaxed problem. Using the usual second-order Taylor expansion of utility around mean income, the prize spread implementing the optimal effort level can be shown to satisfy

$$W_1^*(\sigma) - W_2^*(\sigma) \approx \min\{\frac{\sigma}{\sigma^*}, 1\} \cdot \left(W_1^{\mathrm{FOA}}(\sigma) - W_2^{\mathrm{FOA}}(\sigma)\right), \tag{14}$$

where the approximation is accurate for $\sigma \geq \sigma^*$.[11] Thus, also the prediction of the prize spread may be biased under the first-order approach. In partic-

[11] To see this, note that the necessary first-order condition (5) implies $W_1 - W_2 \approx c\mu\sigma/U'g(0)$ for the respective solutions of the unrelaxed and the relaxed problems.

ular, as σ gets smaller, the optimal prize spread diminishes much faster than the first-order approach would suggest.[12]

The worker's problem has a unique global maximum for $\sigma > \sigma^*$, but there will be at least one additional global maximum at some μ_* for $\sigma \leq \sigma^*$. Even under the assumptions of Proposition 3, the cardinality of the set of maximizers may be large (possibly infinite). However, assuming in addition that g is strictly bell-shaped,[13] one can convince oneself that the worker's objective function allows at most two local maxima. In those cases, the threshold value σ^* may be computed numerically by exploiting the first-order condition at μ_* as well as the worker's indifference between μ_* and μ^*. For example, in the case of a standard normal distribution and risk neutrality, $\sigma^* \approx 0.2 \cdot V/c$.

Notably, constraint (12) may come into play in response to changes in V or c, i.e., even if the information structure does not change. As discussed in the next section, an increase in the number of contestants may have a similar effect.

5 Large tournaments

This section considers an extension to tournaments with more than two contestants. Attention will be restricted to the special case of a single winner.

Denote by F and f the distribution and density functions associated

[12] When the assumptions of Proposition 3 are relaxed, one can still show that $\mu^*(\sigma) = \mu^{\text{FOA}}(\sigma)$ for σ sufficiently large and that $\mu^*(\sigma) \to 0$ as $\sigma \to 0$. Thus, even though the homogeneous relationships reflected in (13) and (14) tend to break down for cost functions that do not exhibit a constant elasticity, the characterization result captures, in its essence, a more general fact.

[13] I.e., there is some $r > 0$ such that $g''(z) \gtrless 0$ if $|z| \gtrless r$.

with an individual error term ε (assumed i.i.d. across players). Considering a tournament between n workers, and provided that all opponents of some given player j exert the same effort level μ, worker j's probability of winning may be represented as

$$G_n(\mu_j, \mu) = \int_{-\infty}^{+\infty} F(\mu_j + \varepsilon - \mu)^{n-1} dF(\varepsilon). \tag{15}$$

The problem of the firm is only slightly modified:

$$\max_{\substack{W_1 \geq W_2 \\ \mu \geq 0}} \quad \frac{U(W_1) + (n-1)U(W_2)}{n} - C(\mu) \tag{16}$$

s.t.

$$\mu V = \frac{W_1 + (n-1)W_2}{n} \tag{17}$$

$$(U(W_1) - U(W_2))G_n(\widehat{\mu}, \mu) - C(\widehat{\mu}) \tag{18}$$

$$\leq (U(W_1) - U(W_2))G_n(\widehat{\mu}, \mu) - C(\mu) \qquad (\widehat{\mu} \geq 0)$$

The optimal tournament satisfies, in particular, the necessary first-order condition for a symmetric pure-strategy Nash equilibrium,

$$U(W_1) - U(W_2) = \Delta_n(\mu) \equiv \frac{C'(\mu)}{g_n}, \tag{19}$$

where

$$g_n = (n-1) \int_{-\infty}^{+\infty} F(\varepsilon)^{n-2} f(\varepsilon)^2 d\varepsilon. \tag{20}$$

An approximation for the solution of the relaxed problem, μ_n^{FOA}, can be found as before. However, as pointed out by McLaughlin (1988, p. 241), it is in general very difficult to tell if the first-order approach is valid for large n.

To side-step the first-order approach, one defines again the "upper envelope," which reads in this case

$$\varphi_n(\mu) = \max_{\widehat{\mu} \geq 0} \left\{ (G_n(\widehat{\mu}, \mu) - G_n(\mu, \mu))\Delta_n(\mu) + C(\mu) - C(\widehat{\mu}) \right\}. \tag{21}$$

Then, as above, one can show that if marginal costs are logconcave, then the optimally implemented effort μ_n^* in the tournament between n workers and the corresponding optimal effort level μ_n^{FOA} in the relaxed problem satisfy $\mu_n^* \leq \mu_n^{\mathrm{FOA}}$. Thus, also in tournaments with more than two contestants, the first-order approach, if invalid, would tend to overstate implemented effort levels.

Additional conclusions can be obtained by focusing, as Nalebuff and Stiglitz (1983) do, on the incentive compatibility condition at the specific effort level $\widehat{\mu} = 0$. In the case of the normal distribution at least, one may then characterize the limit behavior of μ_n^* as follows.

Proposition 4. *Suppose that F is normal. Then, as the number of contestants n increases above all finite bounds, the optimally implemented effort level μ_n^* goes to zero.*

The result above characterizes the limit behavior of a sequence of optimal tournaments in a setting where it is a priori not clear if the first-order approach is applicable. It follows from the proposition that the first-order approach is indeed invalid in large tournaments in the case of risk-neutrality. Even though Proposition 4 holds also under the assumption of risk-aversion, no conclusion is possible about the validity of the first-order approach in large tournaments for the case of risk-aversion. However, this fact only supports the usefulness of the envelope approach because it delivers results also in situations where sufficient conditions for the first-order approach may be difficult to find.

6 Conclusion

In this paper, it has been shown that the first-order approach, if used exclusively, may lead to a positively biased assessment of the efficiency of rank-order tournaments. In particular, tournaments may not be very suitable as compensation schemes when performance is a relatively good signal of effort. Intuitively, prize structure and performance measurement are complements, forcing firms to reduce the former when the latter improves. In the settings studied above, the prize structure is so unrewarding that the avoidance of cheating becomes a binding constraint. Individual contracts such as piece rates may then dominate the optimal tournament even when workers are risk-neutral.[14]

Regarding further research, one issue might be the question of whether the theoretical issues discussed in this paper may constitute a practical reason for not using tournaments. For example, Lazear and Rosen (1981, p. 848) argue that in the case of risk-neutrality, the tie between individual contracts and tournaments is broken by differential costs of information and measurement. The present analysis obviously provides an alternative hypothesis. Another interesting issue would be the extension of the present analysis to more than two prizes or to the case of heterogeneous contestants. Finally, it might be worthwhile to explore whether the comparably simple approach outlined in Section 3 could be applied to other settings in contract theory and mechanism design.

[14] With this type of observation, the present paper takes the same line as, e.g., Chaigneau et al. (2014), who show that the sufficient statistics theorem fails to hold when the first-order approach is dropped in a standard principal-agent problem.

7 Appendix

Proof of Proposition 1. By Jensen's inequality, condition (3) implies $(U(W_1) + U(W_2))/2 \leq U(\mu V)$. Hence, from the Inada conditions, there is a $\overline{\mu} > 0$ such that implementing $\mu \notin [0, \overline{\mu}]$ is never optimal. Via (5) and (3), there is similarly a $\overline{W} > 0$ such that $W_1, W_2 \notin [-\overline{W}, \overline{W}]$ are never optimal. Thus, one may replace the feasible set by the bounded subset $I = \{(W_1, W_2, \mu) \in [-\overline{W}, \overline{W}]^2 \times [0, \overline{\mu}] : (3), (4), \text{ and } W_1 \geq W_2\}$. But $I \neq \varnothing$ for $\overline{W}$ sufficiently large, because then $(U(0), U(0), 0) \in I$. Moreover, I is closed as an intersection of closed sets. □

The following lemma is used in the proof of Proposition 2.

Lemma A.1. *Define*

$$\psi(\mu, \widehat{\mu}) \equiv \frac{\partial}{\partial \mu} \{\Delta(\mu)(G(\widehat{\mu} - \mu) - G(0)) + C(\mu) - C(\widehat{\mu})\} \tag{22}$$

$$= \Delta'(\mu)(G(\widehat{\mu} - \mu) - G(0)) - \Delta(\mu) g(\widehat{\mu} - \mu) + C'(\mu), \tag{23}$$

where $\Delta'(\mu) = C''(\mu)/g(0)$*. Then the family* $\{\psi(\cdot, \widehat{\mu})\}_{\widehat{\mu} \geq 0}$ *is equidifferentiable at any* $\mu \geq 0$.

Proof. Since g is a density with bounded first and second derivatives,

$$\frac{\partial^2 \psi(\mu, \widehat{\mu})}{\partial \mu^2} = \Delta'''(\mu)(G(\widehat{\mu} - \mu) - G(0)) - 3\Delta''(\mu) g(\widehat{\mu} - \mu) \tag{24}$$

$$+ 3\Delta'(\mu) g'(\widehat{\mu} - \mu) - \Delta(\mu) g''(\widehat{\mu} - \mu) + C'''(\mu) \tag{25}$$

exists and is bounded in $\widehat{\mu}$, for any $\mu \geq 0$. It follows that the family $\{\partial \psi(\cdot, \widehat{\mu})/\partial \mu\}_{\widehat{\mu} \geq 0}$ is equicontinuous at any $\mu \geq 0$. Using the Mean Value Theorem, as in Milgrom and Segal (2002, p. 587), $\{\psi(\cdot, \widehat{\mu})\}_{\widehat{\mu} \geq 0}$ is now seen to be equidifferentiable at any $\mu \geq 0$. □

Proof of Proposition 2. Denote by $X(\mu)$ the set of maximizers in problem (10). Using Lemma A.1, it follows from Milgrom and Segal (2002, Th. 1&3) that φ is right-hand differentiable at any $\mu \geq 0$ with

$$\varphi'(\mu+) \equiv \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \left(\varphi(\mu + \varepsilon) - \varphi(\mu) \right) \geq \psi(\mu, \widehat{\mu}), \tag{26}$$

for any $\widehat{\mu} \in X(\mu)$.[15] Moreover, as a consequence of local and global optimality conditions,

$$\Delta(\mu) g(\widehat{\mu} - \mu) - C'(\widehat{\mu}) \leq 0, \tag{27}$$

and

$$\Delta(\mu)(G(\widehat{\mu} - \mu) - G(0)) + C(\mu) - C(\widehat{\mu}) \geq 0, \tag{28}$$

for any $\widehat{\mu} \in X(\mu)$. Suppose $\mu > 0$. Then, using inequalities (27) and (28) to put a lower bound on (23) shows that

$$\varphi'(\mu+) \geq -\frac{C''(\mu)}{C'(\mu)}(C(\mu) - C(\widehat{\mu})) - C'(\widehat{\mu}) + C'(\mu) \equiv \phi(\mu, \widehat{\mu}) \tag{29}$$

for any $\widehat{\mu} \in X(\mu)$. By assumption, C''/C' is weakly decreasing. Therefore, for any $\widehat{\mu} \leq \mu$,

$$\frac{C''(\mu)}{C'(\mu)}(C(\mu) - C(\widehat{\mu})) = \frac{C''(\mu)}{C'(\mu)} \int_{\widehat{\mu}}^{\mu} C'(\widetilde{\mu}) d\widetilde{\mu} \tag{30}$$

$$\leq \int_{\widehat{\mu}}^{\mu} C'(\widetilde{\mu}) \frac{C''(\widetilde{\mu})}{C'(\widetilde{\mu})} d\widetilde{\mu} \tag{31}$$

$$= C'(\mu) - C'(\widehat{\mu}). \tag{32}$$

Hence, $\phi(\mu, \widehat{\mu}) \geq 0$ in this case. Using completely analogous arguments, one shows that, similarly, $\phi(\mu, \widehat{\mu}) \geq 0$ if $\widehat{\mu} > \mu$. Thus, $\varphi'(\mu+) \geq 0$ for any $\mu > 0$. Note also that φ is continuous on $\mathbb{R}_+$, as a consequence of Berge's theorem. It

[15] Intuitively, the value function increases by at least as much as the value at any given global maximum.

follows that φ is monotone increasing (Royden, 1988, Sec. 5). Hence, noting that $\varphi(0) = 0$, the feasible set of problem (11-12) is an interval $[0, \mu^{\#}]$, for some $\mu^{\#} \geq 0$. But μ^{FOA} is a global optimum of $\overline{U}$. Therefore, $\mu^* \leq \mu^{\mathrm{FOA}}$, proving the first assertion. The second assertion is now immediate. $\square$

For the following three lemmas, the assumptions of Proposition 3 are imposed.

Lemma A.2. *$\mu^*(\sigma) \neq \mu^{\mathrm{FOA}}(\sigma)$ for some $\sigma > 0$.*

Proof. As in the proof of Proposition 1, there is a $\overline{\mu} > 0$ such that $\mu^{\mathrm{FOA}}(\sigma) \leq \overline{\mu}$ for any $\sigma > 0$. Hence, the first-order condition

$$U(W_1) - U(W_2) = \frac{C'(\mu)}{g_\sigma(0)} = \frac{\sigma c \mu}{g(0)} \leq \frac{\sigma c \overline{\mu}}{g(0)}, \tag{33}$$

with $g_\sigma(z) \equiv g(z/\sigma)/\sigma$, implies that $W_1^{\mathrm{FOA}}(\sigma) - W_2^{\mathrm{FOA}}(\sigma)$ vanishes as $\sigma \to 0$. As a consequence, the approximation in

$$\mu^{\mathrm{FOA}}(\sigma) \approx \frac{V/c}{1 + s\sigma c/4g(0)^2} \tag{34}$$

becomes arbitrarily accurate, so that $\mu^{\mathrm{FOA}}(\sigma) \to V/c$. On the other hand, from incentive compatibility with respect to a deviation to $\widehat{\mu} = 0$,

$$0 \geq (G_\sigma(-\mu) - G_\sigma(0))\frac{C'(\sigma)}{g_\sigma(0)} + C(\mu) - C(0) \geq -\frac{c\sigma\mu}{g(0)} + \frac{c\mu^2}{2}, \tag{35}$$

where the second inequality follows from $G_\sigma \leq 1$. Hence, $\mu \leq 2\sigma/g(0)$, and therefore, $\mu^*(\sigma) \to 0$ as $\sigma \to 0$. Thus, for any $\sigma > 0$ sufficiently small, $\mu^*(\sigma) \neq \mu^{\mathrm{FOA}}(\sigma)$, which proves the lemma. $\square$

Lemma A.3. *$\overline{U}$ is strongly pseudoconcave in μ.*

Proof. Total differentiation of equations (3) and (5), and subsequently solving the resulting system of linear equations, yields

$$\frac{dW_1}{d\mu} = \frac{2Vu_2' + c/g(0)}{u_1' + u_2'}, \tag{36}$$

$$\frac{dW_2}{d\mu} = \frac{2Vu_1' - c/g(0)}{u_1' + u_2'}, \tag{37}$$

where $u_1' \equiv U'(W_1)$ and $u_2' \equiv U'(W_2)$. Therefore,

$$\frac{\partial \overline{U}}{\partial \mu} = 2V\frac{u_1'u_2'}{u_1' + u_2'} + \frac{c}{2g(0)}\frac{u_1' - u_2'}{u_1' + u_2'} - c\mu. \tag{38}$$

Differentiating (38) with respect to μ, and assuming that $\partial\overline{U}/\partial\mu = 0$, one obtains

$$\frac{\partial^2\overline{U}}{\partial\mu^2} = \frac{2V}{u_1' + u_2'}\left\{u_1''u_2'\frac{dW_1}{d\mu} + u_1'u_2''\frac{dW_2}{d\mu}\right\} + \frac{c/2g(0)}{u_1' + u_2'}\left\{u_1''\frac{dW_1}{d\mu} - u_2''\frac{dW_2}{d\mu}\right\}$$
$$- \frac{c\mu}{u_1' + u_2'}\cdot\left\{u_1''\frac{dW_1}{d\mu} + u_2''\frac{dW_2}{d\mu}\right\} - c, \tag{39}$$

where $u_1'' \equiv U''(W_1)$ and $u_2'' \equiv U''(W_2)$. Hence, using (36-37) and $\partial\overline{U}/\partial\mu = 0$ another time, one arrives at

$$\frac{\partial^2\overline{U}}{\partial\mu^2} = (-2s)\cdot\frac{2V^2u_1'u_2' + c^2/4g(0)^2 - c^2\mu^2}{u_1' + u_2'} - c, \tag{40}$$

where $s = -u_1''/u_1' = -u_2''/u_2' \geq 0$. It follows that $\partial^2\overline{U}/\partial\mu^2 < 0$ if $\mu \leq 1/2g(0)$. Otherwise, i.e., if $\mu > 1/2g(0)$, then $\partial\overline{U}/\partial\mu = 0$ implies

$$2Vu_1'u_2' = c\mu(u_1' + u_2') - \frac{c}{2g(0)}(u_1' - u_2') \tag{41}$$

$$= c(\mu - \frac{1}{2g(0)})u_1' + c(\mu + \frac{1}{2g(0)})u_2' \tag{42}$$

$$\geq c\mu u_2'. \tag{43}$$

Hence, $2Vu_1' \geq c\mu$. Similarly, using $u_2' \geq u_1'$, one finds

$$2Vu_1'u_2' = c(\mu - \frac{1}{2g(0)})u_1' + c(\mu + \frac{1}{2g(0)})u_2' \geq 2c\mu u_1', \tag{44}$$

so that $Vu_2' \geq c\mu$. Multiplying the two inequalities, one arrives at $2V^2u_1'u_2' \geq c^2\mu^2$. It follows that $\partial^2\overline{U}/\partial\mu^2 < 0$, which proves the claim. □

Lemma A.4. $\mu^{\text{FOA}}(\sigma)$ *is continuous and strictly decreasing in* σ.

Proof. Differentiating (38) with respect to σ and exploiting that $\partial\overline{U}/\partial\mu = 0$, one obtains

$$\frac{\partial^2\overline{U}}{\partial\mu\partial\sigma} = \frac{2V}{u_1'+u_2'}\left\{u_1''u_2'\frac{dW_1}{d\sigma} + u_1'u_2''\frac{dW_2}{d\sigma}\right\} + \frac{c\sigma/2g(0)}{u_1'+u_2'}\left\{u_1''\frac{dW_1}{d\sigma} - u_2''\frac{dW_2}{d\sigma}\right\}$$
$$- \frac{c\mu}{u_1'+u_2'}\cdot\left\{u_1''\frac{dW_1}{d\sigma} + u_2''\frac{dW_2}{d\sigma}\right\} + \frac{c}{2g(0)}\frac{u_1'-u_2'}{u_1'+u_2'}. \qquad (45)$$

But, from equations (3) and (33), it is immediate that

$$\frac{dW_1}{d\sigma} = -\frac{dW_2}{d\sigma} = \frac{c\mu}{g(0)(u_1'+u_2')}. \qquad (46)$$

Simplifying the right-hand side of (45) using (46), one arrives at

$$\frac{\partial^2\overline{U}}{\partial\mu\partial\sigma} = -\frac{s\sigma c^2\mu}{2g(0)^2(u_1'+u_2')} - \frac{sc^2\mu^2(u_2'-u_1')}{g(0)(u_1'+u_2')^2} - \frac{c(u_2'-u_1')}{2g(0)(u_1'+u_2')} < 0. \qquad (47)$$

Since $\overline{U}$ is strongly pseudoconcave with respect to μ, the claim follows. □

Proof of Proposition 3. By Lemma A.2, there is a $\widetilde{\sigma} > 0$ such that $\mu^*(\widetilde{\sigma}) \neq \mu^{\text{FOA}}(\widetilde{\sigma})$. Hence, the envelope constraint must be binding in the reformulated problem associated with $G_{\widetilde{\sigma}}$. Since marginal costs are logconcave, it follows from the proof of Proposition 2 that $\mu \leq \mu^*(\widetilde{\sigma})$ is equivalent to

$$(G_{\widetilde{\sigma}}(\widehat{\mu}-\mu) - G_{\widetilde{\sigma}}(0))\frac{C'(\mu)}{g_{\widetilde{\sigma}}(0)} + C(\mu) - C(\widehat{\mu}) \leq 0 \quad (\widehat{\mu} \geq 0). \qquad (48)$$

Let $\sigma > 0$. Then, with $\lambda \equiv \sigma/\widetilde{\sigma}$, purely algebraic manipulation exploiting the homogeneity of the cost function shows that

$$(G_{\widetilde{\sigma}}(\widehat{\mu}-\mu) - G_{\widetilde{\sigma}}(0))\frac{C'(\mu)}{g_{\widetilde{\sigma}}(0)} + C(\mu) - C(\widehat{\mu})$$
$$= \frac{1}{\lambda^2}\left\{(G_\sigma(\widehat{\mu}_\lambda - \mu_\lambda) - G_\sigma(0))\frac{C'(\mu_\lambda)}{g_\sigma(0)} + C(\mu_\lambda) - C(\widehat{\mu}_\lambda)\right\}, \qquad (49)$$

where $\mu_\lambda \equiv \lambda\mu$ and $\widehat{\mu}_\lambda \equiv \lambda\widehat{\mu}$. Hence, $\mu_\lambda \leq \lambda\mu^*(\widetilde{\sigma})$ is equivalent to

$$(G_\sigma(\widehat{\mu}_\lambda - \mu_\lambda) - G_\sigma(0))\frac{C'(\mu_\lambda)}{g_\sigma(0)} + C(\mu_\lambda) - C(\widehat{\mu}_\lambda) \leq 0 \quad (\widehat{\mu}_\lambda \geq 0). \tag{50}$$

Invoking Lemma A.3, it follows that

$$\mu^*(\sigma) = \min\{\frac{\sigma}{\widetilde{\sigma}}\mu^*(\widetilde{\sigma}), \mu^{\mathrm{FOA}}(\sigma)\} \tag{51}$$

for any $\sigma > 0$. By Lemma A.4, there is a unique σ^* such that

$$\frac{\sigma^*}{\widetilde{\sigma}}\mu^*(\widetilde{\sigma}) = \mu^{\mathrm{FOA}}(\sigma^*). \tag{52}$$

Moreover,

$$\mu^*(\sigma) = \frac{\sigma}{\widetilde{\sigma}}\mu^*(\widetilde{\sigma}) = \frac{\sigma}{\sigma^*}\mu^{\mathrm{FOA}}(\sigma^*) \tag{53}$$

if $\sigma \leq \sigma^*$, and $\mu^*(\sigma) = \mu^{\mathrm{FOA}}(\sigma)$ if $\sigma > \sigma^*$. □

Proof of Proposition 4. Consider the specific deviation to $\widehat{\mu} = 0$. For any $\mu \geq 0$, we have

$$\varphi_n(\mu) \geq (G_n(0,\mu) - G_n(\mu,\mu))\frac{C'(\mu)}{g_n} + C(\mu) - C(0) \tag{54}$$

$$\geq -\frac{C'(\mu)}{ng_n} + C(\mu) - C(0), \tag{55}$$

since $G_n(\mu,\mu) = \frac{1}{n}$. For μ_n^* to constitute an equilibrium in the tournament between n workers, it is necessary that $\varphi_n(\mu_n^*) \leq 0$. Hence,

$$\frac{C(\mu_n^*) - C(0)}{C'(\mu_n^*)} \leq \frac{1}{ng_n}. \tag{56}$$

Because $f'(\varepsilon) = -\varepsilon f(\varepsilon)/\sigma^2$ in the case of the normal distribution, integrating by parts yields

$$ng_n = n \int_{-\infty}^{+\infty} (n-1) F(\varepsilon)^{n-2} f(\varepsilon)^2 d\varepsilon \tag{57}$$
$$= -n \int_{-\infty}^{+\infty} F(\varepsilon)^{n-1} f'(\varepsilon) d\varepsilon \tag{58}$$
$$= \frac{n}{\sigma^2} \int_{-\infty}^{+\infty} \varepsilon F(\varepsilon)^{n-1} f(\varepsilon) d\varepsilon \tag{59}$$
$$\backsimeq \frac{1}{\sigma}\sqrt{2\ln n}, \tag{60}$$

where the asymptotic relationship for the mean extreme of n identically and independently distributed normal variables has been taken from David and Naragaja (2003, Sec. 10.5). But, as in the proof of Proposition 1, Jensen's inequality implies

$$\frac{U(W_1) + (n-1)U(W_2)}{n} \leq U(\mu V) \tag{61}$$

for any n. Hence, $\mu_n^* \leq \overline{\mu}$ for any n. Since $ng_n \to \infty$ for $n \to \infty$, it follows from (56) that, indeed, $\mu_n^* \to 0$ for $n \to \infty$. □

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