Economic distributions and primitive distributions in monopolistic competition

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Abstract

We link fundamental technological and taste distributions to endogenous economic distributions of firm size (output, profit) and prices in extensions of canonical IO and trade models. On the IO side, we develop the logit model of monopolistic competition with heterogeneous firms to show that an exponential (resp. normal) distribution generates a Pareto (resp. log-normal) economic size distribution. We formulate general IIA monopolistic competition models with mark-ups that depend on quality-cost, we tie down the demand structure from the output and profit distributions, and we find long-run equilibrium distributions as a function of the primitives: consumer preferences, entry costs, and the distributions of costs and quality-costs. On the Trade side, we provide a parallel analysis for the CES and break the Pareto circle by introducing quality into consumer preferences.

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1 Introduction

Distributions of economic variables have attracted the interest of economists at least since Pareto (1896). In industrial organization, firm size (output, sales, or profit) distributions have been analyzed, while different studies have looked at the distribution of prices within an industry. In international trade, recent advances have enabled studying distributions of sales revenues.[1] The distributions of these "economic" variables are (presumably) jointly determined by the fundamental underlying distributions of tastes and technologies. In this paper we determine the links between the various distributions, both linking the economic ones to each other and to the fundamental ones (which can be uncovered from the economic ones).

The vehicle we use for this purpose is the Logit model of monopolistic competition, which we develop and extend here. This model has several advantages. First, the logit is the workhorse model in structural empirical IO. Second, it readily incorporates taste and cost heterogeneity. Third, it is tractable, where virtually no other model is, and we generalize it to allow for endogenous mark-up functions that depend on quality-cost differences across firms. An alternative model is the logit's close cousin, the CES model, which we extend to allow for quality differences in order to deliver a richer set of distribution patterns than is allowed in the standard version with just cost heterogeneity.

Firm sizes (profitability, say) within industries are wildly asymmetric, and frequently involve a long-tail of smaller firms, which still make positive profits. The idea of the long tail has recently been invoked prominently in studies of Internet Commerce (Anderson, 2006, Oberholzer-Gee, 2012), and particular distributions – mainly the Pareto and log-normal – seem to fit the data well. The monopolistic competition framework has recently been deployed to this end.[2]

Our intent is to provide the theoretical links between the various economic distributions of interest, and the underlying distributions that generate them. We connect distributions of economic variables with those of fundamentals (taste/technology) using the logit monopolistic competition model. We show a three-way relation between two groups of distributions and the quality-to-cost relation: knowing one element from any two of these ties down the third. On one leg, we generate the relation between equilibrium profit dispersion, firm outputs, and the fundamental quality-cost distribution. On a second leg, we show the relation between the cost distribution and the dispersion of equilibrium prices. Knowing any one of the distributions on one leg suffices to determine the others on

[1] See e.g., Eaton, Kortum, and Kramarz (2011).

[2] Ironically, Chamberlin (1933) is best remembered for his symmetric monopolistic competition analysis. Yet he went to great length to point out that he believed asymmetry to be the norm, and that symmetry was a very restrictive assumption. We model both quality and production cost differences across firms.

that leg, and knowing a distribution from each leg allows us to determine what the relation between cost and quality must be on the third leg. Some important equivalences include that normally distributed quality-costs induce log-normal distributions of profits, and that a power distribution of costs along with an exponential distribution of quality-costs leads to a Pareto distribution for profit [check!!]. We then broaden our scope by relaxing the constant mark-up property of the logit in deploying a more general model of monopolistic competition based on the IIA property (shared by Logit and CES models). This model allows us to determine the demand function from mark-up and quality-cost distributions, which can in turn be backed out of ?? distributions.

We provide a parallel analysis for the CES model. The CES representative consumer model is widely used in economics in conjunction with a market structure assumption of monopolistic competition It is used as a theoretical component in the New Economic Geography and Urban Economics, it is the linchpin of Endogenous Growth Theory, Keynesian underpinnings in Macro, and of course, Industrial Organization. The current most intensive use of the model is in International Trade, following Melitz (2003), where it is at the heart of empirical estimation. The convenience of the model stems from its analytic manipulability and simple welfare properties. The CES model delivers equilibrium mark-ups proportional to marginal costs, and so delivers market imperfection (imperfect competition) in a simple way without complex market interaction. The standard models in this vein (following Melitz, 2003) assume that firms' unit production costs are heterogeneous, and thus generate cut-off cost values determining which firms are in the market.

However, when we apply this model to distributions, if one distribution (such as profits) is described as a Pareto distribution, then the distributions of all the economic variables lie in the Pareto family. This we call the "Pareto circle". We break the circle by introducing qualities into the demand model. Doing so implies that there are two fundamental drivers of equilibrium distributions (instead of just one) – the cost distribution and the quality/cost one. Hence even if one distribution is Pareto, then others can take different forms. Most notably, the output distribution depends on the cost distribution (as before) but now also on the quality/cost distribution.

There has been a flurry of recent contributions using the CES and variants thereof (e.g., Dhingra and Morrow, 2013, Zhelobodko, Kokovin, Parenti, and Thisse, 2012, Bertoletti and Etro, 2013 and 2014, etc.). Rather than start with a representative consumer, we instead prefer, as per the IO literature, to build up from discrete choice roots of individuals choosing one of the many options available, and thus generating market demands for differentiated products by aggregating up over individuals.[3]

We argue that the Logit is a reasonable and attractive alternative framework to the CES. Anderson, de Palma, and Thisse (1992) have shown that the CES can be viewed as a form of Logit model. It is shown here that the monopolistic competition version of the Logit delivers simple mark-ups like the CES, and is also amenable to adding other variables of interest, such as quality differences. We first describe the Logit model of monopolistic competition, which is simpler analytically than the oligopoly version and this enables the deeper distributional analysis. We then show how the distribution of quality-costs determines the equilibrium distribution of outputs and profits (our size variables). The Pareto economic distribution is generated from an exponential primitive distribution; a normal fundamental distribution begets a log-normal economic distribution.

2 The Logit Model of Monopolistic Competition

2.1 Set-up

There is a continuum of active firms. Each firm, i, is associated to a distinctive quality v_i, (constant) marginal production cost, c_i, and chooses a price, p_i. Let Ω be the set of active (producing) firms, and let ω denote an element of this set. Total demand is normalized to 1, w.l.o.g. Demand for Firm i is a Logit function of active firms' qualities and prices:[4]

$$y_i = \frac{\exp\left(\frac{v_i - p_i}{\mu}\right)}{\int_{\omega\in\Omega} \exp\left(\frac{v(\omega)-p(\omega)}{\mu}\right) d\omega + \exp\left(\frac{v_0}{\mu}\right)}, \quad i \in \Omega, \tag{1}$$

where $\mu > 0$ measures the degree of product heterogeneity and $v_0 \in (-\infty, \infty)$ measures the attractivity of the outside option (which could also represent a competitive sector). We thus adapt the continuous Logit model (see Ben-Akiva and Watanada, 1981) to monopolistic competition.[5]

Consumer choices are driven by two forces. First, absent product differentiation, consumers want the best quality-price deal (highest $v_i - p_i$). Second, consumers have idiosyncratic tastes for differentiated products. When the level of product differentiation measured by μ is very large, quality-price is unimportant and each differentiated good has the *same* probability of being purchased (because the underlying idiosyncratic tastes are symmetric). In between, there is a trade-off between objec-

[3] Fajgelbaum, Grossman, and Helpman (2009) likewise take a nested multinomial logit approach.

[4] We assume that the integral in the denominator is bounded: conditions are given below.

[5] As shown by Anderson, de Palma, and Thisse (1992), such a demand function can be generated from a representative consumer utility function with entropic form, or else from the more traditional discrete choice theoretic roots as per the usual econometric derivation (see, McFadden, 1978).

tive quality (vertical differentiation, as measured by quality-price), and subjective quality (horizontal differentiation): the best choice reflects both idiosyncratic preferences and quality-prices.

The (gross) profit for Firm i is

$$\pi_i = (p_i - c_i)\, y_i, \qquad i \in \Omega. \tag{2}$$

Under monopolistic competition the own-demand derivative is

$$\frac{dy_i}{dp_i} = \frac{-y_i}{\mu}, \qquad i \in \Omega, \tag{3}$$

because the firm has no impact on the denominator in (1). Hence

$$\frac{d\pi_i}{dp_i} = y_i \left[1 - \frac{(p_i - c_i)}{\mu}\right], \qquad i \in \Omega,$$

and, since the term inside the square brackets is strictly decreasing in p_i while the term outside is positive, the profit function is strictly quasi-concave and the profit-maximizing price of Firm i is[6]

$$p_i^* = c_i + \mu, \qquad i \in \Omega. \tag{4}$$

The absolute mark-up is the same across all firms regardless of quality.[7] In Section 5 we show how generalizations of the Logit model relax this property.

Given the pricing rule (4), the corresponding equilibrium outputs (using (1) and (4)) are

$$y_i = \frac{\exp\left(\frac{v_i - c_i}{\mu}\right)}{\int_{\omega\in\Omega} \exp\left(\frac{v(\omega)-c(\omega)}{\mu}\right) d\omega + \mathcal{V}_0}, \qquad i \in \Omega, \tag{5}$$

where we have defined $\mathcal{V}_0 = \exp\left(\frac{v_0}{\mu} + 1\right) \geq 0$ (and we have suppressed a star on y_i to not encumber the analysis, but it is understood henceforth as the equilibrium output). The relation (5) indicates an output ranking over firms based directly on the level of quality-cost (to be read as quality minus cost):

$$y_i > y_j \quad \text{if and only if} \quad (v_i - c_i) > (v_j - c_j), \qquad i, j \in \Omega.$$

[6] In the oligopoly case, with n firms, the equilibrium prices are the implicit solutions of equations $p_i^* = c_i + \frac{\mu}{1-y_i}$, $i = 1...n$. In the symmetric case, $p^* = c + \frac{\mu n}{n-1}$, which converges to $c + \mu$ as $n \to \infty$ (Anderson, de Palma, and Thisse, 1992, Ch.7).

[7] In comparison, the CES model gives the constant *relative* mark-up property: $p_i^* = c_i(1 + \mu)$, which also holds regardless of the quality variable (see Section 7). The similarity between the Logit and CES is not fortuitous, and has previously been developed by Anderson, de Palma, and Thisse (1992): μ is related to the parameter ρ in CES models by the relation $\mu = \frac{1-\rho}{\rho}$. In particular, both models can be construed as sharing individual discrete choice roots with a Type I Extreme Value of idiosyncratic errors.

The equilibrium (gross) profit is $\pi_i = (p_i - c_i)\, y_i = \mu y_i$, $i \in \Omega$, where y_i is the equilibrium output defined in (5). Hence, outputs and profits are fully characterized by quality-cost levels. The next Proposition summarizes these results.

Proposition 1 *In the Logit model of Monopolistic Competition, all firms set the same absolute mark-up, μ. Higher quality-costs are expressed as higher equilibrium outputs and higher profits.*

A very high quality and a high cost is equivalent (for output and profit) to a low quality/ low cost combination. Hence, all we need to track to determine the distribution of output and profit is the distribution of quality minus cost, a unidimensional variable.[8] Insofar as higher qualities are also higher costs in practice, then they are also higher priced. However, output and profitability may well be highest for medium-quality products (see the further discussion in Section 4.4).

2.2 Quality-Cost, Output, and profit distributions

We now determine how equilibrium output and profit are distributed. We take the distribution of the quality-cost as a primitive, and derive the endogenous consequent distribution of the economic variable, output. As (5) indicates, all that matters is the distribution of the difference between quality and costs. Hence, let x be a one-dimensional parameterization of quality-cost, $v - c$.

Let the distribution of quality-cost be $F_X(x) = \Pr(X < x)$, with density $f_X(\cdot)$ and support $[\underline{x}, \infty)$. The idea is that if we draw at random a quality-cost level, X, from the population, the chance it is below x is $F_X(x)$. We seek the corresponding distribution of equilibrium output, denoted by $F_Y(y)$, and the relation between x and y is[9]

$$y = \frac{\exp\left(\frac{x}{\mu}\right)}{D}, \quad y \geq \underline{y} = \frac{\exp\left(\frac{\underline{x}}{\mu}\right)}{D}, \tag{6}$$

where we assume henceforth that $f_X(\cdot)$ is such that the output denominator D is finite:[10]

$$D = M \int_{u \geq \underline{x}} \exp\left(\frac{u}{\mu}\right) f_X(u)\, du + \mathcal{V}_0, \tag{7}$$

and M represents the total mass of firms, i.e., $M = \|\Omega\|$. For future reference, note that the consumer surplus associated to the logit model above is $V = \mu \ln D - \mu$, which is the continuum version of the classic "log-sum" formula, where the second term is the common mark-up.

The quality-cost distribution determines the distributions of equilibrium output and profit, where equilibrium (gross) profit is $\pi = \mu y \geq \underline{\pi} = \mu \underline{y}$.

[8] To find the distribution of prices, we need to track the distribution of cost. We do this in Section 2.2 below.
[9] Here all firms are active. We later introduce fixed costs to determine endogenously the set of active firms.
[10] This holds true for any finite support as well as for the examples below under the restrictions given.

Theorem 1 *For the Logit model of Monopolistic Competition, the distribution of quality-costs, $F_X(x)$, generates the distribution of equilibrium output $F_Y(y) = F_X(\mu \ln(yD))$ and the distribution of equilibrium profit $F_\Pi(\pi) = F_X\left(\mu \ln\left(\frac{\pi D}{\mu}\right)\right)$, where D is given by (7).*

The key relation underlying the twinning of distributions is the increasing relation between quality-cost and output (or profit) for the Logit. Similar increasing relations hold for other models than logit, namely the general IIA model below, as well as the CES (under some restrictions: see Section 7). The converse result to Theorem 1 uses the increasing relation to describe how quality-costs can be determined from output or profit distributions (where y_{av} and π_{av} denote average output and profit, respectively):

Theorem 2 *The quality-cost distribution for the Logit model of Monopolistic Competition, $F_X(x)$, can be derived from the equilibrium output distribution, $F_Y(y)$, via the relation $F_X(x) = F_Y\left(\frac{\exp\left(\frac{x}{\mu}\right)}{D_y}\right)$, where $D_y = \mathcal{V}_0 / (1 - My_{av})$. It can also be derived from the equilibrium profit distribution as $F_X(x) = F_\Pi\left(\mu \frac{\exp\left(\frac{x}{\mu}\right)}{D_\pi}\right)$, where $D_\pi = \mathcal{V}_0 / \left(1 - \frac{M}{\mu}\pi_{av}\right)$.*

Taking the two theorems together, a specific quality-cost distribution generates a specific output (resp. profit) distribution (Theorem 1). Conversely (by Theorem 2), this output or profit distribution could only have been generated from the initial quality-cost distribution. These two theorems thus show the equivalence between the different ways of describing logit monopolistic competition markets.

2.3 Comparative statics of distributions

We here briefly consider the comparative static properties of the model. Because we are dealing with distributions, we need to compare distributions. The natural way of doing so is to engage first order stochastic dominance (fosd).

Proposition 2 *i) An increase in quality-costs (in the sense of fosd) increases both mean output and mean profits; ii) a mean-preserving spread in the quality-cost distribution increases mean output and mean profit. The induced increases are strict if the market is not fully covered ($\mathcal{V}_0 > 0$).*

Even though the proof of the first part of the Proposition is straightforward, it belies some counteracting effects. While moving up quality-cost mass will move up output mass ceteris paribus, it also has an effect of increasing competition for all the other firms (a D effect), which ceteris paribus reduces their output. Note though mean output does not necessarily rise if mean quality-cost rises.[11]

Notice that because the relation between output and profit distributions does not involve D ($\Pi = \mu Y$ implies $F_{\Pi}(\pi) = F_Y\left(\frac{\pi}{\mu}\right)$), an increase in output (in the sense of fosd) implies an increase in profit, and vice versa. However, a fosd increase in quality-cost does not necessarily lead to a fosd increase in output. Suppose for example that the increase in quality-cost is small for low quality-costs, but large for high ones. Then competition is intensified (an increase in D), and output at the bottom end goes down, while rising at the top end. So then there can be a rotation of $F_Y(.)$ (in the sense of Johnson and Myatt, 2006) without fosd (a similar rotation is delivered in Proposition 3 below). Nevertheless, specific examples do deliver stronger relations, as we show in Section 3.

We next determine how the economic parameters feed through into the endogenous economic distributions. To do so, we first prove an ancillary result of independent economic interest.

Lemma 1 *For the logit model, consumer surplus (at fixed prices) increases with product variety, μ; hence $\mu \ln D$ increases in μ.*

The proof is in the Appendix. We show there that consumer surplus for the logit model increases with μ for any set of (given) product prices. The last statement in the Lemma is just a special case of the result. The proof proceeds by showing that the derivative of consumer surplus is given by the Shannon (1948) measure of information (which is also referred to as entropy), and this is positive for entropy.

Proposition 3 *A higher attractivity of the outside option, v_0, decreases outputs and profits (in the sense of fosd). A higher degree of product differentiation, μ, first-order stochastically increases outputs and profits for low quality-cost and first-order stochastically decreases them for high quality-cost; a lower profit is a sufficient condition for a firm to have a lower output.*

A more attractive outside option clearly reduces outputs and therefore profits of all firms. The impact of higher product heterogeneity is more subtle. When μ goes up, weak (low quality-cost) firms are helped and good ones are hurt. The intuition is as follows. With little perceived product differentiation, consumers tend to buy the best quality-cost products. With more product differentiation (which increases the mark-up), consumers tend to buy more of the low quality-cost goods (which have lower outputs, as per Proposition 1) and less of the high quality-cost goods (which have

[11] For example, mean output rises with a mean-preserving spread in quality-cost (part (ii)) but then if the mean of quality-cost is reduced slightly, mean output can still register an increase, so the two means can move in different directions.

higher outputs). Hence, higher μ evens out demands across options. The fact that output may decrease and profit increase with μ follows because $\pi_i = \mu y_i$. Thus it can happen that doubling μ does not double the profit of the top quality-cost firms, but it more than doubles the profit of the lowest quality-cost firms.

Whether high or low qualities are most profitable depends on whether quality-costs rise or fall with quality. Indeed, more profitable firms could be "in the middle" (see the further discussion in Section 4.4).

3 Specific distributions

We now present several applications of Theorem 1 for specific distributions of quality-costs and the output and profit distributions they generate. In some cases, the variables such as $\underline{\pi}$ or D are in turn functions of the underlying parameters of the quality-cost distribution. In others, as flagged, we are able to get closed form solutions. Bear in mind that the derived distributions still satisfy the comparative static properties with respect to μ and v_0 (see Proposition 3). Some comments deal with truncated versions. Given the key role of the Pareto distribution of firm size, we defer discussion of this distribution to the next sub-section (3.2).

We derive the equilibrium profit distributions: the equilibrium output distributions are analogous (because $\pi = \mu y$). We express all examples from quality-cost distributions to implied profit distributions: from Theorem 2 the reverse relations hold too, and output relations are analogous.

3.1 Normal, uniform, and truncated Pareto

The *normal* distribution is perhaps the most natural primitive assumption to take for quality-costs.

Corollary 1 *Let quality-cost $X \in (-\infty, \infty)$ be normally distributed, $X \sim N(m, \sigma)$. Then profit $\Pi \in (0, \infty)$ is log-normally distributed with parameters $\ln\left(\frac{D}{\mu}\right) - \frac{m}{\mu}$ and $\frac{\sigma}{\mu}$, where $D = M \exp\left(\frac{m}{\mu} + \frac{\sigma^2}{2\mu^2}\right) + \mathcal{V}_0$.*

The proof is in Appendix 2. Notice that increasing m increases quality-costs (in the sense of fosd), and increases mean profit (in accord with Proposition 2). In this case too, profits (and output) also increase (in the sense of fosd).

The resulting log-normal has sometimes been fitted to firm size distribution (see Cabral and Mata, 2001, for a well-cited study of Portuguese firms). Note that a truncated normal begets a truncated log-normal (which is therefore important once we consider free entry equilibria below).

The log-normal distribution is an alternative distribution to the Pareto distribution. It has been used to describe survival of species, giving a potential underpinning for describing the distributions of profits and outputs.

The simplest text-book case is the *uniform* distribution.

Corollary 2 *Let quality-cost be uniformly distributed with support* $[0,1]$*. Then the equilibrium profit* Π *has distribution* $F_{\Pi}(\pi)=\mu\ln\left(\frac{\pi D}{\mu}\right)$ *with support* $\pi\in\left[\frac{\mu}{D},\frac{\mu}{D}\exp\left(\frac{1}{\mu}\right)\right]$*, where* $D=\mu M\left(\exp\left(\frac{1}{\mu}\right)-1\right)$ $+\mathcal{V}_0$*.*

The CDF of profit is increasing and concave, and its density $f_{\Pi}(\pi)=\frac{\mu}{\pi}$ is unit elastic.

We finally consider the celebrated Pareto distribution for quality-cost. The untruncated version leads to an infinite value of D. We therefore consider a truncated Pareto distribution, with CDF $F_X(x)=\frac{1-\left(\frac{x}{\underline{x}}\right)^{-\alpha}}{1-\left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}}$, with $x\in[\underline{x},\overline{x})$ and $\alpha>0$. Recall that the mean and variance of the Pareto distribution are finite as long as $\alpha>1$. The leads, after some renormalization, to the Log-Pareto for profit (or output). We have:

Corollary 3 *Let quality-cost* $X\in[\underline{x},\overline{x})$ *be truncated Pareto distributed with parameter* $\alpha>0$*. Then profit has a truncated Log-Pareto distribution:*[12]

$$F_{\Pi}(\pi)=\frac{1-\left(1+\frac{\mu}{\underline{x}}\ln\left(\frac{\pi}{\underline{\pi}}\right)\right)^{-\alpha}}{1-\left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}},\pi\in\left[\underline{\pi},\overline{\pi}\right].$$

A higher minimum (resp. maximum) quality-cost, $\underline{x}$ *(resp.* $\overline{x}$*) leads to an increase in quality-costs and profits (in the sense of fosd).*

At a simplistic level, Theorem 1 indicates that we just need to find the log-distribution of the seed distribution. However, we still need to match parameters, as done above for the examples, and we also need to find the corresponding expression for D and ensure it is defined. Notice too that the methods described above work for more general demands under monopolistic competition (see Section 5).

3.2 Generating Pareto distributions of profits and outputs

The most successful function to fit the distribution of firm size has been the Pareto. The Pareto distribution for profits implies that there are many low-profit firms and few very profitable ones. We

[12] Equivalently, $F_{\Pi}(\pi)=1-\left(1+\frac{\mu}{\underline{x}}\ln\frac{\pi}{\underline{\pi}}\right)^{-\alpha}$ corresponds to the Generalized Log-Pareto distribution for Π (see Cormann and Reiss, 2009).

wish to know what this implies for the distribution of quality-cost. To do that, we reverse-engineer the problem using Theorem 2 to find the distribution of quality-cost which generates the Pareto distribution for profits. The answer is given next (the proof, in Appendix 2, is based on Theorems 1 and 2):

Proposition 4 *Let quality-cost be exponentially distributed* $F_X(x) = 1 - \exp(-\lambda(x - \underline{x}))$, $\lambda > 0$, $\underline{x} > 0$, $x \in [\underline{x}, \infty)$, *with* $\lambda\mu > 1$. *Then equilibrium profit is Pareto distributed:*

$$F_\Pi(\pi) = 1 - \left(\frac{\underline{\pi}}{\pi}\right)^{\alpha_\pi}, \text{ where } \pi \in [\underline{\pi}, \infty) \text{ and } \alpha_\pi = \lambda\mu > 1,$$

where $\underline{\pi} = \frac{\mu}{D}\exp\left(\frac{\underline{x}}{\mu}\right)$, *with* $D = \frac{\alpha_\pi}{\alpha_\pi - 1} M \exp\left(\frac{\underline{x}}{\mu}\right) + \mathcal{V}_0$. *Equilibrium output is Pareto distributed:*

$$F_Y(y) = 1 - \left(\frac{\underline{y}}{y}\right)^{\alpha_y}, \text{ where } y \in [\underline{y}, \infty) \text{ and } \alpha_y = \lambda\mu > 1,$$

with $\underline{y} = \underline{\pi}/\mu = \frac{1}{D}\exp\left(\frac{\underline{x}}{\mu}\right)$. *Conversely, a Pareto distribution for equilibrium output with* $\alpha_y > 1$ *and* $\underline{y} < M^{-1}\left(1 - \frac{1}{\alpha_y}\right)$ *or profit with* $\alpha_\pi > 1$ *and* $\underline{\pi} < \frac{\mu}{M}\left(1 - \frac{1}{\alpha_\pi}\right)$ *can only be generated by an exponential distribution of quality-costs,* $F_X(x) = 1 - \exp(-\lambda(x - \underline{x}))$, *with* $\lambda = \frac{\alpha_y}{\mu} > 0$ *or with* $\lambda = \frac{\alpha_\pi}{\mu} > 0$, *respectively. The lowest quality-cost is given by* $\underline{x} = -\mu \ln\left[\frac{\mu}{\mathcal{V}_0}\left(\frac{1}{\underline{\pi}} - \frac{\lambda M}{\lambda\mu - 1}\right)\right]$.

In summary, the size distribution of output and profit is Pareto, with shape parameter $\lambda\mu > 1$. Hence the shape parameter, $\alpha_y = \alpha_\pi$, for the endogenous economic distributions of output and profit depends just on the product of the taste heterogeneity and the technology shape parameter. The condition $\lambda\mu > 1$ needed to bound D means that $\mu > 1/\lambda$, which can be interpreted as the requirement that taste heterogeneity is larger than the average quality-cost.

The Pareto example also substantiates the results of Proposition 3 by showing that average profits can go up or down as μ changes. Indeed, using $f_\Pi(\pi) = \lambda\mu\left(\frac{\underline{\pi}}{\pi}\right)^{\lambda\mu}\pi^{-1}$, average profit is $\pi_m = \frac{\lambda\mu}{\lambda\mu - 1}\underline{\pi}$, which tends to μ/M as $\mathcal{V}_0 \to 0$; in this case, average profit is increasing in μ. On the other hand, if the outside good is very competitive ($\mathcal{V}_0$ large), it is readily shown that the opposite result can hold. Two effects are at work here: the beneficial direct mark-up effect, and output rebalancing towards the monopolistically competitive sector if the outside good is relatively attractive ($\mathcal{V}_0$ large).

4 Three-way synthesis

The distributions of quality-cost, output, and profit are determined from any one of them (Theorems 1 and 2). Likewise, price and cost distributions are linked analogously (as we show next). The link between any of the former distributions and either of the latter two is determined by the relation between costs and quality-costs. This section draws together these relations, and shows how the link between distributions can be determined. Conversely, knowing relation between cost and quality-cost and one distribution enables us to tie down the other distributions.

Note some special cases. First, there is no price dispersion if and only if there is no cost dispersion. Second, there is no profit dispersion if and only if there is no quality-cost dispersion: then there is only cost dispersion, with price dispersion mirroring the cost dispersion as per the formula below. The "classic" symmetric case often analyzed in the literature (e.g., Chamberlin, 1933) has neither cost nor profit dispersion.

4.1 Leg #1: Quality-cost, Output, and Profit

As shown in Theorems 1 and 2, knowledge of any one of these distributions ties down the other two.

4.2 Leg #2: Prices and Costs

The distribution of costs $F_C(c)$ and the distribution of prices $F_P(p)$ are related by the shift, $P = C + \mu$, so $F_C(c) = \Pr(P < c + \mu) = F_P(c + \mu)$, with $\underline{p} = \underline{c} + \mu$. Conversely, knowing the price distribution ties down the cost distribution. Suppose that the price distribution follows the Pareto form (which has been suggested as empirically viable) $F_P(p) = 1 - \left(\frac{\underline{p}}{p}\right)^{\alpha_p}$, with $p \in [\underline{p}, \infty)$ and $\alpha_p > 1$. The corresponding cost distribution is

$$F_C(c) = 1 - \left(\frac{\underline{c} + \mu}{c + \mu}\right)^{\alpha_p}, \quad c \in [\underline{c}, \infty), \ \alpha_p > 1. \tag{8}$$

4.3 Link: Quality-costs and Costs

We just showed that knowing one distribution from either leg enables us to determine the other(s) on that leg. We link the distributions across the two legs by postulating a functional relation between tastes and technology, and so we link quality-cost from the first leg with cost from the second leg.

Suppose then that $x = \beta(c)$. Notice that if we know that $x = \beta(c)$, then we can determine the relation between quality, v, and cost as $v = \beta(c) + c$. A priori, several cases are possible. Normally, one might expect that quality should increase with cost, so $\beta'(c) > -1$. Otherwise though, quality might be increasing or decreasing in c or indeed non-monotonic. A hump-shaped relation represents highest quality-costs for intermediate cost levels.

We can either treat $\beta(c)$ as a datum to determine other relations, or else we can infer it if we have knowledge of 2 seed distributions, one from each distribution leg. In the sequel, we treat the case where $\beta'(c) > 0$ so that "better" products have higher costs. Other cases are treated after the main analysis.

4.4 Synthesis

We now synthesize the relations between the different groups of relations. We assume that the logit denominator D is finite.[13]

Theorem 3 *Consider the Logit model of monopolistic competition. Suppose that one element is known from two of the three following groups. Then all elements are known.*

i) a distribution of quality-cost, profit, output (x,π,y);

ii) a distribution of price or cost (p, c);

iii) an increasing relation between any pair of x, v, and c.

The construction of $F_C(c)$ from $F_X(x)$ and $\beta(c)$ is shown in Figure 1. In the upper right panel we have the "seed" distribution $F_X(x)$, and below it is $\beta^{-1}(x)$. Values of x map into values of c via the relation $\beta^{-1}(x)$ in the lower right panel and hence through the lower left panel into values of c in the upper left panel, where the corresponding value from $F_X(x)$ therefore yields the desired value of $F_C(c)$. Note that the Figure also shows the converse constructions.

INSERT FIGURE 1: Relation between the (increasing) cost to quality-cost function and the cost and quality-cost distributions.

The property that $x = \beta(c)$ is monotonically increasing means that quality rises fast enough with cost, so that the highest cost products, which are sold at the highest price, are also the most attractive ones.[14]

4.5 Example: Pareto distributed profits and prices

We proceed with the example of Pareto distributed profits and prices, and show how to derive the implied link $x = v - c = \beta(c)$, which is the relation between quality-cost and cost. First note that if profit follows a Pareto size distribution with shape parameter α_π, then from Proposition 4 above, the quality-cost distribution is exponential and given by (31). Because $\beta(\cdot)$ is increasing, then

$$\begin{aligned} F_C(c) &= \Pr(X < \beta(c)) = F_X(\beta(c)) \\ &= 1 - \exp\left[-\lambda\left(\beta(c) - \beta(\underline{c})\right)\right], \ \lambda = \frac{\alpha_\pi}{\mu} > 0, \ \underline{c} > 0, \ c \in [\underline{c}, \infty), \end{aligned}$$

[13] For example, D exists if distributions are bounded.

[14] One alternative approach is to consider a joint density of x and c. In principle, the above procedure can be extended to characterize such situations.

where the last step uses the specification (31): $F_X(x) = 1 - \exp(-\lambda(x - \underline{x}))$, $\lambda > 0$, $\underline{x} > 0$, $x \in [\underline{x}, \infty)$.

On the other hand, if price follows a Pareto distribution with shape parameter α_p then the corresponding cost distribution is given by (8). These two expressions for $F_C(c)$ can now be equated to determine the form of $\beta(c)$ so $\left(\frac{\underline{c}+\mu}{c+\mu}\right)^{\alpha_p} = \exp\left[-\lambda\left(\beta(c) - \beta(\underline{c})\right)\right]$ and hence

$$\beta(c) = \beta(\underline{c}) + \mu\frac{\alpha_p}{\alpha_\pi}\ln\left(\frac{c+\mu}{\underline{c}+\mu}\right), \ \beta(c) \in [\beta(\underline{c}), \infty].$$

Thus $x = v - c = \beta(c)$ is increasing with c, i.e. $\beta'(c) > 0$ so that valuations rise faster than costs and quality-cost is indeed increasing, as postulated. The lower bound of the distribution $\beta(\underline{c}) = \underline{x}$ is given by Proposition 4:

$$\beta(\underline{c}) = -\mu\ln\left[\frac{1}{\mathcal{V}_0}\left(\frac{\mu}{\underline{\pi}} - \frac{\alpha_\pi M}{\alpha_\pi - 1}\right)\right].$$

We have then proved:

Proposition 5 *Let $F_P(p)$ be Pareto distributed with shape parameter α_p and let $F_\Pi(\pi)$ be Pareto distributed with shape parameter $\alpha_\pi > 1$ and $\underline{\pi} < \frac{\mu}{M}\frac{\alpha_\pi - 1}{\alpha_\pi}$, and suppose that $x = v - c = \beta(c)$ is an increasing function. Then $\beta(c) = \beta(\underline{c}) + \mu\frac{\alpha_p}{\alpha_\pi}\ln\left(\frac{c+\mu}{\underline{c}+\mu}\right)$.*

This is the situation illustrated in Figure 1 (parameterized by $\alpha_p = \alpha_\pi = \mu = 2$, $\beta(\underline{c}) = 0$, $\underline{c} = 0$, and $\underline{x} = 1$).

4.6 Decreasing cost to quality-cost relation

We now consider the reverse case where β is decreasing, and thus the higher quality-cost products are at the lower end of the cost spectrum. This is an important case because it corresponds to the extant literature à la Melitz (2003), which entertains only cost differences (we situate this on its home ground in the CES model in Section 7, and Section 6 considers the endogenous set of product quality-costs).

Now quality-costs decrease with cost, which entails a reversal of the ordering of products. The basic problem though is the same, and so analogous results to those above hold. We state them without proof, and offer a diagrammatic exposition along with one basic statement that shows the fundamental relation.

INSERT FIGURE 2: Relation between the (decreasing) cost to quality-cost function and the cost and quality-cost distributions.

We show first how to construct the function $F_X(x)$ from $F_C(c)$ and given $\beta(c)$ decreasing on the relevant support $[\underline{c}, \bar{c}]$. The steps are

$$F_X(x) = \Pr(\beta(C) < x) = \Pr\left(C > \beta^{-1}(x)\right) = 1 - F_C\left(\beta^{-1}(x)\right).$$

The construction is shown in Figure 2, where we end up with constructing $1 - F_C(c)$ in the upper left quadrant. Thence $F_C(c)$ is readily constructed by subtraction. We can also show how a decreasing β function can be constructed from $F_C(c)$ and $F_X(x)$. Following the earlier proof for increasing β, we postulate that there exists a continuous decreasing function $\beta(C) = X$ and so $F_X(x) = \Pr(\beta(C) < x) = \Pr\left(C > \beta^{-1}(x)\right) = 1 - F_C\left(\beta^{-1}(x)\right)$. Now, since $F_X(x) = 1 - F_C\left(\beta^{-1}(x)\right)$, then $\beta^{-1}(x) = F_C^{-1}(1 - F_X(x))$, where the function $(1 - F_X)$ is decreasing. The function $\beta(c)$ is clearly decreasing and continuous as desired.

We now apply this analysis to the case of cost heterogeneity alone (which parallels our later analysis for the CES). Let v be constant, and write $\beta(c) = \bar{v} - c$. Then

$$F_X(x) = \Pr(\bar{v} - C < x) = \Pr(C > \bar{v} - x) = 1 - F_C(\bar{v} - x).$$

Suppose for illustration that prices are Pareto distributed so that $F_C(c)$ is given by (8). Hence we get the power distribution

$$F_X(x) = \left(\frac{\bar{v} - \underline{x} + \mu}{\bar{v} - x + \mu}\right)^{\alpha_p}, \qquad x \in (-\infty, \underline{x}].$$

This is the case illustrated in Figure 2 (with $\bar{v} = \mu = 1$, $\alpha_p = 2$, and $x \in [-5, 0]$ so $c \in [1, 6]$).

For another example, suppose that prices are uniformly distributed so that costs are too. Write $F_C(c) = c,\ c \in [0, 1]$. Hence (recall $F_X(x) = 1 - F_C(\bar{v} - x)$) $F_X(x) = 1 - (\bar{v} - x)$, so $x \in [\bar{v} - 1, \bar{v}]$. Profits for the uniform are given from Corollary 2 (setting $\bar{v} = 1$ for simplicity) as $F_\Pi(\pi) = \mu \ln\left(\frac{\pi D}{\mu}\right)$ for $\pi \in \left[\frac{\mu}{D}, \frac{\mu}{D}\exp\left(\frac{1}{\mu}\right)\right]$, where $D = \mu M\left(\exp\left(\frac{1}{\mu}\right) - 1\right) + \mathcal{V}_0$.

4.7 Hump-shaped cost to quality-cost relation

Now consider the case that $\beta(c)$ is not monotone. For example, suppose that $\beta(c)$ is increasing from $\underline{c}$ to $\hat{c}$ and decreasing from $\hat{c}$ to $\bar{c}$. In this case, quality rises faster than cost at first, and then rises slower or even falls (if $\beta' < -1$). This case involves highest quality-cost (and hence highest output and profit, by Theorem 1) for middling cost levels.

The cumulative quality-cost distribution is derived from the two pieces. Suppose for illustration that $\beta(c) < \beta(\bar{c})$ for $c \in [\underline{c}, \tilde{c})$ (and so $\beta(\tilde{c}) = \beta(\bar{c})$). Then $F_X(x)$ is derived from $F_C(c)$ via

$$F_X(x) = F_C\left(\beta^{-1}(x)\right) \quad \text{for } x \in [\beta(\underline{c}), \beta(\tilde{c}))$$

Higher x values can come from either the increasing or decreasing part of β, and we need to sum the two contributions.

Define $\beta_I^{-1}(x)$ as the inverse function for β increasing (i.e., corresponding to $c < \hat{c}$) and $\beta_D^{-1}(x)$ as the inverse function for β decreasing (i.e., corresponding to $c > \hat{c}$). Then for such values ($x \in [\beta(\tilde{c}), \beta(\hat{c})]$) we have that $F_X(x)$ is given as the sum of the contributions from the two parts, as per the statement in the second line below: summarizing:

$$\begin{aligned} F_X(x) &= F_C\left(\beta^{-1}(x)\right) && \text{for } x \in [\beta(\underline{c}), \beta(\tilde{c})) \\ F_X(x) &= F_C\left(\beta_I^{-1}(x)\right) + 1 - F_C\left(\beta_D^{-1}(x)\right) && \text{for } x \in [\beta(\tilde{c}), \beta(\hat{c})] \end{aligned}$$

(notice indeed that $F_X(x)$ is increasing, with a kink up at $\beta(\tilde{c})$ and indeed that $F_X(\beta(\hat{c})) = 1$).

5 General mark-ups and alternative demand forms

The logit model has the property that the mark-up is constant, which we have taken as a datum in developing the links between the various distributions. Here we consider a generalization of the logit demand form, which renders it still amenable to monopolistic competition analysis, and is based on the IIA property of the Logit (which property the CES also has). This allows us to find the mark-up as a function of quality-cost. Equivalently, we show that knowing the equilibrium mark-up function enables us to back out the demand function that generated it.

Suppose now that demands are generated from the relation

$$y_i = \frac{h(v_i - p_i)}{\int_{\omega\in\Omega} h(v(\omega) - p(\omega))\, d\omega + \mathcal{V}_0}, \quad i \in \Omega, \tag{9}$$

which generalizes (1). This we refer to as the IIA demand system. Here the "scale value" $h(.)$ is a positive, increasing, strictly (-1)-concave,[15] and differentiable function of quality-price, and $\mathcal{V}_0$ represents the attractivity of the outside option. As before, we assume that the integral converges. The Logit is the case of a log-linear scale.

Firm i's profit is (with D as the denominator in (9))

$$\pi_i = (p_i - c_i)\frac{h(v_i - p_i)}{D} = m_i \frac{h(x_i - m_i)}{D}, \quad i \in \Omega,$$

[15] This is equivalent to $\frac{1}{h(.)}$ strictly convex, and is the minimal condition ensuring a maximum to profit. See Caplin and Nalebuff (1992) for more on ρ-concave functions; and Fassbinder and Weyl (2013) for the properties of pass-through as a function of demand curvature: the analogue to cost pass-through is here the complementary feature of quality-pass-through.

where $m_i = p_i - c_i$ is i's mark-up. Notice (by the envelope theorem) that the maximized value, $\pi_i^* (x_i)$ is increasing in x_i. Under monopolistic competition, the equilibrium mark-up satisfies

$$m_i = \frac{h(x_i - m_i)}{h'(x_i - m_i)}, \qquad i \in \Omega. \tag{10}$$

Theorem 4 *Let $h(.)$ be a positive, increasing, strictly (-1)-concave, and twice differentiable function. Then the associated monopolistic competition mark-up, $\mu(x)$ is the unique solution to (10) and $\mu'(x) < 1$. The associated equilibrium value $h^*(x) \equiv h(x - \mu(x))$ is strictly increasing.*

The only class with constant mark-up (in the IIA family) is the logit, which has $h(\cdot)$ log-linear, and so $\frac{h(x-m)}{h'(x-m)}$ is constant and $\mu(x)$ is constant (as per our earlier sections). When $h(\cdot)$ is strictly log-concave, the mark-up increases with x. Then higher quality-costs always yield higher equilibrium outputs. When $h(\cdot)$ is strictly log-convex, the mark-up decreases: this is analogous to a cost pass-through greater than 1. Notice that the property $\mu'(x) < 1$ is just the property that price never goes down if costs increase.

The converse result to Theorem 4 indicates how the mark-up function can be inverted to determine the form of h^* (and hence $h(.)$).

Theorem 5 *Let there be a mark-up function $\mu(x)$ for $x \in (\underline{x}, \bar{x})$ with $\mu'(x) < 1$. Then there exists a (unique) equilibrium scale function $h^*(\cdot)$ given by (12) defined on $(\underline{x}, \bar{x})$ and an associated primitive scale function $h(\cdot)$, given by (13), which is (-1)-concave defined on support $(\underline{x} - \mu(\underline{x}), \bar{x} - \mu(\bar{x}))$ that generates the observed mark-ups.*

Proof. First note from (35) and (10) that

$$\frac{dh^*(x)/dx}{h^*(x)} = \frac{(1 - \mu'(x))\, h'(x - \mu(x))}{h(x - \mu(x))} = \frac{(1 - \mu'(x))}{\mu(x)} \equiv g(x). \tag{11}$$

Thus $[\ln h^*(x)]' = g(x)$, and so $\ln\left(\frac{h^*(x)}{h^*(\underline{x})}\right) = \int_{\underline{x}}^{x} g(v)\, dv$, which implies

$$h^*(x) = h^*(\underline{x}) \exp \int_{\underline{x}}^{x} g(v)\, dv, \tag{12}$$

which therefore determines $h^*(x)$ up to a positive factor.[16]

We can now use $h^*(x)$ to back out the original function $h(x - m)$ via the following steps. First, define $\phi(x) = x - \mu(x)$ and note that this is monotone increasing because $1 - \mu'(x) > 0$, so the

[16] Here $h^*(x)$ is only determined up to a positive factor because multiplying all scales and $\mathcal{V}_0$ by any positive factor leaves the choice probabilities unchanged.

inverse function $\phi^{-1}(\cdot)$ is increasing. Now, $h(u) = h^*(\phi^{-1}(u))$ with $u \in (\underline{x} - \mu(\underline{x}), \bar{x} - \mu(\bar{x}))$ and thus the function $h(\cdot)$ is recovered on the support. ■

It remains to show that the function $h(\cdot)$ is (-1)-concave on its support. Using (12) with $h(u) = h^*(\phi^{-1}(u))$, we have

$$h(u) = h^*(\underline{x}) \exp \int_{\underline{x}}^{\phi^{-1}(u)} g(v)\, dv, \tag{13}$$

and so (using (11)), and noting that $\phi'(x) = 1 - \mu'(x)$:

$$\frac{h(u)}{h'(u)} = \frac{1}{g(\phi^{-1}(u))\left[\phi^{-1}(u)\right]'} = \frac{\phi'(x)}{\frac{(1-\mu'(x))}{\mu(x)}} = \mu(x).$$

Thus $h(u)$ is strictly (-1)-concave, since:[17]

$$\left[\frac{h(u)}{h'(u)}\right]' = \frac{\mu'(x)}{1-\mu'(x)} > -1.$$

Notice that the function $h(\cdot)$ is tied down only on the support corresponding to the domain on which we have information about the equilibrium value in the market. Outside that support, we know only that $h(\cdot)$ must be consistent with the maximizer $\mu(x)$, which restricts the shape of $h(\cdot)$ to not be "too" convex. The case of positive quality pass-through (which is equivalent to cost pass-through below 100%) is associated to log-concave demand.[18]

5.1 Deriving mark-up and demand forms from output and profit distributions

Suppose we know F_Y and F_Π. We now show how the mark-up function, preference structure (scale values) and quality-cost distribution are uncovered.

For a given type density, $f_X(x)$, and setting $D^h = \int_{u \geq \underline{x}} h^*(u) f_X(u)\, du + \mathcal{V}_0$, then $y = \frac{h^*(x)}{D^h}$, which increases with x (as per Theorem 4). We can derive a first expression for the distribution of quality-cost from the output distribution:

$$F_X(x) = \Pr(X < x) = \Pr\left(Y < \frac{h^*(x)}{D^h}\right) = F_Y\left(\frac{h^*(x)}{D^h}\right) \tag{14}$$

[17] When $h(u)$ is strictly (-1)-concave, then $h(u) h''(u) - 2[h'(u)]^2 < 0$, which rearranges to $\left[\frac{h(u)}{h'(u)}\right]' > -1$.

[18] Indeed, we can also show that if $\mu'(x) \in [0,1)$ and $\frac{(1-\mu'(x))}{\mu(x)}$ is decreasing, then the equilibrium function $h^*(.)$ is log-concave on its support $(\underline{x}, \bar{x})$: because $\ln h^*(x) = \ln h^*(\underline{x}) + \int_{\underline{x}}^{x} g(v)\, dv$, then $[\ln h^*(x)]'' = g'(x)$ which is negative if $\frac{(1-\mu'(x))}{\mu(x)}$ is decreasing, so that $h^*(x)$ is log-concave, as claimed. From (34) we have log-concavity of h implies $\mu' \in (0,1)$, so the extra condition here is decreasing $\frac{(1-\mu'(x))}{\mu(x)}$ to ensure log-concavity of h^*.

where we used the property that equilibrium output increases with x. Inverting, we can recover the equilibrium demand from

$$h^*(x) = D^h F_Y^{-1}(F_X(x)). \tag{15}$$

Equilibrium profit is

$$\pi = \mu(x)\frac{h^*(x)}{D^h}$$

which is also increasing in x.[19] We find a second expression for $F_X(x)$ from the profit distribution, to write

$$F_X(x) = \Pr(X < x) = \Pr\left(\Pi < \frac{\mu(x)\,h^*(x)}{D^h}\right) = F_\Pi\left(\frac{\mu(x)\,h^*(x)}{D^h}\right). \tag{16}$$

From (16) and (14), we can eliminate the common value of $F_X(x)$ to give:[20]

$$F_\Pi\left(\frac{\mu(x)\,h^*(x)}{D^h}\right) = F_Y\left(\frac{h^*(x)}{D^h}\right);$$

thus,

$$\mu(x) = \frac{D^h}{h^*(x)}F_\Pi^{-1}\left(F_Y\left(\frac{h^*(x)}{D^h}\right)\right). \tag{17}$$

Because $h^*(x)$ and $\mu(x)$ are related via Theorems 4 and 5, then (17) is an equation in a single unknown, the function $\mu(x)$. Now notice that once the function $\mu(x)$ has been recovered, then the quality-cost distribution can be recovered through the relation $F_X(x) = \Pr(X < x) = \Pr\left(Y < \frac{h^*(x)}{D^h}\right) = F_Y\left(\frac{h^*(x)}{D^h}\right)$. In summary:

Theorem 6 *Consider a Lucian IIA demand system under monopolistic competition. Assume that the distributions of profit, F_Π, and output, F_Y, are known. Then the mark-up $\mu(x)$ and the equilibrium generating function, $h^*(x)$ obey $F_\Pi\left(\frac{\mu(x)h^*(x)}{D^h}\right) = F_Y\left(\frac{h^*(x)}{D^h}\right)$, with $\frac{h^{*\prime}(x)}{h^*(x)} = \frac{1-\mu'(x)}{\mu(x)}$ and $D^h = \mathcal{V}_0/(1-My_{av})$. The distribution of quality-cost is given by $F_X(x) = F_Y\left(\frac{h^*(x)}{D^h}\right)$.*

We illuminate with two examples.

---

[19] This property follows from the envelope theorem: recall that $\pi = \frac{mh(x-m)}{D^h}$. Then $\frac{d\pi}{dx} = \frac{mh'(x-m)}{D^h}$. By the first-order condition, the numerator is equal to $h(x-m)$ where m is the optimal choice, and hence it is equal to $h^*(x)$, as claimed.

[20] Inverting, and using (15) we can find the mark-up function as

$$\mu(x) = \frac{F_\Pi^{-1}(F_X(x))}{F_Y^{-1}(F_X(x))}.$$

The RHS is the ratio of profit to output at the (common) level in the profit and output distributions. The mark-up can also be written in terms of the percentile firm in the various distributions: define $\theta = F_X(x)$ and so the mark-up of the firm of rank θ in the mark-up distribution, $\tilde{\mu}(\theta)$, is $\tilde{\mu}(\theta) = \frac{F_\Pi^{-1}(\theta)}{F_Y^{-1}(\theta)}$. The RHS is the ratio of profit to output at common rank θ in the distributions.

Example 1. Suppose we know that $F_\Pi(\pi) = 2\sqrt{D\pi}$ and $F_Y(y) = 2Dy$. Then, from (17) we have the equation $F_\Pi\left(\frac{\mu(x)h^*(x)}{D^h}\right) = F_Y\left(\frac{h^*(x)}{D^h}\right)$ as

$$\begin{aligned} \sqrt{\mu(x)h^*(x)} &= h^*(x) \quad \text{and hence} \\ \mu(x) &= h^*(x). \end{aligned}$$

This we can solve from $\frac{dh^*(x)/dx}{h^*(x)} = \frac{(1-\mu'(x))}{\mu(x)}$ (from Theorem 6) so $\mu'(x) = 1/2$, and $\mu(x) = k + \frac{x}{2}$. Thus $h^* = k + \frac{x}{2}$, and the associated h function is $h(v-p) = h(x-m) = x - m$, so this is a form of linear demand system. Folding the constant into the V_0, we can uncover F_X through $F_X(x) = F_Y\left(\frac{h^*(x)}{D^h}\right) = 2D^h\left[\frac{h^*(x)}{D^h}\right] = x$, which implies that $F_X(x)$ is uniform on $[0,1]$.

Example 2. Suppose we know that $F_\Pi(\pi) = k_\pi \pi^\phi$ for $\pi \in \left[0, k_\pi^{-1/\phi}\right]$ and $F_Y(y) = k_y y^\delta$ for $y \in \left[0, k_y^{-1/\delta}\right]$. Then, from (17) we have $F_\Pi\left(\frac{\mu(x)h^*(x)}{D^h}\right) = F_Y\left(\frac{h^*(x)}{D^h}\right)$, so

$$k_\pi\left(\frac{\mu(x)h^*(x)}{D^h}\right)^\phi = k_y\left(\frac{h^*(x)}{D^h}\right)^\delta.$$

2a. If $\delta = \phi$, the mark-up is constant:

$$\mu(x) \equiv \mu = \left(\frac{k_y}{k_\pi}\right)^{\frac{1}{\phi}},$$

and the condition

$$\frac{dh^*(x)/dx}{h^*(x)} = \frac{(1-\mu'(x))}{\mu(x)} = \frac{k_\pi}{k_y}$$

has the solution $h^*(x) = k\exp\left(\frac{x}{\mu}\right)$, which is the Logit model.

2b. Otherwise (if $\delta \neq \phi$), we have

$$h^*(x) = D^h[\mu(x)]^{\frac{\phi}{\delta-\phi}}\left[\frac{k_\pi}{k_y}\right]^{1/(\delta-\phi)}.$$

This we can solve from $\frac{dh^*(x)/dx}{h^*(x)} = \frac{(1-\mu'(x))}{\mu(x)}$ (from Theorem 6). First,

$$\frac{dh^*(x)/dx}{h^*(x)} = \frac{\phi}{\delta-\phi}\frac{\mu'(x)}{\mu(x)}.$$

Thus:

$$\begin{aligned} \frac{h^{*\prime}(x)}{h^*(x)} &= \frac{\phi}{\delta-\phi}\frac{\mu'(x)}{\mu(x)} = \frac{1-\mu'(x)}{\mu(x)}, \text{ so} \\ \mu'(x) &= \frac{\delta-\phi}{\delta} < 1. \text{ Integrating,} \\ \mu(x) &= \left(\frac{\delta-\phi}{\delta}\right)x + \varsigma. \end{aligned}$$

Thence

$$\begin{aligned}
\left[\ln h^*(x)\right]' &= \frac{\phi}{\delta-\phi}\frac{\left(\frac{\delta-\phi}{\delta}\right)}{\left(\frac{\delta-\phi}{\delta}\right)x+\varsigma} \text{ and so, integrating:} \\
\ln\frac{h^*(x)}{h^*(\underline{x})} &= \ln\left[\frac{\varsigma+\left(\frac{\delta-\phi}{\delta}\right)x}{\varsigma+\left(\frac{\delta-\phi}{\delta}\right)\underline{x}}\right]^{\frac{\phi}{\delta-\phi}} \\
h^*(x) &= \frac{h^*(\underline{x})}{\left(\varsigma+\left(\frac{\delta-\phi}{\delta}\right)\underline{x}\right)^{\frac{\phi}{\delta-\phi}}}\left[\varsigma+\left(\frac{\delta-\phi}{\delta}\right)x\right]^{\frac{\phi}{\delta-\phi}}.
\end{aligned}$$

If $\delta > \phi$, this is the "Luce" model (and we have mark-ups etc. for it in a holding section below). Notice that the ς is already in there in the sense that we could add constants to the v without really changing it (?)?. If $\delta < \phi$, this gives us a new form,

Note that $h^{*\prime}(x)$ as desired! also $\mu(x) = \left(\frac{\delta-\phi}{\delta}\right)x+\varsigma$ so that $h^*(x) = h^*(\underline{x})\left[\frac{\mu(x)}{\mu(\underline{x})}\right]^{\frac{\phi}{\delta-\phi}}$.

$\int\frac{1}{Ax+B}dx = \frac{1}{A}\ln\frac{1}{A}(B+Ax)$

$\mu'(x) = 1/2$, and thus $h^* = k+\frac{x}{2}$ (and the associated h?). Folding the constant into the V_0, we can uncover F_X through $F_X(x) = F_Y\left(\frac{h^*(x)}{D^h}\right) = 2D^h\left[\frac{h^*(x)}{D^h}\right] = x$, which implies that $F_X(x)$ is uniform on $[0,1]$.

5.2 Mark-ups and Quality-costs

The mark-up distribution is readily derived from the price and cost distributions. This yields a link between mark-ups and prices or costs analogous to Leg #2 for the simple logit model, except now with non-constant mark-ups. We know that mark-ups rise with quality-cost, so assume too that quality-cost rises with cost and hence mark-ups (and prices) increase with costs (and so there is no quality-cost overturn). Consider some feasible price $\hat{p}$ associated to a cost $\hat{c}$. Then $F_P(\hat{p}) = F_C(\hat{c})$ and the associated mark-up, $\hat{m} = \hat{p}-\hat{c}$ also has the same F.[21] That is, $\hat{m} = \hat{p}-F_C^{-1}(F_P(\hat{p}))$ and we can plug this $\hat{m}$ into $F_M(\hat{m}) = F_P(\hat{p})$.[22]

We can now relate the various economic and fundamental distributions via the pertinent transformation functions. Of key interest now is the link between $F_X(x)$ and the newly introduced mark-up distribution, $F_M(m)$. These are linked through the function $\mu(x)$. Knowing $\mu(x)$ (or the demand relation $h^*(x)$) determines one distribution from the other; or indeed knowing the two distributions

[21] The idea is simple in a diagram: draw $F_P(p)$ and $F_C(c)$ on the same diagram. Choose an equal height: the difference in the graphs horizontally is the corresponding mark-up (and its cdf is the common height).

[22] E.g., let $F_C(c)$ be uniform on $[0,1]$, $F_P(p)$ be uniform on $[1,3]$, then $F_M(m)$ is uniform on $[1,2]$

determines the transformation function $\mu(x)$. That is, we can readily extend the idea of Theorem 1 and provide analogues of our previous results and theorems.

Suppose for example that we know $\mu(x)$ and $F_X(x)$. Then, because $\mu(x)$ is increasing,

$$F_M(m) = \Pr(M < m) = \Pr(\mu(X) < m) = \Pr\left(X < \mu^{-1}(m)\right) = F_X\left(\mu^{-1}(m)\right).$$

Likewise, $F_X(x) = F_M(\mu(x))$. Suppose instead we know $F_X(x)$ and $F_M(m)$. From the relation $F_X(x) = F_M(\mu(x))$ we have the missing link $\mu(x) = F_M^{-1}(F_X(x))$. This means in turn that we can uncover the demand system, $h^*(.)$ consistent with the two underlying distributions.[23]

Theorem 7 *For any pair of distributions of quality-cost and mark-ups, $F_X(x)$ and $F_M(m)$ respectively, satisfying $f_X(x) < f_M\left(F_M^{-1}(F_X(x))\right)$, there exists a log-concave and increasing quasi-demand function $h(v-p)$ such that the IIA monopolistic competition equilibrium is consistent with the pair of distributions.*

Proof. As noted above, $\mu(x) = F_M^{-1}(F_X(x))$, and we wish to show that this mark-up function is consistent with a log-concave $h(.)$. We work via the function $\mu(x)$. We have: $\mu'(x) = \frac{f_X(x)}{f_M(\mu(x))} > 0$. Second, under the condition on the distributions that $f_X(x) < f_M\left(F_M^{-1}(F_X(x))\right)$, then $\mu'(x) < 1$. As shown above, hence we can solve for $h^*(x)$ by integrating the relation $\ln h^*(x)]' = \frac{1-\mu'(x)}{\mu(x)}$ and thence recover $h^*(x)$ and $h(x-m)$. ■

An example follows. Suppose that $h(v-p) = h(x-m) = x - m$. Then $\mu(x) = \frac{x}{2}$ (and $h^*(x) = \frac{x}{2}$), so assume $x > 0$. Then $F_X(x) = F_M\left(\frac{x}{2}\right)$ so that if $F_X(x) = x$ for $x \in [0,1]$, then we have $F_M(m) = 2m$, and $m \in \left[0, \frac{1}{2}\right]$. This is a linear demand model.[24]

5.3 Other links

The other Leg in our earlier synthesis is the third one, which gives the relation between cost and quality-cost. That is, the relation between $F_X(x)$ and $F_C(c)$ can be again described by the transformation function $\beta(c)$, which constitutes a pure technological link.

We can also determine the other links and distributions from the relations above, analogous to our earlier synthesis. For example, consider the price distribution. We have from the above

[23] Because $F_X(x) = F_M(\mu(x))$ then $f_X(x) = \mu'(x) f_M(\mu(x))$, and hence $\mu'(x) = \frac{f_X(x)}{f_M(\mu(x))} > 0$, as desired for a (-1)-concave h. Note also though that $\mu'(x) = \frac{f_X(x)}{f_M(\mu(x))} < 1$ iff $f_X(x) < f_M(\mu(x))$, which is therefore a necessary condition for matching the distributions. Taking it a step further, we need $f_X(x) < f_M\left(F_M^{-1}(F_X(x))\right)$.

[24] Here we have the condition $1 < f_M\left(F_M^{-1}(x)\right)$ is satisfied because it is equivalent to $1 < 2$.

that the mark-up is $\mu(x)$ and we have that cost is $\beta^{-1}(x)$. This means that we can write $p(x) = \mu(x) + \beta^{-1}(x)$ and so we can recover the price distribution from $F_X(x)$. Alternatively, we can write $p(c) = \mu(\beta(c)) + c$ and we can find the price distribution from $F_C(c)$ (so if we define $\chi(c) = \mu(\beta(c)) + c$ then $F_P(p) = F_C(\chi^{-1}(p))$). Or, indeed, we can proceed in similar manner from the mark-up distribution by writing $p(m) = m + c(x(m)) = m + \beta^{-1}(\mu^{-1}(m))$.

6 Long run Logit

Here we develop the long-run analysis of the logit model following recent directions in Trade models, and emphasize the shape of the equilibrium distributions that ensue. A fuller (more general) analysis along the lines of the previous section would follow similar lines, but here we aim for simplicity. We assume in the groove of Melitz (2003) that firms first pay a cost K_1 to get a quality-cost draw, then they pay K_2 to actively produce. We solve backwards.[25] To put in play market size effects, we introduce market size (number of consumers) N (which was normalized to 1 in the analysis so far).

For a given mass, M, of firms that have paid K_1, equilibrium involves all sufficiently good types paying the subsequent fixed cost K_2. All types $x \geq \hat{x}$ will produce (because profits increase in x by Proposition 1). The gross profit of firm with quality-cost x is now

$$\pi(x, \tilde{x}) = \mu N \frac{\exp\left(\frac{x}{\mu}\right)}{D(M, \tilde{x})},$$

where we define the function in the denominator (see (7)) $D(M, \tilde{x}) \equiv M \int_{u \geq \tilde{x}} \exp \frac{u}{\mu} f_X(u)\, du + \mathcal{V}_0$, $\underline{x} \leq \tilde{x}$. $D(M, \tilde{x})$ is decreasing in $\tilde{x}$ so that the profit of the marginal firm, $\hat{\pi} \equiv \pi(\hat{x}, \hat{x})$ is increasing in $\hat{x}$. Hence there is a unique cut-off value $\hat{x}$ such that $\hat{\pi} = K_2$, i.e., $\hat{x}$ satisfies

$$\mu N \frac{\exp\left(\frac{\hat{x}}{\mu}\right)}{D(M, \hat{x})} = K_2, \tag{18}$$

as long as $\pi(\underline{x}, \underline{x}) < K_2$. This is the case we consider: otherwise, all firms enter, and all make strictly positive profits. Thus, there is a unique long-run equilibrium cut-off value for the weakest viable firm.

Once a firm has paid the cost K_1 to get a draw, it has a probability $1 - F_X(\hat{x})$ to get a good enough draw, and to be active. The mass of potentially active firms, M, is determined at the first

[25] Alternatively, if the reader prefers a scenario where X is observed in advance by firms before entry, then only the first part of the analysis should be retained.

step via the zero-profit condition:

$$\mu N \frac{\int_{u\geq\hat{x}} \exp\left(\frac{u}{\mu}\right) f_X(u)\,du}{D(M,\hat{x})} - K_2 \int_{u\geq\hat{x}} f_X(u)\,du = K_1. \tag{19}$$

The first term is the expected gross profit of a firm which has paid the entry cost K_1 to get a draw. The second is the fixed (continuation) cost, to be paid by all firms with a draw of at least $\hat{x}$. Inserting (18) in (19) gives

$$\int_{u\geq\hat{x}} \left(\exp\left(\frac{u-\hat{x}}{\mu}\right) - 1\right) f_X(u)\,du = \frac{K_1}{K_2}. \tag{20}$$

The LHS is monotonically decreasing in $\hat{x}$ so there is a *unique* solution for $\hat{x}$ that depends only on the parameters in (20) – in particular, it is independent of market size N and of $\mathcal{V}_0$ (Etro and Bertoletti, 2013, show a similar neutrality result). Note that the condition for an interior solution is that $\int_{u\geq\underline{x}} \left(\exp\left(\frac{u-\underline{x}}{\mu}\right) - 1\right) f_X(u)\,du < \frac{K_1}{K_2}$, that is

$$\exp\left(\frac{-\underline{x}}{\mu}\right) \int_{u\geq\underline{x}} \exp\left(\frac{u}{\mu}\right) f_X(u)\,du < \frac{K_1+K_2}{K_2} \tag{21}$$

Otherwise, all firms are in the market. Given the solution for $\hat{x}$, we can then determine M from (18) with $D(M,\hat{x})$ and defining $x^c = \max\{\hat{x},\underline{x}\}$:

$$M = \frac{\frac{\mu N}{K_2}\exp\left(\frac{x^c}{\mu}\right) - \mathcal{V}_0}{\int_{u\geq x^c} \exp\left(\frac{u}{\mu}\right) f_X(u)\,du}. \tag{22}$$

Therefore $M > 0$ if

$$Z^c \equiv \frac{\mu N}{K_2}\exp\left(\frac{x^c}{\mu}\right) > \mathcal{V}_0. \tag{23}$$

Otherwise, there is no entry ($M = 0$). Note that condition (23) depends on both K_1 and K_2 since $\hat{x}$ depends on K_1/K_2.

Proposition 6 *(Logit, long-run) Consider the Logit Monopolistic Competition model, with cost K_1 to get a quality-cost draw, and with cost K_2 to actively produce. There is a unique long-run equilibrium. A positive mass of firms enters if (23) holds; some of these entrants do not produce if (21) holds. Then the solution is given from (18) and (22).*

This solution parallels that for the Melitz (2003) model.[26] Some comparative static properties readily follow. The elasticity of M with respect to N is (using (23)):

[26] We can readily include a second threshold for exporting firms in an international trade context.

$$\frac{N}{M}\frac{dM}{dN}=\frac{1}{1-\frac{\mathcal{V}_0}{Z^c}}>1.$$

Hence if $\mathcal{V}_0$ is very small (note that we have a "covered market" as $\mathcal{V}_0 \to 0$), then doubling the market size doubles the number of firms. Otherwise, *it more than doubles* as we push out the outside option (see also Melitz, 2003, and Bertoletti and Etro, 2013, for the converse possibility in the presence of income effects).[27]

We have shown above that the long-run case uniquely determines an $\hat{x}$ value and an M value. From these, we can simply find the long-run cumulative distribution of quality-cost for the model with endogenous entry, which is $\tilde{F}_X(x)=\frac{F_X(x)-F_X(\hat{x})}{1-F_X(\hat{x})}$ for $x\in[\hat{x},\bar{x}]$, and directly apply directly our existing results in Theorems 1, 2, and 3. Moreover, the inheritance properties of the key distributions on which we have focussed still apply. In particular, an exponential distribution for F_X implies $\tilde{F}_X$ is also exponential ($\tilde{F}_X=1-\exp(-\lambda(x-\hat{x}))$). Thus profits and outputs are Pareto, and then we can link to the cost and price distributions as we did before: *the size distribution of output and profit is Pareto, with shape parameter* $\lambda\mu$.

Notice finally that the comparative statics results of Proposition 3 are readily amended. As we showed there, for fixed $\underline{x}$ at the bottom of the support, a higher $\mathcal{V}_0$ decreases profits, while a higher μ raises them for low quality-cost firms and reduces them for high quality-cost firms. Now, when the lower bound is endogenous, it is clear from (20) that $\hat{x}$ is unchanged when $\mathcal{V}_0$ rises, but M falls (from (22)). Thus the effect is just as before, except now milder by the exit of firms. For higher μ, by (22) $\hat{x}$ falls,[28] so that increased taste heterogeneity increases the range of firm types that will stay in the market after their initial draw. However, the number of firms taking the first draw (M) may increase or decrease – high quality firm types get lower profits (cf. Proposition 3), and this may decrease the desire to enter.

7 CES models

The CES model in various guises has recently enjoyed a huge spurt in popularity, most noticeably in the new international trade literature. Here we apply our methods and distributional analysis to the CES. We start with the standard CES monopolistic competition model with heterogeneity only

[27] Other comparative statics can be performed. For example, increasing both K's proportionately leaves $\hat{x}$ the same (by (20)), but decreases $D(M,\hat{x})$ (by (19)) and M decreases.

[28] For example, for the exponential distribution of quality-cost, $\hat{x}=\underline{x}+\frac{1}{\lambda}\ln\left(\frac{K_2}{K_1}\frac{1}{\lambda\mu-1}\right)$, which is decreasing in μ. The corresponding mass of entering firms is $M=\frac{N}{\lambda K_1}-\frac{\mathcal{V}_0}{\lambda\mu}\frac{K_2}{K_1}\exp\left(\frac{-\hat{x}}{\mu}\right)$.

in unit production costs across firms. This is the basic Melitz (2003) approach. We show that all economic distributions are tied down by the distribution of costs.

A central distribution considered in the literature has been the Pareto. We show that all relevant distributions are Pareto if any one is (caveat: for prices and costs it is the distribution of the reciprocal that is Pareto). This result we term the *Pareto circle*. To put this another way, if we posit that the reciprocal of costs is Pareto distributed (equivalently, costs have a power distribution), then so is the reciprocal of prices, and the other variables – output, revenue, and profit – are all Pareto distributed. In particular, it is not possible to have (for example) a Pareto distribution for profits and (another) Pareto distribution for prices in the CES model. Or indeed, we cannot escape the Pareto circle if one element is Pareto.

We therefore introduce a further dimension of heterogeneity, just as we did for the logit model, and again with the interpretation of "quality." As with the logit analysis we link the two distributions via a bridge function that writes quality as a function of cost. Doing this then enables us to get two linked groups of distributions. In one group are profit and revenue, and in the other are costs and prices (and output forms a convex combination). We take as our leading example a particular bridge function that enables us to get Pareto distributions in each group (and hence too a Pareto distribution for output). We also use the bridge function to determine when inverse distributions apply. Before introducing quality differences and deriving the transformation function from the seed distributions, we first develop the analysis for cost heterogeneity alone.

7.1 Standard CES model

Several forms of CES representative consumer utility functions are prevalent in the literature. We nest these into one embracing form. The CES representative consumer involves a sub-utility functional for the differentiated product $\chi = \left(\int_\Omega q(\omega)^\rho d\omega\right)^{1/\rho}$ with $\rho \in (0,1)$ (with $\rho = 1$ being perfect substitutes, and $\rho \to 0$ being independent demands corresponding to a Cobb-Douglas form), and the q's are quantities consumed of the differentiated variants. Common forms of representative consumer formulation are (i) Melitz model (see also Dinghra and Morrow, 2013) where $U = \chi$ so there is only one sector); (ii) the classic Dixit-Stiglitz (1977) case much used in earlier trade theory, $U = \chi q_0^\eta$ with $\eta > 0$, where q_0 is consumption in an outside sector; (iii) $U = \ln \chi + q_0$, which constitutes a partial equilibrium approach in the sense that there are no income effects (see Anderson and de Palma, 2000). The first two involve unit income elasticities, hence their popularity in Trade models. Utility is maximized under the budget constraint $\int_\Omega q(\omega) p(\omega) d\omega + q_0 \leq I$, where I is income.

The next results are quite standard (and are special cases of the CES quality-enhanced model developed in the next sub-section).

For a given set of prices and a set Ω of active firms, Firm i's demand (output) is (recalling $\rho \in (0,1)$):

$$q_i = \frac{\Xi(I)}{p_i} \frac{p_i^{\frac{\rho}{\rho-1}}}{\int_{\omega \in \Omega} p(\omega)^{\frac{\rho}{\rho-1}} d\omega}, \tag{24}$$

where $\Xi(I)$ is I for case (i), $\frac{I}{1+\eta}$ for case (ii) (which clearly nests case (i) for $\eta = 0$); and 1 for the last case. In each case, $\Xi(I)$ is the amount spent on the differentiated commodity in aggregate. In the sequel we follow case (iii); (the others are similarly straightforward.

The monopolistically competitive equilibrium price solves $\max_{p_i} \frac{(p_i - c_i)}{p_i} p_i^{\frac{\rho}{\rho-1}}$, so $p_i = \frac{c_i}{\rho}$. Hence the Lerner index is $\frac{p_i - c_i}{p_i} = (1-\rho)$. Given this pricing rule, Firm i's equilibrium output (from (24)) is

$$q_i = \rho \Xi(I) \frac{c_i^{\frac{\rho}{\rho-1}-1}}{D_C}, \tag{25}$$

where $D_C = M \int c(u)^{\frac{\rho}{\rho-1}} f_C(u)\, du$, and $f_C(.)$ is the density of unit costs. Firm i's equilibrium profit is a constant fraction of its sales revenue ($r_i = p_i q_i$):

$$\begin{aligned} \pi_i &= (1-\rho)\,\Xi(I) \frac{c_i^{\frac{\rho}{\rho-1}}}{D_C} \\ &= (1-\rho)\, r_i. \end{aligned}$$

We can now tie together the various equilibrium distributions with the help of the following straightforward result, which tells us how distributions are modified by powers and multiplicative transformations. These transformations relate profit, revenue, output, price reciprocal ($1/p$), cost reciprocal ($1/c$) in the CES model.

Lemma 2 *(Transformation) Let $F_X(x)$ be the CDF associated to a random variable X. Then the CDF of kX^θ with $k > 0$ is*

$$\begin{aligned} F_{kX^\theta}(x) &= F_X\left[\left(\frac{x}{k}\right)^{\frac{1}{\theta}}\right] \quad \text{for } \theta > 0 \\ F_{kX^\theta}(x) &= 1 - F_X\left[\left(\frac{x}{k}\right)^{\frac{1}{\theta}}\right] \quad \text{for } \theta < 0. \end{aligned}$$

For example, power distributions beget power distributions under positive power transforms and Pareto distributions under negative power transforms. Furthermore, normal distributions beget normal distributions in both cases, due to the symmetry of the normal distribution, etc. We refer to pairs of distributions with the same functional forms but different parameters as being in the same *class* (e.g., Pareto, power, normal distributions are all classes).

Proposition 7 *(CES-circle) For the CES, the distributions of profit, revenue, output, price reciprocal and cost reciprocal are all in the same class.*

Proof. From the analysis above, all these variables for the CES involve positive power transformation and/or multiplication by positive constants. That is, profit and revenue are related by a multiplicative constant $(1-\rho)$. So too are price and cost, and hence so are their reciprocals (the multiplicative constant is then ρ). From (25), we have equilibrium output, q_i, related to the cost reciprocal, $1/c_i$, by a positive power and a positive factor because $q_i = \rho \Xi(I) \frac{\left(\frac{1}{c_i}\right)^{\frac{1}{1-\rho}}}{D_C}$. The other relations follow directly. ■

In particular, if any one of these distributions is Pareto (resp. power), then they all are Pareto (resp. power) family, although they have different parameters. Similarly, if one is normal (resp. log-normal) then all are normal (resp. log-normal). This result we term the *CES-circle*.

We have focussed on distributions of the reciprocals of price and cost to get the CES-circle. Here is where we can invoke the second relation in Lemma 2. If the profit distribution is Pareto, then so too are the revenue and output distributions. But price and cost have power distributions.[29] Conversely, if profit, revenue, and output have power distributions, then price and cost have Pareto distributions.

This means, for example, that the standard CES model with cost heterogeneity alone cannot deliver say Pareto distributions for both profit and prices. Indeed, if profit is Pareto distributed, then price must follow a power distribution. We next introduce quality heterogeneity to break the CES-circle.

7.2 CES quality-enhanced model

We now extend the model to allow for quality differences across products, by writing the sub-utility functional as $\chi = \left(\int_\Omega z(\omega)^\rho d\omega\right)^{1/\rho}$ with $\rho \in (0,1)$ and interpreting $z = vq$ as the quality-adjusted

[29] Indeed, suppose that profits are Pareto distributed: $F_\Pi(\pi) = 1 - \left(\frac{\underline{\pi}}{\pi}\right)^{\alpha_\Pi}$.for $\Pi \in [\underline{\pi}, \infty)$. Write too $\Pi = kC^{\frac{\rho}{\rho-1}}$ (setting $k = \frac{(1-\rho)\Xi(I)}{D_C} > 0$). Then $F_C(c) = \Pr(C < c) = \Pr\left(kC^{\frac{\rho}{\rho-1}} > kc^{\frac{\rho}{\rho-1}}\right) = \Pr\left(\Pi > kc^{\frac{\rho}{\rho-1}}\right) = 1 - F_\Pi\left(kc^{\frac{\rho}{\rho-1}}\right) = \left(\frac{\underline{\pi}}{k}\right)^{\alpha_\Pi} c^{\frac{\alpha_\Pi \rho}{1-\rho}}$, which is indeed a power distribution with $c \in (0, \bar{c}]$ where $\bar{c}$ is the cost level associated to the lowest profit, $\underline{\pi}$. Note that in this model the lowest cost firm has the highest profit, selling an infinite quantity at an infinitesimal price.

consumption. We can readily derive the corresponding demands as:[30]

$$q_i = \frac{\Xi(I)}{p_i} \frac{\hat{p}_i^{\frac{\rho}{\rho-1}}}{\int_{\omega \in \Omega} \hat{p}(\omega)^{\frac{\rho}{\rho-1}} d\omega}, \tag{26}$$

where we have defined $\hat{p}_i = p_i/v_i$ which is interpreted as the price per unit of "quality" and $\Xi(I)$ is as above for the three different cases (the amount spent on the differentiated commodity in aggregate). The key feature of (26) is that p_i enters both with and without quality in the denominator. Note that the standard model (24) ensues when all the v's are the same.

Under monopolistic competition, Firm i's equilibrium price solves $\max_{p_i} \frac{(p_i - c_i)}{p_i} \hat{p}_i^{\frac{\rho}{\rho-1}}$ so it is clear that we still get the well-known pricing solution $p_i = \frac{c_i}{\rho}$, and so $\frac{(p_i - c_i)}{p_i} = (1-\rho)$ at that solution. Hence, using $x_i = v_i/c_i$,[31] which we refer to as quality/cost, the equilibrium profit is[32]

$$\begin{aligned} \pi_i &= (1-\rho)\,\Xi(I) \frac{\hat{p}_i^{\frac{\rho}{\rho-1}}}{\int_{\omega \in \Omega} \hat{p}(\omega)^{\frac{\rho}{\rho-1}} d\omega} && (27) \\ &= (1-\rho)\,\Xi(I) \frac{x_i^{\frac{\rho}{1-\rho}}}{\int_{\omega \in \Omega} x(\omega)^{\frac{\rho}{1-\rho}} d\omega}. && (28) \end{aligned}$$

Here we have still the property that equilibrium profit is a fraction $(1-\rho)$ of revenue (i.e., $\pi_i = (1-\rho)\, r_i$). As per our earlier analysis of the standard CES using Lemma 2, this implies that profit and sales revenue distributions are in the same class (where a class includes transformations by positive factors and powers). Moreover, as long as applying a positive power operator yields a distribution in the same class, then the distribution of quality/costs is in the same class as profit, as can be seen from (28).

A similar classing result holds for prices and costs, but it is no longer necessarily true that reciprocal costs (or reciprocal prices) and profits (or revenues) are linked. This is because introducing quality has brought in a further dimension of heterogeneity.

[30] A quick derivation follows for case (iii) (where $U = \ln \chi + q_0$): the other two are similarly straightforward. The consumer's optimal choice of each consumption level sets marginal utility to price given the presence of the numeraire, so $v_i z_i^{\rho-1} \chi^{-\rho} = p_i$ or $q_i = \frac{1}{p_i} \left(\frac{z_i}{\chi}\right)^{\rho}$. Note that $\int_\Omega p(\omega) q(\omega) d\omega = 1$, so we assume that $I > 1$ to have case (iii); otherwise, we are in case (i). Rearranging the marginal utility condition, $z_i^{\rho-1} = \hat{p}_i \chi^{\rho}$, so $z_i^{\rho} = (\hat{p}_i)^{\frac{\rho}{\rho-1}} \chi^{\frac{\rho^2}{\rho-1}}$. Integrating over all varieties, $\chi^{\rho} = \int_\Omega z^{\rho}(\omega) d\omega = \chi^{\frac{\rho^2}{\rho-1}} \int_\Omega (\hat{p}(\omega))^{\frac{\rho}{\rho-1}} d\omega$. Taking the ratio $\left(\frac{z_i}{\chi}\right)^{\rho}$ then gives (26) (with $\Xi(I) = 1$ as befits case (iii)). Note that $\chi = \left(\int_\Omega \hat{p}^{\frac{\rho}{\rho-1}}(\omega) d\omega\right)^{\frac{\rho-1}{\rho}}$ can be interpreted as the price index for the differentiated variants.

[31] Here the ratio of quality to cost is used whereas the logit used the absolute difference between them.

[32] The qualitative properties of Proposition 1 hold here too, with the key exception that the mark-up for the CES is not additive but is proportional to unit production cost.

Indeed, the bridge between the cost and profit sides and how the distributions are linked is determined by the relation between cost and quality. That is, we can engage the conceptual framework we set forth for the logit model to link the various distributions. In the current context, a functional relation between cost and quality/cost ties down the bridging relation, and the distributions on the "other" side.

A central example of a bridging function is to assume that quality/costs and costs are tied together by the function $x = c^{\beta}$ so that quality/cost is increasing with cost (so quality rises faster than cost) if $\beta > 0$ and it is decreasing if $\beta < 0$. The latter case is embodied in the standard CES model of the preceding sub-section in which $\beta = -1$ and so "better" firms are those with lower costs. The advantage of the constant elasticity bridging function is that it allows us to deploy results (Lemma 2) on applying power transforms to random variables.

Recall now that profits are proportional to $x_i^{\frac{\rho}{1-\rho}}$ (see (28)), and so they are proportional to $c_i^{\beta\frac{\rho}{1-\rho}}$. Hence if $\beta > 0$ profits are in the same distribution class as costs (modulo the stipulation that positive power transformations stay in the class). So then too are revenues and quality-costs. But if $\beta < 0$, profits, revenues and quality-costs are in the "opposite" (or "inverse") class - this is the generalization of the earlier standard CES result. Prices, of course, are in the same class as costs (because prices are proportional to costs), but output is more intricate because it draws its influences from both sides. Indeed, output is proportional to $\frac{x_i^{\frac{\rho}{1-\rho}}}{c_i}$ (see (26)) which equals $c_i^{\beta\frac{\rho}{1-\rho}-1}$ under the constant elasticity formulation. This implies that for $\beta < 1 - \frac{1}{\rho} < 0$ the output distribution is in the inverse class, while otherwise it is in the same class.

For what follows, we define two distributions as in the same class if they have the same functional form. One distribution is the *inverse* of another one if it is the survival function of the other distribution.

A summarizing statement:

Proposition 8 *(breaking the CES circle) Consider the quality-enhanced CES model of monopolistic competition and suppose that quality/cost has constant elasticity β with respect to unit cost. Then:*

i) the equilibrium price distribution is the unit cost distribution up to a positive factor;

ii) equilibrium profits, sales revenue, and quality/cost are in the same distribution class for $\beta < 0$ and in the inverse class for $\beta > 0$;

iii) equilibrium output is in the inverse distribution class for $\beta < 1 - \frac{1}{\rho} < 0$, and in the same distribution class for $\beta > 1 - \frac{1}{\rho}$.

Note that with symmetric distributions such as the Normal, the inverse distribution takes the same form, so then all distributions belong to the same class – once a normal, always a normal.

Take the example of a Pareto distribution for costs. First, prices are also Pareto distributed. Second, profits, revenue, and quality/cost are Pareto distributed for $\beta > 0$ and power distributed for $\beta < 0$ (they are independent of cost if $\beta = 0$). Third, output is power distributed for $\beta < 1 - \frac{1}{\rho} < 0$, Pareto distributed for $\beta > 1 - \frac{1}{\rho}$. If costs are power distributed, Pareto and power are reversed in the above statements. Hence, we resolve the puzzle of getting Pareto distributions for both prices and profits by including the bridge function.

Proposition 8(ii) indicates that quality/cost and profits fall in the same distribution. For example, suppose that the distribution of quality/costs is Pareto: $F_X(x) = 1 - \left(\frac{\underline{x}}{x}\right)^{\lambda}$ and assume that $\lambda \frac{1-\rho}{\rho} > 1$. Then the size distribution of profit is Pareto with tail parameter $\alpha_{\Pi} = \lambda \frac{1-\rho}{\rho}$. The well-known claimed empirical regularity "80-20" rule (that the top 20% of firms account for 80% of sales [profit]) corresponds to a value α_{Π} of 1.161. The result here is that the profit tail parameter is the confluence of a preference parameter and a quality/cost distribution one.[33]

8 Conclusions

The basic ideas here are simple. Market performance depends on the economic fundamentals of tastes and technologies, and how these interact in the market-place.[34] The fundamental distribution of tastes and technologies, embodied in the comparative advantage of firms, feeds through the economic process to generate the endogenous distribution of economic variables, such as prices, outputs, and profits. By invoking the monopolistic competition assumption we get a straight feed-through from fundamental distributions to performance distributions.

The CES model has been the workhorse model of monopolistic competition with asymmetric firms. Yet the model imposes some restrictive properties, such as unit elastic variant demand. The Logit model gives several similar properties, while differing on others. For example, the simple version of the CES has the property that mark-ups are constant percentages; the Logit has constant absolute mark-ups; the latter property is perhaps quite descriptive for cinema movies, DVDs, and CDs. The Logit can be deployed for similar purposes as the CES. The Logit has an established pedigree in terms of its micro-economic underpinnings, and it has a strong econometric backdrop.

Evidently, any simple and tractable formulation must sacrifice much detail for its benefits. Yet

[33] Although why they yield the same constant across settings remains intriguing.

[34] Firm size distributions have recently come to the fore in Chris Anderson's (2006) work on the Long Tail of internet sales.

such a flexible formulation also forms a flexible framework for modification. We do not here attempt further modifications, but we do note that the Logit model is at the heart of much of the structural empirical industrial organization revolution.

Quality (and cost) differences are especially interesting for empirical work, and studying asymmetric firms. Here we link the distribution of the economic fundamentals (qualities and costs) to the economic outcomes (firm size). In particular, a normal quality-cost distribution leads to a log-normal distribution of firm size. While the log-normal has sometimes been proposed as descriptive of various economic phenomena, like income distribution, it is the Pareto distribution that has attracted most attention because it fits well the size distribution of firms. In the CES formulation, matters are straightforward because the assumed distribution of the qualities is also the equilibrium distribution of outputs: Pareto begets Pareto. For the Logit model, we are lead to ask what distribution of primitives begets the Pareto distribution of the outcome variable? Loosely, since the outcome variable is the exponential function of the input variable, then the output distribution is the log distribution of the input distribution. We therefore want to find the distribution (of the input variable) that when the log is taken we find the Pareto: so we want the exponential function of the Pareto. This is just the exponential distribution. Thus an exponential quality-cost distribution generates a Pareto distribution of firm sizes, which is in line with some empirical evidence on the size distribution of firms.

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Appendix 1

Proof of Theorem 1

We first seek the distribution of outputs, $F_Y(y) = \Pr(Y < y)$, that is generated from the primitive distribution of quality-cost. First note from (6) that $Y = \frac{\exp\left(\frac{X}{\mu}\right)}{D}$, so:

$$F_Y(y) = \Pr\left(\frac{\exp\left(\frac{X}{\mu}\right)}{D} < y\right) = F_X(\mu \ln(yD)),$$

where D is given by (7). Because equilibrium profit is proportional to output ($\pi = \mu y$), we have a similar relation for the distribution of profit, $F_\Pi(\pi) = \Pr(\Pi < \pi)$:

$$F_\Pi(\pi) = \Pr\left(\mu \frac{\exp\left(\frac{X}{\mu}\right)}{D} < \pi\right) = F_X\left(\mu \ln\left(\frac{\pi D}{\mu}\right)\right),$$

where D is given by (7).

Proof of Theorem 2

We first determine the distribution of quality-costs consistent with a given observed distribution of output. Suppose that output has a distribution $F_Y(y)$. Applying the increasing transformation $y = \frac{1}{D}\exp\left(\frac{x}{\mu}\right)$, and $Y = \frac{1}{D}\exp\left(\frac{X}{\mu}\right)$, we get:

$$F_X(x) = \Pr\left(\frac{1}{D}\exp\frac{X}{\mu} < \frac{1}{D}\exp\frac{x}{\mu}\right) = \Pr\left(Y < \frac{1}{D}\exp\frac{x}{\mu}\right) = F_Y\left(\frac{1}{D}\exp\frac{x}{\mu}\right).$$

However, D is written in terms of $f_X(x)$, and we want to find the distribution solely in terms of $F_Y(y)$: this means writing D in terms of $f_Y(y)$. The corresponding expression, denoted D_y is given in the Theorem and derived below as (29). Similar reasoning gives the profit expression:

$$F_X(x) = \Pr(X < x) = \Pr\left(\Pi < \frac{\mu}{D}\exp\left(\frac{x}{\mu}\right)\right) = F_\Pi\left(\frac{\mu}{D}\exp\left(\frac{x}{\mu}\right)\right),$$

where the expression for D_π in terms of $f_\Pi(\pi)$ is given in the Theorem and derived below as (30).

We now show here how to write the function D as a function of $f_Y(.)$ or $f_\pi(.)$.

We first find the value of D in terms of the distribution of Y. Recall (7):

$$D = M \int_{u \geq \underline{x}} \exp\left(\frac{u}{\mu}\right) f_X(u)\, du + \mathcal{V}_0.$$

Now, $\Pr(x < X) = \Pr(y < Y)$, where since $y = \frac{1}{D}\exp\left(\frac{x}{\mu}\right)$, so that $F_X(x) = F_Y\left(\frac{1}{D}\exp\left(\frac{x}{\mu}\right)\right)$. Therefore, $f_X(x) = \frac{1}{\mu D}\exp\left(\frac{x}{\mu}\right) f_Y\left(\frac{1}{D}\exp\left(\frac{x}{\mu}\right)\right)$. We now make the change of variable $w =$

$\frac{1}{D}\exp\left(\frac{u}{\mu}\right)$, so that $dw = \frac{1}{\mu D}\exp\left(\frac{u}{\mu}\right)du$, and hence the key term in the output denominator,

$$\begin{aligned}\int_{u\geq\underline{x}}\exp\left(\frac{u}{\mu}\right)f_X(u)\,du &= \int_{u\geq\underline{x}}\frac{1}{\mu D}\exp\left(\frac{u}{\mu}\right)\exp\left(\frac{u}{\mu}\right)f_Y\left(\frac{1}{D}\exp\left(\frac{u}{\mu}\right)\right)du \\ &= D\int_{w\geq\underline{y}} wf_Y(w)\,dw = Dy_{av},\end{aligned}$$

where y_{av} is average firm output. Hence, the expression for D written in terms of $f_Y(.)$ reduces to $D_y = D_y M y_{av} + \mathcal{V}_0$. The solution D_y is given as a function of $f_Y(.)$ as:

$$D_y = \frac{\mathcal{V}_0}{1 - My_{av}}. \tag{29}$$

The denominator in the last expression is necessarily positive because My_{av} is total output, which is less than one when $\mathcal{V}_0 > 0$.

Similarly, $F_X(x) = F_\Pi\left(\frac{\mu}{D}\exp\left(\frac{x}{\mu}\right)\right)$, so that

$$D_\pi = \frac{\mathcal{V}_0}{1 - \frac{M}{\mu}\pi_{av}}, \tag{30}$$

which is now expressed as a function of $f_\Pi(.)$, and where where π_{av} is average firm output. The denominator of the expression for D_π is positive because $M\pi_{av}$ is total profit, which is less than μ because the market is not fully covered (for $\mathcal{V}_0 > 0$).

Proof of Proposition 4

We first calculate the logit denominator, D, from (7), using the CDF for the exponential distribution

$$F_X(x) = 1 - \exp(-\lambda(x - \underline{x})), \quad \lambda > 0, \ \underline{x} > 0, \quad x \in [\underline{x}, \infty), \tag{31}$$

whose density is $f_X(x) = \lambda\exp(-\lambda(x - \underline{x}))$. Assume that $\lambda\mu > 1$, which is required for the integral to be bounded. By integration, we get:

$$D = \frac{M\lambda\mu}{\lambda\mu - 1}\exp\left(\frac{\underline{x}}{\mu}\right) + \mathcal{V}_0, \tag{32}$$

which is positive for any $\mathcal{V}_0$, since $\lambda\mu > 1$. Now, from Theorem 2,

$$F_\Pi(\pi) = 1 - \exp\left(-\lambda\left(\mu\ln\left(\frac{\pi D}{\mu}\right) - \underline{x}\right)\right) = 1 - \left(\frac{\pi D}{\mu}\right)^{-\lambda\mu}\exp(\lambda\underline{x}).$$

The lower bound of the support, $\underline{\pi}$, is the profit of the lowest quality-cost firm and solves $F_\Pi(\underline{\pi}) = 0$, and thus verifies the expected property $\underline{\pi} = \frac{\mu}{D}\exp\left(\frac{\underline{x}}{\mu}\right)$. Inserting this value back into $F_\Pi(\pi)$ gives the expression in the Proposition.

The output distribution follows from the profit distribution:

$$F_Y(y) = \Pr(Y < y) = \Pr\left(\frac{\Pi}{\mu} < y\right) = F_\Pi(\mu y) = 1 - \left(\frac{\underline{\pi}}{\mu y}\right)^{\lambda\mu} = 1 - \left(\frac{\underline{y}}{y}\right)^{\lambda\mu},$$

where in the last step we used the fact that the lowest output, $\underline{y}$, is associated to the lowest profit, $\underline{\pi} = \mu\underline{y}$. The stated result follows.

The last statement follows from Theorem 2: starting with a Pareto distribution for output or profit implies an underlying exponential distribution for quality-cost. The lowest quality-cost is given by the condition $\underline{y} = \frac{\mu}{D}\exp\left(\frac{\underline{x}}{\mu}\right)$, so

$$\underline{\pi} = \mu\underline{y} = \frac{\mu}{\frac{M\lambda\mu}{\lambda\mu-1}\exp\left(\frac{\underline{x}}{\mu}\right) + \mathcal{V}_0}\exp\left(\frac{\underline{x}}{\mu}\right) < \frac{1}{M}\left(\mu - \frac{1}{\lambda}\right). \quad (33)$$

We need the condition $\lambda\mu > 1$ (in order for the logit denominator D to exist). The lowest quality, $\underline{x}$, is given by inverting (33).

Proof of Proposition 2

i) We treat the case of output; profit is analogous. Mean output is:

$$y_{av} = \int_{w \geq \underline{y}} w f_Y(w)\, dw = \frac{M\xi}{M\xi + \mathcal{V}_0},$$

where $\xi = \int_{u \geq \underline{x}} \exp\left(\frac{u}{\mu}\right) f_X(u)\, du = \exp\left(\frac{\bar{x}}{\mu}\right) - \frac{1}{\mu}\int_{\underline{x}}^{\bar{x}} \exp\left(\frac{u}{\mu}\right) F_X(u)\, du$, so that a fosd decrease in $F_X(u)$ (holding $\bar{x}$ constant) raises ξ (strictly if and only if $\mathcal{V}_0 > 0$) and hence raises y_{av}.

Proof. ii) Given that $\exp\left(\frac{u}{\mu}\right)$ is convex and increasing, ξ increases (strictly if and only if $\mathcal{V}_0 > 0$) with a mean-preserving spread in $f_X(.)$. The result follows immediately. ■

Proof of Proposition 3

By Theorem 1, $F_Y(y) = F_X(\mu \ln(yD))$. Because D is increasing in v_0, then $F_Y(y)$ is increasing in v_0, so that output is first-order stochastically dominated when v_0 rises. The argument for profits is analogous because $F_\Pi(\pi) = F_\Pi\left(\mu \ln\left(\frac{\pi D}{\mu}\right)\right)$ by Theorem 1.

The effects on $F_Y(y)$ of an increase in μ are determined by the behavior of $\mu \ln(yD)$ as a function of μ. The derivative is $\ln y + \frac{d(\mu \ln D)}{d\mu}$; the second term is independent of y, and we show in Lemma 1 that it is positive. Note that $y < 1$ (because it is an output and the sum of outputs is below 1), so that $\ln y < 0$. Therefore $F_Y(y)$ goes down with μ for y low enough (or for quality-costs low enough: see Proposition 1), and goes up for y high enough (or for quality-costs high enough); note that either of these might be out of bounds, depending on parameter values. The effects on $F_\Pi(\pi)$ of an increase in μ are determined by when $\mu \ln\left(\frac{\pi D}{\mu}\right)$ is increasing or decreasing in μ. The derivative is $\ln \frac{\pi}{\mu} + \frac{d(\mu \ln D)}{d\mu} - 1$ which equals $\left[\ln y + \frac{d(\mu \ln D)}{d\mu}\right] - 1$ and so for intermediate output (or profit) levels, a higher μ may increase $F_Y(y)$, but decrease $F_\Pi(\pi)$.

Proof of Theorem 3

Suppose we know $F_X(x)$ and the transformation $\beta(c)$ with $\beta'(.) > 0$. Because $F_X(x)$ is continuous and increasing on support $[\underline{x}, \bar{x}]$ and because $\beta^{-1}(x)$ is continuous and increasing on support $[\underline{x}, \bar{x}]$ with $\beta^{-1}(\underline{x}) > 0$. Then $F_C(c) = \Pr(C < c)$ is uniquely defined and continuous and increasing on support $[\underline{c}, \bar{c}]$:

$$F_C(c) = \Pr\left(\beta^{-1}(X) < c\right) = \Pr(X < \beta(c)) = F_X(\beta(c)).$$

The last term is a continuous and increasing function of a continuous and increasing function, so $F_C(c)$ is recovered. The proof for constructing $F_X(x)$ from $F_C(c)$ and $\beta(c)$ is completely analogous.

We now show how to construct a unique increasing $\beta(c)$ from the two distributions: let $F_X(x) = \Pr(X < x)$ and we postulate that there exists a continuous increasing function $\beta(C) = X$ and so $F_X(x) = \Pr(\beta(C) < x) = \Pr\left(C < \beta^{-1}(x)\right)$ which is then equal to $F_C\left(\beta^{-1}(x)\right)$. Now, since $F_X(x) = F_C\left(\beta^{-1}(x)\right)$, then $\beta^{-1}(x) = F_C^{-1}(F_X(x))$ so $\beta(x) = \left[F_C^{-1}(F_X(x))\right]^{-1}$ and $\beta(x) = F_X^{-1}(F_C(x))$. This is clearly increasing and continuous in x as desired.

Because $F_X(x)$ both can be used to construct and can be constructed from the other distributions on its leg (and likewise for $F_C(c)$), then the claim in the Theorem is shown.

Proof of Theorem 4

The solution to (10) is uniquely determined (and strictly positive) when the RHS of (10) has slope less than one, as is implied by $h(.)$ being strictly (-1)-concave.[35] Denote the solution to (10) by $\mu(x)$. Applying the implicit function theorem to (10) shows that

$$\mu'(x) = \frac{\left(\frac{h(x-m)}{h'(x-m)}\right)'}{1 + \left(\frac{h(x-m)}{h'(x-m)}\right)'}, \tag{34}$$

where the numerator is strictly positive when h is (-1)-concave.[36] The mark-up thus has slope less than one. Let $h^*(x) = h(x - \mu(x))$ denote the value of $h(.)$ under the profit-maximizing choice of mark-up. Then

$$dh^*(x)/dx = (1 - \mu'(x))\, h'(x - \mu(x)) > 0, \tag{35}$$

and so $h^*(x)$ is strictly increasing, as claimed.

[35] Modulo a problem that the solution could be infinite m if h/h' asymptotes the 45 - degree line ...

[36] We can have quality rise and mark-up go down immensely near the -1-concave limit: think too of cost pass-through; with a demand 1/p then a zero cost gives a price of zero, but a small cost gives an infinite price.

Proof of Lemma 1

We consider the discrete version of the Logit model, as follows:

$$y_j = \frac{\exp\left(\frac{r_j}{\mu}\right)}{\sum\limits_{l=0...n} \exp\left(\frac{r_l}{\mu}\right)} < 1, \qquad j = 1..n$$

where r_l is an arbitrary scale value independent of μ. Taking the logarithm on both sides, we get:

$$\ln y_j = \frac{r_j}{\mu} - \ln\left(\sum_{l=0...n} \exp\left(\frac{r_l}{\mu}\right)\right).$$

Summing over $j = 0, 1...n$, and using for weights the choice probabilities, y_j (which sum to one), we get:

$$V = \mu \ln\left(\sum_{j=0...n} \exp\left(\frac{r_j}{\mu}\right)\right) = \sum_{j=0...n} r_j y_j - \mu \sum_{j=0...n} y_i \ln y_i \tag{36}$$

According to this expression, the indirect utility function V is equal to the average deterministic utility plus the Shannon measure of information (which is positive, since $y_i < 1$). Thus, the Shannon measure of information provides a measure of the aggregate benefit from variety. Note that the same expression can be obtained by comparing the direct and the indirect utility functions for the representative consumer associated to the Logit model (see its formulation in Chapter 2 of Anderson et al. 1992).

The derivative of V is :

$$\begin{aligned} \frac{dV}{d\mu} &= \ln\left(\sum_{j=0...n} \exp\left(\frac{r_j}{\mu}\right)\right) - \frac{1}{\mu} \sum_{j=0...n} \frac{\exp\left(\frac{r_j}{\mu}\right)}{\sum\limits_{j=0...n} \exp\left(\frac{r_j}{\mu}\right)} r_i \\ &= \ln\left(\sum_{j=0...n} \exp\left(\frac{r_j}{\mu}\right)\right) - \frac{1}{\mu} \sum_{i=0...n} r_i y_i. \end{aligned}$$

Now, using (36) above, the derivative of the indirect utility function is equal (up to a multiplicative factor) to the Shannon measure of information (or entropy, which is positive):

$$\frac{dV}{d\mu} = -\sum_{i=0...n} y_i \ln y_i > 0.$$

Appendix 2: Distribution details [NOT FOR PUBLICATION]

We now prove the Corollaries in Section 3.1: these involve parameter matching for the distribution examples.

Normal: For the normal, $F_X(x) = \frac{1}{\sigma\sqrt{2\tilde{\pi}}}\int_{-\infty}^{x}\exp\left(-\frac{(u-m)^2}{2\sigma^2}\right)du$, where $\tilde{\pi} = 3.1415...$. From Theorem 2, we have

$$F_\Pi(\pi) = F_X\left(\mu\ln\left(\frac{\pi D}{\mu}\right)\right),$$

where $\mu\ln\left(\frac{\pi D}{\mu}\right) \in (-\infty, \infty)$, so

$$F_\Pi(\pi) = F_X\left(\mu\ln\left(\frac{\pi D}{\mu}\right)\right) = \frac{1}{\sigma\sqrt{2\tilde{\pi}}}\int_{-\infty}^{\mu\ln\left(\frac{\pi D}{\mu}\right)}\exp\left(-\frac{(u-m)^2}{2\sigma^2}\right)du.$$

Using the change of variable $\Pi = \frac{\mu}{D}\exp\left(\frac{u}{\mu}\right)$ (so $u = \mu\ln\left(\frac{\Pi D}{\mu}\right)$ and $du = \frac{\mu}{\Pi}d\Pi$) we obtain

$$F_\Pi(\pi) = \frac{\mu}{\sigma\sqrt{2\tilde{\pi}}}\int_0^{\pi}\exp\left(-\frac{\left(\mu\ln\left(\frac{\Pi D}{\mu}\right)-m\right)^2}{2\sigma^2}\right)\frac{d\Pi}{\Pi},$$

which can be written in a standard form as:

$$F_\Pi(\pi) = \frac{1}{\left(\frac{\sigma}{\mu}\right)\sqrt{2\tilde{\pi}}}\int_0^{\pi}\frac{1}{\Pi}\exp\left(-\frac{\left(\ln\Pi-\left(\ln\left(\frac{D}{\mu}\right)-\frac{m}{\mu}\right)\right)^2}{2\left(\frac{\sigma}{\mu}\right)^2}\right)d\Pi.$$

Hence profits are log-normally distributed with parameters $\left[\ln\left(\frac{D}{\mu}\right)-\frac{m}{\mu}\right]$ and $\left(\frac{\sigma}{\mu}\right)$.

Recall: $D = M\int_{u\geq\underline{x}}\exp\left(\frac{u}{\mu}\right)f_X(u)\,du + \mathcal{V}_0$. Then:

$$D = \frac{M}{\sigma\sqrt{2\tilde{\pi}}}\int_{-\infty}^{\infty}\exp\left(\frac{x}{\mu}\right)\exp\left(\frac{-(x-m)^2}{2\sigma^2}\right)dx + \mathcal{V}_0.$$

Routine computation shows that:

$$D = M\exp\left(\frac{m}{\mu}+\frac{\sigma^2}{2\mu^2}\right)+\mathcal{V}_0.$$

Logistic.

A *logistic* distribution for quality-cost has a CDF given by $F_X(x) = \left(1+\exp\left(-\frac{x-m}{s}\right)\right)^{-1}$, $x \in (\underline{x}, \infty)$, with mean m and variance $s^2\pi^2/3$. The PDF is similar in shape to the normal, but it has thicker tails (see the discussion in Fisk, 1961, and the comparison with the Weibull distribution). Indeed, as suggested by Corollary 1:

Corollary 4 *Let quality-cost $X \in (-\infty, \infty)$ be logistically distributed with parameters m and s. For $\mu > s$, profit $\Pi \in (0, \infty)$ is log-logistically distributed with parameters $\left(\frac{D}{\mu}\right) \exp\left(-\frac{m}{\mu}\right)$ and $\frac{\mu}{s}$:*

$$F_{\Pi}(\pi) = \left(1 + \left(\frac{\pi}{\left(\frac{D}{\mu}\right) \exp\left(-\frac{m}{\mu}\right)}\right)^{-\frac{\mu}{s}}\right)^{-1}, \qquad \pi \in [0, \infty).$$

There is no closed form expression for D in this case. However, it can be shown that the condition $\mu > s$ guarantees that the output denominator D exists. The Log-logistic distribution (which provides a one parameter model for survival analysis) is very similar in shape to the log-normal distribution, but it has fatter tails. It has an explicit functional form, in contrast to the Log-normal distribution.

The logistic distribution (with mean m and standard deviation $s\tilde{\pi}/\sqrt{3}$) is given by:

$$F_X(x) = \frac{1}{1 + \exp\left(-\frac{x-m}{s}\right)}, x \in (\underline{x}, \infty).$$

From Theorem 2,

$$F_{\Pi}(\pi) = F_X\left(\mu \ln\left(\frac{\pi D}{\mu}\right)\right) = \frac{1}{1 + \exp\left(-\frac{\mu \ln\left(\frac{\pi D}{\mu}\right) - m}{s}\right)},$$

where $F_{\Pi}(0) = 0$ and $F_{\Pi}(\infty) = 1$. Thus

$$F_{\Pi}(\pi) = \frac{1}{1 + \exp\left(\frac{m}{s}\right) \exp\left(\ln\left(\frac{\pi D}{\mu}\right)^{-\frac{\mu}{s}}\right)} = \frac{1}{1 + \left(\frac{\pi}{\left(\frac{D}{\mu}\right) \exp\left(-\frac{m}{\mu}\right)}\right)^{-\frac{\mu}{s}}}.$$

Recall the log-logistic distribution is defined as:

$$F^{LL}(x; \alpha, \beta) = \frac{1}{1 + \left(\frac{x}{\alpha}\right)^{-\beta}}, \quad x > 0.$$

Thus, the parameter matching is:

$$F_{\Pi}(\pi) = F^{LL}\left(x; \left(\frac{D}{\mu}\right) \exp\left(-\frac{m}{\mu}\right), \frac{\mu}{s}\right).$$

We need to check when D converges, i.e., when $\int_{-\infty}^{\infty} \exp\left(\frac{x}{\mu}\right) f_X(x)\, dx$ converges. Because $f_X(x) = \frac{1}{s} \frac{\exp\left(-\frac{x-m}{s}\right)}{\left(1+\exp\left(-\frac{x-m}{s}\right)\right)^2}$, we need to ensure the convergence of the expression

$$\int_{-\infty}^{\infty} \frac{\exp\left(-x\left(\frac{1}{s} - \frac{1}{\mu}\right)\right)}{\left(1 + \exp\left(-\frac{x-m}{s}\right)\right)^2} dx.$$

Convergence is guaranteed if and only if $\mu > s$.

Pareto: The Pareto distribution is given by:

$$F_X(x) = \frac{1 - \left(\frac{\underline{x}}{x}\right)^{\alpha}}{1 - \left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}}.$$

From Theorem 2

$$F_{\Pi}(\pi) = \frac{1 - \left(\frac{\underline{x}}{\mu \ln\left(\frac{\pi D}{\mu}\right)}\right)^{\alpha}}{1 - \left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}}.$$

Recall that $\pi = \frac{\mu \exp\left(\frac{x}{\mu}\right)}{D}$ or $x = \mu \ln\left(\frac{\pi D}{\mu}\right)$, so that $\underline{x} = \mu \ln\left(\frac{\underline{\pi} D}{\mu}\right)$ and $\overline{x} = \mu \ln\left(\frac{\overline{\pi} D}{\mu}\right)$, so that $F_{\Pi}(\pi) = 0$ and $F_{\Pi}(\overline{\pi}) = 1$. Note that D is bounded because the distribution of quality-cost is bounded.

Consider a log-Pareto distribution with scale parameter σ and shape parameters γ and β:

$$F^{LP}(\pi; \gamma, \beta, \sigma) = \frac{1 - \left(1 + \frac{1}{\beta} \ln\left(1 + \frac{\pi - \underline{\pi}}{\sigma}\right)\right)^{-\frac{1}{\gamma}}}{1 - \left(1 + \frac{1}{\beta} \ln\left(1 + \frac{\overline{\pi} - \underline{\pi}}{\sigma}\right)\right)^{-\frac{1}{\gamma}}}, \qquad \pi > \underline{\pi}.$$

In order to match parameters, of $F_{\Pi}(\pi)$ with $F^{LP}(\pi; \gamma, \beta, \sigma)$ observe that

$$\mu \ln\left(\frac{\pi D}{\mu}\right) = \mu \ln\left(\frac{\pi D}{\mu} \frac{\underline{\pi}}{\underline{\pi}}\right) = \underline{x} + \mu \ln\left(\frac{\pi}{\underline{\pi}}\right) = \underline{x} + \mu \ln\left(1 + \frac{\pi - \underline{\pi}}{\underline{\pi}}\right).$$

Therefore:

$$\begin{aligned} F_{\Pi}(\pi) &= \frac{1 - \underline{x}^{\alpha}\left(\mu \ln\left(\frac{\pi D}{\mu}\right)\right)^{-\alpha}}{1 - \left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}} \\ &= \frac{1 - \left(1 + \frac{\mu}{\underline{x}} \ln\left(1 + \frac{\pi - \underline{\pi}}{\underline{\pi}}\right)\right)^{-\alpha}}{1 - \left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}}. \end{aligned}$$

Thus π obeys a Log-Pareto distribution $F^{LP}\left(\pi; \frac{1}{\alpha}, \frac{\underline{x}}{\mu}, \underline{\pi}\right)$, i.e., $\gamma = \frac{1}{\alpha}, \beta = \frac{\underline{x}}{\mu}, \sigma = \underline{\pi}$.

It remains to check that the normalization factors are equal. Recall that $\overline{\pi}/\underline{\pi} = \exp \frac{\overline{x} - \underline{x}}{\mu}$. Using the specification $\gamma = \frac{1}{\alpha}, \beta = \frac{\underline{x}}{\mu}, \sigma = \underline{\pi}$, we get:

$$\begin{aligned} \left(1 + \frac{\mu}{x} \ln\left(1 + \frac{\overline{\pi} - \underline{\pi}}{\underline{\pi}}\right)\right)^{-\alpha} &= \left(1 + \frac{\mu}{x} \ln\left(\frac{\overline{\pi}}{\underline{\pi}}\right)\right)^{-\alpha} \\ &= \left(1 + \left(\frac{\overline{x} - \underline{x}}{\overline{x}}\right)\right)^{-\alpha} = \left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}. \end{aligned}$$

Figure 1

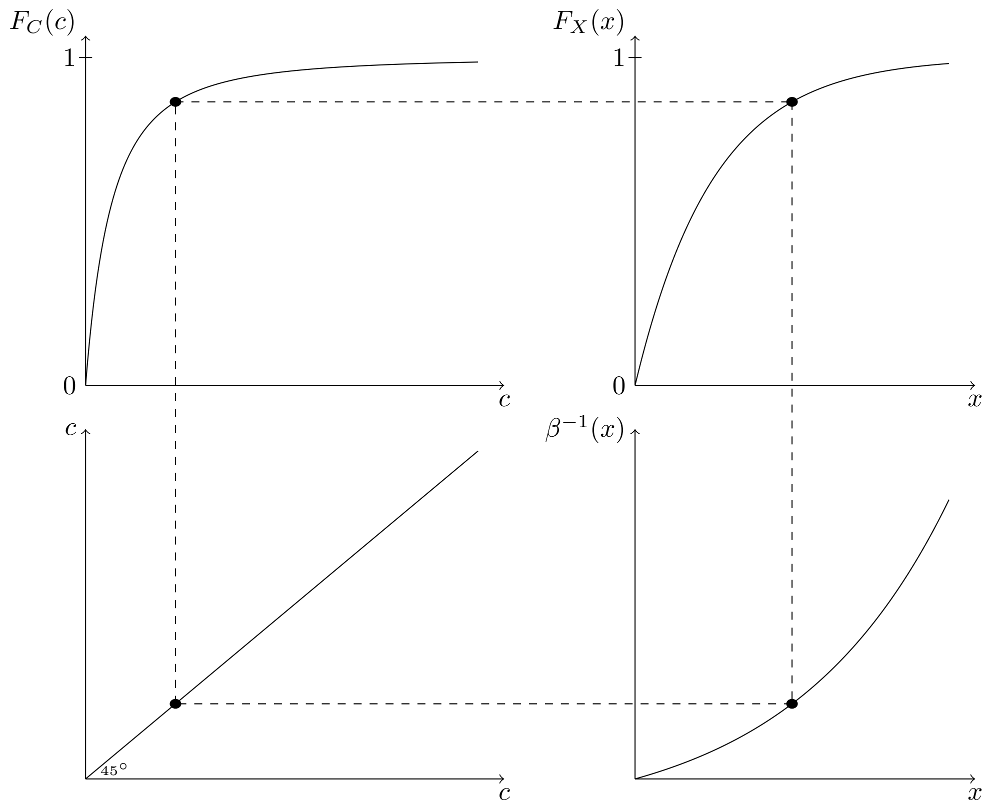

Figure 2